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The Surviving Rate of IC-Planar Graphs

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Abstract: Let k and n be two positive integers. Firefighting is a discrete dynamical process of preventing the spread of fire. Let G be a connected graph G with n vertices. Assuming a fire starts at one of the vertices of G, the firefighters choose k unburned vertices at each step, and then the fire spreads to all unprotected neighbors of the burning vertices. The process continues until the fire stops spreading. The goal is to protect as many vertices as possible. When a fire breaks out randomly at a vertex of G, its k-surviving rate, $\rho_k(G)$, is the expected number of saved vertices. A graph is IC-planar if it has a drawing in which each edge cross once and their endpoints are disjoint. In this paper, we prove that $\rho_4(G) > \frac{1}{124}$ for every IC-planar graph G. This is proven by the discharging method and the locally symmetric of the graph.

Keywords: firefighting; surviving rate; IC-planar graph

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1. Introduction

Firefighting is a discrete-time process that occurs on a connected graph that simulates the spread of fire, virus, or rumors across a network. Such a problem is named after Hartnell [1], who presented it at a combinatorial mathematics and computing conference in 1995. Specifically, given a connected graph G and a positive integer k, initially, assume that the fire breaks out at one of the vertices of G. At each stage, firefighters select a subset of at most k vertices that are not yet on fire for defense. At the same time, the fire spreads to all undefended neighbours. This process ends until the fire stops spreading. Furthermore, unburned vertices can be regarded as saved. The goal is to maximize the number of saved vertices.

The complexity of firefighting is NP-complete for bipartite graphs [2], cubic graphs [3], and even trees of maximum degree three [4]. There is a considerable amount of literature on the algorithms of firefighting and complexity. Further information on firefighting can be found in the latest survey article by Wang and Kong [5]. Moreover, firefighting modeling has many applications in computer science, biology, the spread of epidemics, etc. (see [6–8]).

To demonstrate the defensive capabilities of a graph or a network as a whole, Cai and Wang [9] introduced surviving rate of graphs in 2009. Let $\mathrm{sn}_k(v)$ be the maximum number of vertices that the firefighter can save when a fire breaks out at vertex v. The *k-surviving rate*, denoted by $\rho_k(G)$, is the expectation of the vertices that can be saved when a fire breaks out randomly on one vertex of G with n vertices. That is:

$$\rho_k(G) = \frac{\sum_{v \in V(G)} \mathrm{sn}_k(v)}{n^2},$$

In the probabilistic sense, for any $\varepsilon > 0$, almost all graphs have $\rho_k(G) < \varepsilon$, due to Wang et al. [10]. Let c be a positive constant less than one. It is natural to determine whether a given graph G has $\rho_k(G) \geq c$. For convenience, such graph G is called *k-good*.

Let l be a non-negative integer. A graph which can be embedded in the Euclidean plane with at most l crossings per edge is called an *l-planar* graph, and let $\mathcal{P}_l$ be the family of l-planar graphs. Obviously, $\mathcal{P}_0$ is the family of planar graphs. In particular, a 1-planar graph is *IC-planar* if it admits a 1-planar drawing such that no two crossed edges share an end-vertex. A 1-planar graph is *NIC-planar* if it admits a 1-planar drawing such that any two distinct pairs of crossing edges have at most one end-vertex in common. We use $\mathcal{P}_{IC}$ and $\mathcal{P}_{NIC}$ to denote the classes of IC-planar graphs and NIC-planar graphs, respectively. From the definition, we have:

$$\mathcal{P}_0 \subseteq \mathcal{P}_{IC} \subseteq \mathcal{P}_{NIC} \subseteq \mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \cdots.$$

An example of a planar graph, an IC-planar graph, a NIC-planar graphs, and a 1-planar graph is shown in Figure 1. Observe that all of them are symmetric with respect to their vertices.

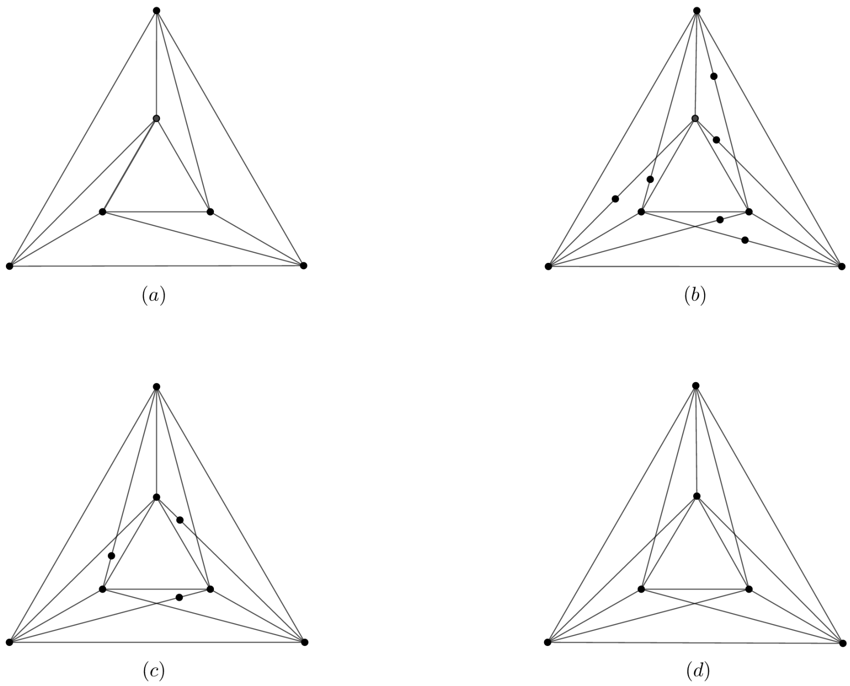

Figure 1. (**a**) A planar graph. (**b**) An IC-planar graph. (**c**) A NIC-planar graph. (**d**) A 1-planar graph.

Note that the smaller the number of edges, the higher the probability of k-good. Most investigations of surviving rate concerning sparse graphs. Table 1 shows all known such results on the graph classes of k-good, up to our own knowledge.

Table 1. The latest results of k-good families.

Graph Family	**Restriction**	***k*-Good [Reference]**
Tree		$k = 1$ [9]
Halin graph [1]		$k = 1$ [9]
Outerplanar graph [2]		$k = 1$ [9]
Sparse graph	$m \leq \frac{15}{11}n$ $m \leq \frac{1}{2}(l + 2 - \frac{1}{l+2})n (l \geq 2)$	$k = 1$ [11] $k = l$ [12]

Table 1. *Cont.*

Graph Family	Restriction	k-Good [Reference]
$\mathcal{P}_0$	$m \leq \frac{9}{4}n$ no 6-cycle no chordal 5-cycle [3]	$k = 2$ [13] $k = 2$ [14] $k = 2$ [15]
$\mathcal{P}_0$	girth [4] ≥ 7 girth ≥ 5	$k = 1$ [16] $k = 1$ (Conjecture [17])
$\mathcal{P}_0$		$k = 5$ [10] $k = 4$ [17,18] $k = 3$ [19,20] $k = 2$ (Conjecture [17])
$\mathcal{P}_{IC}$		$k = 4$ (this paper)
$\mathcal{P}_{NIC}$		$k = 5$ [21]
$\mathcal{P}_1$		$k = 6$ [22]

[1] A Halin graph is a graph $H = T \cup C$, where T is a plane tree on at least four vertices in which no vertex has a degree of two, and C is a cycle connecting the leaves of T in the cyclic order determined by the embedding of T. [2] A graph is outerplanar if it has a planar embedding in which all vertices lie on the boundary of its outer face. [3] A cycle having a chord is called a chordal cycle. [4] The girth of a graph is defined as the length of a shortest cycle.

In this paper, we will show the following result, which implies that IC-planar graphs are 4-good.

Theorem 1. *Let G be an IC-planar graph. Then, $\rho_4(G) > \frac{1}{124}$.*

2. Notation

An *IC-plane graph*, which is a drawing of a IC-planar graph in the Euclidean plane, such that each edge intersects another edge with as few crossings as possible. For an IC-plane graph G, for each pair ab, cd edges that cross each other at a crossing point z, their end vertices are pairwise distinct. Let $X(G)$ be the set of all crossing points and $E_1(G)$ be the non-crossed edges in G. An *associated plane graph* $G^X = \left(V\left(G^X\right), E\left(G^X\right)\right)$of G is the plane graph with:

$$V(G^X) = V(G) \cup X(G); E(G^X) = E_1(G) \cup E_2(G)$$

where $E_2(G) = \{az, bz \mid ab$ is a crossing edge and z is the crossing point on $ab\}$. Thus, each of the crossing point becomes an actual vertex of degree four in G^X. For convenience, we still refer to the crossing vertices in G^X as new vertices, and the edges in $E(G)$ that contain a crossing vertex are called crossing edges. Denote $F\left(G^X\right)$ the set of faces of G^X. Let $V_k(G), V_k^-(G)$ and $V_k^+(G)$ denote the sets of vertices in G with degree k, at most k and at least k, respectively. We say a vertex v is k-vertex (k^--vertex or k^+-vertex) if $v \in V_k\left(v \in V_k^-(G)\right.$ or $\left.v \in V_k^+(G)\right)$. The set of vertices which is adjacent to v in graph G is denoted by $N_G(v)$, and set $N[v] = N(v) \cup \{v\}$.

A *kite* is a graph isomorphic to K_4 with an embedding such that all the vertices are on the boundary of the outer face, the four edges on the boundary are planar, and the remaining two edges cross each other (see Figure 2a). We say that a kite (a, b, c, d) with crossing edges ad and bc is *empty* if it contains no other vertices; that is, the edges ac, ad, and ab are consecutive in the counterclockwise order around a (see Figure 2b,c).

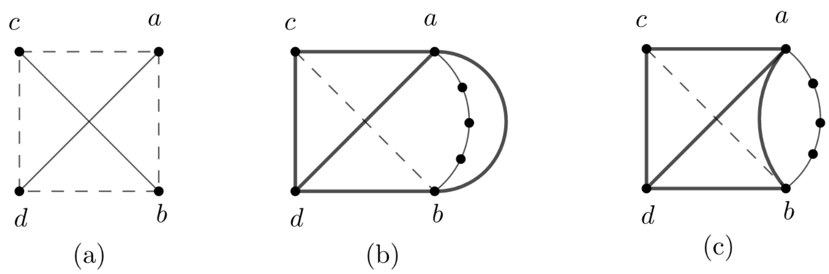

Figure 2. (**a**) A kite. (**b**) The kite (bold lines) is not empty. (**c**) Rerouting edge ab to make the kite empty.

A vertex v is called *good* if $\mathrm{sn}_4(v) \geq n-4$. Otherwise, v is called *bad*. Let V^g and V^b be the set of good vertices and bad vertices, respectively. Moreover, let V_k^b be the set of bad k-vertices, For a vertex $v \in V(G^X)$, we use $n_k^b(v)$ and $n_X(v)$ to denote the number of bad k-vertices and crossing vertices that are adjacent to v in G^X, respectively. Let

$V_X^1 = \{v \in X(G) | n_5^b(v) = 1\}$,
$V_X^i = \{v \in X(G) | n_6^b(v) = 4-i\}$ for $2 \leq i \leq 4$,
$V_7^1 = \{v \in V_7^b | \, vs.$ is adjacent to a vertex in $V_X^2\}$,
$V_7^2 = V_7^b \backslash V_7^1$.

Let G be an IC-plane graph. G is *maximal* if no edge can be added without violating IC-planarity. An IC-planar graph G is maximal if every IC-planar embedding is maximal. If we restrict to IC-plane graphs, we say that an IC-plane graph G is *planar-maximal* if no edge can be added without creating at least an edge crossing on the newly added edge or making the graph not simple.

3. Structural Properties

In this section, we identify some structural properties that lead to good vertices in the plane-maximal IC-plane graph.

Lemma 1 ([23]). *Let $G = (V, E)$ be an IC-plane graph. There exists a plane-maximal IC-plane graph $G^+ = (V, E^+)$ with $E \subseteq E^+$ such that the following conditions hold:*

(1) The four endpoints of each pair of crossing edges induce a kite.
(2) Each kite is empty.
(3) Let C be the set of crossing edges in G^+. Let $C^* \subset C$ be a subset containing exactly one edge for each pair of crossing edges. Then, $G^+ \backslash C^*$ is plane and triangulated.
(4) The outer face of G^+ is a 3-cycle of non-crossed edges.

Corollary 1. *Let G^+ be a plane-maximal IC-plane graph and $v \in V(G^+)$. Then, the neighbors of v in G^+ are adjacent successively.*

Proof. The Lemma 1 states that the edge set of every plane-maximal IC-plane graph can be decomposed into a triangulation and a matching M. Let $v \in V(G^+)$ be a k-vertex with $N_{G^+}(v) = v_0, v_1, \ldots, v_{k-1}$ in clockwise order around v. If v is matched under M, without loss of generality, suppose that $vv_1 \in M$. Then, $v_0v_2, v_iv_{i+1} \in E(G^+)$ by Lemma 1(3), where $2 \leq i \leq k-1$. Since vv_1 and v_0v_2 are crossing edges, $v_0v_1, v_1v_2 \in E(G^+)$ by Lemma 1(1). Otherwise, according to Lemma 1(3), we have $v_iv_{i+1} \in E(G^+)$, where $0 \leq i \leq k-1$. □

Fact 1 ([9]). *Let H be a subgraph of G with $V(H) = V(G)$. Then, $\rho_k(H) \geq \rho_k(G)$.*

Given the fact above, plane-maximal IC-plane graphs have the lowest surviving rates among all IC-plane graphs. For convenience, we always assume that G is a plane-maximal IC-plane graph, and G^X is an associated plane graph of G in the following argument. Furthermore, assume the neighbours of k-vertex v in G in clockwise order around v are $v_0, v_1, \ldots, v_{k-1}$.

Lemma 2. *Every* 4^-*-vertex is good.*

Lemma 3. *Let* v *be a bad 5-vertex. Then,* v *is adjacent to five* 8^+*-vertices in* G*. Moreover, if* v *is incident to a crossing edge, then* v *is adjacent to at least two* 9^+*-vertices in* G*.*

Proof. By contradiction, assume v is adjacent to a 7^--vertex, say v_0. When a fire breaks out at v, we protect $\{v_1, v_2, v_3, v_4\}$ first. By Corollary 1, we have $v_0v_1, v_0v_4 \in E(G)$. Noting that both v_1 and v_4 have been protected in the first turn, we only need to protect all vertices in $N_G(v_0)\backslash\{v, v_1, v_4\}$ in the second turn. Thus, all the vertices in $V\backslash\{v, v_0\}$ have been saved under our strategy. This implies that $sn_4(v) \geq n-2$, that is, v is a good 5-vertex, is a contradiction.

Suppose v is incident to a crossing edge. Without loss of generality, suppose that vv_0 is a crossing edge. By Corollary 1, we obtain that $v_0v_1, v_1v_4, v_0v_4, v_1v_2, v_3v_4 \in E(G)$. If v_1 is a 8^--vertex, then we first protect $\{v_0, v_2, v_3, v_4\}$; next, we protect all vertices in $N_G(v_1)\backslash\{v, v_0, v_2, v_4\}$ when a fire breaks out at v. It follows that $sn_4(v) \geq n-2$, that is, v is a bad 5-vertex, which is impossible. Thus, v_1 is a 9^+-vertex. An analogous argument shows that v_4 is a 9^+-vertex. □

Lemma 4. *Let* $[vuw]$ *be a 3-cycle with* v, u, w *are bad vertices, and let* $x \in (N_G(u) \cup N_G(w)) \setminus N_G[v]$*.*

(1) If v *is a bad 6-vertex, then* $|(N_G(u) \cup N_G(w)) \setminus N_G[v]| \geq 5$*.*
(2) Moreover, if $|(N_G(u) \cup N_G(w)) \setminus N_G[v]| = 5$*, then* $d(x) \geq 8$*.*

Proof. Suppose, by way of contradiction, that $|(N_G(u) \cup N_G(w)) \setminus N_G[v]| \leq 4$. Assume that a fire breaks out at v. We first protect $N_G(v) \setminus \{u, w\}$, then $(N_G(u) \cup N_G(w)) \setminus N_G[v]$. It is easy to inspect that all vertices in $V \setminus \{v, u, w\}$ have been saved under the above strategy. Hence, $sn_4(v) \geq n-3$, that is, v is a good 6-vertex, is a contradiction.

Assume that $|(N_G(u) \cup N_G(w)) \setminus N_G[v]| = 5$. Suppose x is a 7^--vertex. When a fire breaks out at v, we first protect $N_G(v) \setminus \{u, w\}$, then $(N_G(u) \cup N_G(w)) \setminus (N_G[v] \cup \{x\})$, and finally the neighbors of x in G which are not protected. According to Corollary 1, it is not difficult to see that all the vertices in $V \setminus \{v, u, w, x\}$ have been saved under our strategy. Thus, $sn_4(v) \geq n-4$, which contradicts that v is a bad 6-vertex. □

By calculating the cardinality of $N_G(u) \cup N_G(w)) \setminus N_G[v]$, deduce the following corollary from Lemma 4.

Corollary 2. *Let* $[vuw]$ *be a 3-cycle with* $v, w \in V_6^b$ *and* $u \in V_7^1$*. Then, the vertices in* $(N_G(u) \cup N_G(w)) \setminus N_G[v]$ *and* $(N_G(u) \cup N_G(v)) \setminus N_G[w]$ *are* 8^+*-vertices.*

Corollary 3. *There are at most two bad 6-vertices in the four endpoints of each pair of crossing edges.*

Proof. Let ad and bc be crossing edges. By Lemma 1, derive that $ac, cd, db, ab \in E(G)$. We establish the corollary by contradiction. Suppose that there are at least three bad 6-vertices in $\{a, b, c, d\}$, say a, b and c. Hence, there is a 3-cycle $[abc]$, where a, b, c are bad 6-vertices. Since $|(N_G(a) \cup N_G(b)) \setminus N_G[c]| \leq 4$, according to Lemma 4, c is a good 6-vertex, there is a contradiction. □

Lemma 5. *Every 8-vertex is adjacent to at most one bad 5-vertex in* G*.*

Proof. By contradiction. Let u be a 8-vertex which is adjacent to at least two bad 5-vertices in G, say v and w. Because bad 5-vertices are not adjacent by Lemma 3, $uw \notin E(G)$. Suppose a fire breaks out at v, we first protect $N_G(v) \setminus \{u\}$, then $N_G(u) \setminus (N_G[v] \cup \{w\})$, and finally $N_G(w) \setminus \{u\}$. Since $|N_G(u) \setminus (N_G[v] \cup \{w\})| \leq 4$ by Corollary 1, all the vertices in $V \setminus \{v, u, w\}$ have been saved under our strategy. Thus, $sn_4(v) \geq n-3$, which contradicts v, is a bad 5-vertex. □

Lemma 6. *Every vertex in* $V_5^b \cup V_6^b \cup V_7^1$ *is adjacent to a vertex in* $V^g \cup V_7^2 \cup V_8^+$*.*

Proof. If $v \in V_5^b$, then v is adjacent to a 8^+-vertex by Lemma 3. Assume $v \in V_6^b$. If v is adjacent to a 5^--vertex u, then u is a good vertex by Lemma 3. Suppose the neighbors of v are 6^+-vertices. If there are two adjacent bad 6-vertices in $N_G(v)$. Without loss of generality, say v_0 and v_1 are bad 6-vertices, it can readily be checked that $|(N_G(v) \cup N_G(v_1)) \setminus N_G[v_0]| = 5$. It follows that $d(v_i) \geq 8$, where $2 \leq i \leq 4$, by Lemma 4(2). Thus, we may assume v is adjacent to a 7-vertex, say v_0. If $v_0 \in V_7^1$, then there is a 3-cycle $[v_0uw]$ with $u, w \in V_6^b$ by the definition of V_7^1, since $v \in (N_G(u) \cup N_G(v_0)) \setminus N_G[w]$, by Corollary 2, $d(v) \geq 8$, is a contradiction. Hence, $v_0 \in V_7^2$.

Assume $v \in V_7^1$. Then, v is adjacent to a vertex x in V_X^2 with $n_6^b(x) = 2$, by the definition of V_7^1. Without loss of generality, we may suppose that vv_0 crosses v_1v_6 in G at the crossing x. Since $n_6^b(x) = 2$, there is a 3-cycle $[vuw]$, where $\{u, w\} \subseteq \{v_0, v_1, v_6\} \cap V_6^b$. It is not hard to check that $[vuw]$ satisfies the condition of Lemma 4(2). Hence, $d(v_i) \geq 8$, where $3 \leq i \leq 5$. □

4. The Surviving Rate

In this section, we use the Discharging Method in two stages to prove the main result of Theorem 1.

Let G be a plane-maximal IC-plane graph, and G^X be an associated plane graph of G. Assign a charge of $d_{G^X}(v) - 4$ to each vertex $v \in V(G^X)$, and a charge of $d_{G^X}(f) - 4$ to each face $f \in F(G^X)$. Using Euler's Formula, it can be verified that the total charge assigned to vertices and faces is -8.

4.1. First Discharge

In the first stage, we define the following discharging rules (R1) to (R3).

(R1) Every vertex in G^X sends charge $\frac{1}{3}$ to incident 3-face.

(R2) Suppose v is a crossing vertex and u is a neighbor of v in G^X.

- (R2.1) If u is good, then u sends charge $\frac{4}{3}$ to v;
- (R2.2) If $d(u) \geq 8$ and $v \in V_X^1$, then u sends charge $\frac{4}{9}$ to v;
- (R2.3) If $d(u) \geq 7$ and $v \in V_X^2$, then u sends charge $\frac{2}{3}$ to v;
- (R2.4) If $d(u) \geq 7$ and $v \in V_X^3$, then u sends charge $\frac{4}{9}$ to v;
- (R2.5) If $d(u) \geq 7$ and $v \in V_X^4$, then u sends charge $\frac{1}{3}$ to v.

(R3) Suppose v is a bad 5-vertex and u is a neighbor of v in G^X.

- (R3.1) If $d(u) = 8$, then u sends charge $\frac{2}{15}$ to v;
- (R3.2) If $d(u) \geq 9$, then u sends charge $\frac{1}{5}$ to v.

Lemma 7. *Let* w_1 *denote the resultant weight function after the discharging procedure is performed on* G^X *according to the rules (R1) to (R3). Then, we have the following:*

(1) *If* $x \in V^g$*, then* $w_1(x) \geq \frac{2d(x)-16}{3}$*;*
(2) *If* $x \in F(G^X) \cup X(G^X)$*, then* $w_1(x) = 0$*;*
(3) *For each vertex* $x \in V \setminus V^g$*, we have the following:*

- *(3.1)* *If* $x \in V_5^b \cup V_6^b \cup V_7^1$*, then* $w_1(x) \geq 0$*;*
- *(3.2)* *If* $x \in V_7^2$*, then* $w_1(x) \geq \frac{2}{9}$*;*
- *(3.3)* *If* $d(x) = 8$*, then* $w_1(x) \geq \frac{8}{15}$*;*
- *(3.4)* *If* $d(x) \geq 9$*, then* $w_1(x) \geq \frac{17d(x)-140}{30}$*.*

Proof. Assume $x \in V^g$, $w_1(x) \geq d(x) - 4 - d(x) \times \frac{1}{3} - \frac{4}{3} = \frac{2d(x)-16}{3}$ by (R1) and (R2.1). If $x \in F(G^X)$, then $d(x) = 3$ according to Lemma 1. It follows that $w_1(x) = 3 - 4 + 3 \times \frac{1}{3} = 0$ by (R1). Let $x \in X(G^X)$, without loss of generality, and assume that ab cross cd in G at the crossing x. If one of $\{a, b, c, d\}$ is a good vertex, it sends $\frac{4}{3}$ to x by (R2.1). This implies that $w_1(x) = 4 - 4 - 4 \times \frac{1}{3} + \frac{4}{3} = 0$. Otherwise, every vertex in $\{a, b, c, d\}$ is bad. If x is adjacent to a bad 5-vertex, say a, that is $x \in V_X^1$, then each vertex in $\{b, c, d\}$ is a 8^+-vertex by Lemma 3. It follows that $w_1(x) = 4 - 4 - 4 \times \frac{1}{3} + \frac{4}{9} \times 3 = 0$ by (R2.2). According to Corollary 3, there are at most two bad 6-vertices in $\{a, b, c, d\}$. If $n_6^b(x) = 2$, that is, $x \in V_X^2$, then $n_{7^+}(x) = 2$. We have $w_1(x) = 4 - 4 - 4 \times \frac{1}{3} + \frac{2}{3} \times 2 = 0$ by (R2.3). Similarly, according to (R2.4) and (R2.5), if $n_6^b(x) = 1$, then $w_1(x) = 4 - 4 - 4 \times \frac{1}{3} + \frac{4}{9} \times 3 = 0$, if $n_6^b(x) = 0$, then $w_1(x) = 4 - 4 - 4 \times \frac{1}{3} + \frac{1}{3} \times 4 = 0$, , respectively. Therefore, (1) and (2) hold.

Now, we begin with the proof of the statement (3). Let $x \in V_5^b$. If $n_X(x) = 0$, then x is adjacent to five 8^+-vertices in G^X by Lemma 3, which sends at least $\frac{2}{15}$ to x by (R3.1) and (R3.2). Hence, $w_1(x) \geq 5 - 4 - 5 \times \frac{1}{3} + \frac{2}{15} \times 5 = 0$. If $n_X(x) = 1$, then x is adjacent to at least two 9^+-vertices in G by Lemma 3. Therefore, $w_1(x) \geq 5 - 4 - 5 \times \frac{1}{3} + \frac{2}{15} \times 2 + \frac{1}{5} \times 2 = 0$ by (R3.1) and (R3.2). Assume $x \in V_6^b$. Then, $w_1(x) = 6 - 4 - 6 \times \frac{1}{3} = 0$ by (R1). Assume $x \in V_7^1$. Then x is adjacent to a vertex in V_X^2, which gets $\frac{2}{3}$ from x by (R2.3). Hence, $w_1(x) = 7 - 4 - 7 \times \frac{1}{3} - \frac{2}{3} = 0$. Hence, (3.1) holds.

If $x \in V_7^2$, then $w_1(x) \geq 7 - 4 - 7 \times \frac{1}{3} - n_X(x) \times \max\{\frac{4}{9}, \frac{1}{3}\} \geq \frac{2}{9}$ by (R1), (R2.4) and (R2.5).

If $d(x) = 8$, then $n_5^b(x) \leq 1$ by Lemma 5. Therefore, $w_1(x) \geq 8 - 4 - 8 \times \frac{1}{3} - \frac{2}{15} \times n_5(x) - \max\{\frac{4}{9}, \frac{2}{3}, \frac{1}{3}\} \times n_X(x) \geq \frac{8}{15}$ by (R1), (R2), and (R3.1).

If $d(x) \geq 9$, then $n_5^b(x) \leq \lfloor \frac{d(x)}{2} \rfloor$ as bad 5-vertices are not adjacent. Consequently, $w_1(x) \geq d(x) - 4 - d(x) \times \frac{1}{3} - \frac{1}{5} \times n_5^b(x) - \max\{\frac{4}{9}, \frac{2}{3}, \frac{1}{3}\} \times n_X(x) \geq \frac{17d(x) - 140}{30}$. □

4.2. Second Discharge

We now need to perform a second stage of discharging. To do this, we define an additional discharging rule R′ as follows:
(R′) Every vertex in $V^g \cup V_7^2 \cup V_8^+$ sends $\frac{1}{36}$ to every adjacent vertex in $V_5^b \cup V_6^b \cup V_7^1$.

Let w_2 denote the resultant weigh function after the second discharging procedure is performed on G^X according to the rule (R′).

Lemma 8.

(1) If $x \in V^g$, then $w_2(x) \geq -\frac{123}{36}$;
(2) If $x \in V(G) \backslash V^g$, then $w_2(x) \geq \frac{1}{36}$;
(3) If $x \in F(G^X) \cup X(G^X)$, then $w_2(x) = 0$.

Proof. If $x \in V^g$, then it is obvious that $w_2(x) = w_1(x) - \frac{1}{36}d(x) \geq \frac{2d(x) - 16}{3} - -\frac{1}{36}d(x) = \frac{23d(x) - 192}{36} \geq -\frac{123}{36}$ as $\delta(G^X) \geq 3$.

Let $x \in V(G) \backslash V^g$. Then, $d(x) \geq 5$. By Lemma 7, $w_1(x) \geq 0$. If $x \in V_5^b \cup V_6^b \cup V_7^1$, then x is adjacent to a vertex in $V^g \cup V_7^2 \cup V_8^+$ by Lemmas 3 and 6. Hence, $w_2(x) \geq w_1(x) + \frac{1}{36} \geq \frac{1}{36}$ by Lemma 7 (3.1) and (R′). If $x \in V_7^2$, then $w_2(x) \geq w_1(x) - \frac{1}{36} \times 7 \geq \frac{2}{9} - \frac{7}{36} = \frac{1}{36}$. If $x \in V_8$, then $w_2(x) \geq w_1(x) - \frac{1}{36} \times 8 \geq \frac{8}{15} - \frac{8}{36} = \frac{14}{45}$. If $x \in V_9^+$, then $w_2(x) \geq w_1(x) - \frac{1}{36} \times d(x) \geq \frac{17d(x) - 140}{30} - \frac{d(x)}{36} = \frac{97d(x) - 840}{180} \geq \frac{11}{60}$.

If $x \in F(G^X) \cup X(G^X)$, then $w_2(x) = w_1(x) = 0$ by Lemma 7. □

4.3. The Main Result

By Lemma 8, let $n^g = |V^g|$, we can derive the following.

$$
\begin{aligned}
0 > -8 &= \sum_{x \in V(G^X)} w_2(x) + \sum_{x \in F(G^X)} w_2(x) \\
&= \sum_{x \in C(G^X)} w(x) + \sum_{x \in V^g} w(x) + \sum_{x \in V(G) \setminus V^g} w(x) + \sum_{x \in F(G^X)} w(x) \\
&\geq \sum_{x \in V^g} w(x) + \sum_{x \in V(G) \setminus V^g} w(x) \\
&\geq -\frac{123}{36} n^g + \frac{1}{36}(n - n^g)
\end{aligned}
$$

Thus:

$$n^g > \frac{1}{124} n.$$

It is easy to see that when a fire breaks out at a vertex in $V(G) \setminus V^g$, the firefighter can save at least four vertices. Thus:

$$
\begin{aligned}
\sum_{x \in V(G)} \mathrm{sn}_4(x) &= \sum_{x \in V^g} \mathrm{sn}_4(x) + \sum_{x \in V(G) \setminus V^g} \mathrm{sn}_4(x) \\
&\geq (n-4)n^g + 4(n - n^g) \\
&= (n-8)n^g + 4n \\
&> \frac{n(n-8)}{124} + 4n \\
&> \frac{1}{124} n^2
\end{aligned}
$$

Therefore:

$$\rho_4(G) = \frac{\sum_{x \in V(G)} \mathrm{sn}_4(x)}{n^2} > \frac{1}{124}.$$

It is interesting to improve the lower bound for the class of IC-planar graphs, as this is not the best possible. However, we are mostly concerned with the following question.

Question 1. *What is the smallest integer k^* such that all IC-planar graphs are k^*-good ?*

Note that $K_{2,n}$ is a planar graph, which is also an IC-planar graph. It is easy to compute that $\rho(K_{2,n}) = \frac{2}{n+2}$; therefore, $\lim_{n \to \infty} \rho(K_{2,n}) = 0$. This fact and our Theorem 1 imply that $2 \leq k^* \leq 4$. It is conjectured that planar graphs are 2-good in [17], and we conjecture that IC-planar graphs are also 2-good.

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