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Symmetries, Reductions and Exact Solutions of a Class of $(2k+2)$th-Order Difference Equations with Variable Coefficients

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Abstract: We perform a Lie analysis of $(2k+2)$th-order difference equations and obtain $k+1$ non-trivial symmetries. We utilize these symmetries to obtain their exact solutions. Sufficient conditions for convergence of solutions are provided for some specific cases. We exemplify our theoretical analysis with some numerical examples. The results in this paper extend to some work in the recent literature.

Keywords: difference equation; group invariant solutions; periodicity; reduction; symmetry

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1. Introduction

Rational ordinary difference equations have been studied in the literature by many researchers [1–8]. Recently, the well-known symmetry methods for differential equations have been extended to difference equations [9–11]. The concept is similar to the one developed for differential equations (see [12]), and it consists of solving the difference equations using the group of transformations that leave the equation invariant. Hydon [10] developed a systematic method that enables one to find the group of transformations for difference equations. Though his algorithm is valid for any given difference equation, he mainly applied it to second-order difference equations. This could be due to the fact that for higher-order difference equations, the calculations are cumbersome. For more references on recurrence equations via the symmetry approach, see [13–15].

In this paper, we investigate the solutions of difference equations of the form

$$x_{n+1} = \frac{x_{n-(2k+1)}}{a_n + b_n x_{n-k} x_{n-(2k+1)}}, \tag{1}$$

for some arbitrary sequences a_n and b_n using a symmetry-based method.

Special cases of the above equation exist in the recent literature. In [1], Cinar studied the positive solutions of the special case where $a_n = 1,\ b_n = 1$, and $k = 0$:

$$x_{n+1} = \frac{x_{n-1}}{1 + x_{n-1} x_n}. \tag{2}$$

In [2], the author studied the special case where $a_n = -1,\ b_n = 1$, and $k = 0$:

$$x_{n+1} = \frac{x_{n-1}}{-1 + x_{n-1} x_n}. \tag{3}$$

In [5], the author studied the special case where $a_n = 1,\ b_n = a$, and $k = 0$:

$$x_{n+1} = \frac{x_{n-1}}{1 + a x_{n-1} x_n}. \tag{4}$$

In [4], the author studied the special case where $a_n = -1,\ b_n = a$, and $k = 0$:

$$x_{n+1} = \frac{x_{n-1}}{-1 + a x_{n-1} x_n}. \tag{5}$$

In [3], the author studied the special case where $a_n = 1/a,\ b_n = b/a$, and $k = 0$:

$$x_{n+1} = \frac{a x_{n-1}}{1 + b x_{n-1} x_n}. \tag{6}$$

In [15], the author obtained the solutions of Equation (6) with less restrictions on the initial conditions.

In [6], the author studied the special case where $a_n = a,\ b_n = -1$, and $k = 0$:

$$x_{n+1} = \frac{x_{n-1}}{a - x_{n-1} x_n}. \tag{7}$$

In [7], the case where $a_n = \pm 1,\ b_n = \pm 1$, and k = 1 such that

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-1} x_{n-3}}, \tag{8}$$

was studied by Elsayed, and exact solutions were obtained.

In [8], the author studied and obtained exact solutions of the special case where $a_n = -1,\ b_n = 1$, and $k = 2$:

$$x_{n+1} = \frac{x_{n-5}}{-1 + x_{n-3} x_{n-5}}. \tag{9}$$

In [16], the author studied the positive solutions and attractivenesss of the special case where $a_n = 1$ and $b_n = 1$:

$$x_{n+1} = \frac{x_{n-(2k+1)}}{1 + x_{n-k} x_{n-(2k+1)}}. \tag{10}$$

In [17], the author studied the solutions of the special case where $a_n = -1$ and $b_n = 1/a$:

$$x_{n+1} = \frac{a x_{n-(2k+1)}}{-a + x_{n-k} x_{n-(2k+1)}}. \tag{11}$$

For more work on difference equations, please see [18].

For definiteness, we study the difference equation

$$u_{n+2k+2} = \frac{u_n}{A_n + B_n u_{n+k+1} u_n}, \tag{12}$$

instead of Equation (1). To derive the solutions for Equation (12) using a symmetry-based method, we will first find the Lie group of point transformations of Equation (12). Then, we will reduce the order via the invariants and construct the solutions. Furthermore, we will explain how the solutions for Equation (1) can be obtained from the solutions for Equation (12). Finally, we will show how one can obtain the results in the literature using our results.

Preliminaries

In this section, we provide background to Lie symmetry analysis of difference equations. In this paper, we adopt the same notation as that in [10].

Definition 1. *Let G be a local group of transformations acting on a manifold M. A subset $\mathcal{S} \subset M$ is called G-invariant, and G is called a symmetry group of $\mathcal{S}$. If whenever $x \in \mathcal{S}$, and $g \in G$ is such that $g \cdot x$ is defined, then $g \cdot x \in \mathcal{S}$ [12].*

Definition 2. *Let G be a connected group of transformations acting on a manifold M. A smooth real-valued function $\zeta : M \to \mathbb{R}$ is an invariant function for G if and only if*

$$X(\zeta) = 0 \qquad \textit{for all} \qquad x \in M,$$

and every infinitesimal generator X of G [12].

Definition 3. *A parameterized set of point transformations*

$$\Gamma_\varepsilon : x \mapsto \hat{x}(x; \varepsilon),$$

where $x = x_i, i = 1, \ldots, p$ are continuous variables is a one-parameter local Lie group of transformations if the following conditions are satisfied [10]:

1. *Γ_0 is the identity map if $\hat{x} = x$ when $\varepsilon = 0$;*
2. *$\Gamma_a \Gamma_b = \Gamma_{a+b}$ for every a and b sufficiently close to 0;*
3. *Each $\hat{x}_i$ can be represented as a Taylor series (in a neighborhood of $\varepsilon = 0$ that is determined by x), and therefore*
$$\hat{x}_i(x : \varepsilon) = x_i + \varepsilon \xi_i(x) + O(\varepsilon^2), i = 1, \ldots, p.$$

Consider the ordinary difference equation

$$u_{n+2k+2} = \omega(n, u_n, u_{n+1}, \ldots, u_{n+2k+1}), \quad n \in D \tag{13}$$

for some smooth function ω and a regular domain $D \subset \mathbb{Z}$. To find a symmetry group of Equation (13), we consider the group of point transformations given by

$$G_\varepsilon : (n, u_n) \mapsto (n, u_n + \varepsilon Q(n, u_n)), \tag{14}$$

where ε is the parameter and Q is a continuous function which we shall refer to as a characteristic. Let

$$\begin{aligned} X =& Q(n, u_n)\frac{\partial}{\partial u_n} + Q(n+1, u_{n+1})\frac{\partial}{\partial u_{n+1}} + Q(n+2, u_{n+2})\frac{\partial}{\partial u_{n+2}} + \cdots \\ &\cdots + Q(n+2k+1, u_{n+2k+1})\frac{\partial}{\partial u_{n+2k+1}} \end{aligned}$$

be the corresponding infinitesimal of G_ε. The substitution of the new variable (obtained using G_ε given in Equation (14)) in Equation (13) yields the linearized symmetry condition

$$\mathcal{S}^{(2k+2)} Q(n, u_n) - X\omega = 0 \tag{15}$$

whenever Equation (13) is true. The shift operator S acts on n as follows: $S : n \to n+1$. Once the characteristic $Q = Q(n, u_n)$ is known, the invariant V_n may be obtained by solving the characteristic system [12]

$$\frac{du_n}{Q} = \frac{du_{n+1}}{SQ} = \cdots = \frac{du_{n+2k+1}}{S^{n+2k+1}Q} \left(= \frac{dV_n}{0} \right)$$

or by introducing the canonical coordinate [19]

$$s_n = \int \frac{du_n}{Q(n, u_n)}.$$

We use the standard conventions

$$\prod_{j=s}^{l}\theta_j = 1 \text{ when } s > l \text{ and } \sum_{j=s}^{l}\theta_j = 0 \text{ when } s > l.$$

2. Symmetries

Consider the $(2k+2)$th-order difference Equation (12), which is

$$u_{n+2k+2} = \frac{u_n}{A_n + B_n u_{n+k+1} u_n}.$$

To obtain the symmetries, we impose the infinitesimal criterion of invariance (Equation (15)) to obtain

$$\begin{aligned} &Q(n+2k+2, u_{n+2k+2}) + \frac{B_n {u_n}^2}{(B_n u_n u_{n+k+1} + A_n)^2} Q(n+k+1, u_{n+k+1}) \\ &- \frac{A_n}{(B_n u_n u_{n+k+1} + A_n)^2} Q(n, u_n) = 0. \end{aligned} \tag{16}$$

The latter is a functional equation for the characteristic Q, making Equation (16) difficult to solve. To eliminate the first undesirable argument u_{n+2k+2}, we differentiate implicitly with respect to u_n (keeping u_{n+2k+2} fixed and regarding u_{n+k+1} as a function of u_n and u_{n+2k+2}). This leads to

$$\begin{aligned} &\frac{1}{(B_n u_n u_{n+k+1} + A_n)^2} \frac{\partial}{\partial u_{n+k+1}} Q(n+k+1, u_{n+k+1}) - \\ &\frac{1}{(B_n u_n u_{n+k+1} + A_n)^2} \frac{\partial}{\partial u_n} Q(n, u_n) + \frac{2}{u_n (B_n u_n u_{n+k+1} + A_n)^2} Q(n, u_n) = 0 \end{aligned}$$

After simplification, we obtain

$$\frac{\partial}{\partial u_{n+k+1}} Q(n+k+1, u_{n+k+1}) - \frac{\partial}{\partial u_n} Q(n, u_n) + \frac{2}{u_n} Q(n, u_n) = 0.$$

To eliminate the second undesirable variable u_{n+k+1}, we differentiate with respect to u_n, and we obtain the following second-order differential equation involving only the argument u_n:

$$\frac{\mathrm{d}^2}{\mathrm{d}{u_n}^2} Q(n, u_n) - \frac{2}{u_n} \frac{\mathrm{d}}{\mathrm{d}u_n} Q(n, u_n) + \frac{2}{{u_n}^2} Q(n, u_n) = 0. \tag{17}$$

The general solution of Equaiton (17) is then given by

$$Q = f(n) u_n + g(n) {u_n}^2, \tag{18}$$

where f and g are arbitrary functions of n. To eliminate the dependency between these two functions, we substitute Equation (18) into Equation (16) and simplify the resulting equation to obtain

$$\begin{aligned} &B_n g(n+k+1) {u_{n+k+1}}^2 {u_n}^2 + B_n f(n+k+1) u_{n+k+1} {u_n}^2 + B_n f(n+2k+2) u_{n+k+1} {u_n}^2 \\ &- A_n g(n) {u_n}^2 - A_n f(n) u_n + A_n f(n+2k+2) u_n + g(n+2k+2) {u_n}^2 = 0. \end{aligned}$$

We separate by powers of shifts of u_n, and we readily reduce the resulting over-determining system to obtain

$$\begin{aligned} &f(n) + f(n+k+1) = 0, \\ &g(n) = 0. \end{aligned} \tag{19}$$

The solutions for Equation (19) are

$$\begin{cases} \exp\left[\pm i\left(\frac{(2s+1)n\pi}{k+1}\right)\right], & 0 \le s \le (k-1)/2 \quad \text{if } k \text{ odd} \\ \\ (-1)^n, \quad \exp\left[\pm i\left(\frac{(2s+1)n\pi}{k+1}\right)\right], & 0 \le s \le (k-2)/2 \quad \text{if } k \text{ even.} \end{cases}$$

This means that we have $k+1$ non-trivial characteristics

$$\begin{aligned} Q_s =& (\beta_s)^n u_n, \\ \hat{Q}_s =& (\bar{\beta}_s)^n u_n, \qquad 0 \le s \le \frac{k-1}{2} \text{ if } k \text{ odd} \end{aligned}$$

and

$$\begin{aligned} Q_{-1} =& (-1)^n u_n, \\ Q_s =& (\beta_s)^n u_n, \\ \hat{Q}_s =& (\bar{\beta}_s)^n u_n, \qquad 0 \le s \le \frac{k-2}{2} \text{ if } k \text{ even}, \end{aligned}$$

where $\beta_s = \exp\left[i\left(\frac{(2s+1)\pi}{k+1}\right)\right]$. We will refer to X_i as the corresponding symmetry generator of Q_i.

3. Exact Solutions

Here, we introduce the canonical coordinate (i.e., the variable S_n such that $XS_n = 1$). We use the well-known choice [19]

$$S_n = \int \frac{du_n}{Q(n, u_n)}.$$

Using any one of the symmetry generators, such as X_s, we have that

$$S_n = \int \frac{du_n}{Q_s(n, u_n)} = \frac{1}{(\beta_s)^n} \ln |u_n|.$$

Using Equation (19), we have proven that

$$X_s\left[(\beta_s)^{n+k+1} S_{n+k+1} + (\beta_s)^n S_n\right] = 0 \tag{20}$$

Therefore, the following is an invariant function of X_s:

$$r_n = (\beta_s)^{n+k+1} S_{n+k+1} + (\beta_s)^n S_n$$

Since the equation under study is a rational difference equation, it is favorable to use

$$|\tilde{r_n}| = \exp(-r_n), \tag{21}$$

In other words, $\tilde{r}_n = \pm 1/u_n u_{n+k+1}$. It turns out (using the plus sign together with Equation (12)) that

$$\tilde{r}_{n+k+1} = A_n \tilde{r}_n + B_n. \tag{22}$$

It is easy to verify that the solution for Equation (22) in closed form is given by

$$\tilde{r}_{(k+1)n+j} = \tilde{r}_j \left(\prod_{k_1=0}^{n-1} A_{(k+1)k_1+j} \right) + \sum_{l=0}^{n-1} \left(B_{(k+1)l+j} \prod_{k_2=l+1}^{n-1} A_{(k+1)k_2+j} \right), \tag{23}$$

for $j = 0, 1, \ldots, k$. By going up the hierarchy created by the change of variables, we obtain the solution for Equation (12) in a unified manner as follows:

- For k being odd:

$$\begin{aligned} |u_n| =& \exp\left\{ \sum_{s=0}^{\frac{k-1}{2}} (c_s \beta_s{}^n + \tilde{c}_s \bar{\beta}_s^n) + \sum_{s=0}^{\frac{k-1}{2}} \left(\sum_{k_s=0}^{n-1} \frac{1}{k+1} \beta_s{}^n \bar{\beta}_s^{k_s} \ln \tilde{r}_{k_s} \right) \right. \\ & \left. + \sum_{s=0}^{\frac{k-1}{2}} \left(\sum_{k_s=0}^{n-1} \frac{1}{k+1} \bar{\beta}_s{}^n \beta_s{}^{k_s} \ln \tilde{r}_{k_s} \right) \right\} \\ |u_n| =& \Gamma_n \exp\left\{ \sum_{s=0}^{\frac{k-1}{2}} \left(\sum_{k_s=0}^{n-1} \frac{2}{k+1} \mathrm{Re}[\gamma_s(n,k_s)] \ln \tilde{r}_{k_s} \right) \right\} \end{aligned} \tag{24}$$

- For k being even:

$$\begin{aligned} |u_n| = \quad & \theta_n \exp\left\{ \sum_{s=0}^{\frac{k-2}{2}} \left(\sum_{k_s=0}^{n-1} \tfrac{2}{k+1} \mathrm{Re}[\gamma_s(n,k_s)] \ln \tilde{r}_{k_s} \right) \right. \\ & \left. + (-1)^n \textstyle\sum_{j=0}^{n-1} \tfrac{1}{k+1} (-1)^j \ln \tilde{r}_{(k+1)j} \right\} \end{aligned} \tag{25}$$

where

$$\begin{aligned} \Gamma_n &= \exp\left[\sum_{s=0}^{\frac{k-1}{2}} (c_s \beta_s{}^n + \tilde{c}_s \bar{\beta}_s^n) \right], \\ \theta_n &= \exp\left[c(-1)^n + \sum_{s=0}^{\frac{k-2}{2}} (c_s \beta_s{}^n + \tilde{c}_s \bar{\beta}_s^n) \right], \\ \gamma_s(n,k_s) &= \beta_s{}^n \bar{\beta}_s^{k_s}. \end{aligned}$$

The properties of $\Gamma_n,\ \theta_n$ and $\gamma(n,k)$ are as follows:

$$\begin{aligned} &\Gamma_{(2k+2)n+j} = \Gamma_j = |u_j|, && 0 \le j \le k \\ &\Gamma_{(2k+2)n+k+1+j} = \frac{1}{\Gamma_j} = \frac{1}{|u_j|}, && 0 \le j \le k \end{aligned}$$

$$\begin{aligned} &\theta_{(2k+2)n+j} = \theta_j = |u_j|, && 0 \le j \le k \\ &\theta_{(2k+2)n+k+1+j} = \frac{1}{\theta_j} = \frac{1}{|u_j|}, && 0 \le j \le k \end{aligned}$$

$$\begin{aligned} &\gamma_s((2k+2)n, j) = \gamma_s(0,j), \\ &\gamma_s(n+k+1, j) = \gamma_s(n, j+k+1) = -\gamma_s(n,j), \\ &\gamma_s(j,j) = 1. \end{aligned}$$

In regard to these properties, one can further simplify the solution through Equations (24) and (25). We have that

$$
\begin{aligned}
|u_{(2k+2)n}| =&\Gamma_{(2k+2)n}\exp\left\{\sum_{s=0}^{\frac{k-1}{2}}\left(\sum_{k_s=0}^{(2k+2)n-1}\frac{2}{k+1}\mathrm{Re}[\gamma_s((2k+2)n,k_s)]\ln\tilde{r}_{k_s}\right)\right\}\\
=&u_0\exp\left\{\sum_{s=0}^{\frac{k-1}{2}}\left(\sum_{k_s=0}^{(2k+2)n-1}\frac{2}{k+1}\mathrm{Re}[\gamma_s(0,k_s)]\ln\tilde{r}_{k_s}\right)\right\}\\
=&u_0\exp\left\{\frac{2}{k+1}\sum_{k_0=0}^{(2k+2)n-1}\mathrm{Re}\left[\exp\left(-\frac{k_0}{k+1}i\pi\right)+\exp\left(-\frac{3k_0}{k+1}i\pi\right)\right.\right.\\
&\left.\left.+\dots+\exp\left(-\frac{kk_0}{k+1}i\pi\right)\right]\ln\tilde{r}_{k_0}\right\} \qquad (26)\\
=&u_0\exp\left\{\frac{2}{k+1}\sum_{k_0=0}^{2n-1}\mathrm{Re}\left[\exp(-k_0 i\pi)+\exp(-3k_0 i\pi)+\dots\right.\right.\\
&\left.\left.+\exp(-kk_0 i\pi)\right]\ln\tilde{r}_{k_0}\right\}\\
=&u_0\exp\left\{\sum_{k_0=0}^{2n-1}(-1)^{k_0}\ln\tilde{r}_{k_0(k+1)}\right\}\\
=&\left|u_0\prod_{k_0=0}^{n-1}\frac{\tilde{r}_{2k_0(k+1)}}{\tilde{r}_{(2k_0+1)(k+1)}}\right|.
\end{aligned}
$$

Note: It can be shown, using the expression of $\tilde{r}_n$ given in Equation (21), that one can write Equation (26) without using the symbol of the absolute value.

On this note, and in the same way, we have proven that

$$
u_{(2k+2)n+j} = u_j\prod_{k_0=0}^{n-1}\frac{\tilde{r}_{2k_0(k+1)+j}}{\tilde{r}_{(2k_0+1)(k+1)+j}},\qquad 0\le j\le k, \tag{27}
$$

$$
u_{(2k+2)n+k+1+j} = u_{k+1+j}\prod_{k_0=0}^{n-1}\frac{\tilde{r}_{(2k_0+1)(k+1)+j}}{\tilde{r}_{(2k_0+2)(k+1)+j}},\qquad 0\le j\le k. \tag{28}
$$

We therefore obtain, by combining Equations (23), (27) and (28) and by replacing $\tilde{r}_i$ with $1/(u_i u_{i+k+1})$, the solution for Equation (12) in closed form as

$$
u_{(2k+2)n+j} = u_j\prod_{k_0=0}^{n-1}\frac{\left(\prod_{k_1=0}^{2k_0-1}A_{(k+1)k_1+j}\right)+u_ju_{j+k+1}\sum_{l=0}^{2k_0-1}\left(B_{(k+1)l+j}\prod_{k_2=l+1}^{2k_0-1}A_{(k+1)k_2+j}\right)}{\left(\prod_{k_1=0}^{2k_0}A_{(k+1)k_1+j}\right)+u_ju_{j+k+1}\sum_{l=0}^{2k_0}\left(B_{(k+1)l+j}\prod_{k_2=l+1}^{2k_0}A_{(k+1)k_2+j}\right)}, \tag{29}
$$

$$
u_{(2k+2)n+k+1+j} = u_{k+1+j}\prod_{k_0=0}^{n-1}\frac{\left(\prod_{k_1=0}^{2k_0}A_{(k+1)k_1+j}\right)+u_ju_{j+k+1}\sum_{l=0}^{2k_0}\left(B_{(k+1)l+j}\prod_{k_2=l+1}^{2k_0}A_{(k+1)k_2+j}\right)}{\left(\prod_{k_1=0}^{2k_0+1}A_{(k+1)k_1+j}\right)+u_ju_{j+k+1}\sum_{l=0}^{2k_0+1}\left(B_{(k+1)l+j}\prod_{k_2=l+1}^{2k_0+1}A_{(k+1)k_2+j}\right)}, \tag{30}
$$

where $0\le j\le k$ and provided that

$$
-u_ju_{j+k+1}\sum_{l=0}^{m}\left(B_{(k+1)l+j}\prod_{k_2=l+1}^{m}A_{(k+1)k_2+j}\right)\neq\prod_{k_1=0}^{m}A_{(k+1)k_1+j}, \tag{31}
$$

where $m < 2n$. Recall that we enacted the shift operator $2k+1$ times in Equation (1) to obtain Equation (12), and therefore we can write the solutions for Equation (1) as follows

$$x_{(2k+2)n-(2k+1)+j} = x_{j-2k-1}\prod_{k_0=0}^{n-1}\frac{\left(\prod_{k_1=0}^{2k_0-1}a_{(k+1)k_1+j}\right)+x_{j-2k-1}x_{j-k}\sum_{l=0}^{2k_0-1}\left(b_{(k+1)l+j}\prod_{k_2=l+1}^{2k_0-1}a_{(k+1)k_2+j}\right)}{\left(\prod_{k_1=0}^{2k_0}a_{(k+1)k_1+j}\right)+x_{j-2k-1}x_{j-k}\sum_{l=0}^{2k_0}\left(b_{(k+1)l+j}\prod_{k_2=l+1}^{2k_0}a_{(k+1)k_2+j}\right)}, \tag{32}$$

$$x_{(2k+2)n-k+j} = x_{j-k}\prod_{k_0=0}^{n-1}\frac{\left(\prod_{k_1=0}^{2k_0}a_{(k+1)k_1+j}\right)+x_{j-2k-1}x_{j-k}\sum_{l=0}^{2k_0}\left(b_{(k+1)l+j}\prod_{k_2=l+1}^{2k_0}a_{(k+1)k_2+j}\right)}{\left(\prod_{k_1=0}^{2k_0+1}a_{(k+1)k_1+j}\right)+x_{j-2k-1}x_{j-k}\sum_{l=0}^{2k_0+1}\left(b_{(k+1)l+j}\prod_{k_2=l+1}^{2k_0+1}a_{(k+1)k_2+j}\right)}, \tag{33}$$

provided that

$$-x_{j-2k-1}x_{j-k}\sum_{l=0}^{m}\left(b_{(k+1)l+j}\prod_{k_2=l+1}^{m}a_{(k+1)k_2+j}\right)\neq\prod_{k_1=0}^{m}a_{(k+1)k_1+j},\ m<2n. \tag{34}$$

3.1. Case with $(K+1)$-Periodic Sequences (A_n) and (B_n)

Let $(a_n) = (a_0, a_1, \ldots, a_k, a_0, a_1, \ldots)$, $(b_n) = (b_0, b_1, \ldots, b_k, b_0, b_1, \ldots)$ and $\Phi_j = x_{j-2k-1}x_{j-k}b_j$. The solutions given in Equations (32), (33) and (34) simplify to

$$x_{(2k+2)n+j-(2k+1)} = x_{j-2k-1}\prod_{k_0=0}^{n-1}\frac{(a_j)^{2k_0}+\Phi_j\sum_{l=0}^{2k_0-1}(a_j)^l}{(a_j)^{2k_0+1}+\Phi_j\sum_{l=0}^{2k_0}(a_j)^l}, \tag{35}$$

$$x_{(2k+2)n-k+j} = x_{j-k}\prod_{k_0=0}^{n-1}\frac{(a_j)^{2k_0+1}+\Phi_j\sum_{l=0}^{2k_0}(a_j)^l}{(a_j)^{2k_0+2}+\Phi_j\sum_{l=0}^{2k_0+1}(a_j)^l}, \tag{36}$$

with

$$\Phi_j\sum_{l=0}^{2k_0}(a_j)^l\neq-(a_j)^{2k_0+1},\quad k_0<n, \Phi_j\sum_{l=0}^{2k_0+1}(a_j)^l\neq-(a_j)^{2k_0+2},\ k_0<n,\ j\leq k. \tag{37}$$

Case with $A_j \neq 1$

The solutions given in (35), (36) and (37) become

$$x_{(2k+2)n+j-(2k+1)} = x_{j-2k-1}\prod_{k_0=0}^{n-1}\frac{(a_j)^{2k_0}+\Phi_j\left(\frac{1-(a_j)^{2k_0}}{1-a_j}\right)}{(a_j)^{2k_0+1}+\Phi_j\left(\frac{1-(a_j)^{2k_0+1}}{1-a_j}\right)} \tag{38}$$

$$x_{(2k+2)n-k+j} = x_{j-k}\prod_{k_0=0}^{n-1}\frac{(a_j)^{2k_0+1}+\Phi_j\left(\frac{1-(a_j)^{2k_0+1}}{1-a_j}\right)}{(a_j)^{2k_0+2}+\Phi_j\left(\frac{1-(a_j)^{2k_0+2}}{1-a_j}\right)}, \tag{39}$$

with

$$\Phi_j\left(1-(a_j)^{2k_0+1}\right) \neq -(1-a_j)(a_j)^{2k_0+1}, \tag{40}$$

$$\Phi_j\left(1-(a_j)^{2k_0+2}\right) \neq -(1-a_j)(a_j)^{2k_0+2},\ k_0<n,\ j\leq k. \tag{41}$$

3.2. Case with One-Periodic Sequences (A_n) and (B_n)

Let $(a_n)=(a_0,a_0,\dots)$, $(b_n)=(b_0,b_0,\dots)$ and $\Phi_j=x_{j-2k-1}x_{j-k}b_0$. The solutions given in Equations (35), (36) and (37) simplify to

$$x_{(2k+2)n+j-(2k+1)} = x_{j-2k-1}\prod_{k_0=0}^{n-1}\frac{(a_0)^{2k_0}+\Phi_j\sum_{l=0}^{2k_0-1}(a_0)^l}{(a_0)^{2k_0+1}+\Phi_j\sum_{l=0}^{2k_0}(a_0)^l} \tag{42}$$

$$x_{(2k+2)n-k+j} = x_{j-k}\prod_{k_0=0}^{n-1}\frac{(a_0)^{2k_0+1}+\Phi_j\sum_{l=0}^{2k_0}(a_0)^l}{(a_0)^{2k_0+2}+\Phi_j\sum_{l=0}^{2k_0+1}(a_0)^l}, \tag{43}$$

with

$$\Phi_j\sum_{l=0}^{2k_0}(a_0)^l \neq -(a_0)^{2k_0+1}, \Phi_j\sum_{l=0}^{2k_0+1}(a_0)^l \neq -(a_0)^{2k_0+2},\ k_0<n,\ j\leq k. \tag{44}$$

3.2.1. Case with $A_0 \neq 1$

The solutions given in Equations (42), (43) and (44) become

$$x_{(2k+2)n+j-(2k+1)} = x_{j-2k-1}\prod_{k_0=0}^{n-1}\frac{(a_0)^{2k_0}+\Phi_j\left(\frac{1-(a_0)^{2k_0}}{1-a_0}\right)}{(a_0)^{2k_0+1}+\Phi_j\left(\frac{1-(a_0)^{2k_0+1}}{1-a_0}\right)} \tag{45}$$

$$x_{(2k+2)n-k+j} = x_{j-k}\prod_{k_0=0}^{n-1}\frac{(a_0)^{2k_0+1}+\Phi_j\left(\frac{1-(a_0)^{2k_0+1}}{1-a_0}\right)}{(a_0)^{2k_0+2}+\Phi_j\left(\frac{1-(a_0)^{2k_0+2}}{1-a_0}\right)}, \tag{46}$$

with

$$\Phi_j\left(1-(a_0)^{2k_0+1}\right) \neq -(1-a_0)(a_0)^{2k_0+1}, \Phi_j\left(1-(a_0)^{2k_0+2}\right) \neq -(1-a_0)(a_0)^{2k_0+2}, \tag{47}$$

$k_0<n,\ j\leq k.$

3.2.2. Case with $A_0 = 1$

The solutions given in Equations (42), (43) and (44) become

$$x_{(2k+2)n+j-(2k+1)} = x_{j-2k-1}\prod_{k_0=0}^{n-1}\frac{1+(2k_0)\Phi_j}{1+(2k_0+1)\Phi_j} \tag{48}$$

$$x_{(2k+2)n-k+j} = x_{j-k}\prod_{k_0=0}^{n-1}\frac{1+(2k_0+1)\Phi_j}{1+(2k_0+2)\Phi_j}, \tag{49}$$

with

$$(2k_0+1)\Phi_j \neq -1, \quad (2k_0+2)\Phi_j \neq -1,\ k_0<n,\ j\leq k. \tag{50}$$

4. Results

In this section, we verify the results in [1–8,16,17] by utilizing different combinations of values of a_n and b_n in Equation (1):

- If we set $k = 0,\ b_0 = 1$ in Equations (42)–(44), we obtain the result in [1] (see Theorem 2.1) for Equation (2). However, the restriction (x_{-1} and x_0 are positive real numbers) in [1] is a special case of our restrictions ($(2k_0+1)x_{-1}x_0 \neq -1$ and $(2k_0+1)x_0x_1 \neq -1,\ k_0 < n$);
- If we set $k = 0,\ b_0 = 1$ in Equations (45)–(47), we obtain the result in [2] for Equation (3) (see Theorem 2.1). The restriction ($x_{-1}x_0 \neq 1$) in [2] coincides with our restriction;
- If we set $k = 0,\ b_0 = a$ in Equations (48)–(50), we obtain the result in [5] for Equation (4) (see Theorem 2.1). However, the restriction (x_{-1} and x_0 are nonnegative real numbers) in [5] is a special case of our restrictions ($(2k_0+1)ax_{-1}x_0 \neq -1$ and $(2k_0+2)ax_{-1}x_0 \neq -1,\ k_0 < n$);
- If we set $k = 0$ and $b_0 = a$ in Equations (45)–(47), we obtain the result in [4] for Equation (5) (see Theorem 2.1), and the restriction ($ax_{-1}x_0 \neq 1$) in [4] coincides with our restriction;
- If we set $k = 0,\ a_0 = (1/a)$, and $b_0 = (b/a)$ in Equations (42)–(44) we obtain the result in [3] for Equation (6) (see Theorem 2.1). However, the restriction ($a,\ b,\ x_{-1}$, and x_0 are nonnegative real numbers) in [3] is a special case of our restrictions ($bx_{-1}x_0 \sum\limits_{l=0}^{2k_0} a^l \neq -1$ and $bx_{-1}x_0 \sum\limits_{l=0}^{2k_0+1} a^l \neq -1,\ k_0 < n$);
- If we set $k = 0,\ a_0 = a$, and $b_0 = -1$ in Equations (42)–(44), we obtain the result (for the case $a \neq 1$) in [6] for Equation (7) (see Theorem 2), and the restriction ($x_{-1}x_0 \neq a^j(1-a)/(1-a^j)$) in [6] coincides with our restrictions ($x_{-1}x_0\left(1-a^{2k_0+1}\right) \neq (1-a)a^{2k_0+1},\ x_{-1}x_0\left(1-a^{2k_0+2}\right) \neq (1-a)a^{2k_0+2}$, and $k_0 < n$). Additionally, the solution for the case $a = 1$ (see Theorem 5 in [6]) corresponds to our solution with the same restrictions on the initial conditions;
- If we set $k = 1,\ a_0 = 1$, and $b_0 = 1$ (resp $b_0 = -1$) in Equations (48)–(50), we obtain the result in [7] for Equation (8) (see Theorem 1 (resp. Theorem 4)). However, the restriction (x_{-1} and x_0 are nonzero positive real numbers (resp. $jx_0x_{-2} \neq 1, jx_{-1}x_{-3} \neq 1$ for $j = 1, 2, 3, \ldots$)) in [7] is a special case of our restrictions ($(2k_0+1)x_{-1}x_{-3} \neq -1,\ (2k_0+2)x_0x_{-2} \neq -1$ (resp. $[2k_0+1]x_{-1}x_{-3} \neq 1$, and $[2k_0+2]x_0x_{-2} \neq 1$)). On the other hand, if $a_0 = -1$ and $b_0 = \pm 1$, the results are the same as in [7] (see Theorems 6 and 9) with the same restrictions on the initial conditions ($x_0x_{-2} \neq 1$ and $x_{-1}x_{-3} \neq 1$);
- If we set $k = 2,\ a_0 = -1$, and $b_0 = 1$ in Equations (45)–(47), we obtain the result in [8] for Equation (9) (see Theorem 1), and the restrictions ($x_0x_{-3} \neq 1,\ x_{-1}x_{-4} \neq 1$, and $x_{-5}x_{-2} \neq 1$) in [8] coincide with our restriction;
- If we set $a_n = 1$ and $b_n = 1$ in Equations (32)–(34), we obtain the result in [16] for Equation (10) (see Theorem 1). The author in [16] restricted himself to positive solutions. However, our result is valid for negative solutions as well, provided that Equation (34) is satisfied.
- If we set $a_0 = -1$ and $b_0 = 1/a$ in Equations (45)–(47), we obtain the result in [17] for Equation (11) (see Theorem 1). However, the restriction ($x_0x_{-(k+1)} \neq a;\ x_{-1}x_{-(k+2)} \neq a;\ x_2{-}2x_{-(k+3)} \neq a, \ldots, x_{-k}x_{-(2k+1)} \neq a$) in [17] is a special case of our restriction ($x_{j-2k-1}x_{j-k} \neq a,\ j \leq k$).

5. Numerical Examples

Below are some graphs that show the behavior of the solutions.

Example 1. *For the case with* $A_n = 1$, $B_n = B$ *and* $k = 0$, *where* $u_{n+2} = \frac{u_n}{1+Bu_nu_{n+1}}$, *setting* $A_n = 1$, $B_n = B$, *and* $k = 0$ *in Equations (29) and (30) yields*

$$u_{2n} = u_0 \prod_{k_0=0}^{n-1} \frac{1 + 2k_0 B u_0 u_1}{1 + (2k_0 + 1) B u_0 u_1}$$

and

$$u_{2n+1} = u_1 \prod_{k_0=0}^{n-1} \frac{1 + (2k_0 + 1) B u_0 u_1}{1 + (2k_0 + 2) B u_0 u_1}.$$

A sufficient condition for this solution to converge is given by $1 + 2k_0Bu_0u_1 < 1 + (2k_0 + 1)Bu_0u_1$ *and* $1 + (2k_0 + 1)Bu_0u_1 < 1 + (2k_0 + 2)Bu_0u_1$ *for* $0 \leq k_0 \leq n - 1$. *This implies that* $Bu_0u_1 > 0$. *It is then clear that if* $Bu_0u_1 > 0$, *then the solution converges. On the other hand, a sufficient condition for this solution to diverge is given by* $1 + 2k_0Bu_0u_1 > 1 + (2k_0 + 1)Bu_0u_1$ *or* $1 + (2k_0 + 1)Bu_0u_1 > 1 + (2k_0 + 2)Bu_0u_1$ *for* $0 \leq k_0 \leq n - 1$. *This implies that* $Bu_0u_1 < 0$. *It is then clear that if* $Bu_0u_1 < 0$, *then the solution diverges.*

Figure 1 shows the graph of $u_{n+2} = \frac{u_n}{1+u_nu_{n+1}}$ *with* $u_0 = 2$ *and* $u_1 = 5$. *The solution converges as expected, since* $Bu_0u_1 > 0$.

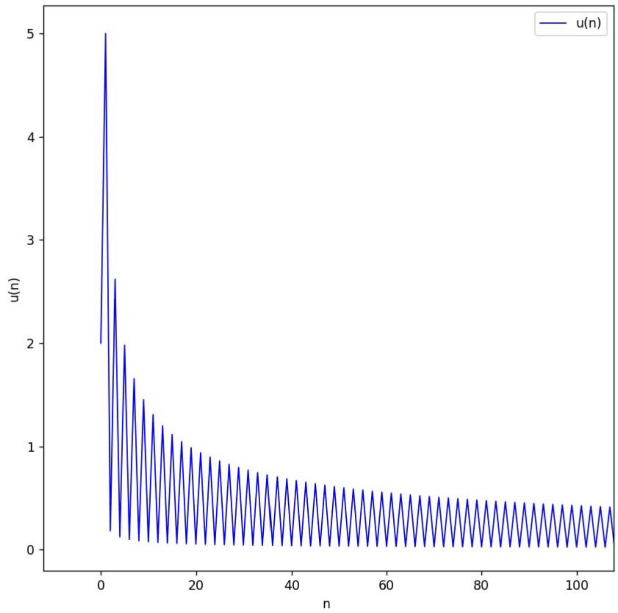

Figure 1. The graph of $u_{n+2} = \frac{u_n}{1+u_nu_{n+1}}$.

Figure 2 shows the graph of $u_{n+2} = \frac{u_n}{1-0.28u_nu_{n+1}}$ *with* $u_0 = 0.08$ *and* $u_1 = 0.67$. *The solution diverges as expected, since* $Bu_0u_1 < 0$.

Example 2. *In the case with* $A_n = A \neq 1$, $B_n = B$, *and* $k = 1$ *such that* $u_{n+4} = \frac{u_n}{A+Bu_nu_{n+2}}$, *setting* $A_n = A$, $B_n = B$, *and* $k = 1$ *in Equations (29) and (30) yields*

$$u_{4n} = \frac{u_0}{A + Bu_0u_2} \prod_{k_0=1}^{n-1} \frac{A^{2k_0} + Bu_0u_2 \sum_{l=0}^{2k_0-1} A^l}{A^{2k_0+1} + Bu_0u_2 \sum_{l=0}^{2k_0} A^l}, \tag{51}$$

$$u_{4n+1} = \frac{u_1}{A + Bu_1u_3} \prod_{k_0=1}^{n-1} \frac{A^{2k_0} + Bu_1u_3 \sum_{l=0}^{2k_0-1} A^l}{A^{2k_0+1} + Bu_1u_3 \sum_{l=0}^{2k_0} A^l}, \tag{52}$$

$$u_{4n+2} = u_2 \prod_{k_0=0}^{n-1} \frac{A^{2k_0+1} + Bu_0u_2 \sum_{l=0}^{2k_0} A^l}{A^{2k_0+2} + Bu_0u_2 \sum_{l=0}^{2k_0+1} A^l}, \tag{53}$$

$$u_{4n+3} = u_3 \prod_{k_0=0}^{n-1} \frac{A^{2k_0+1} + Bu_1u_3 \sum_{l=0}^{2k_0} A^l}{A^{2k_0+2} + Bu_1u_3 \sum_{l=0}^{2k_0+1} A^l}. \tag{54}$$

Similarly, a sufficient condition for this solution to converge is for the numerators in Equations (51)–(54) to be less than their corresponding denominators. This implies that $Bu_0u_2 > 1 - A$ and $Bu_1u_3 > 1 - A$, where $A > 0$. It is then clear that if $Bu_0u_2 > 1 - A$ and $Bu_1u_3 > 1 - A$, then the solution converges. On the other hand, a sufficient condition for this solution to diverge is for at least one of the numerators in Equations (51)–(54) to be greater than its corresponding denominator. This implies that $Bu_0u_2 < 1 - A$ or $Bu_1u_3 < 1 - A$, where $A > 0$. It is then clear that if $Bu_0u_2 < 1 - A$ or $Bu_1u_3 < 1 - A$, then the solution diverges.

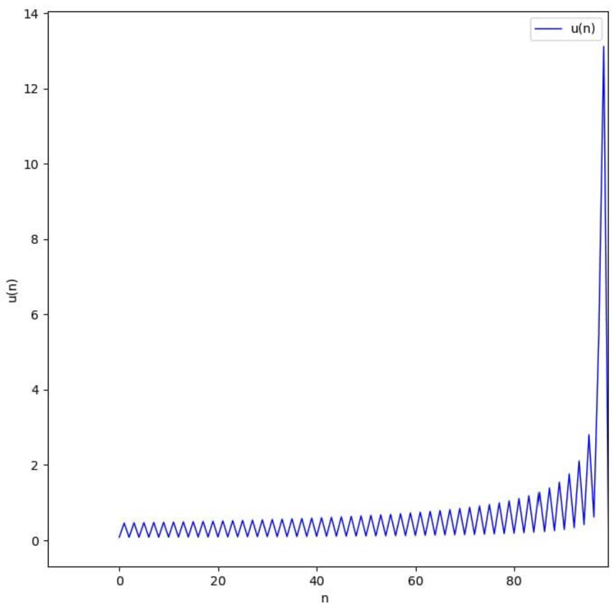

Figure 2. The graph of $u_{n+2} = \frac{u_n}{1-0.28u_nu_{n+1}}$.

Figure 3 shows the graph of $u_{n+4} = \frac{u_n}{2-2u_nu_{n+2}}$ with $u_0 = 0.63$, $u_1 = 0.33$, $u_2 = 0.72$, and $u_3 = 0$. The solution converges as expected, since $Bu_0u_2 > 1 - A$ and $Bu_1u_3 > 1 - A$.

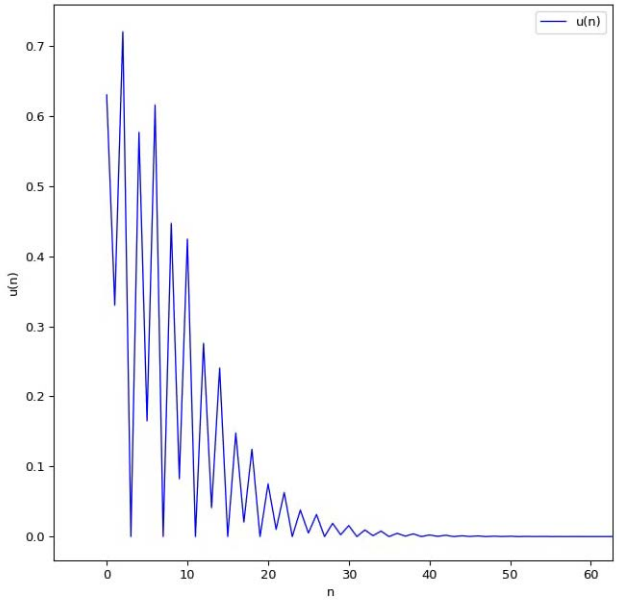

Figure 3. The graph of $u_{n+4} = \frac{u_n}{2-2u_nu_{n+2}}$.

Figure 4 shows the graph of $u_{n+4} = \frac{u_n}{0.78+0.18u_nu_{n+2}}$ *with* $u_0 = 0.36$, $u_1 = 0.63$, $u_2 = 0.01$, *and* $u_3 = 0$. *The solution diverges as expected, since* $Bu_0u_2 < 1 - A$.

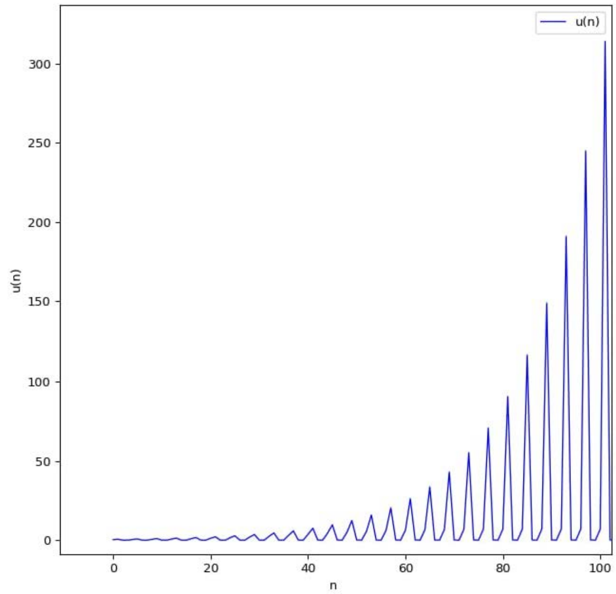

Figure 4. The graph of $u_{n+4} = \frac{u_n}{0.78+0.18u_nu_{n+2}}$.

6. Conclusions

We obtained all the Lie point symmetries of the difference equation (Equation (12)). We used a symmetry-based method to derive its solutions. The solutions were given in 'single' form and then separated into $(2k+2)$ categories. We explained how the solutions of Equation (1) can be obtained using those of Equation (12) to show that the results in the recent literature can be obtained as special cases of our generalized solutions.

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