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Analysis of Solutions to a Parabolic System with Absorption

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Abstract: In this paper, we investigate the blow-up rate and global existence of solutions to a parabolic system with absorption under the homogeneous Dirichlet boundary. By using the comparison principle and super-sub solution method, we obtain some sufficient conditions for the global existence and blow-up in finite time of solutions and establish some estimates of the upper and lower bounds of the blow-up rates. For the special case, if the domain is symmetric, for example, if it is a ball, the results of this paper also hold.

Keywords: global existence; blow-up rate; blow up; absorption

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1. Introduction and Main Results

In this paper, we consider the following problem:

$$\begin{cases} u_t - \Delta u^m = v^p - au^r, & x \in \Omega, \ t > 0, \\ v_t - \Delta v^n = u^q - bv^s, & x \in \Omega, \ t > 0, \\ u(x,t) = 0, \ v(x,t) = 0, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where $m, n > 1$, $p, q, r, s, a, b > 0, \Omega \subset \mathbb{R}^N$ is a bounded and smooth domain. The initial conditions are $u_0, v_0 \in C^{2+v}(\bar{\Omega})$, with $0 < v < 1, u_0, v_0 \geq, \not\equiv 0$.

Problem (1) can be described as the cooperative interaction of two diffusing biological species [1]. Suppose that each species can find its habitat from the activities of another species (represented by the reaction terms v^p and u^q), corresponding, for example, to the actions of invasive species or to overcrowding (represented by the absorption terms $-u^r$ and $-v^s$). We also refer to [2–5]. For the case $p, q, r, s > 1$, the existence and uniqueness of non-negative solutions of problem (1) can be obtained using the standard contraction mapping theorem [3,4]. If one of the parameters is less than 1, this method is not directly applicable. However, by using a smooth nonlinear approximation, the local existence can still be easily proved [6]. If $p, q > 1$, due to the dissipative property of the absorption terms, the solutions are still unique even if $r < 1$ or $s < 1$.

Problem (1) with $a = b = 0$ was studied in [7,8] (see also [9,10]), and the following conclusions were drawn:

(a) If $pq < mn$, then all solutions of (1) with continuous bounded initial values are global;
(b) If $pq > mn$, then there exist both nontrivial global solutions and nonglobal solutions of (1);
(c) If $pq = mn$ and the diameter of the domain is sufficiently small, all solutions of (1) are global.

It is interesting and important to study the properties of solutions in the critical case $pq = mn$ [9,11]. For other porous equations without absorption, we can consult [12,13].

Problem (1) with $m = n = 1$ was investigated by Bedjaoui and Souplet in [1]. They divided the (p, q, r, s) parameter region into three regions: (i) $pq > \max\{r, 1\} \max\{s, 1\}$;

(ii) $pq = \max\{r,1\}\max\{s,1\}$; and (iii) $pq < \max\{r,1\}\max\{s,1\}$. They discussed the parameters of the absorption terms and reaction terms for global solutions and blow-up solutions. Xiang et al. [14] also considered the case, and estimates of the blow-up rates were established when the blow-up case occurred.

In 2021, the authors of [15] considered the following fourth-order parabolic problem with absorption

$$\begin{cases} u_t + \Delta^2 u = -u^p, & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x), & \text{in } \mathbb{R}^N. \end{cases} \tag{2}$$

They used the relationships to study the asymptotic behavior of solutions and found that complicated asymptotic behavior can occur in the solutions of this equation with absorption.

For a single parabolic equation

$$u_t = \Delta u - f(u), \;\; x \in \Omega, \; t > 0$$

with $f(u) = \lambda u^\alpha$, $\lambda > 0$, $\alpha \in (0,1)$, the phenomenon called extinction of the solution is observed. This means that the solutions corresponding to bounded non-negative initial data will vanish identically after a finite time. This was observed first by A. S. Kalashnikov in 1974 (see [16,17] and the references therein). Furthermore, for the case $f(u) = c(x,t)u^p$, the authors of [18] discussed the parabolic equation with nonlinear nonlocal Neumann boundary conditions, in 2017. They proved a comparison principle and the existence of a local solution, and they also studied the solutions for non-uniqueness and uniqueness.

In 2020, Dao [19] studied the instantaneous shrinking of the support for solutions to parabolic problems.

$$\begin{cases} u_t = A(u) - u^{-\beta}\chi_u, & \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \mathbb{R}^N, \end{cases} \tag{3}$$

where $\beta \in (0,1)$, χ_u is the characteristic function and $A(u)$ is a diffusion term, expressing Δu, $(|u_x|^{p-2}u_x)_x$ or $(u^m)_{xx}$ in one dimension. Some conditions were given for any solution u of (3) to be compactly supported for all time $t > 0$ under these three cases. In 2021, Palencia [20] discussed porous-medium problems with advection, and he showed the existence, uniqueness and regularity of weak solutions.

The global solutions and blow-up problems for parabolic systems such as problem (1) have been studied extensively (see [21–28] and the references therein). Motivated by such papers, we discuss two aspects of problem (1). Firstly, we study the optimal conditions for the global existence and blow-up solutions to (1). Secondly, we obtain an precise estimate of the blow-up rate when the blow-up phenomenon occurs, by using the scaled variable.

We start with the notions of a subsolution and supersolution. Throughout this paper, we denote $Q_T = \Omega \times (0,T)$.

Definition 1. *We say that a non-negative function $(u,v) \in [C^{2,1}(Q_T) \cap C(Q_T)]^2$ is a supersolution of problem (1) in Q_T if*

$$\begin{cases} u_t - \Delta u^m \geq v^p - au^r, & x \in \Omega, \;\; t > 0, \\ v_t - \Delta v^n \geq u^q - bv^s, & x \in \Omega, \;\; t > 0, \\ u(x,t) \geq 0, \; v(x,t) \geq 0, & x \in \partial\Omega, \;\; t > 0, \\ u(x,0) \geq u_0(x), v(x,0) \geq v_0(x), & x \in \Omega, \end{cases} \tag{4}$$

and $(u,v) \in [C^{2,1}(Q_T) \cap C(Q_T)]^2$ is a subsolution if $(u,v) \geq 0$ satisfies (4) in the reverse order. We say that (u,v) is a solution of problem (1) in Q_T if (u,v) is both a subsolution and a supersolution of (1) in Q_T. Furthermore, we say that (u,v) is a global solution of problem (1) if it is a solution of (1) in Q_T for any $T > 0$. And if $T < \infty$, then any solution (u,v) blows up in the sense of the L^∞ norm:

$$\lim_{t \to T} (\|u(t,\cdot)\|_\infty + \|v(t,\cdot)\|_\infty) = \infty.$$

Our main results in detail are as follows.

Theorem 1. *If $pq < \max\{r, m\} \max\{s, n\}$, then all non-negative solutions of (1) are global. Moreover, if $r \geq m, s \geq n$ (hence $pq < rs$), then solutions are uniformly bounded.*

Theorem 2. *Let $pq = \max\{r, m\} \max\{s, n\}$. Then:*

(i) If $r \geq m, s \geq n$, and a and b are sufficiently large, then all non-negative solutions are global and uniformly bounded.
(ii) If $r < m$ or $s < n$, then all non-negative solutions are global.
(iii) If $r \geq m, s \geq n$, and a and b are sufficiently small, then there exists a non-negative blow-up solution of (1).

Theorem 3. *If $pq > \max\{r, m\} \max\{s, n\}$, then any non-negative solution of (1) with large initial data blows up in finite time.*

Theorem 4. *Under the assumptions of Theorem 3, assume $r \leq p\frac{q+1}{p+1}$ and $s \leq q\frac{p+1}{q+1}$. Let (u, v) be a non-negative blow-up solution of (1) in finite time T. Then, there exists a positive constant c that satisfies*

$$\max_{\bar{\Omega}\times[0,t]} u(x, \tau) \geq c(T - t)^{-\alpha}, \max_{\bar{\Omega}\times[0,t]} v(x, \tau) \geq c(T - t)^{-\beta}, \tag{5}$$

where $\alpha = \frac{p+1}{pq-1}$ and $\beta = \frac{q+1}{pq-1}$. Moreover, if $r < p < n - 1, s < q < m - 1$, then there exists a positive constant C that satisfies

$$\max_{\bar{\Omega}\times[0,t]} u(x, \tau) \leq C(T - t)^{-\alpha}, \max_{\bar{\Omega}\times[0,t]} v(x, \tau) \leq C(T - t)^{-\beta}. \tag{6}$$

Corollary 1. *Suppose that all the hypotheses of Theorem 4 hold. Let (u, v) be a solution of (1) that blows up in finite time T. Then, there exist two constants $c, C > 0$ such that*

$$c(T - t)^{-\alpha} \leq \max_{\bar{\Omega}\times[0,t]} u(x, \tau) \leq C(T - t)^{-\alpha}, \ c(T - t)^{-\beta} \leq \max_{\bar{\Omega}\times[0,t]} v(x, \tau) \leq C(T - t)^{-\beta}.$$

This paper is organized as follows. The comparison principle is established in Section 2. Section 3 is devoted to proving Theorems 1–3. In Section 4, we establish estimates of blow-up rates for blow-up solutions in a finite time, which is Theorem 4. In Section 5, some conclusions and observations are discussed.

2. Comparison Principle

Firstly, we give a comparison principle that plays an important role in the study of problem (1).

Lemma 1. *Suppose that $(\bar{u}, \bar{v})$ and $(\underline{u}, \underline{v})$ are a supersolution and a subsolution of (1), respectively. Then, $(\underline{u}, \underline{v}) \leq (\bar{u}, \bar{v})$ in $\bar{\Omega} \times [0, T)$, if $(\underline{u}(x, 0), \underline{v}(x, 0)) \leq (\bar{u}(x, 0), \bar{v}(x, 0))$ and $(\bar{u}, \bar{v})$ has a positive lower bound.*

Proof. The technique for the proof of the comparison principle with respect to degenerate parabolic equations is standard [29–31]. Here, we will give a brief proof. From the definition of supersolutions and subsolutions, we have

$$\begin{cases} (\underline{u} - \bar{u})_t - (\Delta\underline{u}^m - \Delta\bar{u}^m) \leq (\underline{v}^p - \bar{v}^p) - a(\underline{u}^r - \bar{u}^r), \\ (\underline{v} - \bar{v})_t - (\Delta\underline{v}^n - \Delta\bar{v}^n) \leq (\underline{u}^q - \bar{u}^q) - b(\underline{v}^s - \bar{v}^s). \end{cases} \tag{7}$$

Define a class of test functions,

$$\Psi = \{\psi \in C(\bar{Q}_t) | \psi_t, \Delta\psi \in C(Q_t) \cap L^2(Q_t); \psi \geq 0;\ \psi|_{\partial\Omega} = 0\}.$$

Multiplying by $\varphi(x,t) \in \Psi$ in the first equation of (7) and integrating over Q_t for $t \in (0,T)$, we obtain

$$\begin{aligned}
& \int_\Omega (\underline{u}(x,t) - \bar{u}(x,t))\varphi(x,t)\mathrm{d}x \\
\leq\ & \int_\Omega (\underline{u}(x,0) - \bar{u}(x,0))\varphi(x,0)\mathrm{d}x \\
& + \int\int_{Q_t} (\underline{u} - \bar{u})\varphi_\tau + (\underline{u}^m - \bar{u}^m)\Delta\varphi + (\underline{v}^p - \bar{v}^p)\varphi - a(\underline{u}^r - \bar{u}^r)\varphi \mathrm{d}x\mathrm{d}\tau, \\
\leq\ & \int_\Omega (\underline{u}(x,0) - \bar{u}(x,0))\varphi(x,0)\mathrm{d}x \\
& + \int\int_{Q_t} (\underline{u} - \bar{u})\left[\varphi_\tau + \Phi(x,\tau)\Delta\varphi\right] + a|(\underline{u} - \bar{u})G(x,s)\varphi| + (\underline{v} - \bar{v})D(x,\tau)\varphi \mathrm{d}x\mathrm{d}\tau,
\end{aligned}$$

where

$$\begin{aligned}
\Phi(x,s) &= \int_0^1 m(\theta\bar{u} + (1-\theta)\underline{u})^{m-1}\mathrm{d}\theta, \\
G(x,s) &= \int_0^1 r(\theta\bar{u} + (1-\theta)\underline{u})^{r-1}\mathrm{d}\theta, \\
D(x,s) &= \int_0^1 p(\theta\bar{v} + (1-\theta)\underline{v})^{p-1}\mathrm{d}\theta.
\end{aligned}$$

Since $(\underline{u},\underline{v})$ and $(\bar{u},\bar{v})$ on Q_t are bounded, it follows from $m,r,p \geq 1$ that $\Phi(x,s), G(x,s)$ and $D(x,s)$ are bounded non-negative functions. Now, if $r,p < 1$, then we have

$$G(x,s) \leq \delta^{r-1}, D(x,s) \leq \delta^{p-1}.$$

Thus, an appropriate test function may be chosen exactly as in [29] (pp. 118–123) to obtain

$$\int_\Omega (\underline{u} - \bar{u})_+ \mathrm{d}x \leq \|\varphi\|_\infty \int_\Omega (\underline{u}(x,0) - \bar{u}(x,0))_+ \mathrm{d}x + C\int\int_{Q_t} (\underline{u} - \bar{u})_+ + (\underline{v} - \bar{v})_+ \mathrm{d}x\mathrm{d}\tau, \tag{8}$$

where $w_+ = \max\{w, 0\}$ and $C > 0$ is a bounded constant. Similarly, we can prove

$$\int_\Omega (\underline{v} - \bar{v})_+ \mathrm{d}x \leq \|\psi\|_\infty \int_\Omega (\underline{v}(x,0) - \bar{v}(x,0))_+ \mathrm{d}x + C\int\int_{Q_t} (\underline{u} - \bar{u})_+ + (\underline{v} - \bar{v})_+ \mathrm{d}x\mathrm{d}\tau, \tag{9}$$

where $\psi(x,t) \in \Psi$. Combining (8) with (9), it follows from the Gronwall lemma that $\underline{u} \leq \bar{u}$, $\underline{v} \leq \bar{v}$, since $\underline{u}(x,0) \leq \bar{u}(x,0)$, $\underline{v}(x,0) \leq \bar{v}(x,0)$. □

3. Global Existence and Blow-Up

In this section, we will give the proof of global existence and blow-up solutions using the comparison principle. Firstly, we will give the proof of the global existence of Theorem 1.

Proof of Theorem 1. Suppose that $pq < \max\{r,m\}\max\{s,n\}$.

Case 1: $m \geq r, n \geq s$. This implies $pq < mn$. By the comparison principle, we have $u \leq w$ and $v \leq z$, where (w,z) satisfies

$$\begin{cases}
w_t - \Delta w^m = z^p, & x \in \Omega,\ \ t > 0, \\
z_t - \Delta z^n = w^q, & x \in \Omega,\ \ t > 0, \\
w(x,t) = 0,\ z(x,t) = 0, & x \in \partial\Omega,\ \ t > 0, \\
w(x,0) = u_0(x), z(x,0) = v_0(x), & x \in \Omega,
\end{cases} \tag{10}$$

It follows from [7,8] that (w,z) is global, and so is (u,v).

Case 2: $m \leq r, n \leq s$. This implies $pq < rs$. For any $C > 0$, our aim is to find a constant supersolution of (1), i.e., $(\bar{u},\bar{v}) = (A,B)$ satisfies

$$\min\{A,B\} \geq \max\{C, \max_{\bar{\Omega}} u_0(x),\ \max_{\bar{\Omega}} v_0(x)\}.$$

Since $(\bar{u}, \bar{v})$ is a supersolution,

$$\begin{cases} aA^r \geq B^p, \\ bB^s \geq A^q, \end{cases}$$

that is,

$$B^{pq} \leq a^q A^{qr} \leq a^q b^r B^{rs}. \tag{11}$$

Since $rs > pq$, then there is a suitable A, B satisfying (11). By the comparison principle, such inequalities imply that the global solutions are uniformly bounded.

Case 3: $m \leq r, n \geq s$. Clearly, we have $pq < nr$. Let $\psi(x)$ be the unique solution of

$$-\Delta\psi = 1, \text{in } \Omega; \psi(x) = 0, \text{ on } \partial\Omega. \tag{12}$$

Now, we will seek a global supersolution of (1) in the following form:

$$(\bar{u}, \bar{v}) = \left((h(t) + c_0)^n \psi^{\frac{1}{m}}, C(h(t) + c_0)^q \psi^{\frac{1}{n}}\right), \tag{13}$$

where $h(t)$ solves $h'(t) = -\delta(h(t) + c_0)^{mn-n+1}$ with $0 < h(0) = h_0 < 1$, $c_0 > 0$ and $0 < c_0 + h_0 < 1$, where δ and C satisfy

$$\begin{cases} b\psi^{\frac{s-1}{n}} c_0^{\max\{q(s-1)-n(m-1),\, 0\}} \geq \delta q C^{1-s}, \\ \psi^{\frac{q}{m}} \leq C^n, \\ \delta \leq \frac{1}{n}\|\psi\|_\infty^{-\frac{1}{m}}, \\ ac_0^{nr-pq}\psi^{\frac{r}{m}-\frac{p}{n}} \geq C^p. \end{cases}$$

The existence of a suitable δ, C is clear if a, b are sufficiently large.

From (13), a direct computation gives

$$\begin{cases} (h + c_0)^{mn}\left(1 - n\delta\psi^{\frac{1}{m}}\right) + (h + c_0)^{pq}\left(-C^p\psi^{\frac{p}{n}} + a(h + c_0)^{nr-pq}\psi^{\frac{r}{m}}\right) \geq 0, \\ \left(-\delta Cq(h + c_0)^{q+mn-n}\psi^{\frac{1}{n}} + bC^s(h + c_0)^{qs}\psi^{\frac{s}{n}}\right) + (h + c_0)^{nq}\left(C^n - \psi^{\frac{q}{m}}\right) \geq 0. \end{cases} \tag{14}$$

Noticing that $nr > pq$, $\delta \leq \frac{1}{n}\|\psi\|_\infty^{-\frac{1}{m}}$ and $ac_0^{nr-pq}\psi^{\frac{r}{m}-\frac{p}{n}} \geq C^p$, the first equation of (14) holds. The conditions $\psi^{\frac{q}{m}} \leq C^n$ and $b\psi^{\frac{s-1}{n}} c_0^{\max\{q(s-1)-n(m-1),\, 0\}} \geq \delta q C^{1-s}$ guarantee the second equation of (14). Indeed, if $s < 1$, we only need to require $b\psi^{\frac{s-1}{n}} \geq \delta q C^{1-s}$. If $1 \leq s \leq n$ and $q(s-1) - n(m-1) \geq 0$, we only need to require $b\psi^{\frac{s-1}{n}} c_0^{q(s-1)-n(m-1)} \geq \delta q C^{1-s}$. If $1 \leq s \leq n$ and $q(s-1) - n(m-1) < 0$, we need to require $b\psi^{\frac{s-1}{n}} \geq \delta q C^{1-s}$.

Case 4: $m \geq r, n \leq s$. We see that $pq < ms$. Exchanging the roles of u and v, we can obtain the result.

Combining these four cases, the results of Theorem 1 hold. □

Proof of Theorem 2. Theorem 2 (i) can be proved as in case 2 in Theorem 1.

To prove Theorem 2 (ii), we assume that $(\bar{u}, \bar{v}) = (e^{\alpha t}, ce^{q\alpha t})$, with $c = a^{1/p}$ and $\alpha = q^{-1}a^{-1/p}$. It is obvious that $(\bar{u}, \bar{v})$ is the supersolution of (1). For more details, see p. 204 in [1], since $m \geq 1, n \geq 1$.

Next, we will prove Theorem 2 (iii). It suffices to establish a blow-up subsolution of problem (1). Without loss of generality, assume that Ω contains the origin. Denote

$$\underline{u}(x,t) = (T-t)^{-\alpha_1} U\left(\frac{|x|}{(T-t)^{\beta_1}}\right), \quad \underline{v}(x,t) = (T-t)^{-\alpha_2} V\left(\frac{|x|}{(T-t)^{\beta_2}}\right),$$

where $U(y) = (A^2 - y^2)_+^{\frac{1}{m-1}}$, $V(\tilde{y}) = (A^2 - \tilde{y}^2)_+^{\frac{1}{n-1}}$, $\beta_1 = \frac{1-(m-1)\alpha_1}{2} > 0$, $\beta_2 = \frac{1-(n-1)\alpha_2}{2} > 0$ and $\alpha_1, \alpha_2, A, T > 0$ are to be determined later. Note that $B_{AT^\beta}(0)$ contains the support of $\underline{u}(x,t)$ and $\underline{v}(x,t)$, where $\beta = \max\{\beta_1, \beta_2\}$ if $T > 1$; $\beta = \min\{\beta_1, \beta_2\}$ if $T \le 1$, which is included in Ω if T is sufficiently small.

Calculating directly,

$$\begin{array}{rcl} \underline{u}_t & = & (T-t)^{\alpha_1+1}(\alpha_1 U + \beta_1 y U'), \\ \nabla \underline{u}^m & = & (T-t)^{-(\alpha_1 m+\beta_1)}(U^m)'\nabla|x|, \\ \triangle \underline{u}^m & = & (T-t)^{-(\alpha_1 m+2\beta_1)}((U^m)'' + \frac{N-1}{y}(U^m)'), \end{array}$$

where $y = \frac{|x|}{(T-t)^{\beta_1}}$. We need to find suitable parameters such that

$$\begin{array}{rl} & (T-t)^{-(\alpha_1+1)}\left[\alpha_1 U + \beta_1 y U' - (U^m)'' - \frac{N-1}{y}(U^m)'\right] + a(T-t)^{-\alpha_1 r}U^r \\ \le & (T-t)^{-\alpha_2 p}V^p \end{array} \tag{15}$$

and

$$\begin{array}{rl} & (T-t)^{-(\alpha_2+1)}\left[\alpha_2 V + \beta_2 \widetilde{y} V' - (V^n)'' - \frac{N-1}{\widetilde{y}}(V^n)'\right] + b(T-t)^{-\alpha_2 s}V^s \\ \le & (T-t)^{-\alpha_1 q}U^q, \end{array} \tag{16}$$

where $\widetilde{y} = \frac{|x|}{(T-t)^{\beta_2}}$.

Note that U, V is continuous for C^2 except for $y = A$, where U', V' has a positive jump. Therefore, to obtain a subsolution of (1), we will prove (15) and (16) pointwise for $y > 0$, with $y \ne A$.

It is easy to see that

$$\begin{array}{rcl} U'(y) & = & -\frac{2}{m-1}(A^2-y^2)_+^{\frac{2-m}{m-1}}y, \\ (U^m)'(y) & = & -\frac{2m}{m-1}(A^2-y^2)_+^{\frac{1}{m-1}}y, \\ (U^m)''(y) & = & \frac{4m}{(m-1)^2}(A^2-y^2)_+^{\frac{2-m}{m-1}}y^2 - \frac{2m}{m-1}(A^2-y^2)_+^{\frac{1}{m-1}}, \end{array}$$

and (15) is trivial for $y \ge A$. A simple computation shows that (15) is satisfied. We distinguish two steps for $0 < y \le \theta A$ and $\theta A < y < A$, where

$$\theta = \left(\frac{\alpha_1 + \frac{2mN}{m-1}}{\alpha_1 + \frac{2mN}{m-1} + \frac{4m}{(m-1)^2}}\right)^{\frac{1}{2}} < 1. \tag{17}$$

Step 1. For $\theta A < y < A$,

$$\begin{array}{rl} & \alpha_1 U + \beta_1 y U' - (U^m)'' - \frac{N-1}{y}(U^m)' \\ = & (A^2-y^2)_+^{\frac{2-m}{m-1}}\left[(\alpha_1 + \frac{2mN}{m-1})A^2 - (\alpha_1 + \frac{2mN}{m-1} + \frac{2\beta_1}{m-1} + \frac{4m}{(m-1)^2})y^2\right] \\ \le & (A^2-y^2)_+^{\frac{2-m}{m-1}}\left[(\alpha_1 + \frac{2mN}{m-1})A^2 - (\alpha_1 + \frac{2mN}{m-1} + \frac{2\beta_1}{m-1} + \frac{4m}{(m-1)^2})\theta^2 A^2\right] \\ \le & -\frac{2\beta_1}{m-1}\theta^2 A^2 (A^2-y^2)_+^{\frac{2-m}{m-1}}, \end{array}$$

Step 2. For $0 < y \le \theta A$, we have

$$U(y) \ge (1-\theta^2)^{\frac{1}{m-1}}A^{\frac{2}{m-1}} > 0, \quad V(y) \ge (1-\theta^2)^{\frac{1}{n-1}}A^{\frac{2}{n-1}} > 0.$$

It follows from $\alpha_2 p > \alpha_1 + 1$ that

$$(T-t)^{-(\alpha_1+1)}\left[\alpha_1 U + \beta_1 y U' - (U^m)'' - \frac{N-1}{y}(U^m)'\right] \le \frac{1}{2}(T-t)^{-\alpha_2 p}V^p,$$

if T is sufficiently small. If

$$a(T-t)^{-\alpha_1 r}U^r \leq \frac{1}{2}(T-t)^{-\alpha_2 p}V^p. \tag{18}$$

then (15) holds.

Similarly, if

$$b(T-t)^{-\alpha_2 s}V^s \leq \frac{1}{2}(T-t)^{-\alpha_1 q}U^q, \tag{19}$$

$\alpha_1 q > \alpha_2 + 1$ and T is sufficiently small, then (16) holds.

Next, we choose an appropriate α_1, α_2 to satisfy (18) and (19). It is clear that there exist $\alpha_1 > \frac{p+1}{pq-1}$ and $\alpha_2 > \frac{q+1}{pq-1}$, solving the inequalities

$$\alpha_2 p > \alpha_1 + 1, \qquad \alpha_1 q > \alpha_2 + 1. \tag{20}$$

If $pq = rs$, then we can choose some large α_1 and α_2 to satisfy (20), and $p = \frac{\alpha_1}{\alpha_2} r$, hence $q = \frac{\alpha_2}{\alpha_1} s$. Therefore, by (18) and (19), we have

$$\begin{cases} aU^r \leq \frac{1}{2}V^p, \\ bV^s \leq \frac{1}{2}U^q, \end{cases} \tag{21}$$

that is,

$$2^{q+r}a^q b^r V^{rs} \leq 2^q a^q U^{rq} \leq V^{pq},$$

is clearly true, provided $2^{q+r}a^q b^r < 1$, since a and b are sufficiently small.

Therefore, $(\underline{u}, \underline{v})$ is a blow-up subsolution of problem (1) with appropriately large (u_0, v_0). This completes the proof of Theorem 2 (iii). □

Proof of Theorem 3. Firstly, we consider case (i) $r \geq m$ and $s \geq n$, hence $pq > rn$. The discussion for the parameters is similar to that in [1], but here we give more details. In order to guarantee that (21) still holds, we require

$$\begin{cases} \alpha_2 p > \alpha_1 + 1, \qquad \alpha_2 p > \alpha_1 r \\ \alpha_1 q > \alpha_2 + 1 > \alpha_2 s. \end{cases} \tag{22}$$

Set $\lambda = \alpha_1/\alpha_2$, then by (22),

$$\frac{s}{q} < \lambda < \frac{p}{r}, \quad s - 1 < \frac{1}{\alpha_2} < \min\{p - \lambda, \lambda q - 1\}.$$

If $\lambda \leq \frac{p+1}{q+1}$, then $\min\{p-\lambda, \lambda q - 1\} = \lambda q - 1$. Without loss of generality, we assume that

$$\frac{s}{q} < \frac{p+1}{q+1}. \tag{23}$$

Indeed, since $pq > rs$, (23) holds or

$$\frac{r}{p} < \frac{q+1}{p+1}. \tag{24}$$

If (24) holds, we can just exchange the roles of u and v in problem (1). Thus, we only have to guarantee that (23) holds.

To satisfy (22), we need to find a suitable λ satisfying

$$\frac{s}{q} < \lambda < \min\{\frac{p+1}{q+1}, \frac{p}{r}\},$$

which is possible since $\frac{s}{q} < \frac{p}{r}$ and $\alpha_2 > 0$, such that

$$0 < s - 1 < \frac{1}{\alpha_2} < \lambda q - 1.$$

Therefore, (21) holds. Thus, $(\underline{u}, \underline{v})$ is a subsolution of (1) for the case $r \geq m$ and $s \geq n$.

Next, we consider case (ii) $r < m$ or case (iii) $s < n$. Assume, for instance, that $r < m$ and $s \geq n$, hence $pq > ms$. Any solution of (1) is a supersolution of

$$\begin{cases} u_t - \Delta u^m \geq v^p - au^m - a, & x \in \Omega, \ t > 0, \\ v_t - \Delta v^n \geq u^q - bv^s, & x \in \Omega, \ t > 0, \end{cases} \tag{25}$$

with homogeneous Dirichlet boundary conditions. Since $pq > ms$, similarly to the above proof, we can see that $(\underline{u}, \underline{v})$ is still a subsolution of (25) for suitable u_0 and v_0, which implies that (u, v) blows up. The cases $r \geq m, s < n$ and $r < m, s < n$ can be treated in a similar way. □

4. Blow-Up Rate

In this section, we always assume $pq > \max\{r, m\} \max\{s, n\}$. Fix $t \in (0, T)$ such that $M_u(t) = \sup_{\bar{\Omega} \times (0,t]} u(x, \tau) > 1$, $M_v(t) = \sup_{\bar{\Omega} \times (0,t]} v(x, \tau) > 1$.

Now, we give a lemma which shows the relationship between $M_u(t)$ and $M_v(t)$ near the blow-up time T. The lemma also implies that the blow-up phenomena occur at the same time. The proof is similar to [14], and we omit the details.

Lemma 2. *Assume $r \leq p\frac{q+1}{p+1}$ and $s \leq q\frac{p+1}{q+1}$. Let (u, v), the solution of (1), blow up in finite time T. Then, there exists $\eta \in (0, 1)$ such that*

$$\eta \leq M_u^{-\frac{1}{2\alpha}} M_v^{\frac{1}{2\beta}} \leq \eta^{-1}, \qquad t \in (\frac{T}{2}, T).$$

Proof of Theorem 4. Denote

$$\varphi_{M_u}(y, \tau) = \frac{1}{M_u(t)} u(a_1 y, b_1 \tau + t), \ \ \psi_{M_v}(y, \tau) = \frac{1}{M_v(t)} v(a_2 y, b_2 \tau + t), \ \text{ in } B_{\tilde{a}R}(0) \times (0, S),$$

where $a_1 = M_u^{\frac{m}{2}} M_v^{-\frac{p}{2}}, a_2 = M_v^{\frac{n}{2}} M_u^{-\frac{q}{2}}, b_1 = M_u M_v^{-p}, b_2 = M_v M_u^{-q}, \tilde{a} = \max\{a_1^{-1}, a_2^{-1}\}$ and $S = \min\{M_u^{-1} M_v^p (T - t), M_v^{-1} M_u^q (T - t)\}$.

It is clear that $(\varphi_{M_u}, \psi_{M_v})$ blows up at $\tau = S$. Moreover, it is a solution of the following problem:

$$\begin{cases} (\varphi_{M_u})_\tau - \Delta \varphi_{M_u}^m = \psi_{M_v}^p - aM_v^{-p} M_u^r \varphi_{M_u}^r, & (y, \tau) \in B_{\tilde{a}R}(0) \times (0, S), \\ = \varphi_{M_u}^q - bM_u^{-q} M_v^s \psi_{M_v}^s, & (y, \tau) \in B_{\tilde{a}R}(0) \times (0, S), \\ \varphi_{M_u}(y, \tau) = 0, \ \ \psi_{M_v}(y, \tau) = 0, & (y, \tau) \in \partial B_{\tilde{a}R}(0) \times (0, S), \\ \varphi_{M_u}(y, 0) = \frac{1}{M_u} u(a_1 y, t), \psi_{M_v}(y, 0) = \frac{1}{M_v} v(a_2 y, t), & y \in B_{\tilde{a}R}(0). \end{cases} \tag{26}$$

Step 1: We prove (5). Set

$$\bar{W}(y, \tau) = (S_1 - s)^{-\alpha} f(\xi_1), \qquad \bar{Z}(y, \tau) = (S_1 - s)^{-\beta} g(\xi_2),$$

where $f(\xi_1) = (L + \delta(L - \xi_1)_+)^{\frac{1}{m-1}}$, $g(\xi_2) = (L + \delta(L - \xi_2)_+)^{\frac{1}{n-1}}$, $\xi_1 = \frac{|y|}{(S_1 - \tau)^{\beta_1}}$, $\xi_2 = \frac{|y|}{(S_1 - \tau)^{\beta_2}}$, δ, L, S_1 to be determined later, and $\alpha, \beta, \beta_1, \beta_2$ satisfy

$$\alpha + 1 = \alpha m + 2\beta_1 = \beta p, \tag{27}$$

$$\beta + 1 = \beta n + 2\beta_2 = \alpha q. \tag{28}$$

Moreover, (27) and (28) imply

$$\alpha=\frac{p+1}{pq-1},\ \ \beta=\frac{q+1}{pq-1},\ \ \beta_1=\frac{1-(m-1)\alpha}{2},\ \ \beta_2=\frac{1-(n-1)\beta}{2}. \qquad (29)$$

After a direct computation, for $0<\xi_1,\xi_2<L$, we have

$$\begin{aligned}
\bar W_\tau(y,\tau)= \ & \alpha(S_1-\tau)^{-(\alpha+1)}(L+\delta(L-\xi_1)_+)^{\frac{1}{m-1}}\\
& -\tfrac{\delta\beta_1}{m-1}(S_1-\tau)^{\alpha+1}\xi_1(L+\delta(L-\xi_1)_+)^{\frac{2-m}{m-1}},\\
\bar Z_\tau(y,\tau)= \ & \beta(S_1-\tau)^{-(\beta+1)}(L+\delta(L-\xi_2)_+)^{\frac{1}{n-1}}\\
& -\tfrac{\delta\beta_2}{n-1}(S_1-\tau)^{\beta+1}\xi_2(L+\delta(L-\xi_2)_+)^{\frac{2-n}{n-1}},\\
\Delta\bar W^m(y,\tau)= \ & (S_1-\tau)^{-(\alpha m+2\beta_1)}\big[\tfrac{m\delta^2}{(m-1)^2}(L+\delta(L-\xi_1)_+)^{\frac{2-m}{m-1}}\\
& -\tfrac{m\delta}{m-1}(L+\delta(L-\xi_1)_+)^{\frac{1}{m-1}}\tfrac{N-1}{\xi_1}\big],\\
\Delta\bar Z^n(y,\tau)= \ & (S_1-\tau)^{-(\beta n+2\beta_2)}\big[\tfrac{n\delta^2}{(n-1)^2}(L+\delta(L-\xi_2)_+)^{\frac{2-n}{n-1}}\\
& -\tfrac{n\delta}{n-1}(L+\delta(L-\xi_2)_+)^{\frac{1}{n-1}}\tfrac{N-1}{\xi_2}\big].
\end{aligned}$$

Then, by (27)

$$\begin{aligned}
& \bar W_s-\Delta\bar W^m-\bar Z^p+aM_v^{-p}M_u^r\bar W^r\\
\geq\ & (S_1-\tau)^{-(\alpha+1)}(L+\delta(L-\xi_1)_+)^{\frac{2-m}{m-1}}\cdot\big[\alpha(L+\delta(L-\xi_1)_+)-\tfrac{\delta\beta_1}{m-1}L\\
& -\tfrac{m\delta^2}{(m-1)^2}-(L+\delta(L-\xi_1)_+)^{\frac{m-2}{m-1}}(L+\delta(L-\xi_2)_+)^{\frac{p}{n-1}}\big]\\
\geq\ & (S_1-\tau)^{-(\alpha+1)}(L+\delta(L-\xi_1)_+)^{\frac{2-m}{m-1}}\\
& \cdot\big[\alpha L-\tfrac{\delta\beta_1}{m-1}L-\tfrac{m\delta^2}{(m-1)^2}-L^{\frac{m-2}{m-1}}(c_1L)^{\frac{p}{n-1}}\big]\\
=\ & (S_1-\tau)^{-(\alpha+1)}(L+\delta(L-\xi_1)_+)^{\frac{2-m}{m-1}}\\
& \cdot\big[(\tfrac{\alpha L}{4}-\tfrac{\delta\beta_1}{m-1}L)+(\tfrac{\alpha L}{4}-\tfrac{m\delta^2}{(m-1)^2})+(\tfrac{\alpha L}{2}-L^{\frac{m-2}{m-1}+\frac{p}{n-1}}c_1^{\frac{p}{n-1}})\big]\\
\geq\ & 0,
\end{aligned}$$

where $c_1=1$ if $p<n-1$; $c_1=2$ if $p\geq n-1$, and δ,L satisfy

$$0<\delta<\min\{1,\frac{(m-1)\alpha}{4\beta_1},\frac{m-1}{2}(\frac{\alpha L}{m})^{\frac12}\},\quad 0<2L^{\frac{m-2}{m-1}+\frac{p}{n-1}-1}c_1^{\frac{p}{n-1}}<\alpha. \qquad (30)$$

Similarly, for $0<\xi_2<L$, by (28), we have

$$\bar Z_s-\Delta\bar Z^n-\bar W^q+bM_u^{-q}M_v^s\bar Z^s\geq 0,$$

and δ,L also satisfy

$$0<\delta<\min\{1,\frac{(n-1)\beta}{4\beta_2},\frac{n-1}{2}(\frac{\beta L}{n})^{\frac12}\},\quad 0<2L^{\frac{n-2}{n-1}+\frac{q}{m-1}-1}c_2^{\frac{q}{m-1}}<\beta. \qquad (31)$$

where $c_2=1$ if $q<m-1$; $c_2=2$ if $q\geq m-1$.

Clearly, $\bar W_s-\Delta\bar W^m-\bar Z^p+aM_v^{-p}M_u^r\bar W^r\geq 0$, $\bar Z_s-\Delta\bar Z^n-\bar W^q+bM_u^{-q}M_v^s\bar Z^s\geq 0$, for $\xi_1,\xi_2>L$, $\bar W(y,s)>0$, $\bar Z(y,s)>0$ on $\partial B_{\tilde aR}(0)\times(0,S_1)$ and $\bar W(y,0)\geq\varphi_{M_u}(y,0)$, $\bar Z(y,0)\geq\psi_{M_v}(y,0)$, provided that S_1 satisfies, in $B_{\tilde aR}(0)$

$$\min\{S_1^{-\alpha}L^{\frac{1}{m-1}},\ S_1^{-\beta}L^{\frac{1}{n-1}}\}>1.$$

Then, we obtain a supersolution independent of M_u, M_v. Therefore, the blow-up time of $\varphi_{M_u}, \psi_{M_v}$ is more than S_1, That is, $\min\{M_u^{-1}M_v^p(T-t), M_v^{-1}M_u^q(T-t)\} \geq S_1$. Without loss of generality, we assume

$$M_u^{-1}M_v^p(T-t) = \min\{M_u^{-1}M_v^p(T-t), M_v^{-1}M_u^q(T-t)\} \geq S_1.$$

From Lemma 2 and (29), there exists some $c > 0$, depending only on S_1, such that

$$M_u^{\frac{1}{\alpha}}(T-t) > c, \text{ and, } M_v^{\frac{1}{\beta}}(T-t) > c.$$

That is

$$\max_{\bar{\Omega}\times[0,t]} u(x,\tau) > c(T-t)^{-\alpha}, \quad \max_{\bar{\Omega}\times[0,t]} v(x,\tau) > c(T-t)^{-\beta}.$$

Step 2: We prove (6). Set

$$\underline{W}(y,\tau) = (S_2-s)^{-\alpha}U(\xi_1), \qquad \underline{Z}(y,\tau) = (S_2-s)^{-\beta}V(\xi_2),$$

where $U(\xi_1) = (A^2-\xi_1^2)^{\frac{1}{m-1}}$, $\xi_1 = \frac{|y|}{(S_2-\tau)^{\beta_1}}$, $V(\xi_2) = (A^2-\xi_2^2)^{\frac{1}{n-1}}$, $\xi_2 = \frac{|y|}{(S_2-\tau)^{\beta_2}}$, $\alpha, \beta, \beta_1, \beta_2$ are given by (29) and S_2 is given later. After some computations, we have

$$\begin{aligned}
&\underline{W}_s = (S_2-\tau)^{-(\alpha+1)}(\alpha U + \beta_1\xi_1 U'), \qquad \underline{Z}_s = (S_2-\tau)^{-(\beta+1)}(\beta V + \beta_2\xi_2 V'),\\
&\Delta\underline{W}^m = (S_2-\tau)^{-(\alpha m+2\beta_1)}((U^m)'' + \tfrac{N-1}{\xi_1}(U^m)'),\\
&\Delta\underline{Z}^n = (S_2-\tau)^{-(\beta n+2\beta_2)}((v^n)'' + \tfrac{N-1}{\xi_2}(V^n)').
\end{aligned}$$

Thus,

$$\begin{aligned}
&\underline{W}_s - \Delta\underline{W}^m - \underline{Z}^p + aM_v^{-p}M_u^r\underline{W}^r\\
=\ &(S_2-\tau)^{-(\alpha+1)}(\alpha U + \beta_1\xi_1U') - (S_2-\tau)^{-(\alpha m+2\beta_1)}((U^m)'' + \tfrac{N-1}{\xi_1}(U^m)')\\
&-(S_2-\tau)^{p\beta}V^p + aM_v^{-p}M_u^r(S_2-\tau)^{\alpha r}U^r,
\end{aligned}$$

$$\begin{aligned}
&\underline{Z}_s - \Delta\underline{Z}^n - \underline{W}^q + bM_u^{-q}M_v^s\underline{Z}^s\\
=\ &(S_2-\tau)^{-(\beta+1)}(\beta V + \beta_2\xi_2V') - (S_2-\tau)^{-(\beta n+2\beta_2)}((V^n)'' + \tfrac{N-1}{\xi_2}(V^n)')\\
&-(S_2-\tau)^{q\alpha}V^q + bM_u^{-q}M_v^s(S_2-\tau)^{\beta s}V^s,
\end{aligned}$$

Denote $M = \max\{M_u(t), M_v(t)\}$, and note (27) and (28) and the assumption $r < p < n-1$. Similarly to the proof of Theorem 3 for $\theta A \leq \xi_1 < A$ (θ given by (17)), we have

$$\begin{aligned}
&\underline{W}_s - \Delta\underline{W}^m - \underline{Z}^p + aM_v^{-p}M_u^r\underline{W}^r\\
\leq\ &(S_2-\tau)^{-(\alpha+1)}\big[\alpha U + \beta_1\xi_1U' - (U^m)'' - \tfrac{N-1}{\xi_1}(U^m)'\big] + aM^{r-p}(S_2-\tau)^{\alpha r}U^r\\
\leq\ &(S_2-\tau)^{-(\alpha+1)}(A^2-\xi_1^2)_+^{\frac{2-m}{m-1}}\big[-\tfrac{2\beta_1}{m-1}\theta^2A^2 + aM^{r-p}(A^2-\xi_1^2)_+^{\frac{m+r-2}{m-1}}(S_2-\tau)^{p\beta-r\alpha}\big]\\
\leq\ &(S_2-\tau)^{-(\alpha+1)}A^2(A^2-\xi_1^2)_+^{\frac{2-m}{m-1}}\big[-\tfrac{2\beta_1}{m-1}\theta^2 + aM^{r-p}S_2^{p\beta-r\alpha}A^{\frac{2(r-1)}{m-1}}\big](note: r<p)\\
\leq\ &0,
\end{aligned}$$

where $M \geq \frac{m-1}{2\beta}\theta^{-2}aS_2^{p\beta-r\alpha}A^{\frac{2(r-1)}{m-1}}$.

For $0 < \xi_1 < \theta A$, by the assumption $r < p < n-1$, we have

$$\begin{aligned}
&\underline{W}_s - \Delta\underline{W}^m - \underline{Z}^p + aM_v^{-p}M_u^r\underline{W}^r\\
\leq\ &(A^2-\xi_1^2)_+^{\frac{1}{m-1}}(S_2-\tau)^{-(\alpha+1)}\big[(\alpha + \tfrac{2mN}{m-1}) - (A^2-\xi_2^2)_+^{\frac{p}{n-1}}(A^2-\xi_1^2)_+^{\frac{-1}{m-1}}\\
&+aM^{r-p}(A^2-\xi_1^2)_+^{\frac{r-1}{m-1}}(S_2-\tau)^{p\beta-r\alpha}\big]\\
\leq\ &(A^2-\xi_1^2)_+^{\frac{1}{m-1}}(S_2-\tau)^{-(\alpha+1)}\big[(\alpha + \tfrac{2mN}{m-1}) - A^{\frac{2p}{n-1}}(1-\theta^2)^{-\frac{1}{m-1}}A^{\frac{-2}{m-1}}\\
&+aM^{r-p}c_3^{\frac{r-1}{m-1}}A^{\frac{2(r-1)}{m-1}}S_2^{p\beta-r\alpha}\big]\\
\leq\ &0,
\end{aligned}$$

where $c_3 = 1 - \theta^2$ if $r < m$; $c_3 = 1$ if $r \geq m$. In addition, we require that $r < p < n - 1$ and A and M satisfy

$$
\begin{aligned}
&A^{\frac{2(p-1)}{n-1}} > (1-\theta^2)^{\frac{1}{m-1}}(\alpha + \tfrac{2mN}{m-1}),\\
&M \geq \big[(A^{\frac{2(p-1)}{n-1}}(1-\theta^2)^{-\frac{1}{m-1}} - (\alpha + \tfrac{2mN}{m-1}))^{-1} a c_3^{\frac{r-1}{m-1}} S_2^{p\beta - r\alpha} A^{\frac{2(r-1)}{m-1}}\big]^{\frac{1}{p-r}}.
\end{aligned}
$$

Therefore, we combine the two cases $\theta A \leq \xi_1 < A$ and $0 < \xi_1 < \theta A$, and we require that M satisfies

$$
M \geq \max\Big\{ \quad (\tfrac{m-1}{2\beta}\theta^{-2} a S_2^{p\beta - r\alpha} A^{\frac{2(r-1)}{m-1}}), \\
\big[(A^{\frac{2(p-1)}{n-1}}(1-\theta^2)^{-\frac{1}{m-1}} - (\alpha + \tfrac{2mN}{m-1}))^{-1} a c_3^{\frac{r-1}{m-1}} S_2^{p\beta - r\alpha} A^{\frac{2(r-1)}{m-1}}\big]^{\frac{1}{p-r}} \Big\}.
$$

Thus, we have proved

$$\underline{W}_s - \Delta \underline{W}^m - \underline{Z}^p + a M_v^{-p} M_u^r \underline{W}^r \leq 0.$$

Similarly, we can prove

$$\underline{Z}_s - \Delta \underline{Z}^n - \underline{W}^q + b M_u^{-q} M_v^s \underline{Z}^s \leq 0.$$

Finally, we choose some large initial data $u_0(x)$ and $v_0(x)$, such that $u_0 \geq \underline{W}(y,0)$ and $v_0 \geq \underline{Z}(y,0)$. Then, we have a subsolution independent of M. Thus, the blow-up time of $\varphi_{M_u}, \psi_{M_v}$ is less than S_2, i.e.,

$$\max\{M_u^{-1} M_v^p (T-t), M_v^{-1} M_u^q (T-t) \leq S_2.$$

Without loss of generality, we may assume

$$M_u^{-1} M_v^p (T-t) = \max\{M_u^{-1} M_v^p (T-t), M_v^{-1} M_u^q (T-t)\} \leq S_2.$$

From Lemma 2 and (29), there exists some $C > 0$, depending only on S_2, such that

$$M_u^{\frac{1}{\alpha}}(T-t) < C, \quad \text{and}, \quad M_v^{\frac{1}{\beta}}(T-t) < C.$$

That is,

$$\max_{\bar{\Omega}\times[0,t]} u(x,\tau) \leq C(T-t)^{-\alpha}, \quad \max_{\bar{\Omega}\times[0,t]} v(x,\tau) \leq C(T-t)^{-\beta}.$$

In view of step 1 and step 2, the conclusions of Theorem 4 hold. □

5. Conclusions and Observations

In this paper, we presented a novel method for deriving the blow-up rate and global existence of solutions to a parabolic system with absorption. Using the classification relationship between parameters, some interesting results were obtained by constructing appropriate auxiliary functions, using the comparison principle and the super and lower solution method. The three cases $pq < \max\{r,m\}\max\{s,n\}$, $pq = \max\{r,m\}\max\{s,n\}$ and $pq > \max\{r,m\}\max\{s,n\}$ are discussed in Theorems 1–3, respectively. For the case $pq < \max\{r,m\}\max\{s,n\}$, all non-negative solutions of (1) are global. For the case $pq = \max\{r,m\}\max\{s,n\}$, the global solution and blow-up solution of (1) are obtained. For the case $pq > \max\{r,m\}\max\{s,n\}$, any non-negative solution of (1) with a large initial condition blows up in finite time. However, it is unclear that the solution of (1) with a small initial condition will blow up in finite time or that it has global existence.

We also obtained the result that the solutions u and v blow-up simultaneously when the blow-up case occurs. More importantly, we also analyzed the blow-up rates. For the special case, if the domain is symmetric, for example if it is a ball ($N \geq 2$) or a segment ($N = 1$), the results of this paper also hold. In this case, more interesting results may be obtained due to the symmetry of the ball and the segment. The relevant proofs are left to the reader.

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