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Diagonals of Rational Functions: From Differential Algebra to Effective Algebraic Geometry

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Abstract: We show that the results we had previously obtained on diagonals of 9- and 10-parameter families of rational functions in three variables x, y, and z, using creative telescoping, yielding modular forms expressed as pullbacked ${}_2F_1$ hypergeometric functions, can be obtained much more efficiently by calculating the j-invariant of an elliptic curve canonically associated with the denominator of the rational functions. These results can be drastically generalized by changing the parameters into arbitrary rational functions of the product $p = xyz$. In other cases where creative telescoping yields pullbacked ${}_2F_1$ hypergeometric functions, we extend this algebraic geometry approach to other families of rational functions in three or more variables. In particular, we generalize this approach to rational functions in more than three variables when the denominator can be associated to an algebraic variety corresponding to products of elliptic curves, or foliations in elliptic curves. We also extend this approach to rational functions in three variables when the denominator is associated with a genus-two curve such that its Jacobian is a split Jacobian, corresponding to the product of two elliptic curves. We sketch the situation where the denominator of the rational function is associated with algebraic varieties that are not of the general type, having an infinite set of birational automorphisms. We finally provide some examples of rational functions in more than three variables, where the telescopers have pullbacked ${}_2F_1$ hypergeometric solutions, because the denominator corresponds to an algebraic variety that has a selected elliptic curve.

Keywords: diagonal of a rational function; pullbacked hypergeometric function; modular form; Hauptmodul; creative telescoping; telescoper; elliptic curve; j-invariant; K3 surface; split Jacobian; extremal rational surface; birational automorphism; algebraic variety of general type

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1. Introduction

In previous papers [1,2], using creative telescoping [3], we have obtained *diagonals* (for the introduction of the concept of diagonals of rational functions, see [4–11]) of 9- and 10-parameter families of rational functions, given by (classical) modular forms expressed as pullbacked ${}_2F_1$ hypergeometric functions [12]. The natural emergence of diagonals of rational functions (the lattice Green functions are the simplest examples of such diagonals of rational functions [13–18]) in lattice statistical mechanics is explained in [19,20]. This can be seen as the reason for the frequent occurrence of *modular forms* and *Calabi–Yau operators* in lattice statistical mechanics [21–27]. In other previous papers [17,18] dedicated to Heun functions that are either diagonals of simple rational functions, or only solutions of *telescopers* [28,29] of simple rational functions of three or four variables, we have obtained many order-three telescopers having squares of Heun functions as solutions that turn out to be squares of pullbacked ${}_2F_1$ hypergeometric functions corresponding to *classical modular forms* and even *Shimura automorphic forms* [30,31], strongly reminiscent of periods of *extremal rational surfaces* [32,33], and other foliations of K3 surfaces in elliptic curves. In other words, one finds experimentally that the ${}_2F_1$ hypergeometric functions emerging in the calculation of the diagonals of rational functions, or of the solutions of the telescopers of rational functions, seem to be only special ${}_2F_1(a, b; c; x)$ hypergeometric functions with a selected set of parameters a, b, c (see the list (B.1) in Appendix B of [17], corresponding to classical modular forms) (see Felix Klein's connection of the ${}_2F_1(1/12, 5/12; 1; x)$ Gauss hypergeometric function with modular forms; for instance, in the very pedagogical and heuristic paper, [12]), together with a finite set of parameters, such as $7/24, 11/24, 5/4$, corresponding to Shimura automorphic forms [30,31]), pullbacked by selected pullbacks. This last paper [17] also underlined the difference between the diagonal of a rational function $Diag(R)$, and the solutions of the telescoper of the same rational function.

These results strongly suggested to find an algebraic geometry interpretation for all of these exact results, and more generally, they suggested to provide an alternative algebraic geometry approach of the results emerging from creative telescoping (the reader may refer to [34] for an extensive survey of "creative telescoping" approaches).

This is the purpose of the present paper. In particular, we are going to show that most of these pullbacked ${}_2F_1$ hypergeometric functions can be obtained efficiently through algebraic geometry calculations, thus providing a more intrinsic algebraic geometry interpretation of the creative telescoping calculations that are typically differential algebra calculations [28,29,34,35].

Creative telescoping [28,29,34,36] is a methodology to deal with parametrized symbolic sums and integrals that yields differential/recurrence equations for such expressions. This methodology became popular in computer algebra in the past 25 years. By the "telescoper" of a rational function, say $R(x, y, z)$, we here refer to the output of the creative telescoping program [3], applied to the *transformed* rational function $\tilde{R} = R(x/y, y/z, z)/(yz)$. Such a telescoper is a linear differential operator T in x and $\frac{\partial}{\partial x}$, such that $T + \frac{\partial}{\partial y} \cdot U + \frac{\partial}{\partial z} \cdot V$ annihilates $\tilde{R}$, where the so-called "certificates" U, V are rational functions in x, y, z. In other words, the telescoper T represents a linear ODE that is satisfied by $Diag(R)$.

The paper is essentially dedicated to the solutions of telescopers of rational functions that are *not necessarily* diagonals of rational functions. These solutions correspond to *periods* [37] of algebraic varieties over some cycles that are not necessarily vanishing [38] cycles (in French: "cycles évanescents" [5,39]) as with the case of the diagonals of rational functions. The reader who is interested in the connection between the process of taking diagonals, calculating telescopers, and the notion of periods, de Rham cohomology (i.e., differential forms) and other Picard–Fuchs equations can read in detail the thesis of Pierre Lairez [35] (see also [40]). We sketch just some of these ideas in Appendix A.

The purpose of this paper is not to give an introduction to creative telescoping [28,29], but to provide many pedagogical (non-trivial) examples of telescopers using (one can obtain these telescopers using Chyzak's algorithm [41] or Koutschan's semi-algorithm [3,42] (the termination is not proven). For the examples displayed in this paper, Koutschan's package [3] is more efficient) the HolonomicFunctions Mathematica package extensively [3].

The paper is organized as follows. We first recall in Section 2 the exact results of [1,2] for the 9- and 10-parameter families of rational functions using creative telescoping, yielding modular forms expressed as pullbacked ${}_2F_1$ hypergeometric functions. We show in Section 3 that these exact results can be obtained, much more efficiently, by calculating the j-invariant of an elliptic curve that is canonically associated with the denominator of the rational function, and we underline the fact that one can drastically generalize these results, the parameters becoming quite arbitrary rational functions. Section 4 generalizes the previous calculations to denominators of the rational functions of more than three variables, corresponding to the products (or foliations) of elliptic curves. In Section 5, we show how modular forms expressed as pullbacked $_2F_1$ hypergeometric functions occur for rational functions in three variables when the denominator is associated with a genus-two curve such that its Jacobian is a split Jacobian corresponding to the product of two elliptic curves. In Section 6, we sketch the situation where the denominator of the rational function is associated with algebraic varieties of low Kodaira dimension, having an infinite set of birational automorphisms. We finally provide some examples of rational functions in more than three variables, where the telescopers have pullbacked $_2F_1$ hypergeometric solutions, the denominator corresponding to an algebraic variety having a selected elliptic curve in the variety explaining these pullbacked $_2F_1$ solutions.

2. Classical Modular Forms and Diagonals of 9- and 10-Parameter Families of Rational Functions

In previous papers [1,2], using creative telescoping [3], we have obtained diagonals of 9- and 10-parameter families of rational functions, given by (classical) modular forms expressed as pullbacked $_2F_1$ hypergeometric functions. Let us recall these results.

2.1. Nine-Parameter Rational Functions Giving Pullbacked $_2F_1$ Hypergeometric Functions for Their Diagonals

Let us recall the nine-parameter rational function in three variables, x, y and z:

$$\frac{1}{a \; + b_1\, x + b_2\, y + b_3\, z \; + c_1\, y\, z + c_2\, x\, z + c_3\, x\, y \; + d\, y^2\, z \; + e\, z\, x^2}. \tag{1}$$

Calculating (using the HolonomicFunctions Mathematica package [3]) the telescoper of this rational function (1), one obtains an order-two linear differential operator annihilating the diagonal of the rational function (1). The diagonal of the rational function (1) can be written [1,2] as a pullbacked hypergeometric function:

$$\frac{1}{P_4(x)^{1/4}} \cdot {}_2F_1\Big(\frac{1}{12}, \frac{5}{12}; 1; \; 1 \, - \, \frac{P_6(x)^2}{P_4(x)^3}\Big), \tag{2}$$

where $P_4(x)$ and $P_6(x)$ are two polynomials of degree four and six in x, respectively. The Hauptmodul pullback in (2) has the form:

$$\mathcal{H} \; = \; \frac{1728}{j} \; = \; 1 \, - \, \frac{P_6(x)^2}{P_4(x)^3} \; = \; \frac{1728 \cdot x^3 \cdot P_8(x)}{P_4(x)^3}, \tag{3}$$

where j is the j-invariant and $P_8(x)$ is a polynomial of degree eight in x. Such a pullbacked $_2F_1$ hypergeometric function (2) corresponds to a classical modular form [1,2].

2.2. Ten-Parameter Rational Functions Giving Pullbacked $_2F_1$ Hypergeometric Functions for Their Diagonals

Let us recall the 10-parameter rational function in three variables, x, y and z:

$$R(x, y, z) \; = \; \frac{1}{a \; + b_1\, x + b_2\, y + b_3\, z \; + c_1\, y\, z + c_2\, x\, z + c_3\, x\, y \; + d_1\, x^2\, y \; + d_2\, y^2\, z + \, d_3\, z^2\, x}. \tag{4}$$

Calculating the telescoper of this rational function (4), one obtains an order-two linear differential operator annihilating the diagonal of the rational function (4). The diagonal of the rational function (4) can be written [1,2] as a pullbacked hypergeometric function:

$$\frac{1}{P_3(x)^{1/4}} \cdot {}_2F_1\Big(\frac{1}{12}, \frac{5}{12}; 1; \; 1 \, - \, \frac{P_6(x)^2}{P_3(x)^3}\Big), \tag{5}$$

where $P_3(x)$ and $P_6(x)$ are two polynomials of degree three and six in x, respectively. Furthermore, the Hauptmodul pullback in (5) is seen to be of the form:

$$\mathcal{H} \,=\, \frac{1728}{j} \,=\, 1 \,-\, \frac{P_6(x)^2}{P_3(x)^3} \,=\, \frac{1728 \cdot x^3 \cdot P_9(x)}{P_3(x)^3}. \tag{6}$$

where $P_9(x)$ is a polynomial of degree nine in x. Again, (5) corresponds to a classical modular form [1,2].

3. Deducing Creative Telescoping Results from Effective Algebraic Geometry

Obtaining the previous pullbacked hypergeometric results (2) and (5) required [1,2] an accumulation of creative telescoping calculations, and a lot of "guessing", using all of the symmetries of the diagonals of these rational functions (1) and (4). We are looking for a more efficient and intrinsic way of obtaining these exact results. These two pullbacked hypergeometric results (2) and (5), are essentially "encoded" by their *Hauptmodul* pullbacks (3) and (6), or, equivalently, their corresponding *j-invariants*. The interesting question, which will be addressed in this paper, is whether it is possible to canonically associate elliptic curves whose j-invariants correspond precisely to these Hauptmoduls $\mathcal{H} \,=\, \frac{1728}{j}$.

3.1. Revisiting the Pullbacked Hypergeometric Results in an Algebraic Geometry Perspective

One expects such an elliptic curve to correspond to the singular part of the rational function, namely, the denominator of the rational function. Let us recall that the diagonal of a rational function in (for example) three variables is obtained through its multi-Taylor expansion [19,20]:

$$R(x,\, y,\, z) \,=\, \sum_m \sum_n \sum_l \, a_{m,\,n,\,l} \cdot x^m \, y^n \, z^l, \tag{7}$$

by extracting the "diagonal" terms, i.e., the powers of the product $p \,=\, xyz$:

$$Diag\Big(R(x,\, y,\, z)\Big) \,=\, \sum_m \, a_{m,\,m,\,m} \cdot x^m. \tag{8}$$

Consequently, it is natural to consider the algebraic curve corresponding to the intersection of the surface defined by the vanishing condition $D(x,\, y,\, z) \,=\, 0$ of the denominator $D(x,\, y,\, z)$ of these rational functions (1) and (4), with the hyperbola $p \,=\, x\,y\,z$ (where p is seen, here, as a constant). This amounts, for instance, to eliminating the variable z, substituting $z \,=\, \frac{p}{x\,y}$ in $D(x,\, y,\, z) \,=\, 0$.

3.1.1. Nine-Parameter Case

In the case of the rational functions (1), this corresponds to the (planar) algebraic curve:

$$\begin{aligned} a \,\, + b_1\, x \,\, + b_2\, y \,\, + b_3\, \frac{p}{x\,y} \,\, + c_1\, y\, \frac{p}{x\,y} \,\, + c_2\, x\, \frac{p}{x\,y} \,\, + c_3\, x\, y \\ + d\, y^2\, \frac{p}{x\,y} \,\, + e\, \frac{p}{x\,y}\, x^2 \,\, = \,\, 0, \end{aligned} \tag{9}$$

which can be rewritten as a (general, nine-parameter) *biquadratic*:

$$\begin{aligned} a\,x\,y \,\, + b_1\, x^2\, y \, + b_2\, x\, y^2 \, + b_3\, p \,\, + c_1\, p\, y \,\, + c_2\, p\, x \,\, + c_3\, x^2\, y^2 \\ + d\, p\, y^2 \,\, + e\, p\, x^2 \,\, = \,\, 0. \end{aligned} \tag{10}$$

Using formal calculations (namely, using with(algcurves) in Maple, and in particular, the command j_invariant) one can easily calculate the genus of the planar algebraic curve (10), and find that it is actually an elliptic curve (of genus one). Furthermore, one can (almost instantaneously) find the exact expression of the j-invariant of this elliptic curve as a rational function of the nine parameters $a, b_1, b_2, \cdots, e$ in (1). One actually finds that this j-invariant is *precisely* the j, such that the Hauptmodul $\mathcal{H} = \frac{1728}{j}$ is the exact expression (3). In other words, the classical modular form result (2) could have been obtained, almost instantaneously, by calculating the j-invariant of an elliptic curve that is canonically associated with the denominator of the rational function (1). The algebraic planar curve (10) corresponds to the most general biquadratic of two variables, which depends on nine homogeneous parameters. Such a general biquadratic is known to be an elliptic curve for *generic values* of the nine parameters. (So many results in integrable models correspond to this most general biquadratic: the Bethe ansatz of the Baxter model [43,44], the elliptic curve foliating the 16-vertex model [44], so many QRT birational maps [45], etc.)

Thus, the nine-parameter exact result (2) can be seen as a simple consequence of the fact that the most general nine-parameter biquadratic is an elliptic curve.

3.1.2. Ten-Parameter Case

In the case of the rational function (4), substituting $z = \frac{p}{x\,y}$ in $D(x, y, z) = 0$, one obtains the 10-parameter bicubic:

$$a\,x\,y^2 \;+ b_1\,x^2\,y^2 \,+ b_2\,x\,y^3 \,+ b_3\,p\,y \;+ c_1\,p\,y^2 \,+ c_2\;p x\,y \,+ c_3\,x^2\,y^3 \\ + d_1\,x^3\,y^3 \;+\; d_2\;y^3 \;+\; d_3\,p^2 \;=\; 0. \tag{11}$$

As before, we find that this planar algebraic curve is actually an elliptic curve (Generically, the most general planar bicubic is *not* a genus-one algebraic curve. It is a genus-four curve.) and that the exact expression of its j-invariant is precisely the j of the Hauptmodul $H = 1728/j$ in (6).

Thus, this 10-parameter result (5) can again be seen as a simple consequence of the fact that there exists a family of 10-parameter bicubics (see (11)) which are elliptic curves for generic values of the 10 parameters.

These preliminary calculations are a strong incentive to attempt to replace the differential algebra calculations of creative telescoping with more intrinsic algebraic geometry calculations, or at least, to perform effective algebraic geometry calculations to provide an algebraic geometry interpretation of the exact results obtained from creative telescoping.

3.2. *Finding Creative Telescoping Results from j-Invariant Calculations*

One might think that these results are a consequence of the simplicity of the denominators of the rational functions (1) or (4), being associated with biquadratics or selected bicubics. In fact, these results are very general. Let us, for instance, consider a nine-parameter family of planar algebraic curves that are not biquadratics or (selected) bicubics:

$$a_1\,x^4 \;+ a_2\,x^3 \;+ a_3\,x^2 \;+ a_4\,x \;+ a_5 + a_6\,x^2\,y \,+ a_7\,y^2 \;+ a_8\,y \;+ a_9\,x\,y \;=\; 0. \tag{12}$$

One can easily calculate the genus of this planar curve and see that this genus is actually one for *arbitrary* values of the a_n's. Thus the planar curve (12) is an elliptic curve for generic values of the nine parameters $a_1, \cdots, a_9$. It is straightforward to see that the algebraic surface $S(x, y, z) = 0$, corresponding to:

$$z \cdot\,(a_1\,x^4 \;+ a_2\,x^3 \;+ a_3\,x^2 \;+ a_4\,x \;+\; a_5 + a_6\,x^2\,y \;+ a_7\,y^2 \;+ a_8\,y) \;+\; a_9\,p \;=\; 0, \tag{13}$$

will automatically be such that its intersection with the hyperbola $p = x\,y\,z$ gives back the elliptic curve (12).

Using this kind of "reverse engineering" suggests that we should consider the rational function in three variables x, y, and z:

$$R(x, y, z) \;=\; \\ \frac{1}{1 \;+ z \cdot\,(a_1\,x^4 \;+\; a_2\,x^3 \;+\; a_3\,x^2 \;+\; a_4\,x \;+\; a_5 +\; a_6\,x^2\,y \;+\; a_7\,y^2 \;+\; a_8\,y)}, \tag{14}$$

which will be such that its denominator is canonically associated with an elliptic curve. Again, we can immediately calculate the j-invariant of that elliptic curve. If one calculates the telescoper of this eight-parameter family of rational functions (14), one finds that this telescoper is an order-two linear differential operator with pullbacked hypergeometric solutions of the form:

$$\mathcal{A}(x) \cdot {}_2F_1\Big(\frac{1}{12}, \frac{5}{12}; 1; \mathcal{H}\Big), \tag{15}$$

where $\mathcal{A}(x)$ is an algebraic function, and where again, the pullback-Hauptmodul, $\mathcal{H} = 1728/j$, precisely corresponds to the j-invariant of the elliptic curve.

More generally, in seeking for planar elliptic curves, one can, for the given values of two integers M and N, look for planar algebraic curves:

$$\sum_{n=0}^{n=N} \sum_{m=0}^{m=M} a_{m,n} \cdot x^n y^m \quad = \quad 0, \tag{16}$$

defined by the set of $a_{m,n}$'s that are equal to zero, apart from $\mathcal{N}$ homogeneous parameters $a_{m,n}$ being, as in (10), (11), or (13), *independent parameters*. Finding such an $\mathcal{N}$-parameter family of (planar) elliptic curves automatically provides an $\mathcal{N}$-parameter family of rational functions such that their telescopers have a pullbacked ${}_2F_1$ hypergeometric solution, which we can simply deduce from the j-invariant of that elliptic curve.

Recalling the results of Section 2.2, the natural question to ask now is whether it is possible to find families of such (planar) elliptic curves that depend on more than 10 independent parameters.

Before addressing this question, let us recall the concept of birationally equivalent elliptic curves. Let us consider the monomial transformation:

$$(x, y) \qquad \longrightarrow \qquad (x^M y^N, \; x^P y^Q), \tag{17}$$

where M, N, P, Q are integers, such that $M \cdot Q - P \cdot N = 1$; then, its compositional inverse is the monomial transformation:

$$(x, y) \qquad \longrightarrow \qquad \Big(\frac{x^Q}{y^N}, \; \frac{y^M}{x^P}\Big). \tag{18}$$

This monomial transformation (17) is thus a *birational* (This transformation is rational and its compositional inverse is also rational (here, monomial)) transformation. A birational transformation transforms an elliptic curve, such as (12), into another elliptic curve with the same j-invariant: these two elliptic curves are said to be birationally equivalent. In the case of the birational and monomial transformation (17), the elliptic curve (12) is changed into (one can easily verify for particular values of the M, N, P, Q, and a_k's, using with(algcurves) in Maple, that the j-invariants of (12) and (19) are actually equal.):

$$\begin{aligned} & a_1 \cdot x^{4M} y^{4N} \; + a_2 \cdot x^{3M} y^{3N} \; + a_3 \cdot x^{2M} y^{2N} \; + a_4 \cdot x^M y^N \; + a_5 \\ & \quad + a_6 \cdot x^{2M+P} y^{2N+Q} \; + a_7 \cdot x^{2P} y^{2Q} + a_8 \cdot x^P y^Q + a_9 \cdot x^{M+P} y^{N+Q} \; = \; 0. \end{aligned} \tag{19}$$

With this kind of birational monomial transformation (17), we see that one can obtain families of elliptic curves (19) of *arbitrary large* degrees in x and y. Consequently, one can find 9- or 10-parameter families of rational functions of arbitrary large degrees yielding pullbacked ${}_2F_1$ hypergeometric functions. There is no constraint on the degree of the planar algebraic curves (19): the only relevant question concerns the maximum number of (linearly) independent parameters in families of planar elliptic curves, which is shown to be 10. The demonstration (we thank Josef Schicho for providing this demonstration) is sketched in Appendix B.

3.3. Pullbacked $_2F_1$ Functions for Higher Genus Curves: Monomial Transformations

Let us recall another important point. We have already remarked in [1,2] that once we have an exact result for a diagonal of a rational function of three variables $R(x, y, z)$, we immediately obtain another exact result for the diagonal of the rational function $R(x^n, y^n, z^n)$ for any positive integer n. As a result, we obtain a new expression for the diagonal changing x into x^n. In fact, this is also a result on the telescoper of the rational function $R(x, y, z)$: the telescoper of the rational function $R(x^n, y^n, z^n)$ is the $x \rightarrow x^n$ pullback of the telescoper of the rational function $R(x, y, z)$. Having a pullbacked $_2F_1$ solution for the telescoper of the rational function $R(x, y, z)$ (respectively, the diagonal of the rational function $R(x, y, z)$), we will immediately deduce a pullbacked $_2F_1$ solution for the telescoper of the rational function $R(x^n, y^n, z^n)$ (respectively, the diagonal of the rational function $R(x^n, y^n, z^n)$).

Along this line, let us change, in the rational function (1), (x, y, z) into (x^2, y^2, z^2):

$$R_2(x, y, z) \; = \; \tag{20}$$
$$\frac{1}{a \; + b_1\, x^2 \, + b_2\, y^2 \, + b_3\, z^2 \; + c_1\, y^2\, z^2 \, + c_2\, x^2\, z^2 \, + c_3\, x^2\, y^2 \; + \; d\, y^4\, z^2 \; + \; e\, z^2\, x^4}.$$

The diagonal of this new rational function (20) will be the pullbacked $_2F_1$ exact expression (2), where we change $x \rightarrow x^2$. The intersection of the algebraic surface corresponding to the vanishing condition of the denominator of the new rational function (20), with the hyperbola $p = x\,y\,z$ (i.e., $z = \frac{p}{x\,y}$), is nothing but Equation (10), where we have changed $(x, y; p)$ into $(x^2, y^2; p^2)$:

$$\begin{aligned} &a\, x^2\, y^2 \; + b_1\, x^4\, y^2 \, + b_2\, x^2\, y^4 \, + b_3\, p^2 \; + c_1\, p^2\, y^2 \; + c_2\, p^2\, x^2 \; + c_3\, x^4\, y^4 \\ &\qquad + d\, p^2\, y^4 \; + e\, p^2\, x^4 \; = \; 0, \end{aligned} \tag{21}$$

which is *no longer* (if we perform the same calculations with the 10-parameter rational function, (4) we get an algebraic curve of genus 10 instead of 9) an elliptic curve, but a curve of genus 9.

With that example, we see that classical modular form results, or pullbacked $_2F_1$ exact expressions such as (2), can actually emerge from higher genus curves like (21). As far as these diagonals, or telescopers, of rational function calculations are concerned, higher genus curves like (21) must in fact be seen as "almost" elliptic curves up to an $x \rightarrow x^n$ covering.

Such results for monomial transformations like $(x, y, z) \rightarrow (x^n, y^n, z^n)$ can, in fact, be generalized to more general (not birational (in contrast with transformations such as (17))) monomial transformations. This is sketched in Appendix C.

3.4. Changing the Parameters into Functions of the Product $p = x\,y\,z$

All these results for the parameterized families of rational functions can be drastically generalized when one remarks that allowing any of these parameters to be a rational function of the product $p = x\,y\,z$ also yields the previous pullbacked $_2F_1$ exact expression, as in (2), where the parameter is changed into that rational function of x (see [1]). Let us consider a simple (two-parameter) illustration of this general result. Let us consider a subcase of the previous 9- or 10-parameter families, introducing, for example, the two-parameter rational function:

$$\frac{1}{1 \; + 2\,x \, + b_2 \cdot y \; + 5\,y\,z \; + x\,z \; + c_3 \cdot x\,y}. \tag{22}$$

The diagonal of this rational function (22) is the pullbacked hypergeometric function:

$$\frac{1}{P_2(x)^{1/4}} \cdot {}_2F_1\Big(\frac{1}{12}, \frac{5}{12}; 1; \; 43200 \cdot x^4 \cdot \frac{P_4(x)}{P_2(x)^3}\Big), \tag{23}$$

where:

$$P_2(x) \;=\; 1 \;\; - 8 \cdot (b_2 + 10) \cdot x \;\; + 8 \cdot (2\,b_2^2 - 20\,b_2 + 15\,c_3 + 200) \cdot x^2, \tag{24}$$

and:

$$\begin{aligned} P_4(x) \;=\; & -675\,c_3^4 \cdot x^4 \;\; + 4\,c_3^2 \cdot (b_2 + 10) \cdot (4\,b_2^2 - 100\,b_2 + 45\,c_3 + 400) \cdot x^3 \\ & +(64\,b_2^4 - 32\,b_2^3\,c_3 - 8\,b_2^2\,c_3^2 - 1280\,b_2^3 + 1280\,b_2^2\,c_3 \\ & \quad -460\,b_2\,c_3^2 - 5\,c_3^3 + 6400\,b_2^2 - 3200\,b_2\,c_3 - 800\,c_3^2) \cdot x^2 \\ & \quad\quad -(b_2 + 10) \cdot (32\,b_2^2 - 16\,b_2\,c_3 - c_3^2) \cdot x \;\; + 2\,b_2 \cdot (2\,b_2 - c_3), \end{aligned} \tag{25}$$

Let us now consider the previous rational function (22) where the two parameters b_2 and c_3 become some rational functions of the product, $p = x\,y\,z$, for instance:

$$b_2(p) \;=\; \frac{1 + 3\,p}{1 + 7\,p^2}, \qquad c_3(p) \;=\; \frac{1 + p^2}{1 + 2\,p} \qquad \text{where:} \qquad p \;=\; x\,y\,z. \tag{26}$$

The new corresponding rational function becomes more involved, but one can easily calculate the telescoper of this new rational function of three variables x, y and z, and find that it is, *again*, an order-two linear differential operator having the pullbacked hypergeometric solution (23) where b_2 and c_3 are now replaced by (p is now x) the functions:

$$b_2(x) \;=\; \frac{1 + 3\,x}{1 + 7\,x^2}, \qquad c_3(x) \;=\; \frac{1 + x^2}{1 + 2\,x}. \tag{27}$$

In that case, one obtains a diagonal that is the pullbacked hypergeometric solution:

$$\begin{aligned} & (1 + 2\,x)^{1/4} \cdot (1 + 7\,x^2)^{1/4} \cdot q_8^{-1/4} \\ & \quad \times {}_2F_1\Big(\frac{1}{12}, \frac{5}{12}; 1; \; \frac{43200 \cdot x^4 \cdot (1 + 7\,x^2)^2 \cdot q_{20}}{(1 + 2\,x) \cdot q_8^3}\Big), \end{aligned} \tag{28}$$

where q_8 and q_{20} are two polynomials with integer coefficients of degree 8 and 20 in x. The exact expression (28) is nothing but (23) (with (24) and (25)), where b_2 and c_3 have been replaced by the rational functions (27). Similar calculations can be performed for the more general rational functions (1) or (4), when all the (9 or 10) parameters are more involved rational functions.

When performing our creative telescoping symbolic calculations using the HolonomicFunctions package [3], such results may look quite impressive. From the algebraic geometry viewpoint, it is almost tautological (An algebraic geometer will probably see this as a trivial remark: diagonalization is an algebraic procedure and nothing really happens to the coefficients. Therefore if one replaces the coefficients by anything else, one will find those replaced coefficients in the end result.), if one takes for granted the result of our previous Sections 3.1 and 3.2, namely, that the pullbacked hypergeometric solution of the telescoper corresponds to the Hauptmodul $1728/j$, where j is the j-invariant of the elliptic curve corresponding to the intersection of the algebraic surface corresponding to the vanishing condition of the denominator, with the hyperbola $p = x\,y\,z$: this calculation of the j-invariant is performed for p fixed, and arbitrary (9 or 10) parameters a, b_1, $\cdots$. It is clearly possible to force the parameters to be functions (The functions should be rational functions if one wants to stick with the diagonals and telescopers of rational functions, but the result remains valid for algebraic functions, or even transcendental functions with reasonable Taylor series expansions at $x = 0$: for instance, for ${}_2F_1$ hypergeometric functions, one obtains a differentially algebraic function corresponding to the composition of ${}_2F_1$ hypergeometric functions.) of p, the j-invariant being changed accordingly. Of course, in that case, the parameters in the rational function are the same functions, but of the product $p = x\,y\,z$.

One thus obtains pullbacked hypergeometric solutions (classical modular forms) for an (unreasonably) large set of rational functions in three variables, namely, the families of the rational functions (1) or (4), but now where the 9 or 10 parameters are 9 or 10 totally arbitrary rational functions (with Taylor series expansions) of the product $p = xyz$.

We see experimentally that changing the parameters of the rational function into functions actually works for the diagonals of rational functions, as well as for the solutions of telescopers of rational functions, depending on the parameters.

4. Creative Telescoping on Rational Functions of More than Three Variables Associated with Products or Foliations of Elliptic Curves

Let us show that such an algebraic geometry approach to creative telescoping can be generalized to the rational functions of *more than three* variables, when the vanishing condition of the denominator can be associated with the products of elliptic curves, or more generally, algebraic varieties with foliations in elliptic curves.

- The telescoper of the rational function in the four variables x, y, z and w:

$$\frac{xyz}{(1+z)^2 \; - x \cdot (1-x) \cdot (x - xyzw) \cdot y \cdot (1-y) \cdot (y - xyzw)}, \tag{29}$$

gives an order-three *self-adjoint* linear differential operator which is, thus, the *symmetric square* of an order-two linear differential operator. The latter has the pullbacked hypergeometric solution:

$$\begin{aligned} \mathcal{S}_1(x) \;&=\; (1 - x + x^2)^{-1/4} \cdot {}_2F_1\Big(\frac{1}{12}, \frac{5}{12}; 1; \; \frac{27}{4} \cdot \frac{x^2 \cdot (1-x)^2}{(x^2 - x + 1)^3}\Big) \\ &=\; {}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; 1; \; x\Big). \end{aligned} \tag{30}$$

In [18], we underlined the difference between the diagonal of a rational function and solutions of the telescoper of the same rational function. In this case, the diagonal of the rational function (29) is zero (The reason for this is that the integration takes place over a cycle that is homologically equivalent to the trivial cycle. The cycle becomes trivial after taking the limit $p \rightarrow 0$. Integrals over non-vanishing cycles usually give logarithms of p, as with the second solution to the hypergeometric function ${}_2F_1(1/2, 1/2; 1; x)$.) , and is thus different from the pullbacked hypergeometric solution (30), which is a "period" [37] of the algebraic variety corresponding to the denominator over some (*non-vanishing* (diagonals of rational functions correspond to periods over vanishing cycles [5,38])) cycle. From now on, we will have a similar situation in most of the following examples of this paper.

This example is a simple illustration of what we expect for the products of elliptic curves, or of algebraic varieties with foliations in elliptic curves. Introducing the product $p = xyzw$, the vanishing condition of the denominator of the rational function (29) yields the surface $S(x, y, z) = 0$:

$$(1+z)^2 \; - x \cdot (1-x) \cdot (x-p) \cdot y \cdot (1-y) \cdot (y-p) \;=\; 0. \tag{31}$$

For fixed p and fixed y, Equation (31) can be seen as an algebraic curve:

$$\begin{aligned} (1+z)^2 \; - \lambda \cdot x \cdot (1-x) \cdot (x-p) \;&=\; 0 \\ \text{with:} \qquad \lambda \;&=\; y \cdot (1-y) \cdot (y-p). \end{aligned} \tag{32}$$

For fixed p and fixed y, λ can be considered as a constant, the algebraic curve (32) being an elliptic curve with an obvious Weierstrass form:

$$Z^2 \; - \; x \cdot (1-x) \cdot (x-p) \;=\; 0 \qquad \text{where:} \qquad Z \;=\; \frac{1+z}{\sqrt{\lambda}}. \tag{33}$$

The j-invariant of (32), or (a shift $z \rightarrow \, z+1$ or a rescaling $z^2 \rightarrow \, \frac{z^2}{\lambda}$ does not change the j-invariant of the Weierstrass elliptic form) (33), is well-known and yields the Hauptmodul $\mathcal{H}$:

$$\mathcal{H} \, = \, \frac{1728}{j} \, = \, \frac{27}{4} \cdot \frac{p^2 \cdot (1-p)^2}{(p^2-p+1)^3} \tag{34}$$

For fixed p and fixed x, Equation (31) can be seen as an algebraic curve:

$$\begin{aligned} (1+z)^2 \, - \mu \cdot y \cdot (1-y) \cdot (y-p) \, &= \, 0 \\ \text{for:} \qquad \mu \, = \, x \cdot (1-x) \cdot (x-p)&, \end{aligned} \tag{35}$$

which is also an elliptic curve with an obvious Weierstrass form and the *same* Hauptmodul (34). This Hauptmodul is precisely the one that occurs in the pullbacked hypergeometric solution (30).

More generally, the rational function of the four variables x, y, z and w:

$$\frac{x\,y\,z}{(1+z)^2 \, - x \cdot (1-x) \cdot (x-R_1(p)) \cdot y \cdot (1-y) \cdot (y-R_2(p))}, \tag{36}$$

where $p \, = \, x\,y\,z\,w$, and where $R_1(p)$ and $R_2(p)$ are two arbitrary rational functions (with Taylor series expansions) of the product $p \, = \, x\,y\,z\,w$, yields a telescoper that has an order-four linear differential operator that is the symmetric product (As the present paper belongs to the literature on symbolic computation and not on pure mathematics for algebraic geometers, we use the standard Maple (DEtools) terminology of symmetric powers and symmetric products of linear differential operators [46]. Note that "symmetric product" is not a proper mathematical name for this construction on the solution space; it is a homomorphic image of the tensor product. The (Maple/DEtools) reason for choosing the name symmetric_product is the resemblance with the function symmetric_power.) of two order-two linear differential operators having, respectively, the pullbacked hypergeometric solutions (30) where x is replaced by $R_1(x)$ and $R_2(x)$. These two hypergeometric solutions thus have the two Hauptmodul pullbacks:

$$\mathcal{H}_1 \, = \, \frac{1728}{j_1} \, = \, \frac{27}{4} \cdot \frac{R_1(p)^2 \cdot (1-R_1(p))^2}{(R_1(p)^2-R_1(p)+1)^3}, \tag{37}$$

$$\mathcal{H}_2 \, = \, \frac{1728}{j_2} \, = \, \frac{27}{4} \cdot \frac{R_2(p)^2 \cdot (1-R_2(p))^2}{(R_2(p)^2-R_2(p)+1)^3}, \tag{38}$$

obtained by calculations similar to the ones previously performed on (31) but, now, for the Weierstrass form corresponding to the denominator (36).

A solution of the telescoper of (36) is thus the product of these two pullbacked hypergeometric functions. Let us give a simple illustration of this general result, with the next example.

- The telescoper of the rational function in the four variables x, y, z and w:

$$\frac{x\,y\,z}{(1+z)^2 \, - x \cdot (1-x) \cdot (x-x\,y\,z\,w) \cdot y \cdot (1-y) \cdot (y-3\,x\,y\,z\,w)}, \tag{39}$$

corresponding to (36), with $R_1(p) \, = \, p$ and $R_2(p) \, = \, 3\,p$, gives an order-four linear differential operator that is the symmetric product of two order-two operators having, respectively, the pullbacked hypergeometric solution (30) and the solution (30) where the variable x has been changed into $3\,x$:

$$\begin{aligned} \mathcal{S}_2(x) \, &= \, \mathcal{S}_1(3\,x) \\ &= \, (1-3\,x+9\,x^2)^{-1/4} \cdot {}_2F_1\Big(\frac{1}{12}, \frac{5}{12}; 1; \, \frac{243}{4} \cdot \frac{x^2 \cdot (1-3\,x)^2}{(1-3\,x+9\,x^2)^3}\Big). \end{aligned} \tag{40}$$

Creative Telescoping on Rational Functions of Five Variables Associated with Products or Foliations of Three Elliptic Curves

Let us now introduce the rational function in five variables x, y, z, v and w:

$$\frac{x\,y\,z\,v}{D(x,\,y,\,z,\,v,\,w)}, \tag{41}$$

where the denominator $D(x, y, z, v, w)$ reads:

$$\begin{aligned} D_p \,\,&= \\ (1+v)^2 \,\,& - x\cdot(1-x)\cdot(x-p)\cdot y\cdot(1-y)\cdot(y-3\,p)\cdot z\cdot(1-z)\cdot(z-5\,p), \\ &\text{where:} \qquad\qquad p \,=\, x\,y\,z\,v\,w. \end{aligned} \tag{42}$$

The telescoper of the rational function (41) of five variables gives (Such a creative telescoping calculation requires "some" computing time to achieve the result.) an order-eight linear differential operator, which is the symmetric product of three order-two linear differential operators having, respectively, the pullbacked hypergeometric solution (30), which is the solution (30) where x has been changed into $3\,x$, namely (40), and the solution (30), where x has been changed into $5\,x$:

$$\begin{aligned} \mathcal{S}_3(x) \,&=\, \mathcal{S}_1(5\,x) \\ &=\, (1-5\,x+25\,x^2)^{-1/4}\cdot {}_2F_1\Big(\frac{1}{12},\,\frac{5}{12};\,1;\,\frac{675}{4}\cdot\frac{x^2\cdot(1-5\,x)^2}{(1-5\,x+25\,x^2)^3}\Big). \end{aligned} \tag{43}$$

In other words, the order-eight telescoper of the rational function (41) has the product $\mathcal{S} \,=\, \mathcal{S}_1\cdot\mathcal{S}_2\cdot\mathcal{S}_3$, of (30), (40), and (43) as a solution. From an algebraic geometry viewpoint, this is a consequence of the fact that, for fixed p, the algebraic variety $D_p \,=\, 0$, where D_p is given by (42), can be seen, for fixed y and z, as an elliptic curve $\mathcal{E}_1$ of equation $D_{y,z,p}(v,\,x) \,=\, 0$, for fixed x and z, as an elliptic curve $\mathcal{E}_2$ of equation $D_{x,z,p}(v,\,y) \,=\, 0$, and also, for fixed x and y, as an elliptic curve $\mathcal{E}_3$ of equation $D_{x,y,p}(v,\,z) \,=\, 0$, the j-invariants j_k, $k \,=\, 1, 2, 3$, of these three elliptic curves $\mathcal{E}_k$ yielding (in terms of p), precisely, the three Hauptmoduls $\mathcal{H}_k \,=:\, \frac{1728}{j_k}$

$$\frac{27}{4}\cdot\frac{x^2\cdot(1-x)^2}{(x^2-x+1)^3}, \qquad \frac{243}{4}\cdot\frac{x^2\cdot(1-3\,x)^2}{(1-3\,x+9\,x^2)^3}, \qquad \frac{675}{4}\cdot\frac{x^2\cdot(1-5\,x)^2}{(1-5\,x+25\,x^2)^3}, \tag{44}$$

occurring as pullbacks in the three $\mathcal{S}_k$'s of the solution $\mathcal{S} \,=\, \mathcal{S}_1\cdot\mathcal{S}_2\cdot\mathcal{S}_3$, of the telescoper of (41).

Remark 1. *Other examples of rational functions of three, four, and five variables where the denominators also correspond to Weierstrass (respectively, Legendre) forms, are displayed in Appendix D. They provide simple illustrations of rational functions where the denominator is associated with K3 surfaces (See the emergence of the product of elliptic curves from a Shioda–Inose structure on surfaces with Picard number 19 in [47]. In [47], Ling Long considers one-parameter families of K3 surfaces with generic Picard number 19. The existence of a Shioda–Inose structure implies that there is a one-parameter family of elliptic curves.), or Calabi–Yau threefolds. In these cases, the algebraic varieties have simple foliations in terms of two or three families of elliptic curves, and the solutions of the corresponding telescopers can be selected* $_3F_2$ *and* $_4F_3$ *hypergeometric functions (see (A28) in Appendix D), naturally associated with K3 surfaces and Calabi–Yau operators [27].*

5. Creative Telescoping of Rational Functions in Three Variables Associated with Genus-Two Curves with Split Jacobians

In papers [17,18], dedicated to Heun functions that are solutions of telescopers of simple rational functions of three and four variables, we have obtained (See Equation (83) in Section 2.2 of [18].) an order-four telescoper of a rational function of three variables, which is the direct sum of two order-two linear differential operators, each having classical modular form solutions that can be written as pullbacked ${}_2F_1$ hypergeometric solutions. Unfortunately, the intersection of the algebraic surface corresponding to the denominator of the rational function with the $p = xyz$ hyperbola yields a genus-two algebraic curve.

Let us try to understand, in this section, how a genus-two curve can yield two classical modular forms. Let us first recall the results in Section 2.2 of [18].

5.1. Periods of Extremal Rational Surfaces

Let us recall the rational function in just three variables [18]:

$$R(x, y, z) \quad = \quad \frac{1}{1 \ + x + y + z \ + xy + yz \ - x^3 yz}. \tag{45}$$

Its telescoper is actually an order-four linear differential operator L_4, which not only factorizes into two order-two linear differential operators, but is actually the *direct sum* (LCLM) of two (These two order-two linear differential operators, L_2 and M_2, are not homomorphic) order-two linear differential operators, $L_4 = L_2 \oplus M_2$. These two (non-homomorphic) order-two linear differential operators contain, respectively, the two pullbacked hypergeometric solutions:

$$\begin{aligned} \mathcal{S}_1 \ = \ & (1 + 9x)^{-1/4} \cdot (1 + 3x)^{-1/4} \cdot (1 + 27x^2)^{-1/4} \\ & \times {}_2F_1\Big(\frac{1}{12}, \frac{5}{12}; 1; \frac{1728 \cdot x^3 \cdot (1 + 9x + 27x^2)^3}{(1 + 3x)^3 \cdot (1 + 9x)^3 \cdot (1 + 27x^2)^3}\Big), \end{aligned} \tag{46}$$

and:

$$\begin{aligned} \mathcal{S}_2 \ = \ & \frac{1}{(1 + 4x - 2x^2 - 36x^3 + 81x^4)^{1/4}} \\ & \times {}_2F_1\Big(\frac{1}{12}, \frac{5}{12}; 1; \frac{1728 \cdot x^5 \cdot (1 + 9x + 27x^2) \cdot (1 - 2x)^2}{(1 + 4x - 2x^2 - 36x^3 + 81x^4)^3}\Big). \end{aligned} \tag{47}$$

The diagonal of (45) is actually the half-sum of the two series (46) and (47):

$$Diag\Big(R(x, y, z)\Big) \quad = \quad \frac{\mathcal{S}_1 + \mathcal{S}_2}{2}. \tag{48}$$

As far as our algebraic geometry approach is concerned, the intersection of the algebraic surface corresponding to the denominator of the rational function (45) with the hyperbola $p = xyz$ gives the planar algebraic curve (corresponding to the elimination of the z variable by the substitution $z = \frac{p}{xy}$):

$$1 \ + x + y + \frac{p}{xy} \ + xy \ + y\,\frac{p}{xy} \ - x^3 y\,\frac{p}{xy} \quad = \quad 0. \tag{49}$$

One easily finds that this algebraic curve is (for p fixed) a genus-two curve, and that this higher genus situation *does not* correspond to the "almost elliptic curves" described in Section 3.2, namely, an elliptic curve transformed by a monomial transformation. How can a "true" genus-two curve yield two j-invariants, namely, a telescoper with two Hauptmodul pullbacked ${}_2F_1$ solutions? We are going to see that the answer is that the Jacobian of this genus-two curve (An algebraic geometer will probably recall that it is very well-known that a genus-two curve *may* have Jacobian isogenous to a product of elliptic curves. This is not the case in general. The genus-two curves that have a (non-constant) map to an elliptic curve have this property. Our purpose in Section 5.3 is to perform a creative telescoping calculation in such a selected situation.) is in fact isogenous to a product $\mathcal{E} \times \mathcal{E}'$ of two elliptic curves (split Jacobian).

5.2. Split Jacobians

Let us first recall the concept of *split Jacobian* with a simple example. In [48], one has a crystal-clear example of a genus-two curve C:

$$y^2 \; - \; (x^3 + 420\,x - 5600) \cdot (x^3 + 42\,x^2 + 1120) \;\; = \;\; 0, \tag{50}$$

such that its Jacobian $J(C)$ is isogenous to a product of elliptic curves with j-invariants $j_1 = -2^7 \cdot 7^2 = -6272$ and $j_2 = -2^5 \cdot 7 \cdot 17^3 = -1100512$, corresponding to the following two values of the Hauptmodul, $\mathcal{H} = \frac{1728}{j}$: $\mathcal{H}_1 = -27/98$ and $\mathcal{H}_2 = -54/34391$. Let us consider the genus-one elliptic curve:

$$v^2 \; = \; u^3 \; + 4900\,u^2 \; + 7031500\,u \; + 2401000000, \tag{51}$$

of j-invariant $j = j_2 = -2^5 \cdot 7 \cdot 17^3$. We consider the following transformation (This transformation is rational but not birational. If it were birational, then it would preserve the genus. Here, one goes from genus one to genus two):

$$u \; = \; -\,\frac{882000 \cdot (x-14)}{x^3 + 420\,x - 5600}, \qquad v \; = \; \frac{49000 \cdot (x^3 - 21\,x^2 - 140)}{(x^3 + 420\,x - 5600)^2} \cdot y. \tag{52}$$

This change of variables (52) actually transforms the elliptic curve (51) in u and v into the genus-two curve (50) in x and y. This provides a simple example of a genus-two curve with split Jacobian through K3 surfaces.

More generally, let us consider the Jacobian of a genus-two curve C. The Jacobian is simple if it does not contain a proper abelian subvariety, otherwise the Jacobian is reducible, or decomposable or "split". For this latter case, the only possibility for a genus-two curve is that its Jacobian is isogenous to a product $\mathcal{E} \times \mathcal{E}'$ of two elliptic curves (along these lines, see also the concepts of Igusa–Clebsch invariants and Hilbert modular surfaces [48–51]). Equivalently, there is a degree-n map $C \rightarrow \mathcal{E}$ to some elliptic curves. Classically, such pairs (One also has an anti-isometry Galois invariant $\mathcal{E}' \simeq \mathcal{E}$ under Weil pairing. The decomposition corresponds to real multiplication by the quadratic ring of discriminant n^2) C, $\mathcal{E}$ arose in the reduction of hyperelliptic integrals to elliptic ones [48]. The j-invariants correspond, here, to the two elliptic subfields: see [48].

5.3. Creative Telescoping on Rational Functions in Three Variables Associated with Genus-Two Curves with Split Jacobians: A Two-Parameter Example

Let us now consider the example with two parameters, a and b, given in Section 4.5 on page 12 of [48]. Let us substitute the rational parameterization (see also [52], Section 6, page 48):

$$u \; = \; \frac{x^2}{x^3 \, + a\,x^2 \, + b\,x \, + 1}, \qquad v \; = \; \frac{y \cdot (x^3 \, - b\,x \, - 2)}{(x^3 + a\,x^2 \, + b\,x \, + 1)^2}, \tag{53}$$

in the elliptic curve:

$$R \cdot v^2 \; = \; R \cdot u^3 \; + 2 \cdot (ab^2 \, - 6\,a^2 \, + 9\,b) \cdot u^2 \; + (12\,a - b^2) \cdot u \; - 4, \tag{54}$$

where:

$$R \; = \; 4 \cdot (a^3 + b^3) \; - a^2b^2 \; - 18\,ab \; + 27. \tag{55}$$

This gives the genus-two curve $C_{a,b}(x,y) = 0$, with:

$$C_{a,b}(x,y) \; = \; R \cdot y^2 \; + (4\,x^3 \, + b^2\,x^2 \, + 2\,b\,x \, + 1) \cdot (x^3 \, + a\,x^2 \, + b\,x \, + 1). \tag{56}$$

The j-invariant of the elliptic curve (54) gives the following exact expression for the Hauptmodul $\mathcal{H} = \frac{1728}{j}$:

$$\mathcal{H} \,\,\, = \,\,\, \frac{108 \cdot (b-3)^3 \cdot (4\,a^3 + 4\,b^3 - a^2b^2 - 18\,ab + 27)^2 \cdot (b^2 + 3\,b + 9)^3}{(a^2b^4 + 12\,b^5 - 126\,ab^3 + 216\,ba^2 + 405\,b^2 - 972\,a)^3}. \qquad (57)$$

Let us consider the telescoper of the rational function of three variables $x\,y/D_a(x,\,y,\,z)$, where the denominator $D_a(x,\,y,\,z)$ is $C_{a,\,b}(x,\,y)$, given by (56), but for $b \, = \, 3 + \, x\,y\,z$:

$$\begin{aligned} D_a(x,\,y,\,z) \,\,\, &= \,\,\, C_{a,3\,+\,xyz}(x,\,y) \\ &= \,\,\, x^6y^3z^3 \, + x^7y^2z^2 \, + 4\,x^3y^5z^3 + 9\,x^5y^2z^2 + 6\,x^6yz + 3\,x^4y^2z^2 + 36\,y^4x^2z^2 \\ &\quad +6\,x^5yz + 4\,x^6 + 27\,x^4yz + 9\,x^5 + 18\,x^3yz + 108\,xy^3z + 18\,x^4 + 3\,x^2yz \\ &\quad +32\,x^3 + 27\,x^2 + 135\,y^2 \, + 9\,x \, + 1 \\ &\quad + (x^6y^2z^2 + 6\,x^5yz + 2\,x^4yz + 4\,x^5 - 18\,xy^3z + 9\,x^4 + 6\,x^3 + x^2 - 54\,y^2) \cdot a \\ &\quad -y^2 \cdot (xyz \, + 3)^2 \cdot a^2 \,\,\, + 4\,y^2 \cdot a^3. \end{aligned} \qquad (58)$$

This telescoper of the rational function:

$$R_a(x,\,y,\,z) \,\,\, = \,\,\, \frac{x\,y}{D_a(x,\,y,\,z)}, \qquad (59)$$

is an order-four linear differential operator L_4, which is actually the direct sum, $L_4 = LCLM(L_2,\,M_2) \, = \, L_2 \oplus M_2$, of two order-two linear differential operators, having two pullbacked hypergeometric solutions. One finds out that one of the two pullbacks precisely corresponds to the Hauptmodul $\mathcal{H}$ given by (57) for $b \, = \, 3 + x$.

Let us consider the $a \, = \, 3$ subcase (the discriminant in b of $4\,a^3 + 4\,b^3 - a^2b^2 - 18\,ab + 27$ reads: $(a-3)^3 \cdot (a^2 \, + 3\,a \, + 9)^3$; consequently, the exact expressions are simpler at $a = 3$). For $a \, = \, 3$, the Hauptmodul $\mathcal{H} \, = \, \frac{1728}{j}$, given by (57) becomes for $b \, = \, 3 \, + x$:

$$\mathcal{H} \,\,\, = \,\,\, \frac{4 \cdot \, x \cdot \, (27 + 4\,x)^2 \cdot \, (x^2 + 9\,x + 27)^3}{(9+x)^3 \cdot \, (4\,x^2 + 27\,x + 27)^3}. \qquad (60)$$

The telescoper of the rational function (59), with $D_a(x,\,y,\,z)$ given by (58) for $a \, = \, 3$, is an order-four linear differential operator, which is the direct sum of two order-two linear differential operators $L_4 \, = \, LCLM(L_2,\,M_2) \, = \, L_2 \oplus M_2$, with these two order-two linear differential operators having the pullbacked hypergeometric solutions:

$$(27 \, + \, 4\,x)^{-1/2} \cdot \, x^{-5/4} \cdot \, {}_2F_1\Big(\frac{1}{12},\, \frac{5}{12};\, 1;\, 1 \, + \, \frac{27}{4\,x}\Big), \qquad (61)$$

for L_2, and:

$$\begin{aligned} & \frac{3+x}{(9+x)^{1/4} \cdot \, (4\,x^2 + 27\,x + 27)^{1/4} \cdot \, x^{3/2} \cdot (27 + 4\,x)^{1/2}} \\ & \quad \times \, {}_2F_1\Big(\frac{1}{12},\, \frac{5}{12};\, 1;\,\, \frac{4 \cdot \, x \cdot \, (27\, + 4\,x)^2 \cdot \, (x^2 \, + 9\,x \, + 27)^3}{(9+x)^3 \cdot \, (4\,x^2 \, + 27\,x \, + 27)^3}\Big), \end{aligned} \qquad (62)$$

for M_2, where we see clearly that the Hauptmodul in (62) is precisely the Hauptmodul (60). The Jacobian of the genus-two curve is a split Jacobian corresponding to the product $\mathcal{E}_1 \, \times \mathcal{E}_2$ of two elliptic curves; the j-invariant of the second elliptic curve corresponds to the Hauptmodul $\mathcal{H} \, = \, \frac{1728}{j}$ given by (57) when the j-invariant of the first elliptic curve reads:

$$j_1 \,\,\, = \,\,\, \frac{6912\,x}{27 \, + 4\,x}, \qquad (63)$$

corresponding to the Hauptmodul $\frac{1728}{j_1} = 1 + \frac{27}{4x}$ in (61). This second invariant is, as it should, exactly the j-invariant of the second elliptic curve $\mathcal{E}'$, given page 48 in [52]:

$$j(\mathcal{E}') = \frac{256 \cdot (3b - a^2)^3}{4a^3c - a^2b^2 - 18abc + 4b^3 + 27c^2}, \tag{64}$$

for $c = 1$, $a = 3$ and $b = 3 + x$.

5.4. Creative Telescoping on Rational Functions of Three Variables Associated with Genus-Two Curves with Split Jacobians: A Simple Example

Another simpler example of a genus-two curve with pullbacked ${}_2F_1$ solution (not a product of pullbacked ${}_2F_1$) of the telescoper can be given if one considers the genus-two algebraic curve $C_p(x, y) = 0$ given in Lemma 7 of [53] (see also [54,55]):

$$C_p(x, y) = x^5 + x^3 + p \cdot x - y^2. \tag{65}$$

Let us introduce the rational function $xy/D(x, y, z)$, where the denominator $D(x, y, z)$ is given by:

$$D(x, y, z) = C_{(p=xyz)}(x, y) = x^5 + x^3 + x^2yz - y^2. \tag{66}$$

The telescoper of this rational function is an order-two linear differential operator that has the two hypergeometric solutions:

$$x^{-1/4} \cdot {}_2F_1\Big(\frac{1}{8}, \frac{5}{8}; \frac{3}{4}; 4x\Big) \tag{67}$$

which is a Puiseux series at $x = 0$ and:

$$x^{-1/4} \cdot {}_2F_1\Big(\frac{1}{8}, \frac{5}{8}; 1; 1 - 4x\Big). \tag{68}$$

These two hypergeometric solutions can be rewritten as (the fact that ${}_2F_1\big(\frac{1}{8}, \frac{5}{8}; 1; z\big)$ can be rewritten as ${}_2F_1\big(\frac{1}{12}, \frac{5}{12}; 1; \mathcal{H}(z)\big)$ where the Hauptmodul $\mathcal{H}(z)$ is solution of a quadratic equation is given in Equation (H.14) of Appendix H of [18]):

$$\mathcal{A}(x) \cdot {}_2F_1\Big(\frac{1}{12}, \frac{5}{12}; 1; \frac{1728}{J}\Big), \tag{69}$$

where the j-invariant J, in the Hauptmodul $\frac{1728}{J}$ in (69), corresponds exactly to the degree-two elliptic subfields:

$$J^2 - 128 \cdot \frac{(2000x^2 + 1440x + 27)}{(1 - 4x)^2} \cdot J - 4096 \cdot \frac{(100x - 9)^3}{(1 - 4x)^3} = 0, \tag{70}$$

given in the first equation of page 6 of [53].

Remark 2. *In contrast to the previous example of Section 5.3, where we had two j-invariants corresponding to the two order-two linear differential operators L_2 and M_2 of the direct-sum decomposition of the order-four telescoper, here, we have just one order-two telescoper, which is enough to "encapsulate" the two j-invariants (70), since they are Galois-conjugate.*

6. Rational Functions with Tri-Quadratic Denominator and N-Quadratic Denominator

We attempt to find telescopers of rational functions corresponding to (factors of) linear differential operators of "small" orders; for instance, order-two linear differential operators with pullbacked ${}_2F_1$ hypergeometric functions, classical modular forms, or their modular generalizations (order-four Calabi–Yau linear differential operators [27], etc.). As we saw in the previous sections, this corresponds to the fact that the denominator of these rational functions is associated with an elliptic curve, or the products of elliptic curves, with K3 surfaces or with threefold Calabi–Yau manifolds corresponding to algebraic varieties with foliations in elliptic curves (Even though K3 surfaces or threefold Calabi-Yau manifolds are *not* abelian varieties, the Weierstrass–Legendre forms introduced in Appendix D amount to saying that K3 surfaces can be "essentially viewed" (as far as creative telescoping is concerned) as foliations in two elliptic curves, and threefold Calabi–Yau manifolds as foliations in three elliptic curves), since this paper attempts to reduce the *differential algebra* creative telescoping calculations to *effective algebraic geometry* calculations. (One has birational automorphisms in projective spaces [56,57], but since this paper is dedicated to (efficient) formal calculations, we work exclusively in affine coordinates (see for instance (A41), (A42), and (A43) below.) For algebraic geometers, an elliptic curve is a smooth complete genus-one curve with a choice of a base point. Here our elliptic curves are, in fact, an affine piece of a genus-one curve with no base point, but this does not really matter, because the j-invariant, which is all we care about in these kinds of creative telescoping calculations, is determined by that much information and structures, we want to focus on rational functions with denominators that correspond to *selected* algebraic varieties [44,58], beyond algebraic varieties corresponding to products of elliptic curves or foliations in elliptic curves (K3 surfaces, threefold Calabi–Yau manifolds, and higher curves with split Jacobian corresponding to products of elliptic curves, . . .), namely, algebraic varieties with an infinite number of birational automorphisms (the best explicit illustration of this situation emerges in integrable models [44,58–60]). This infinite number of birational symmetries excludes algebraic varieties of the "general type" (with a finite number. (There are even precise bounds for the number of automorphisms. The upper bound is $84\,(g-1)$ for curves of genus g, and these bounds have been extensively studied in higher dimensions [61–63]) of birational symmetries.) For algebraic surfaces, this amounts to discarding the surfaces of "general type" that have Kodaira dimension two, focusing on Kodaira dimension one (elliptic surfaces), Kodaira dimension zero (abelian surfaces, hyperelliptic surfaces, K3 surfaces, and Enriques surfaces), or even Kodaira dimension $-\infty$ (ruled surfaces and rational surfaces).

In contrast with algebraic curves, where one can easily and very efficiently calculate the genus of the curves to discard algebraic curves of higher genus and, in the case of genus one, obtain the j-invariant using formal calculations (use with(algcurves) in Maple, and the commands "genus" and "j_invariant"), it is, in practice, quite difficult to see for higher dimensional algebraic varieties, that the algebraic variety is not of the "general type", because it has an infinite number of birational symmetries. For these (low Kodaira dimension) "selected cases" that we are interested in, calculating the generalization of the j-invariant (Igusa–Shioda invariants, etc.) is quite hard.

Along this line, we want to underline that there exists a remarkable set of algebraic surfaces, namely the algebraic surfaces corresponding to tri-quadratic equations:

$$\sum_{m=0,1,2}\ \sum_{n=0,1,2}\ \sum_{l=0,1,2} a_{m,n,l}\cdot x^m\, y^n\, z^l \quad = \quad 0, \tag{71}$$

depending on $27 = 3^3$ parameters $a_{m,n,l}$. More generally, one can introduce algebraic varieties corresponding to N-quadratic equations:

$$\sum_{m_1=0,1,2}\ \sum_{m_2=0,1,2}\ \cdots \sum_{m_N=0,1,2} a_{m_1,m_2,\cdots,m_N}\cdot x_1^{m_1}\, x_2^{m_2}\,\cdots\, x_N^{m_N} \quad = \quad 0. \tag{72}$$

With these tri-quadratic (71), or N-quadratic (72) equations, we will see, in Appendices E.1 and E.2, that we have *automatically* (selected) algebraic varieties that are not of the "general type", having an infinite number of birational symmetries, which is precisely our requirement for the denominator of rational functions with remarkable telescopers. (Telescopers with factors of "small enough" order, possibly yielding classical modular forms, Calabi–

Yau operators, etc. Rational functions with denominators of the "general type" will yield telescopers of very large orders.)

Let us first, as a warm-up, consider, in the next subsection, a remarkable example of tri-quadratic (71), where the underlying foliation in elliptic curves is crystal clear.

6.1. Rational Functions with Tri-Quadratic Denominator Simply Corresponding to Elliptic Curves

Let us first recall the tri-quadratic equation in three variables x, y and z:

$$\begin{aligned} x^2y^2z^2 \quad - 2 \cdot M \cdot xyz \cdot (x + y + z) \quad + 4 \cdot M \cdot (M + 1) \cdot xyz \\ + M^2 \cdot (x^2 + y^2 + z^2) \quad - 2\, M^2 \cdot (xy + xz + yz) \quad = \quad 0, \end{aligned} \tag{73}$$

already introduced in Appendix C of [64]. This algebraic surface, symmetric in x, y, and z, can be seen for z (respectively, x or y) fixed, as an elliptic curve whose j-invariant is independent of z, yielding the corresponding Hauptmodul:

$$\mathcal{H} \;=\; \frac{1728}{j} \;=\; \frac{27 \cdot M^2 \cdot (M-1)^2}{4 \cdot (M^2 - M + 1)^3}. \tag{74}$$

This corresponds to the fact that this algebraic surface (73) can be seen as a product of two instances of the same elliptic curve with the Hauptmodul (74). This is a consequence of the fact that, introducing $x \,=\, sn(u)^2$, $y \,=\, sn(v)^2$ and $z \,=\, sn(u+v)^2$, and $M \,=\, 1/k^2$, this algebraic surface (73) corresponds to the well-known formula for the addition on an elliptic sine (see Equation (C.3) in Appendix C of [64]):

$$sn(u \,+\, v) \quad = \quad \frac{sn(u)\, cn(v)\, dn(v) \;+\; sn(v)\, cn(u)\, dn(u)}{1 \;- k^2\, sn(u)^2\, sn(v)^2}. \tag{75}$$

For $M \,=\, x\,y\,z\,w$, the LHS of the tri-quadratic Equation (73) yields a polynomial of four variables x, y, z, and w, which we denote $T(x,\, y,\, z,\, w)$:

$$\begin{aligned} T(x,\, y,\, z,\, w) \;\; = & \\ x^2y^2z^2 \quad - 2 \,\cdot\, x^2y^2z^2\, w \cdot\, (x + y + z) \quad + 4 \cdot\, (xyzw + 1) \cdot\, x^2y^2z^2\, w \\ + x^2y^2z^2w^2 \cdot\, (x^2 + y^2 + z^2) \quad - 2\, x^2y^2z^2w^2 \cdot\, (xy + xz + yz). \end{aligned} \tag{76}$$

The telescoper of the rational function in four variables x, y, z, and w,

$$\frac{x\,y\,z}{T(x,\, y,\, z,\, w)}, \tag{77}$$

is an order-three (self-adjoint) linear differential operator that is the symmetric square of the order-two linear differential operator, having the following pullbacked ${}_2F_1$ hypergeometric solution:

$$\begin{aligned} x^{-1/2} \cdot\, (x^2 - x + 1)^{-1/4} \\ \times\, {}_2F_1\Big(\frac{1}{12},\, \frac{5}{12};\, 1;\, \frac{27 \cdot x^2 \cdot (x-1)^2}{4 \,\cdot\, (x^2 - x + 1)^3}\Big). \end{aligned} \tag{78}$$

As it should, the Hauptmodul in (78) is the same as the Hauptmodul (74). The algebraic surface (73) can be seen as the product of two instances of the same elliptic curve with the Hauptmodul (74): As expected, the solution of the order-three telescoper is the square of the pullbacked ${}_2F_1$ hypergeometric function (78) with that Hauptmodul.

More generally, we can also consider another tri-quadratic equation of three variables x, y, and z, and two parameters M and N:

$$\begin{aligned} x^2y^2z^2 \quad - 2\, M \cdot\, xyz \cdot\, (x + y + z) \quad + N \cdot\, xyz \\ + M^2 \cdot\, (x^2 + y^2 + z^2) \quad - 2\, M^2 \cdot\, (xy + xz + yz) \quad = \quad 0. \end{aligned} \tag{79}$$

This surface, symmetric in x, y, and z, can be seen for z (respectively, x or y) fixed as an elliptic curve whose j-invariant is again independent of z, yielding the corresponding Hauptmodul:

$$\mathcal{H} \,\,=\,\, \frac{1728}{j} \,\,=\,\, \frac{1728 \cdot M^6 \cdot (64\,M^3 - N^2)}{(48\,M^3 - N^2)^3}. \tag{80}$$

Let us consider the following change of variables $M = m^2$ and $N = 8 \cdot m^3 + p$ in (79). For $p = x\,y\,z\,w$, the LHS of the tri-quadratic Equation (79) yields a polynomial in four variables x, y, z, and w, which we denote $\mathcal{T}_m(x, y, z, w)$:

$$\begin{aligned} \mathcal{T}_m(x, y, z, w) \,\,=\,\, & \\ x^2y^2z^2 \,\, - 2\,m^2 \cdot xyz \cdot (x+y+z) \,\, + (8 \cdot m^3 + \, x\,y\,z\,w) \cdot xyz & \\ +m^4 \cdot (x^2+y^2+z^2) \,\, - 2\,m^4 \cdot (xy+xz+yz). & \end{aligned} \tag{81}$$

For z (respectively, x or y) fixed, the corresponding Hauptmodul (80) reads:

$$\mathcal{H} \,\,=\,\, \frac{1728 \cdot m^{12} \cdot p \cdot (16\,m^3 + p)}{(16\,m^6 + 16\,m^3 \cdot p + p^2)^3}. \tag{82}$$

The telescoper of the rational function in four variables x, y, z, and w,

$$\frac{x\,y\,z}{\mathcal{T}_m(x, y, z, w)}, \tag{83}$$

is an order-three (self-adjoint) linear differential operator that is the symmetric square of an order-two linear differential operator, having the following pullbacked ${}_2F_1$ hypergeometric solution:

$$\begin{aligned} & (16\,m^6 + 16\,m^3 \cdot x + x^2)^{-1/4} \cdot \\ & \quad \times \, {}_2F_1\Big(\frac{1}{12}, \frac{5}{12}; 1; \frac{1728 \cdot m^{12} \cdot x \cdot (16\,m^3 + x)}{(16\,m^6 + 16\,m^3 \cdot x + x^2)^3}\Big). \end{aligned} \tag{84}$$

As it should, the Hauptmodul in (84) is the same as the Hauptmodul (82). The algebraic surface (79) can be seen as the product of two instances of the same elliptic curve with the Hauptmodul (80) (or (82)). As expected, the solution of the order-three telescoper is the square of the pullbacked ${}_2F_1$ hypergeometric function (84) with the Hauptmodul (82).

Remark 3. *Let us perform some (slight) deformations of the rational function (77), changing the first -2 coefficient in (76) into a -3 coefficient. One thus considers the polynomial $T(x, y, z, w)$:*

$$\begin{aligned} T(x, y, z, w) \,\,=\,\, & \\ x^2y^2z^2 \,\, - 3 \cdot x^2y^2z^2\,w \cdot (x+y+z) \,\, + 4 \cdot (xyzw+1) \cdot x^2y^2z^2\,w & \\ +x^2y^2z^2w^2 \cdot (x^2+y^2+z^2) \,\, - 2 \cdot x^2y^2z^2w^2 \cdot (xy+xz+yz). & \end{aligned} \tag{85}$$

The telescoper of the rational function in four variables,

$$\frac{x\,y\,z}{T(x, y, z, w)}, \tag{86}$$

is an (irreducible) linear differential operator L_4 of (only) order four, which is non-trivially homomorphic to its adjoint. (Its exterior square has a rational solution. However this order-four linear differential operator is not MUM (maximum unipotent monodromy [27,65,66]).) A priori, we cannot exclude the fact that L_4 could be homomorphic to the symmetric cube of a second-order linear differential operator, or to a symmetric product of two second-order operators. Furthermore, it could also be, in principle, that these second-order operators admit classical modular forms as solutions *(pullbacks of special ${}_2F_1$ hypergeometric functions). However, these options can both be excluded by using some results from differential Galois theory [67], specifically from [68] (Prop. 7, p. 50) for the symmetric cube case, and from [68] (Prop. 10, p. 69) for the symmetric product case; see also [69] (§3). Indeed, if L_4 were either a symmetric cube or a symmetric product of order-two operators, then its symmetric square would contain a (direct) factor of order 3 or 1. This is ruled out by a factorization procedure which shows that the symmetric square of L_4 is (LCLM-)irreducible.*

This example does not correspond to an addition formula such as (75), but the polynomial $T(x, y, z, w)$ still corresponds to a tri-quadratic. Consequently, it is an algebraic variety with an infinite number of birational automorphisms, as shown in Appendix E.1.

6.2. Rational Functions with Tri-Quadratic Denominator: Fricke Cubics Examples Associated with Painlevé VI Equations

Let us consider other simple examples of tri-quadratic surfaces that occur in different domains of mathematics and physics.

Among the *Fricke families* of cubic surfaces, the family [70–72]:

$$x\,y\,z \;\; + x^2 + y^2 + z^2 \;\; + b_1\,x + b_2\,y + b_3\,z \;\; + c \;=\; 0, \tag{87}$$

of affine cubic surfaces parameterized by the four constants (b_1, b_2, b_3, c) is known [71] to be a deformation of a D_4 singularity that occurs at the symmetric (Manin's) case $b_1 = b_2 = b_3 = -8$, $c = 28$.

Among the symmetric $b_1 = b_2 = b_3$ cases, some selected sets of the four constants (b_1, b_2, b_3, c) emerge: the Markov cubic $b_1 = b_2 = b_3 = c = 0$, Cayley's nodal cubic $b_1 = b_2 = b_3 = 0$, $c = -4$, Clebsch's diagonal cubic $b_1 = b_2 = b_3 = 0$, $c = -20$, and Klein's cubic $b_1 = b_2 = b_3 = -1$, $c = 0$.

Some of these symmetric cubics can be seen as the monodromy manifold of the *Painlevé VI equation* (see Equation (1.7) in [73], see also Equations (1.2) and (1.4) in [72]): the Picard–Hitchin cases $(0, 0, 0, 4)$, $(0, 0, 0, -4)$, $(0, 0, 0, -32)$, Kitaev's cases $(0, 0, 0, 0)$, $(-8, -8, -8, -64)$, and especially Manin's case $(-8, -8, -8, 28)$.

Let us consider the (symmetric) rational function in three variables, x, y, and z [71]:

$$R(x, y, z) \;=\; \frac{1}{x^2 + y^2 + z^2 \;\; + x\,y\,z \;\; + c}, \tag{88}$$

that takes into account the other Picard–Hitchin cases (as well as the Markov cubic $b_1 = b_2 = b_3 = c = 0$, Cayley's nodal cubic $b_1 = b_2 = b_3 = 0$, $c = -4$, and Clebsch's diagonal cubic $b_1 = b_2 = b_3 = 0$, $c = -20$ cases) $(0, 0, 0, 4)$, $(0, 0, 0, -4)$, $(0, 0, 0, 32)$. The rational function (88) has an order-two telescoper that has a simple pullbacked hypergeometric solution:

$$\begin{aligned}
&\frac{1}{x + c} \cdot {}_2F_1\Big(\frac{1}{3}, \frac{2}{3}; 1; \; -\frac{27\,x^2}{(x + c)^3}\Big) \\
&\quad = \;\; (x + c)^{-1/4} \cdot q_3(x)^{-1/4} \cdot {}_2F_1\Big(\frac{1}{12}, \frac{5}{12}, 1; \; -\frac{1728 \cdot x^6 \cdot p_3(x)}{(x + c)^3 \cdot q_3(x)^3}\Big),
\end{aligned} \tag{89}$$

where (the values $c = 0$ and $c = -4$ are the only values such that the discriminant in x of $p_3(x)$ can be zero):

$$\begin{aligned}
p_3(x) \;&=\; x^3 \; + 3 \cdot (c + 9) \cdot x^2 \; + 3 \cdot c^2 \cdot x \; + c^3, \\
q_3(x) \;&=\; x^3 \; + 3 \cdot (c + 8) \cdot x^2 \; + 3 \cdot c^2 \cdot x \; + c^3,
\end{aligned}$$

Eliminating $z = \frac{p}{x\,y}$ in the denominator of (88) gives the genus-four algebraic curve:

$$x^2y^2 \cdot (x^2 \,+\, y^2) \;\; + (p + c) \cdot x^2y^2 \;\; + p^2 \;\;=\;\; 0. \tag{90}$$

Again, the question is to see whether the Jacobian of this genus-four algebraic curve (88) could also correspond to a split Jacobian, with a j-invariant corresponding to the Hauptmodul in (89).

7. Telescopers of Rational Functions of Several Variables

Let us consider the rational function in four variables x, y, z, u:

$$R(x, y, z, u) \,=\, \frac{1}{1 \,+3\,y \,+z \,+9\,y\,z \,+11\,z^2\,y \,+3\,u\,x}. \tag{91}$$

The telescoper of this rational function of four variables is an order-two linear differential operator L_2 that has the pullbacked hypergeometric solution:

$$\begin{aligned} &(1 \,-\, 2592\,x^2)^{-1/4} \\ &\times {}_2F_1\Big(\frac{1}{12}, \frac{5}{12}; 1; \, -\frac{419904 \cdot x^3 \cdot (5 - 12\,x - 19440\,x^2 + 2665872\,x^3)}{(1 \,-\, 2592\,x^2)^3}\Big). \end{aligned} \tag{92}$$

The diagonal of (91) is the expansion of this pullbacked hypergeometric function (92):

$$\begin{aligned} &1 \,+648\,x^2 \,-72900\,x^3 \,+1224720\,x^4 - 330674400\,x^5 \,+23370413220\,x^6 \\ &-1276733858400\,x^7 \,+180019474034400\,x^8 \,-\, 12013427240614800\,x^9 \,+\, \cdots \end{aligned} \tag{93}$$

If one considers the intersection of the vanishing condition of the denominator of (91) with the hyperbola $p \,=\, x\,y\,z\,u$, eliminating, for instance, $u \,=\, \frac{p}{x\,y\,z}$ in the vanishing condition of the denominator of (91), one gets a condition, independent of x, that corresponds to a genus-one curve:

$$11\,y^2z^3 \,+9\,y^2z^2 \,+3\,y^2z \,+yz^2 \,+yz \,+3\,p \,=\, 0. \tag{94}$$

The Hauptmodul of this elliptic curve (94) reads:

$$\mathcal{H} \,=\, -\frac{419904 \cdot p^3 \cdot (5 \,-12\,p \,-19440\,p^2 \,+2665872\,p^3)}{(1 \,-\, 2592\,p^2)^3}, \tag{95}$$

which corresponds precisely to the Hauptmodul pullback in (92).

Remark 4. *The expansion (93) of (92) is not only the diagonal of the rational function* $R(x, y, z, u)$ *in four variables (91); it is also the diagonal of the rational function of three variables* $R(x, y, z, 1)$*. Actually, using Section 3, one sees easily that eliminating* $x \,=\, \frac{p}{y\,z}$ *in the vanishing condition of the denominator of* $R(x, y, z, 1)$ *gives exactly the same elliptic curve (94).*

Let us now generalize the rational function (91) of four variables x, y, z, u, by introducing the rational function of $N+3$ variables $x, y, z, u_1, u_2, \cdots, u_N$:

$$\begin{aligned} &R(x, y, z, u_1, u_2, \cdots, u_N) \\ &\quad= \frac{1}{1 \,+3\,y \,+z \,+9\,y\,z \,+11\,z^2\,y \,+3\,x \cdot u_1\,u_2 \,\cdots\, u_N}. \end{aligned} \tag{96}$$

The telescoper of this rational function of $N+3$ variables is the same order-two telescoper as for (91), which has the pullbacked hypergeometric solution (92). Again, one can verify that the diagonal of (96) is the expansion (93) of the pullbacked hypergeometric function (a pure algebraic geometer will probably consider this result as trivial from the computational point of view, saying that the variety is a fiber bundle over a family of elliptic curves with constant fiber (see also below)) (92). If one considers the intersection of the vanishing condition of the denominator of (96) with the hyperbola $p \,=\, x\,y\,z\,u_1\,u_2 \,\cdots\, u_N$, eliminating, for instance, $u_N \,=\, \frac{p}{x\,y\,z\,u_1 \cdots u_{N-1}}$ in the vanishing condition of the denominator of (96), one again obtains a condition, independent of x but also of $u_1, \ldots, u_N$, that corresponds to the genus-one curve (94):

$$11\, y^2 z^3 \; + 9\, y^2 z^2 \; + 3\, y^2 z \; + y z^2 \; + y z \; + 3\, p \;\; = \;\; 0. \tag{97}$$

The Hauptmodul of this elliptic curve (97) is again equal to the Hauptmodul (95), which corresponds precisely to the Hauptmodul pullback in (92).

Other examples, corresponding to the simple polynomial deformations of (91), such that their diagonal is the pullbacked $_2F_1$ hypergeometric function (92), are displayed in Appendix F. This (infinite) family of rational functions corresponds to a different algebraic geometry scenario: the "canonical" algebraic surface corresponding to the intersection of the vanishing condition of the denominator of the rational function with the hyperbola $p = xyz$, is foliated in (generically high genus) algebraic curves depending on the variable x. One sees (experimentally) that the Hauptmodul of the pullbacked $_2F_1$ hypergeometric functions corresponds to the Hauptmodul of the $x = 0$ algebraic curve, which is an elliptic curve (the algebraic curves for other values of x are not necessarily elliptic curves, they can be algebraic curves of quite large genus). In contrast to the other examples and results of this paper, we have no algebraic geometry interpretation of this experimental result yet.

8. Conclusions

Diagonals of rational functions emerge quite naturally in lattice statistical mechanics [19,20]. This explains the frequent occurrence of modular forms, represented as pullbacked $_2F_1$ hypergeometric functions [1,2] in lattice statistical mechanics [21–27].

We have shown that the results that we had obtained on diagonals of 9- and 10-parameter families of rational functions in three variables, using creative telescoping yielding classical modular forms expressed as pullbacked $_2F_1$ hypergeometric functions [1,2], can be obtained much more efficiently by calculating the j-invariant of an elliptic curve that is canonically associated with the denominator of the rational functions. In the case where creative telescoping yields pullbacked $_2F_1$ hypergeometric functions, we generalize this result to other families of rational functions of three, and even more than three, variables, when the denominator can be associated with products of elliptic curves or foliations in terms of elliptic curves, or when the denominator is associated with a genus-two curve with a split Jacobian corresponding to the products of elliptic curves.

We have seen different scenarios. In the first cases, we have considered denominators corresponding to products of elliptic curves: in these cases, the solutions of the telescoper were products of pullbacked $_2F_1$ hypergeometric functions. We have also considered denominators corresponding to genus-two curves with split Jacobians isogenous to the products of two elliptic curves, and in these cases, the solutions of the telescoper were the sums of two pullbacked $_2F_1$ hypergeometric functions, sometimes with one pullbacked $_2F_1$ hypergeometric function being enough to describe the two Galois-conjugate j-invariants (see Section 5.4). We also considered the denominators corresponding to algebraic varieties where the Hauptmodul pullback in the pullbacked $_2F_1$ hypergeometric functions emerges from a selected ($x = 0$, see Appendices F.1 and F.2) elliptic curve of the algebraic variety. We also encountered denominators corresponding to algebraic manifolds with an infinite set of birational automorphisms and elliptic curves foliation, no longer yielding classical modular forms represented as pullbacked $_2F_1$ hypergeometric functions, but more general modular structures associated with selected linear differential operators such as Calabi–Yau linear differential operators [27,65] and their generalizations.

The creative telescoping method on a rational function is an efficient way to find the periods of an algebraic variety over *all possible cycles* (not only the vanishing cycles [5,38] corresponding to the diagonals of rational functions). The fact that the solution of the telescoper corresponds to "periods" [37] over all possible cycles is a simple consequence of the fact that creative telescoping corresponds to purely differential algebraic manipulations on the integrand independently of the cycles, thus being blind to analytical details. In this paper, we show that the final result emerging from such differential algebra procedures (which can be cumbersome when the result depends on 9 or 10 parameters) can be obtained almost instantaneously from a more fundamental intrinsic pure algebraic geometry approach, calculating, for instance, the j-invariant of some canonical elliptic curve. This corresponds to a shift Analysis → Differential Algebra → Algebraic Geometry. In algebraic geometry studies of more involved algebraic varieties than the product of elliptic curves, foliation in elliptic curves (Calabi–Yau manifolds, etc.) is often a tedious and/or difficult task (finding Igusa–Shioda invariants, etc.), and formal calculation tools are not always available or user-friendly. Ironically, for such involved algebraic varieties, creative telescoping may then become a simple and efficient tool to perform effective algebraic geometry studies.

Some work remains to be done. While in the present paper, we have focused on studying the relationship between the denominator of a rational function and its diagonal, it would be interesting to explore the impact that variations of the numerator have on the telescoper or on the diagonal. Then, one may expect that all these algebraic–geometric findings could be exploited in the design of novel and more efficient creative telescoping algorithms for multivariate rational functions.

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Appendix A. Diagonals of Rational Functions and Picard–Fuchs Equations

For simplicity, let us consider the rational functions of three variables and double integrals [74]. The diagonal of a rational function of three variables is obtained through its multi-Taylor expansion [19,20]:

$$R(x, y, z) \quad = \quad \sum_{m} \sum_{n} \sum_{l} \, a_{m,n,l} \cdot x^m \, y^n \, z^l, \qquad \text{(A1)}$$

by extracting the "diagonal" terms, i.e., the powers of the product $p \, = \, xyz$:

$$Diag\Big(R(x, y, z)\Big) \quad = \quad \sum_{m} \, a_{m,m,m} \cdot p^m. \qquad \text{(A2)}$$

Such diagonals are closely related to the integrals of rational functions. For example, $Diag\Big(R(x,\, y,\, z)\Big)$ is the constant term (in y, z) in the infinite expansion:

$$R\Big(\frac{p}{y\,z},\, y,\, z\Big) \quad = \quad \sum_{m,\,n,\,l\,\geq\, 0} a_{m,\,n,\,l} \cdot p^m \, y^{n-m} \, z^{l-m}, \qquad \text{(A3)}$$

which can be represented by the integral [35]:

$$\frac{1}{(2\,\pi\,i)^2} \oint \oint R\Big(\frac{p}{y\,z},\, y,\, z\Big) \, \frac{dy}{y} \wedge \frac{dz}{z}. \qquad \text{(A4)}$$

The diagonal (A2) is also the constant term (in y, z) of:

$$R\Big(\frac{p}{y},\, \frac{y}{z},\, z\Big) \quad = \quad \sum_{m,\,n,\,l\,\geq\, 0} a_{m,\,n,\,l} \cdot p^m \, y^{n-m} \, z^{l-n}, \qquad \text{(A5)}$$

which is of the form:

$$\frac{1}{(2\,\pi\,i)^2} \oint \oint \frac{N_p(y,\, z)}{D_p(y,\, z)} \, \frac{dy}{y} \wedge \frac{dz}{z}, \qquad \text{(A6)}$$

where the numerator $N_p(y,\, z)$ and the denominator $D_p(y,\, z)$ are polynomials. It is well-known that such integrals satisfy a linear differential equation with respect to p having rational functions in p as coefficients, called the Picard–Fuchs equation. (The order of this linear differential equation is generally equal to the rank of the algebraic de Rham cohomology of $D_p(y,\, z) \, = \, 0$. For curves of genus g, this rank is $2\,g$.) The problem of determining such linear differential equations was commenced by Griffiths [75] with the assumption that the variety $D_p(y,\, z) \, = \, 0$ is smooth, but later techniques were developed to include examples with singular points [35,40]. The linear differential equations (Gauss–Manin systems and telescopers) occurring in integrable models [16,23,24] are of an order that is much larger than order two (Since it is well-known that the Picard–Fuchs equation corresponding to the (Weierstrass) elliptic curve corresponds to the hypergeometric function ${}_2F_1(1/12, 5/12; 1; 1/J)$) and almost never correspond to smooth varieties. Creative telescoping (for a detailed introduction to creative telescoping [36], see for instance [34]), and more specifically, the programs [3] corresponding to a fast approach to creative telescoping [42], are a powerful way to find these linear differential operators annihilating these diagonals of rational functions in the cases emerging naturally in theoretical physics, integrable models, and enumerative combinatorics, for which the order of the linear differential operators is quite large [16,23,24] and the variety $D_p(y,\, z) \, = \, 0$ is (most of the time) not a smooth one. All the pedagogical (but non-trivial) examples of telescopers displayed in this paper can be viewed by an algebraic geometer as a presentation of examples of families of varieties and their Picard–Fuchs equations.

Appendix B. Maximum Number of Parameters for Families of Planar Elliptic Curves

We have seen in Section 3 that the previous results on the diagonals of 9- or 10-parameter families of rational functions of three variables being pullbacked ${}_2F_1$ hypergeometric functions (and in fact, classical modular forms) can actually be seen as corresponding to the fact (well-known in integrable models and integrable mappings) that the most general biquadratic corresponding to elliptic curves is a 9-parameter family, and that the most general ternary cubic corresponding to elliptic curves is a 10-parameter family. One can, for instance, recall page 238 of [76], which amounts to a consideration of the collection of all cubic curves in $\mathbb{C}P_2$ with the homogeneous equation:

$$\begin{aligned} &a\,x^3 \, + b\,x^2\,y \, + c\,x\,y^2 \, + d\,y^3 \, + e\,x^2\,z \, + f\,x\,z^2 \, + g\,y^2\,z \\ &\quad + h\,y\,z^2 \, + i\,z^3 \, + j\,x\,y\,z \quad = \quad 0, \end{aligned} \qquad \text{(A7)}$$

and the associated problems of passing through nine given points. One can also recall the ternary cubics in [77,78] and other problems of elliptic curves of high rank [79] (see the concept of Neron–Severi rank).

Since the rational functions of three variables that we consider are essentially encoded by the denominator of these rational functions, and in the cases we have considered, the emergence of pullbacked ${}_2F_1$ hypergeometric functions (and in fact, classical modular forms) corresponds to the fact that the intersection of these denominators with the hyperbola $p \, = \, x\,y\,z$ corresponds to elliptic curves, one sees that these rational functions are essentially classified by the possible n-parameter families $P(x,\, y) \, = \, 0$ of elliptic curves.

If one considers a polynomial:

$$P(x,y) \quad = \quad \sum_m \sum_n a_{m,n} \cdot x^m \, y^n, \tag{A8}$$

with generic coefficients $a_{m,n} \in \mathbb{C}$, then the genus of the algebraic curve defined by P is determined by the support $supp(P) = \{(m,n) \in \mathbb{N}^2 : a_{m,n} \neq 0\}$. More precisely, the genus equals the number of interior integer lattice points inside the convex hull of $supp(P)$ [80] (see also the discussion in [81]). For example, the support of the 10-parameter family (11) consists of the following 10 points in $\mathbb{N}^2$:

$$(0,0), \ (0,1), \ (0,2), \ (0,3), \ (1,1), \ (1,2), \ (1,3), \ (2,2), \ (2,3), \ (3,3)$$

which forms a right triangle of side length 3. Only one of these points is an interior point, namely $(1,2)$; hence, the genus is 1.

Therefore, we may ask: which integer lattice polytopes exist that have exactly one interior point, and what is the largest such polytope? Not surprisingly, the answer is known: there are (up to transformations like translation, rotation, and shearing) exactly 16 different polytopes with a single interior point [82] (see also Figure 5, page 548 in [83]), the above-mentioned right triangle being the one with the highest total number of lattice points.

This shows that there cannot be a family of elliptic curves with more than 10 parameters.

Appendix C. Monomial Transformations Preserving Pullbacked Hypergeometric Results

More generally, recalling Section 4.2 in [2], and Section 4.2, page 17, in [1], let us consider the monomial transformation:

$$\begin{aligned} (x,\, y,\, z) \quad \longrightarrow \quad & M(x,\, y,\, z) \;\; = \;\; (x_M,\, y_M,\, z_M) \\ = \;\; & \Big(x^{A_1} \cdot y^{A_2} \cdot z^{A_3}, \;\; x^{B_1} \cdot y^{B_2} \cdot z^{B_3}, \;\; x^{C_1} \cdot y^{C_2} \cdot z^{C_3} \Big), \end{aligned} \tag{A9}$$

where the A_i's, B_i's, and C_i's are positive integers, such that $A_1 \, = \, A_2 \, = \, A_3$ is excluded (as well as $B_1 \, = \, B_2 \, = \, B_3$ and $C_1 \, = \, C_2 \, = \, C_3$), and that the determinant (Note a typo in Footnote 28, page 17, of [1], as well as in the second footnote, page 18, in [2]. The sentence has been truncated. One should read: For $n \, = \, 1$, the 3×3 matrix (A10) is stochastic and transformation (A9) is a birational transformation if the determinant of the matrix (A10) is ± 1) of the $3 \, \times \, 3$ matrix [1,2]:

$$\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}, \tag{A10}$$

is not equal to zero (We want the rational function $\tilde{\mathcal{R}} \, = \, \mathcal{R}(M(x, \, y, \, z))$ deduced from the monomial transformation (A9) remains a rational function of three variables, and not of two, or one, variables), and that:

$$A_1 \, + B_1 \, + C_1 \, \, = \, \, A_2 \, + B_2 \, + C_2 \, \, = \, \, A_3 \, + B_3 \, + C_3. \tag{A11}$$

We will denote by $n \, = \, A_i \, + B_i \, + C_i$ the integer in these three equal sums (A11). Condition (A11) is introduced in order to impose that the product (Recall that taking the diagonal of a rational function of three variables extracts, in the multi-Taylor expansion, only the terms that are n-th powers of the product $x\,y\,z$) of $x_M \, y_M \, z_M$ is an integer power of the product of $x\,y\,z$: $x_M \, y_M \, z_M \, = \, (x\,y\,z)^n$.

If we take a rational function $\mathcal{R}(x, \, y, \, z)$ in three variables and perform such a monomial transformation (A9) $(x, \, y, \, z) \, \rightarrow \, M(x, \, y, \, z)$, on this rational function $\mathcal{R}(x, \, y, \, z)$, we obtain another rational function that we denote by $\tilde{\mathcal{R}} \, = \, \mathcal{R}(M(x, \, y, \, z))$. Now, the diagonal of $\tilde{\mathcal{R}}$ is the diagonal of $\mathcal{R}(x, \, y, \, z)$ where we have changed x into x^n:

$$\Phi(x) \, = \, Diag\Big(\mathcal{R}\Big(x, \, y, \, z\Big)\Big), \qquad Diag\Big(\tilde{\mathcal{R}}\Big(x, \, y, \, z\Big)\Big) \, = \, \Phi(x^n). \tag{A12}$$

Appendix D. Weierstrass and Legendre Forms

The telescoper of the rational function in three variables:

$$\frac{x\,y}{(1+y)^2 \, - x \cdot (1-x) \cdot (x \, - x\,y\,z)}, \tag{A13}$$

associated (the diagonal extracts the terms function of the product $p \, = \, x\,y\,z$ in the multi-Taylor series) with the elliptic curve in a *Weierstrass form*:

$$(1 \, + y)^2 \, - x \cdot (1-x) \cdot (x \, - p) \, = \, 0, \tag{A14}$$

is the order-two linear differential operator:

$$L_2 \, = \, -1 \, + 4 \cdot (1 - 2\,x) \cdot D_x \, + 4 \cdot x \cdot (1 \, - x) \cdot D_x^2, \tag{A15}$$

which has the hypergeometric solution:

$$\begin{aligned} &{}_2F_1\Big(\frac{1}{2}, \, \frac{1}{2}; \, 1; \, x\Big) \\ &\quad = \, (1 \, - x \, + x^2)^{-1/4} \cdot {}_2F_1\Big(\frac{1}{12}, \, \frac{5}{12}; \, 1; \, \frac{27}{4} \cdot \frac{x^2 \cdot (1 \, - x)^2}{(1 \, - x \, + x^2)^3}\Big). \end{aligned} \tag{A16}$$

The elliptic curve (A14) has the Hauptmodul:

$$\mathcal{H} \, = \, \frac{27}{4} \cdot \frac{p^2 \cdot (1 \, - p)^2}{(1 \, - p \, + p^2)^3}. \tag{A17}$$

in agreement with the pullback in (A16).

Appendix D.1. K3 Surfaces as Products or Foliations of Two Elliptic Curves

The examples of Section 4 correspond to denominators which are algebraic varieties that can be seen as Weierstrass elliptic curves for fixed values of all the variables except two. Let us show other simple telescopers for rational functions with denominators which are algebraic varieties with some foliation in elliptic curves (such as K3 surfaces or Calabi–Yau threefolds).

The telescoper of the rational function in four variables:

$$\frac{x\,y\,z}{(1+z)^2 \, - x \cdot (1-x) \cdot y \cdot (x \, - y) \cdot (y \, - x\,y\,z\,w)}, \tag{A18}$$

associated with the K_3 surface written in a *Legendre form* (along this line, see the first equation, page 19 of [84])

$$(1+z)^2 \;\; - x \cdot (1-x) \cdot y \cdot (x-y) \cdot (y-p) \;\; = \;\; 0, \tag{A19}$$

is an order-three *self-adjoint* (the order-three linear differential operator is thus the symmetric square of an order-two linear differential operator) linear differential operator, L_3:

$$L_3 \;\; = \;\; x \cdot (2\,\theta + 1)^3 \;\; - 8 \cdot \theta^3, \tag{A20}$$

which has the following ${}_3F_2$ solution (which is also, because of Clausen's formula, the square of a ${}_2F_1$ function):

$${}_3F_2\Big(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; \; x\Big) \;\; = \;\; {}_2F_1\Big(\frac{1}{4}, \frac{1}{4}; 1; \; x\Big)^2. \tag{A21}$$

The K_3 surface (A19) can be seen as being associated with the product of two Weierstrass elliptic curves (K_3 surfaces are not abelian varieties, but they are "close" to abelian varieties: from a creative telescoping viewpoint, they essentially can be seen as products of two elliptic curves) of Hauptmoduls, respectively:

$$\mathcal{H}_x \;\; = \;\; \frac{27}{4} \cdot \frac{p^2 \cdot (1-p)^2}{(1-p+p^2)^3}, \qquad \mathcal{H}_y \;\; = \;\; \frac{27}{4} \cdot \frac{y^2 \cdot (1-y)^2}{(1-y+y^2)^3}. \tag{A22}$$

This order-three linear differential operator L_3 is the symmetric square of the order-two linear differential operator:

$$M_2 \; = \;\; -1 \;\; + 8 \cdot (2-3\,x) \cdot D_x \;\; + 16 \cdot x \cdot (1-x) \cdot D_x^2, \tag{A23}$$

which has the hypergeometric solutions:

$${}_2F_1\Big(\frac{1}{4}, \frac{1}{4}; 1; \; x\Big) \;\; = \;\; \Big(1 - \frac{x}{4}\Big)^{-1/4} \cdot {}_2F_1\Big(\frac{1}{12}, \frac{5}{12}; 1; \; -\frac{27 \cdot x^2}{(x-4)^3}\Big). \tag{A24}$$

Appendix D.2. Calabi–Yau Threefolds as Foliation in Three Elliptic Curves

The telescoper of the rational function in five variables x, y, z, v, and w:

$$\frac{x\,y\,z\,v}{(1+w)^2 \;\; - x \cdot (1-x) \cdot y \cdot (x-y) \cdot z \cdot (y-z) \cdot (z - x\,y\,z\,v\,w)}, \tag{A25}$$

associated (the diagonal extracts the terms function of the product $p \; = \; x\,y\,z\,v\,w$ in the multi-Taylor series) with the Calabi–Yau threefold written in a Legendre form:

$$(1+w)^2 \;\; - x \cdot (1-x) \cdot y \cdot (x-y) \cdot z \cdot (y-z) \cdot (z-p) \;\; = \;\; 0, \tag{A26}$$

is an order-four (self-adjoint) linear differential operator L_4:

$$L_4 \;\; = \;\; 16 \cdot \theta^4 \;\; - x \cdot (2\,\theta + 1)^4, \tag{A27}$$

which is a Calabi–Yau operator (This linear differential operator is self-adjoint, its exterior square is of order five, and it is MUM (maximum unipotent monodromy [27,65,66])) with the ${}_4F_3$ solution:

$${}_4F_3\Big(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1, 1; \; x\Big). \tag{A28}$$

For y and z fixed, the Calabi–Yau threefold (A26) is foliated in genus-one curves:

$$(1+w)^2 \;\; - \lambda \cdot x \cdot (1-x) \cdot (x-y) \;\; = \;\; 0, \tag{A29}$$

where λ is the constant expression (p is fixed):

$$\lambda \,\, = \,\, y \cdot \, z \cdot \, (y \, - z) \cdot \, (z \, - p). \tag{A30}$$

The Hauptmodul of these genus-one curves is independent of p and z, reading:

$$\mathcal{H}_{y,z} \,\, = \,\, \frac{27}{4} \cdot \frac{y^2 \cdot \, (1 \, - y)^2}{(1 \, - y \, + y^2)^3}. \tag{A31}$$

Similarly for x and z fixed, the Calabi–Yau threefold (A26) is foliated in genus-one curves:

$$(1 \, + w)^2 \,\, - \mu \cdot \, y \cdot \, (x \, - y) \cdot \, (y \, - z) \,\, = \,\, 0, \tag{A32}$$

where μ is the constant expression (p is fixed):

$$\mu \,\, = \,\, x \cdot \, z \cdot \, (1 \, - x) \cdot \, (z \, - p). \tag{A33}$$

The genus-one curves (A32) can be written in a simpler Weierstrass form:

$$(1 \, + w)^2 \,\, - \rho \cdot \, Y \cdot \, \Big(1 - Y\Big) \cdot \, \Big(Y \, - \, \frac{z}{x}\Big) \,\, = \,\, 0, \tag{A34}$$

where the constant ρ reads $\rho \, = \, \mu \cdot \, x^3$, and the variable y has been rescaled into $Y \, = \, y/x$. The Hauptmodul of these genus-one curves (A32) is the same as the Hauptmodul of the genus-one curves (A29), and corresponds to expression (A31) where y has been changed into z/x (see the canonical form (A34)), namely:

$$\mathcal{H}_{x,z} \,\, = \,\, \frac{27}{4} \cdot \frac{x^2 \cdot \, z^2 \cdot \, (x \, - z)^2}{(x^2 \, - x\, z \, + z^2)^3}. \tag{A35}$$

Similarly for x and y fixed, the Calabi–Yau threefold (A26) is foliated in genus-one curves,

$$(1 \, + w)^2 \,\, - \nu \cdot \, z \cdot \, (y \, - z) \cdot \, (z \, - p) \,\, = \,\, 0, \tag{A36}$$

where ν reads:

$$\nu \,\, = \,\, x \cdot \, (1 - x) \cdot \, y \cdot \, (x \, - y). \tag{A37}$$

A reduction to a canonical Weierstrass form similar to (A34) gives immediately the Hauptmodul of the genus-one curve (A36), which reads:

$$\mathcal{H}_{x,y} \,\, = \,\, \frac{27}{4} \cdot \frac{y^2 \cdot \, p^2 \cdot \, (y \, - p)^2}{(y^2 \, - y\, p \, + p^2)^3}. \tag{A38}$$

The Calabi–Yau threefold (A26) thus has a foliation in a triple of elliptic curves $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$.

Appendix E. Rational Functions with Tri-Quadratic and N-Quadratic Denominators

Appendix E.1. Rational Functions with Tri-Quadratic Denominators

Let us consider the most general tri-quadratic surface:

$$\sum_{m=0,1,2} \, \sum_{n=0,1,2} \, \sum_{l=0,1,2} \, a_{m,n,l} \cdot \, x^m \, y^n \, z^l \,\, = \,\, 0, \tag{A39}$$

depending on $27 = 3^3$ parameters $a_{m,n,l}$. It can be rewritten as:

$$A(x, \, y) \cdot \, z^2 \,\, + B(x, \, y) \cdot \, z \,\, + C(x, \, y) \,\, = \,\, 0. \tag{A40}$$

It is straightforward to see that condition (A40) is preserved by the *birational involution* I_z:

$$I_z: \qquad \Big(x, \, y, \, z\Big) \qquad \longrightarrow \qquad \Big(x, \, y, \, \frac{C(x, \, y)}{A(x, \, y)} \cdot \frac{1}{z}\Big), \tag{A41}$$

and we have, of course, two other similar birational involutions, I_x and I_y, that single out x and y, respectively. The (generically) *infinite-order* birational transformations $K_x = I_y \cdot I_z$, $K_y = I_z \cdot I_x$, and $K_z = I_x \cdot I_y$ are birational symmetries of the surface (A39) or (A40). They are related by $K_x \cdot K_y \cdot K_z = identity$. Note that the birational transformation K_x preserves x. The iteration of the (generically) infinite-order birational transformation K_x gives elliptic curves. Since Equations (A39) or (A40) is preserved by K_x, which also preserves x, the equation of the elliptic curves corresponding to the iteration (The birational transformation K_x maps the elliptic curve onto itself (self-map). One can use the iteration of the birational transformation K_x to actually visualize the elliptic curve [44,85]) of K_x is actually (A39) for fixed values of x. Equation (A39), for fixed values of x, is a (general) biquadratic curve in y and z, and is thus an elliptic curve depending on x. Therefore, one has a canonical foliation of the algebraic surface (A39) in elliptic curves. Of course, the iteration of K_y (resp. K_z) also yields elliptic curves, and similarly yields two other foliations in elliptic curves.

We have a foliation in two families of elliptic curves $\mathcal{E}$ and $\mathcal{E}'$ of the surface. Consequently, this tri-quadratic surface (A39), having an *infinite set* of birational automorphisms and an infinite set of birational symmetries, cannot be of the "general type" (it has Kodaira dimension of less than 2).

Appendix E.2. Rational Functions with N-Quadratic Denominators

The calculations of Appendix E.1 can straightforwardly be generalized to N-quadratic equations, writing the N-quadratic (72) as:

$$\begin{aligned} A(x_1, \, x_2, \, \cdots, \, x_{N-1}) \cdot \, x_N^2 \quad + B(x_1, \, x_2, \, \cdots, \, x_{N-1}) \cdot \, x_N \\ +C(x_1, \, x_2, \, \cdots, \, x_{N-1}) \quad = \quad 0, \end{aligned} \tag{A42}$$

and introducing the birational involution I_N:

$$\begin{aligned} I_N: \qquad & \Big(x_1, \, x_2, \, \cdots, \, x_N\Big) \\ & \longrightarrow \qquad \Big(x_1, \, x_2, \, \cdots, \, x_{N-1}, \, \frac{C(x_1, \, x_2, \, \cdots, \, x_{N-1})}{A(x_1, \, x_2, \, \cdots \, x_{N-1})} \cdot \frac{1}{x_N}\Big). \end{aligned} \tag{A43}$$

Similarly to Appendix E.1, we can introduce N involutive birational transformations I_m and consider the products of two such involutive birational transformations $K_{m,n} = I_m \cdot I_n$. These $K_{m,n}$'s are (generically) infinite-order birational transformations preserving the $N-2$ variables that are not x_m and x_n.

Using such remarkable N-variable algebraic varieties, with an infinite set of birational automorphisms, one can build rational functions of $N+1$ variables, any of the parameter of the algebraic variety, becoming an arbitrary rational (or even an arbitrary algebraic) function of the product $p = x_1 \, x_2 \, \cdots \, x_N$, with a Taylor series expansion at $p = 0$, the diagonal of rational functions becoming diagonal of algebraic functions of the product $p = x_1 \, x_2 \, \cdots \, x_N$ in order to build the denominator of the rational function. The telescopers of such rational functions are seen (experimentally using creative telescoping) to be of substantially smaller order than the ones for rational functions whose denominators are, after reduction by $p = x_1 \, x_2 \, \cdots \, x_N$, associated with algebraic varieties of "general type".

Appendix F. Telescopers of Rational Functions of Several Variables: Some Examples

Let us consider here the following family of rational functions in four variables:

$$R(x, y, z, u) \;=\; \frac{1}{1 \;+3y \;+z \;+9yz \;+11z^2y \;+3ux \;+x \cdot P(x, y, z)}, \tag{A44}$$

where $P(x, y, z)$ is an *arbitrary polynomial* of the three variables x, y, and z.

Appendix F.1. Telescopers of Rational Functions of Several Variables: A Second Example with Four Variables

Let us now consider the rational function in four variables x, y, z, u:

$$R(x, y, z, u) \;=\; \frac{1}{1 \;+3y \;+z \;+9yz \;+11z^2y + 3ux \;+9x \;+2xy \;+5xz \;+7x^2y}. \tag{A45}$$

which corresponds to $P(x, y, z) \,=\, 9 \;+2y \;+5z \;+7xy$. The telescoper of this rational function of four variables is the same order-two linear differential operator L_2 as for the telescoper of (91). It has the same pullbacked hypergeometric solution (92). The diagonal of the rational function (A45) is the expansion of (92), namely (93).

Performing the intersection of the codimension-one algebraic variety:

$$1 \;+3y \;+z \;+9yz \;+11z^2y \;+3ux \;+9x \;+2xy \;+5xz \;+7x^2y \;=\; 0,$$

corresponding to the denominator of (A45), with the hyperbola $p \,=\, xyzu$, amounts to eliminating, for instance, u (writing $u \,=\, \frac{p}{xyz}$). This gives $P_u \,=\, 0$, where P_u reads:

$$P_u \;=\; 7x^2y^2z \;+2xy^2z + 5xyz^2 + 9xyz \;+11y^2z^3 \;+9y^2z^2 \\ +3y^2z \;+yz^2 \;+yz \;+3p. \tag{A46}$$

Assuming x to be constant, the previous condition $P_u(y, z) \,=\, 0$ is an algebraic curve. Calculating its genus, one finds immediately that it has genus one. Calculating its j-invariant, one deduces the expression of the Hauptmodul $\mathcal{H}_{p,x} \,=\, \frac{1728}{J}$ as a rational expression of p and x:

$$\mathcal{H}_{p,x} \;=\; \frac{1728}{J} \;=\; -\,\frac{46656\,p^3 \cdot (7x^2 + 2x + 3)^2 \cdot N}{D^3}, \tag{A47}$$

where N is a polynomial expression of degree eight in w and three in p, and D is a polynomial expression of degree four in w and two in p. In the $x \,\rightarrow\, 0$ limit of the Hauptmodul $\mathcal{H}_{p,x} \,=\, \frac{1728}{J}$, one finds:

$$\mathcal{H}_p \;=\; -\,\frac{419904 \cdot p^3 \cdot (5 \,-12p \,-19440p^2 \,+2665872p^3)}{(1 \,-\, 2592p^2)^3}, \tag{A48}$$

which is actually the Hauptmodul in (92). In other words, the exact expression of the diagonal of the rational function (A45), which is (92), and is essentially encapsulated in the Hauptmodul in (92), could have been obtained from the $x \,=\, 0$ selection of the Hauptmoduls $\mathcal{H}_{p,x}$.

Appendix F.2. Telescopers of Rational Functions of Several Variables: A Third Example with Four Variables

Let us consider the rational function in four variables x, y, z, u:

$$R(x, y, z, u) \; = \; \frac{1}{1 \; + 3\,y \; + z \; + 9\,y\,z \; + 11\,z^2\,y \; + 3\,u\,x \; + x \cdot (y^2\,z^2 \, + \, x\,y^3)}, \tag{A49}$$

which corresponds to $P(x,y,z) = y^2\,z^2 \, + \, x\,y^3$ in the family (A44). Again, the telescoper of this rational function of four variables is the same order-two linear differential operator L_2 as for the telescoper of (91). It has the same pullbacked hypergeometric solution (92). Actually, the diagonal of the rational function (91) is the expansion (93) of the pullbacked hypergeometric function (92). In this case (A49), the elimination of $u \; = \; \frac{p}{x\,y\,z}$ in the vanishing condition of the denominator (A49) gives the algebraic curve:

$$x^2\,y^4\,z \; + x\,y^3\,z^3 \; + 11\,y^2\,z^3 \; + 9\,y^2\,z^2 \; + 3\,y^2\,z \; + y\,z^2 \; + y\,z \; + 3\,p \; = \; 0. \tag{A50}$$

For x fixed (and of course, p fixed) this algebraic curve (A50) is a genus-five curve, but, of course, in the $x \; = \; 0$ case, it reduces to the same genus-one curve as for the first example (91), namely:

$$11\,y^2\,z^3 \; + 9\,y^2\,z^2 \; + 3\,y^2\,z \; + y\,z^2 \; + y\,z \; + 3\,p \; = \; 0. \tag{A51}$$

which corresponds to the Hauptmodul (A48).

The generalization of this result is straightforward. Let us consider the rational function in four variables $x, y, z,$ and u:

$$R(x, y, z, u) \; = \; \frac{1}{1 \; + 3\,y \; + z \; + 9\,y\,z \; + 11\,z^2\,y \; + 3\,u\,x \; + x \cdot P(x, y, z)}, \tag{A52}$$

where $P(x, y, z)$ is an arbitrary polynomial of the three variables $x, y,$ and z. On a large set of examples, one verifies that the diagonal of (A52) is actually the expansion (93) of the pullbacked hypergeometric function (92):

$$\begin{aligned} &1 \; + 648\,x^2 \; - 72900\,x^3 \; + 1224720\,x^4 \; - 330674400\,x^5 \; + 23370413220\,x^6 \\ &-1276733858400\,x^7 \; + 180019474034400\,x^8 \; - 12013427240614800\,x^9 \; + \; \cdots \end{aligned} \tag{A53}$$

However, as far as creative telescoping calculations are concerned (using the HolonomicFunctions package [3]), the telescoper corresponding to different polynomials $P(x, y, z)$ quickly becomes a quite large *non-minimal* linear differential operator. For instance, even for the simple polynomial $P(x, y, z) \; = \; x + y$, one obtains a quite large order-10 telescoper. Of course, since this telescoper has the pullbacked hypergeometric function (92) as a solution, it is not minimal; it is right-divisible by the order-two linear differential operator having (92) as a solution. It is straightforward to see that the previous elimination of $u \; = \; \frac{p}{x\,y\,z}$ in the vanishing condition of the denominator (A52) gives an algebraic curve (of arbitrary large genus for increasing degrees of the polynomial $P(x, y, z)$):

$$11\,y^2\,z^3 \; + 9\,y^2\,z^2 \; + 3\,y^2\,z \; + y\,z^2 \; + y\,z \; + 3\,p \; + y\,z \cdot P(x, y, z) = \; 0. \tag{A54}$$

which reduces again, in the $x \; = \; 0$ case, to the same genus-one curve (A51).

With that general example (A52), we see that there is an infinite set of rational functions depending on an arbitrary polynomial $P(x, y, z)$ of three variables whose diagonals are actually a pullbacked ${}_2F_1$ hypergeometric solution, namely (92).

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