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Burr XII Distribution for Disease Data Analysis in the Presence of a Partially Observed Failure Mode

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Abstract: Modeling competing failure modes is an important problem in engineering and survival analyses. Competing failure modes are partially observed in many applications and often pose a modeling challenge. This study discusses the inference for partially observed failure modes assuming a Burr XII distribution. In particular, we consider two failure modes, and the failure time data are collected under a hybrid type I censoring scheme. The model parameters are estimated using maximum likelihood and Bayesian methods under a symmetric squared error loss function, whereas the intervals estimation is done with three methods: asymptotic and credible confidence intervals. Besides a simulation study, a real-life data set is taken from individuals who live in an environment with several diseases to present the utility of the work. Additionally, a simulation study is constructed to measure and compare different estimation methods.

Keywords: Burr XII distribution; partially observed failure modes; hybrid type I censoring scheme; maximum likelihood; symmetric squared error

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1. Introduction

The lifetime data of individuals are collected in complete or censoring forms from a life testing experiment. Due to budget or time constraints, we commonly observe censored data rather than complete data. The oldest and most widely used censoring schemes are type I and type II. In the case of type I censoring, a fixed test time is assumed and a random number of failures are observed. Similarly, in type II censoring, a fixed number of failure are observed with a random test time. However, these two schemes do not allow to remove individuals from the test other than the test termination. Contrary to these schemes, progressive censoring schemes are more flexible **censoring schemes proposed to improve the efficiency of the experiment. Different schemes can be considered as a special case of progressive censoring schemes; see** [1–3]. The scheme parameter τ (ideal test time) and m (number of failed units) are fixed in advance, and the resulting scheme is known as hybrid censoring. A hybrid censoring scheme (HCS) combined with type I and type II censoring schemes results in type I and type II HCSs.

For a type I HCS, suppose n units are put on a life testing experiment. Prior to the test, the experimenter decides a suitable number of failures m required for statistical inference and the ideal test time τ. The failure times are recorded until the $\min(T_m, \tau)$ is observed [4–6]. Similarly, for a type II HCS, with n units and (m, τ), the test is stopped at the $\max(T_m, \tau)$ [7].

The presence of competing failure modes results in the latent failure time models in the literature. The life testing data with competing risks consist of failure time and the corresponding failure indicator. The competing risks model can be studied with dependent causes of failure (see [8–11]) or independent causes of failure for simplicity. In the problem of designing systems, independent failure is a significant factor for safety, especially in mechanical systems. In some situations, independent failure might be an optimal design choice to prevent the system from ever shutting down. The competing risks model and its properties were discussed recently by [12–14]. Therefore, in this paper, we study an independent latent failure time model [15] assuming the Burr XII distribution. Additionally, some applicatiosn of distribution have been introduced by Kayal et al. [16]. More details about the competing risks model can be seen in [17–29], and the references cited therein.

Burr [30] introduced a system of twelve types of distributions. These distributions may have a variety of shapes and are widely used in different branches of sciences such as medical sciences, chemical engineering, business, quality control, and reliability studies. One member of this family is known as the Burr XII distribution. Such a model has been proposed by [31]. The Burr XII distribution includes many commonly used distributions such as gamma, lognormal and loglogistic distributions and has two asymptotic limiting cases, Weibull and Pareto Type I. Additionally, this Burr distribution can fit a wide range of empirical data in different branches of science, such as finances, hydrology, reliability to model and failure time data. The parameter values of the Burr XII distribution cover a broad set of skewness and kurtosis. The random variable $T = T_{ij}$ is distributed as a Burr XII distrbution if its cumulative distribution function (CDF) can be written as

$$F_j(t) = 1 - (1 + t^{\theta})^{-\alpha_j},\ t > 0,\ \alpha_j,\ \theta > 0, j = 1, 2. \tag{1}$$

where α_j and θ are the shape parameters. **The assumption of a common shape parameter is desired for making computation easy and hence, testing different groups of the data (see McCool [32])**. The failure rate function of the Burr XII distribution is unimodal at $\theta > 1$ or decreasing at $\theta \leq 1$ and its shape is not effected by the shape parameters α. Different authors have discussed competing Burr XII distributions, **the accelerated competing Burr XII distributions when failure times and causes of failure are observed by** [33]. Recently, ref. [34] **have discussed the competing Burr XII distributions for products in competing duration (mean two-sample case). However, in this paper, we are adopt this model when the causes of failure are partially observed (meaning the cause of failure is undetected for some failure units in the test)**.

The Burr XII distribution density function, survival function $S_j(.)$, and failure rate function $h_j(.)$, respectively, are given by

$$f_j(t) = \alpha_j \theta t^{\theta-1}(1 + t^{\theta})^{-(\alpha_j+1)}, \tag{2}$$

$$S_j(t) = (1 + t^{\theta})^{-\alpha_j}, \tag{3}$$

and

$$h_j(t) = \alpha_j \theta t^{\theta-1}(1 + t^{\theta})^{-1}, \tag{4}$$

where $j = 1, 2$ denotes the number of failure modes in the populations.

For the failure of the experimental units, two different risk factors competing is considered of a lifetime experiment. The data which are obtained under a type I HCS for such a competing risk model present the Burr XII lifetime of the failed units and the corresponding indicator variable, which denotes the modes of failure. Our aim is to develop estimation procedures under a competing risks Burr XII model with two failure modes. The model parameters are estimated with classical and Bayes methods under a type I HCS. In addition, the asymptotic confidence interval as well as the credible intervals are constructed. Moreover, real-life data obtained from a life experiment for individuals under different diseases are analyzed and discussed in detail.

The study is divided into the following sections. The model formulation and likelihood function is discussed in Section 2. The point and interval maximum likelihood estimation is discussed in Section 3. Section 4 discusses the Bayesian estimation, whereas a Monte Carlo simulation study is presented in Section 5. A real-life data under two diseases is analyzed in Section 6, while concluding comments are given in Section 7.

2. Model Formulation

Suppose that n units are randomly selected from a life product to put under test, such that any unit of product failure occurs under one of two activated independent causes of failure. Additionally, suppose the latent failure time of n units are denoted by $T_1, T_2, \ldots, T_n$, where $T_i = \min(T_{i1}, T_{i2})|_{i=1,2,\ldots,n}$ and T_{ij} denotes the i-th failure time under cause j. Under consideration of the type I HCS, let τ and m denote the ideal test time. The numbers needed in statistical inference are previously selected. When the experiment is running, the failure times and the corresponding cause of failure, say (T_i, δ_i), where $i = 1, 2, \ldots, r$ and r denote the numbers of failure to reach the $\min(T_m, \tau)$ and $1 \leq r \leq m$, are recorded. The experiment is continued until $\min(T_m, \tau)$ is observed. Therefore, the observed type I HCS competing risks are defined by $\mathbf{t} = \{(t_{1;n}, \delta_1), (t_{2;n}, \delta_2) \ldots, (t_r, \delta_r)\}$. Under partially observed causes of failure, the indicator δ_i for $i = 1, 2, \ldots, r$ takes three different values; the first values are $\delta_i = 1$ or 2, which denotes a failure with the first or second cause. However, if the cause of failure is not clear, then we use $\delta_i = *$. Therefore, the joint likelihood function under the above-described scenario of type I HCS competing risks $\mathbf{t} = \{(t_{1;n}, \delta_1), (t_{2;n}, \delta_2) \ldots, (t_r, \delta_r)\}$ is formulated by

$$L(\underline{\mathbf{t}}|\underline{\eta}) = \frac{n!}{(n-r)!}\left(\prod_{i=1}^{r}[f_1(t_i)S_2(t_i)]^{\omega(\delta_i=1)}[f_2(t_i)S_1(t_i)]^{\omega(\delta_i=2)}[\mathbf{f}(x_i)]^{\omega(\delta_i=*)}\right)\mathbf{S}(\mu)^{n-r}, \quad (5)$$

where $\underline{\eta} = \{\alpha_1, \alpha_2, \theta\}$, $S_j(t) = P(T_j > t), j = 1, 2$ and $\mathbf{S}(t) = P(T_1 > t, T_2 > t) = S_1(t)S_2(t)$ is the survival function with density $\mathbf{f}(.)$. Additionally, for the latent failure time $T_i = \min(T_{i1}, T_{i2})$,

$$\mu = \begin{cases} \tau, & r < m \\ t_m, & r = m \end{cases}, \omega(\delta_i = k) = \begin{cases} 1, & \delta_i = k, \\ 0, & \text{else} \end{cases}, \quad k = 1, 2, *, \quad (6)$$

and

$$n_j = \sum_{i=1}^{r} \omega(\delta_i = j), \; j = \{1, \; 2, \; *\}. \quad (7)$$

We assume a Burr XII distribution under the type I HCS competing risks model for the latent failure time. Furthermore, one shape parameter θ is fixed, and another shape parameter $\alpha_j = 1, 2$ has the following assumptions:

1. Some individuals fail with an unknown cause, and the latent failure time has a Burr XII distribution with shape parameters $\alpha_1 + \alpha_2$ and θ;
2. A binomial random variable is taken for variables n_1 and n_2 that fail under the first and second causes of failure, respectively, with sample size $(r - n_3)$ and a probability of success $\frac{\alpha_1}{\alpha_1+\alpha_2}$ and $\frac{\alpha_2}{\alpha_1+\alpha_2}$, respectively;
3. The Bernoulli distribution is taken for n_3 with masking probability, $0 < p < 1$. Therefore, the Bernoulli random variable with a value of 1 means that the cause of failure is unknown, and 0 denotes the known cause of failure.

3. Estimations under Maximum Likelihood

For a given type I HCS $\mathbf{t} = \{(t_{1;n}, \delta_1), (t_{2;n}, \delta_2) \ldots, (t_r, \delta_r)\}$, where the competing risks sample is drawn from Burr XII distributions, the likelihood function of α_1, α_2 and θ is formulated by

$$L(\alpha_1,\alpha_2,\theta|\underline{\mathbf{t}}) \propto \alpha_1^{n_1}\alpha_2^{n_2}(\alpha_1+\alpha_2)^{n_3}\theta^r\left(1+\mu^\theta\right)^{-(n-r)(\alpha_1+\alpha_2)}\prod_{i=1}^{r}t_i^{\theta-1}\left(1+t_i^\theta\right)^{-(\alpha_1+\alpha_2)-1} \quad (8)$$

The corresponding logarithmic likelihood function (8) can be written as

$$\begin{aligned}\ell(\alpha_1,\alpha_2,\theta|\underline{\mathbf{t}}) &= n_1\log\alpha_1 + n_2\log\alpha_2 + n_3\log(\alpha_1+\alpha_2) + r\log\theta - (n-r)(\alpha_1+\alpha_2)\\ &\times\log\left[1+\mu^\theta\right] + (\theta-1)\textstyle\sum_{i=1}^{r}\log t_i - (\alpha_1+\alpha_2+1)\sum_{i=1}^{r}\log\left(1+t_i^\theta\right).\end{aligned} \quad (9)$$

The point and asymptotic confidence intervals of model parameters under the competing risk type I HCS are discussed in two cases: known θ and unknown θ. The details for each case are given in the following section.

3.1. Estimation with Known θ

The partial derivatives of Equation (9) with respect to parameters α_1 and α_2 are

$$\frac{n_1}{\alpha_1} + \frac{n_3}{\alpha_1+\alpha_2} - (n-r)\log\left[1+\mu^\theta\right] - \sum_{i=1}^{r}\log\left(1+t_i^\theta\right) = 0 \quad (10)$$

and

$$\frac{n_2}{\alpha_2} + \frac{n_3}{\alpha_1+\alpha_2} - (n-r)\log\left[1+\mu^\theta\right] - \sum_{i=1}^{r}\log\left(1+t_i^\theta\right) = 0. \quad (11)$$

Therefore, the estimators $\hat{\alpha}_1(\theta)$ and $\hat{\alpha}_2(\theta)$ are computed by the following theorem.

Theorem 1. *For a given θ and $n_1, n_2 > 0$, the conditional ML estimators of parameters α_1 and α_2 are computed from*

$$\hat{\alpha}_j(\theta) = \frac{n_j r}{(n_1+n_2)\left[(n-r)\log\left[1+\mu^\theta\right] + \sum_{i=1}^{r}\log\left(1+t_i^\theta\right)\right]},\ j=1,2. \quad (12)$$

Proof. The determinant of the Hessian matrix $H(\alpha_1, \alpha_2)$ is used to prove this theorem. In particular,

$$D(H(\alpha_1,\alpha_2)) = \frac{n_1n_2}{\alpha_1\alpha_2} + \frac{n_1n_3}{\alpha_1^2(\alpha_1+\alpha_2)^2} + \frac{n_2n_3}{\alpha_2^2(\alpha_1+\alpha_2)^2} > 0, \quad (13)$$

and has the property of positivity for all values α_1 and α_2. Therefore the proof agrees with the argument given in [35]. □

Remark 1.

1. *There is no information regarding the parameter α_j if $n_j = 0$, $j = 1, 2$ which means that there are no failures due to cause j.*
2. *The results of the partially observed causes of failure of the competing risks model reduce to the usual competing risks model if $m_3 = \{1, 2\}$.*

3.2. Estimation with Unknown θ

The first partial derivative of (9) with respect to θ is

$$\frac{r}{\theta} + \sum_{i=1}^{r}\log t_i - \frac{(n-r)(\alpha_1+\alpha_2)\mu^\theta\log\mu}{1+\mu^\theta} - (\alpha_1+\alpha_2+1)\sum_{i=1}^{r}\frac{t_i^\theta\log t_i}{1+t_i^\theta} = 0, \quad (14)$$

Equation (14) yields that the estimator of θ is not in closed form. Thus, we use an iterative method such as the Newton–Raphson or fixed point method. The following theorem describes these operations.

Theorem 2. *The estimators of θ under the ML method is given by the iteration*

$$\theta^{(i+1)} = g(\theta^{(i)}), \tag{15}$$

where

$$g(\theta) = \frac{r}{\frac{(n-r)(\alpha_1+\alpha_2)\mu^\theta \log \mu}{1+\mu^\theta} + (\alpha_1+\alpha_2+1)\sum_{i=1}^{r} \frac{t_i^\theta \log t_i}{1+t_i^\theta} - \sum_{i=1}^{r} \log t_i}, \tag{16}$$

where $\alpha_1(\theta)$ and $\alpha_2(\theta)$ are defined in (12). The profile log-likelihood function of θ is obtained from (9) after replacing parameters $\alpha_1(\theta)$ and $\alpha_2(\theta)$ as follows:

$$\begin{aligned} Z(\theta|\underline{\mathbf{t}}) \quad &= (\theta-1)\sum_{i=1}^{r} \log t_i + \sum_{j=1}^{2} n_j \log\left(\frac{n_j r/(n_1+n_2)}{\left[(n-r)\log\left[1+\mu^\theta\right]+\sum_{i=1}^{r}\log\left(1+t_i^\theta\right)\right]}\right) + n_3 \\ &\times \log\left(\frac{r/(n-r)}{\log\left[1+\mu^\theta\right]+\sum_{i=1}^{r}\log\left(1+t_i^\theta\right)}\right) - \frac{r(n-r)\log\left[1+\mu^\theta\right]}{(n-r)\log\left[1+\mu^\theta\right]+\sum_{i=1}^{r}\log\left(1+t_i^\theta\right)} \\ &-\left(\frac{r}{(n-r)\log\left[1+\mu^\theta\right]+\sum_{i=1}^{r}\log\left(1+t_i^\theta\right)}+1\right)\sum_{i=1}^{r}\log\left(1+t_i^\theta\right) + r\log\theta. \end{aligned} \tag{17}$$

For the fixed point iteration method, we stop the algorithm if $|\theta^{(i+1)} - \theta^{(i)}|$ is sufficiently small.

3.3. Interval Estimation

From the log-likelihood function (9), the second derivatives for the model parameters $\underline{\eta} = \{\alpha_1, \alpha_2, \theta\}$ are given by

$$\frac{\partial^2 \ell(\underline{\eta}|\underline{\mathbf{t}})}{\partial \alpha_1^2} = \frac{-n_1}{\alpha_1^2} - \frac{n_3}{(\alpha_1+\alpha_2)^2}, \tag{18}$$

$$\frac{\partial^2 \ell(\underline{\eta}|\underline{\mathbf{t}})}{\partial \alpha_2^2} = \frac{-n_1}{\alpha_2^2} - \frac{n_3}{(\alpha_1+\alpha_2)^2}, \tag{19}$$

$$\frac{\partial^2 \ell(\underline{\eta}|\underline{\mathbf{t}})}{\partial \theta^2} = \frac{-r}{\theta^2} - \frac{(n-r)(\alpha_1+\alpha_2)(1-\mu^\theta)\mu^\theta \log^2 \mu}{\left(1+\mu^\theta\right)^2} - (\alpha_1+\alpha_2+1)\sum_{i=1}^{r} \frac{\left(1-t_i^\theta\right)t_i^\theta \log^2 t_i}{\left(1+t_i^\theta\right)^2}, \tag{20}$$

$$\frac{\partial^2 \ell(\underline{\eta}|\underline{\mathbf{t}})}{\partial \alpha_1 \partial \alpha_2} = \frac{\partial^2 \ell(\underline{\eta}|\underline{\mathbf{t}})}{\partial \alpha_2 \partial \alpha_1} = \frac{-n_3}{(\alpha_1+\alpha_2)^2}, \tag{21}$$

$$\frac{\partial^2 \ell(\underline{\eta}|\underline{\mathbf{t}})}{\partial \alpha_1 \partial \theta} = \frac{\partial^2 \ell(\underline{\eta}|\underline{\mathbf{t}})}{\partial \theta \partial \alpha_1} = \frac{\partial^2 \ell(\underline{\eta}|\underline{\mathbf{t}})}{\partial \alpha_2 \partial \theta} = \frac{\partial^2 \ell(\underline{\eta}|\underline{\mathbf{t}})}{\partial \theta \partial \alpha_2} = \frac{-(n-r)\mu^\theta \log \mu}{1+\mu^\theta} - \sum_{i=1}^{r} \frac{t_i^\theta \log t_i}{1+t_i^\theta}. \tag{22}$$

The Fisher information matrix is defined as the minus expectation of Equations (18)–(22), which does not yield a closed form solution. Hence, the approximated information matrix is used to obtain the standard errors. To this end, suppose that $\Psi\left(\underline{\eta}\right)$ defines the approximated observed information matrix under the MLE of model parameters $\hat{\Psi}\left(\underline{\hat{\eta}}\right)$, where

$$\hat{\Psi}\left(\underline{\hat{\eta}}\right) = \left[-\frac{\partial^2 \ell(\underline{\eta}|\underline{\mathbf{t}})}{\partial \eta_i \partial \eta_j}\right]_{\underline{\hat{\eta}}}. \tag{23}$$

Therefore, the $100(1-2\epsilon)\%$ interval estimators of model parameters $\underline{\hat{\eta}}$ with mean $\underline{\eta}$ and variance-covariance matrix $\Psi_0^{-1}\left(\underline{\hat{\eta}}\right)$ are given by

$$\hat{\alpha}_j \mp z_\epsilon \sqrt{\hat{\Psi}_{jj}},\ j = 1, 2 \tag{24}$$

and

$$\hat{\theta} \mp z_{\epsilon}\sqrt{\hat{\Psi}_{33}}, \tag{25}$$

where $\hat{\Psi}_{jj}$, $j = 1, 2, 3\gamma$ are the diagonal elements of the observed information matrix and z_{ϵ} follows a normal (0, 1) distribution with tailed probability ϵ. Equations (24) and (25) can have a negative lower bound, and to avoid it, we adopted the logarithmic transformation of model parameters defined by $\log \eta_i$, $i = 1, 2, 3$. Then, the delta method was applied [35,36]. The pivotal quantity $F = \frac{\log \hat{\eta}_i - \log \eta_i}{\mathrm{Var}(\hat{\eta}_i)}$ is distributed with standard normal distribution and the approximate interval estimate of $\underline{\eta} = \{\alpha_1, \alpha_2, \theta\}$, such that $100(1-2\epsilon)\%$ is

$$\hat{\eta}_i \exp\left(\mp z_{\epsilon}\sqrt{\mathrm{Var}(\log \hat{\eta}_i)}\right), i = 1,\ 2,\ 3, \tag{26}$$

where $\mathrm{Var}(\log \hat{\eta}_i) = \frac{\mathrm{Var}(\hat{\eta}_i)}{\hat{\eta}_i}$.

4. Bayes Estimation

This section discusses the Bayesian approach for point and the corresponding interval estimation of the model parameters. We suppose an independent gamma prior for the parameter vectors $\underline{\eta} = \{\alpha_1, \alpha_2, \theta\}$, given as

$$\pi^*(\eta_i) \propto \eta_i^{a_i - 1} \exp(-b_i \eta_i). \tag{27}$$

Then, the joint prior density can be written as

$$\pi^*(\underline{\eta}) \propto \prod_{i=1}^{3} \eta_i^{a_i - 1} \exp(-b_i \eta_i). \tag{28}$$

The two functions, the likelihood function (9) and the joint prior density (28), are used to calculate the posterior distribution.

$$\begin{aligned} \pi(\underline{\eta}|\underline{\mathbf{t}}) \propto & \alpha_1^{n_1 + a_1 - 1} \alpha_2^{n_2 + a_2 - 1} \theta^{r + a_3 - 1} \exp\{n_3 \log(\alpha_1 + \alpha_2) + \theta \textstyle\sum_{i=1}^{r} \log t_i - (n - r) \\ & \times (\alpha_1 + \alpha_2) \log\left(1 + \mu^{\theta}\right) - (\alpha_1 + \alpha_2 + 1) \textstyle\sum_{i=1}^{r} \log\left(1 + t_i^{\theta}\right) - b_1\alpha_1 - b_2\alpha_2 - b_3\theta\}. \end{aligned} \tag{29}$$

The joint posterior density defined by (29) needs a normalization problem that involves complicated integrals. Different approximation techniques can be applied to overcome this problem, such as numerical integration, Lindley approximations, and the Markov chain Monte Carlo (MCMC) approach. This study adopts MCMC with an importance sample technique for computing Bayes estimators under the symmetric squared error loss function (SELF); see [37].

4.1. Posterior Distribution under Importance Sample Technique

The joint posterior distribution, Equation (29), can be formulated as

$$\pi(\underline{\eta}|\underline{\mathbf{t}}) = \mathbf{h}(\underline{\eta}|\underline{\mathbf{t}}) \prod_{i=1}^{3} h_i(\eta_i), \tag{30}$$

where

$$h_i(\alpha_i) \propto Q_i^{n_i + a_i} \alpha_i^{\left(n_i + a_i\right) - 1} \exp\{-Q_i \alpha_i\}, i = 1,\ 2, \tag{31}$$

$$h_3(\theta) \propto \theta^{r + a_3 - 1} \exp\left\{\theta \sum_{i=1}^{r} \log t_i - 2 \sum_{i=1}^{r} \log\left(1 + t_i^{\theta}\right) - \theta b_3\right\}, \tag{32}$$

$$\mathbf{h}(\underline{\eta}|\underline{\mathbf{t}}) \propto \left(Q_1^{n_1 + a_1} Q_2^{n_2 + a_2}\right)^{-1} \exp\left\{n_3 \log(\alpha_1 + \alpha_2) + \sum_{i=1}^{r} \log\left(1 + t_i^{\theta}\right)\right\}, \tag{33}$$

and

$$Q_i = b_i + \sum_{i=1}^{r} \log\left(1 + t_i^{\theta}\right) + (n-r)\log\left(1+\mu^{\theta}\right). \tag{34}$$

The proposed full conditional posterior distributions are used to generate a sample from the posterior distribution and hence parameters estimates.

4.2. Point Estimation

The point estimates and the corresponding variances are computed under the following importance sample algorithm.

1. Let $\hat{\eta} = \{\hat{\alpha}_1, \hat{\alpha}_2, \hat{\theta}\}$ as the initial values of an iteration with $I = 1$.
2. Generate $\alpha_1^{(I)}$ and $\alpha_2^{(I)}$ from the gamma densities given in (31).
3. Under the normal distribution as a proposal, the Metropolis–Hasting (MH) algorithm generates $\theta^{(I)}$.
4. For given $\alpha_1^{(I)}$, $\alpha_2^{(I)}$ and $\theta^{(I)}$, compute $\mathbf{h}^{(I)} = \mathbf{h}(\alpha_1^{(I)}, \alpha_2^{(I)}, \theta^{(I)}|\underline{\mathbf{t}})$ and update I by $I+1$.
5. The steps from (2) to (4) are repeated $\mathbf{M}$ times.
6. Suppose that $\mathbf{M}^*$ is a burn-in required to satisfy the stationary distribution.
7. Compute the uniform values $w^{(i)} = \left(\mathbf{h}^{(i)} / \sum_{i=\mathbf{M}^*+1}^{\mathbf{M}} \mathbf{h}^{(i)}\right)$, $i = \mathbf{M}^*+1, \mathbf{M}^*+1, \ldots, M$.

Under the SELF, the point estimator of any function of parameters vectors Ω is given by

$$\hat{\Omega}_{\mathrm{B}} = \sum_{i=\mathbf{M}^*+1}^{\mathbf{M}} \Omega^{(i)} w^{(i)}. \tag{35}$$

and the corresponding posterior variance of Ω is

$$V(\Omega) = \sum_{i=\mathbf{M}^*+1}^{\mathbf{M}} (\Omega^{(i)} - \hat{\Omega}_{\mathrm{B}})^2 w^{(i)}, \tag{36}$$

where Ω may be α_1, α_2, θ or a function of the parameters.

4.3. Interval Estimation

This subsection computes the credible or highest posterior density (HPD) interval estimators by using the MCMC sample generated using importance sampling [38]. The credible intervals of the parameter $\eta_i|_{i=1,2,3} = \alpha_1, \alpha_2$ or θ are given as follows:

1. For the parameter η_i, use $(\alpha_1^{(k)}, \alpha_2^{(k)}, \theta^{(k)})$ of the MCMC sample to find the ϵ-th quantile of η_i by $\eta_i^{(\epsilon)}$ as
$$\eta_i^{(\epsilon)} = \inf\{\eta_i : \ \pi(\eta_i|\underline{\mathbf{t}}) \geq \epsilon\}, \tag{37}$$
where $\pi(\eta_i|\underline{\mathbf{t}})$ is the marginal cumulative posterior distribution.
2. The generated values $\eta_i^{(k)}, k = \mathbf{M}^*+1, \mathbf{M}^*+1, \ldots, M$ are ordered to obtain $\eta_i^{[k]}, k = 1, 2, \ldots, M - \mathbf{M}^*$.
3. Without loss of generality for η_1, we define the value w_i by
$$w_1^{(i)} = \frac{\mathbf{h}(\eta_1^{[k]}, \eta_2^{(k)}, \eta_2^{(k)}|\underline{\mathbf{t}})}{\sum_{i=S^*+1}^{S} \mathbf{h}(\eta_1^{[k]}, \eta_2^{(k)}, \eta_2^{(k)}|\underline{\mathbf{t}})}, \tag{38}$$
where $\mathbf{h}$ is defined by (33).
4. For the ordered pairs (w_1, η_1), the ϵ-th quantile of η_1 of the marginal posterior is
$$\hat{\eta}_1^{(\epsilon)} = \begin{cases} \eta_1^{[k]}, \text{ if } \epsilon = 0 \\ \eta_1^{[k]}, \text{ if } \sum_{i=1}^{k-1} w_1^{(i)} < \epsilon < \sum_{i=1}^{k} w_1^{(i)} \end{cases}. \tag{39}$$

5. The $100(1-2\epsilon\%)$ credible intervals of $\eta_1^{[k]}$ are given by

$$(\eta_1^{(\epsilon)}, \eta_1^{(1-\epsilon)}). \tag{40}$$

The credible interval for η_1 and η_2 can be obtained similarly.

5. Simulation Studies

The performances of the developing results in this paper are assessed and compared in this section through building the Monte Carlo method and analyzing random choices of the parameters' values. Therefore, for given prior information, we generate a random sample of size 10 from the gamma density, and the true parameter values are considered as the mean of this sample. Thus, for $\{(a_i, b_i)\}$ = {(3, 2),(4, 3),(4, 2)}, the parameter vector is equal to $\underline{\eta} = \{\alpha_1, \alpha_2, \theta\}$ = {1.2, 1.7, 2.2}. For $\{(a_i, b_i)\}$ = {(1, 1),(1, 2),(2, 3)}, the parameter vector is equal to $\underline{\eta} = \{\alpha_1, \alpha_2, \theta\}$ = {0.9, 0.6, 1.1}. For each choice of scheme parameters n, m, τ generated 1000 samples of the Burr XII distribution with shapes $\alpha_1 + \alpha_2$ and θ. Additionally, two integers n_1 and n_2 are generated from binomial distributions with the probability of success given by $\frac{\alpha_j}{\alpha_1+\alpha_2}$, $j = 1, 2$, respectively. Furthermore, n_3 is generated from a Bernoulli distribution with a masking probability p. For each sample set, we computed the point and interval estimate with MLE and Bayesian estimations. The point estimate was measured under average (AV) and mean squared error (MSE). Additionally, the interval estimate was measured under mean interval length (MIL) and coverage percentage (CP). The numerical results study the effect of changing the sample size n, m, τ, p and parameter values. In the Bayesian approach, we considered changes with 11,000 iterations, discarding the first 1000 iterations. The numerical results of the simulation study are presented in Tables 1–4.

Table 1. AVs and MSEs in the bracket for the estimators under $\underline{\eta} = \{1.2, 1.7, 2.2\}$.

p	(τ, n, m)	MLE			Bayes		
		α_1	α_2	θ	α_1	α_2	θ
0.1	(0.5, 40, 15)	1.415	2.011	2.485	0.961	1.817	2.321
	(0.305)	(0.397)	(0.522)	(0.207)	(0.308)	(0.371)	
	(0.5, 40, 30)	1.399	1.907	2.416	1.227	1.789	2.337
	(0.253)	(0.348)	(0.448)	(0.161)	(0.267)	(0.316)	
	(0.5, 50, 30)	1.392	1.927	2.443	1.222	1.804	2.337
	(0.259)	(0.351)	(0.443)	(0.161)	(0.260)	(0.312)	
	(1.0, 40, 15)	1.351	1.890	2.401	1.221	1.788	2.321
	(0.249)	(0.338)	(0.447)	(0.152)	(0.259)	(0.288)	
	(1.0, 40, 30)	1.302	1.825	2.384	1.225	1.801	2.314
	(0.201)	(0.285)	(0.375)	(0.101)	(0.211)	(0.227)	
	(1.0, 50, 30)	1.313	1.829	2.380	1.221	1.798	2.311
	(0.294)	(0.268)	(0.361)	(0.098)	(0.192)	(0.204)	
0.2	(0.5, 40, 15)	1.454	2.046	2.514	0.998	1.852	2.024
	(0.325)	(0.415)	(0.541)	(0.221)	(0.325)	(0.387)	
	(0.5, 40, 30)	1.401	1.922	2.430	1.241	1.803	2.356
	(0.271)	(0.362)	(0.465)	(0.175)	(0.281)	(0.331)	
	(0.5, 50, 30)	1.414	1.932	2.441	1.235	1.811	2.350
	(0.276)	(0.365)	(0.461)	(0.179)	(0.275)	(0.325)	
	(1.0, 40, 15)	1.365	1.901	2.407	1.232	1.805	2.339
	(0.249)	(0.338)	(0.447)	(0.152)	(0.259)	(0.288)	
	(1.0, 40, 30)	1.302	1.825	2.384	1.225	1.801	2.314
	(0.201)	(0.285)	(0.375)	(0.101)	(0.211)	(0.227)	
	(1.0, 50, 30)	1.313	1.829	2.380	1.221	1.798	2.311
	(0.207)	(0.281)	(0.379)	(0.107)	(0.208)	(0.221)	

Table 1. *Cont.*

p	(τ, n, m)	MLE			Bayes		
		α_1	α_2	θ	α_1	α_2	θ
0.3	(0.5, 40, 15)	1.485	2.122	2.565	1.050	1.899	2.074
	(0.350)	(0.442)	(0.566)	(0.247)	(0.339)	(0.399)	
	(0.5, 40, 30)	1.430	1.955	2.474	1.267	1.838	2.381
	(0.292)	(0.383)	(0.481)	(0.191)	(0.298)	(0.348)	
	(0.5, 50, 30)	1.445	1.959	2.469	1.267	1.840	2.378
	(0.291)	(0.384)	(0.479)	(0.194)	(0.278)	(0.339)	
	(1.0, 40, 15)	1.388	1.927	2.441	1.259	1.832	2.357
	(0.264)	(0.354)	(0.462)	(0.170)	(0.268)	(0.294)	
	(1.0, 40, 30)	1.327	1.851	2.399	1.252	1.828	2.341
	(0.219)	(0.298)	(0.391)	(0.119)	(0.224)	(0.245)	
	(1.0, 50, 30)	1.340	1.855	2.399	1.248	1.641	2.347
	(0.218)	(0.297)	(0.395)	(0.124)	(0.229)	(0.241)	

Table 2. MILs and CPs in the bracket for the estimators under $\underline{\eta} = \{1.2, 1.7, 2.2\}$.

p	(τ, n, m)	MLE			Bayes		
		α_1	α_2	θ	α_1	α_2	θ
0.1	(0.5, 40, 15)	3.124	3.425	4.852	2.842	3.015	4.052
	(0.89)	(0.87)	(0.89)	(0.90)	(0.91)	(0.89)	
	(0.5, 40, 30)	3.015	3.285	4.599	2.687	2.841	3.911
	(0.90)	(0.91)	(0.89)	(0.93)	(0.91)	(0.90)	
	(0.5, 50, 30)	3.029	3.260	4.590	2.692	2.838	3.921
	(0.91)	(0.89)	(0.90)	(0.93)	(0.93)	(0.94)	
	(1.0, 40, 15)	3.075	3.381	4.801	2.790	2.984	4.003
	(0.90)	(0.89)	(0.89)	(0.90)	(0.91)	(0.92)	
	(1.0, 40, 30)	2.958	3.241	4.557	2.655	2.802	3.871
	(0.90)	(0.92)	(0.90)	(0.93)	(0.94)	(0.93)	
	(1.0, 50, 30)	2.982	3.214	4.547	2.645	2.801	3.887
	(0.92)	(0.90)	(0.90)	(0.93)	(0.93)	(0.92)	
0.2	(0.5, 40, 15)	3.191	3.484	4.915	2.898	3.081	4.112
	(0.90)	(0.87)	(0.88)	(0.90)	(0.89)	(0.90)	
	(0.5, 40, 30)	3.074	3.325	4.666	2.746	2.899	3.972
	(0.90)	(0.90)	(0.89)	(0.90)	(0.91)	(0.93)	
	(0.5, 50, 30)	3.092	3.310	4.651	2.754	2.898	3.979
	(0.90)	(0.90)	(0.91)	(0.93)	(0.92)	(0.91)	
	(1.0, 40, 15)	3.130	3.439	4.864	2.845	3.038	4.069
	(0.92)	(0.90)	(0.89)	(0.90)	(0.94)	(0.92)	
	(1.0, 40, 30)	3.025	3.298	4.680	2.715	2.864	3.918
	(0.91)	(0.92)	(0.96)	(0.93)	(0.91)	(0.92)	
	(1.0, 50, 30)	3.041	3.269	4.592	2.698	2.858	3.941
	(0.91)	(0.92)	(0.90)	(0.93)	(0.92)	(0.91)	
0.3	(0.5, 40, 15)	3.280	3.581	4.999	2.960	3.174	4.202
	(0.86)	(0.89)	(0.88)	(0.90)	(0.89)	(0.91)	
	(0.5, 40, 30)	3.159	3.414	4.741	2.829	2.975	4.050
	(0.89)	(0.89)	(0.90)	(0.90)	(0.91)	(0.89)	
	(0.5, 50, 30)	3.178	3.400	4.732	2.835	2.975	4.030
	(0.90)	(0.92)	(0.91)	(0.93)	(0.92)	(0.94)	
	(1.0, 40, 15)	3.215	3.521	4.945	2.935	3.120	4.148
	(0.90)	(0.92)	(0.91)	(0.94)	(0.92)	(0.92)	
	(1.0, 40, 30)	3.111	3.379	4.760	2.810	2.941	4.045
	(0.93)	(0.91)	(0.96)	(0.93)	(0.92)	(0.92)	
	(1.0, 50, 30)	3.123	3.345	4.680	2.781	2.935	4.022
	(0.92)	(0.92)	(0.91)	(0.93)	(0.95)	(0.91)	

Table 3. AVs and MSEs in the bracket for the estimators under $\underline{\eta} = \{0.9, 0.6, 1.1\}$.

p	(τ, n, m)	MLE			Bayes		
		α_1	α_2	θ	α_1	α_2	θ
0.1	(0.5, 40, 15)	1.118	0.819	1.417	1.049	0.748	1.235
		(0.191)	(0.124)	(0.269)	(0.124)	(0.057)	(0.252)
	(0.5, 40, 30)	1.084	0.772	1.375	1.003	0.715	1.202
		(0.174)	(0.103)	(0.230)	(0.100)	(0.031)	(0.235)
	(0.5, 50, 30)	1.089	0.791	1.380	1.000	0.708	1.202
		(0.171)	(0.100)	(0.236)	(0.091)	(0.019)	(0.240)
	(1.5, 40, 15)	1.074	0.778	1.371	1.002	0.705	1.200
		(0.172)	(0.101)	(0.235)	(0.102)	(0.042)	(0.233)
	(1.5, 40, 30)	1.041	0.733	1.338	0.974	0.677	1.171
		(0.150)	(0.081)	(0.209)	(0.076)	(0.004)	(0.217)
	(1.5, 50, 30)	1.045	0.742	1.336	0.961	0.662	1.155
		(0.151)	(0.082)	(0.214)	(0.082)	(0.003)	(0.221)
0.2	(0.5, 40, 15)	1.147	0.845	1.444	1.074	0.778	1.268
		(0.214)	(0.142)	(0.278)	(0.142)	(0.074)	(0.280)
	(0.5, 40, 30)	1.111	0.803	1.407	1.029	0.741	1.232
		(0.191)	(0.122)	(0.252)	(0.115)	(0.049)	(0.254)
	(0.5, 50, 30)	1.118	0.812	1.402	1.022	0.735	1.229
		(0.188)	(0.120)	(0.255)	(0.111)	(0.040)	(0.257)
	(1.5, 40, 15)	1.104	0.803	1.400	1.028	0.731	1.224
		(0.191)	(0.118)	(0.254)	(0.119)	(0.057)	(0.252)
	(1.5, 40, 30)	1.067	0.761	1.362	1.000	0.707	1.201
		(0.168)	(0.100)	(0.228)	(0.092)	(0.019)	(0.234)
	(1.5, 50, 30)	1.071	0.771	1.362	0.987	0.687	1.184
		(0.168)	(0.100)	(0.233)	(0.100)	(0.018)	(0.241)
0.3	(0.5, 40, 15)	1.191	0.887	1.481	1.112	0.821	1.312
		(0.230)	(0.157)	(0.282)	(0.159)	(0.091)	(0.299)
	(0.5, 40, 30)	1.152	0.851	1.459	1.071	0.779	1.271
		(0.208)	(0.144)	(0.271)	(0.133)	(0.068)	(0.271)
	(0.5, 50, 30)	1.160	0.853	1.445	1.059	0.771	1.270
		(0.205)	(0.139)	(0.276)	(0.129)	(0.058)	(0.274)
	(1.5, 40, 15)	1.145	0.841	1.439	1.066	0.769	1.271
		(0.209)	(0.141)	(0.271)	(0.141)	(0.075)	(0.271)
	(1.5, 40, 30)	1.309	0.799	1.398	1.032	0.745	1.238
		(0.187)	(0.122)	(0.247)	(0.110)	(0.041)	(0.255)
	(1.5, 50, 30)	1.122	0.812	1.397	1.020	0.721	1.217
		(0.190)	(0.124)	(0.251)	(0.119)	(0.037)	(0.260)

Table 4. MILs and CPs in the bracket for the estimators under $\underline{\eta} = \{0.9, 0.6, 1.1\}$.

p	(τ, n, m)	MLE			Bayes		
		α_1	α_2	θ	α_1	α_2	θ
0.1	(0.5, 40, 15)	2.235	1.754	2.578	2.100	1.584	2.411
		(0.88)	(0.88)	(0.89)	(0.93)	(0.91)	(0.89)
	(0.5, 40, 30)	2.185	1.709	2.541	2.065	1.546	2.382
		(0.89)	(0.91)	(0.90)	(0.93)	(0.91)	(0.96)
	(0.5, 50, 30)	2.192	1.701	2.541	2.071	1.541	2.375
		(0.90)	(0.90)	(0.93)	(0.93)	(0.95)	(0.94)
	(1.5, 40, 15)	2.141	1.667	2.501	2.019	1.503	2.336
		(0.92)	(0.89)	(0.89)	(0.94)	(0.91)	(0.93)
	(1.5, 40, 30)	2.102	1.631	2.465	1.852	1.462	2.300
		(0.92)	(0.92)	(0.92)	(0.93)	(0.90)	(0.93)
	(1.5, 50, 30)	2.113	1.625	2.454	1.847	1.452	2.294
		(0.90)	(0.90)	(0.90)	(0.94)	(0.93)	(0.95)

Table 4. *Cont.*

p	(τ, n, m)	MLE			Bayes		
		α_1	α_2	θ	α_1	α_2	θ
0.2	(0.5, 40, 15)	2.275	1.791	2.610	2.138	1.614	2.447
	(0.89)	(0.88)	(0.87)	(0.93)	(0.91)	(0.89)	
	(0.5, 40, 30)	2.322	1.745	2.579	2.099	1.581	2.412
	(0.89)	(0.92)	(0.90)	(0.92)	(0.91)	(0.94)	
	(0.5, 50, 30)	2.234	1.734	2.577	2.099	1.572	2.415
	(0.91)	(0.90)	(0.90)	(0.92)	(0.95)	(0.91)	
	(1.5, 40, 15)	2.177	1.698	2.534	2.051	1.532	2.367
	(0.90)	(0.89)	(0.89)	(0.94)	(0.91)	(0.91)	
	(1.5, 40, 30)	2.137	1.662	2.499	1.887	1.491	2.336
	(0.91)	(0.92)	(0.92)	(0.93)	(0.90)	(0.92)	
	(1.5, 50, 30)	2.147	1.661	2.487	1.879	1.482	2.315
	(0.92)	(0.92)	(0.91)	(0.95)	(0.94)	(0.95)	
0.3	(0.5, 40, 15)	2.299	1.815	2.639	2.161	1.642	2.470
	(0.89)	(0.85)	(0.89)	(0.90)	(0.91)	(0.89)	
	(0.5, 40, 30)	2.357	1.784	2.614	2.132	1.615	2.441
		(0.90)	(0.90)	(0.90)	(0.90)	(0.91)	(0.92)
	(0.5, 50, 30)	2.261	1.760	2.597	2.127	1.597	2.444
	(0.91)	(0.92)	(0.90)	(0.92)	(0.90)	(0.91)	
	(1.5, 40, 15)	2.210	1.729	2.564	2.082	1.564	2.381
	(0.91)	(0.89)	(0.90)	(0.92)	(0.91)	(0.92)	
	(1.5, 40, 30)	2.158	1.689	2.532	1.915	1.524	2.359
	(0.92)	(0.92)	(0.91)	(0.93)	(0.93)	(0.95)	
	(1.5, 50, 30)	2.171	1.692	2.500	1.896	1.509	2.337
	(0.91)	(0.92)	(0.94)	(0.92)	(0.94)	(0.93)	

6. Disease Data Analysis

This section discusses two examples of lifetime data recorded under two diseases to illustrate the applications of the proposed method. The first example data set is taken from [39], which represents about male mice exposed to 300 Roentgens of radiation over the period of 5–6 weeks. This data set has been analyzed by several authors [19,40–42] under different lifetime models. Example two takes a simulated data set to show the method's practicality.

6.1. Example 1: A Disease Data Set

The summary of the first set is given in Table 5, where the diseases reticulum cell sarcoma and thymic lymphoma were considered cause 1 while other diseases were considered cause 2. Without loss of information, the data were divided by 1000. Assuming n = 99, m = 50 and $\tau = 0.7$, the competing risks of the type I HCS are summarized by

$\underline{\mathbf{t}}$ = {(0.04, 2), (0.042, 2), (0.051, 2), (0.062, 2), (0.159, 1), (0.163, 2), (0.179, 2), (0.189, 1), (0.191, 1), (0.198, 1), (0.200, 1), (0.206, 2), (0.207, 1), (0.220, 1), (0.222, 2), (0.228, 2), (0.235, 1), (0.245, 1), (0.249, 2), (0.250, 1), (0.252, 2), (0.256, 1), (0.261, 1), (0.265, 1), (0.266, 1), (0.28, 1), (0.282, 2), (0.317, 1), (0.318, 1), (0.324, 2), (0.333, 2), (0.341, 2), (0.343, 1), (0.356, 1), (0.366, 2), (0.383, *), (0.385, *), (0.399, *), (0.403, 1), (0.407, 2), (0.414, 1), (0.420, 2), (0.428, 1), (0.431, 2), (0.432, 1), (0.441 2), (0.461, 2), (0.462, 2), (0.482, 2), (0.495, 1)}}. That is, the type I HCS competing risks have (n_1, n_2, n_3, r) = (24, 23, 3, 50).

From the profile log-likelihood function (17), Figure 1, the ML estimator can be seen to have an initial guess value of 1.8 for θ. A noninformative prior was considered by fixing $a_i = b_i = 0.0001$, $i = 1, 2, 3$. The estimation results obtained from the MLE and Bayes methods are presented in Table 6. We considered 11,000 iterations and discarded the first 1000 iterations as the burn-in period to compute Bayes estimates. The convergence of the posterior chains is depicted in Figures 2–4. Under the estimated values of the parameters

in Table 6, the survival probability when $t = 0.3$ for MLE is given by S_{1-MLE} = 0.861756, $S2_{2-MLE}$ = 0.867113, and for a Bayes estimate, it is S_{1-B} = 0.863052 and $S2_{2-B}$ = 0.84316.

Table 5. Real life data presented by Hoel (1972) for radiated male mice.

cause1	159	189	191	198	200	207	220	235	245	250	256
	261	265	266	280	317	318	343	356	383	399	403
	414	428	432	495	525	536	549	552	554	558	571
	596	605	612	621	628	631	636	643	647	648	649
	586	594	596	661	663	666	670	695	697	700	705
	712	713	738	748	753						
cause2	40	42	51	62	163	179	206	222	228	249	252
	282	324	333	341	366	385	407	420	431	441	461
	462	482	517	517	524	564	567	586	619	620	621
	622	647	651	686	761	763					

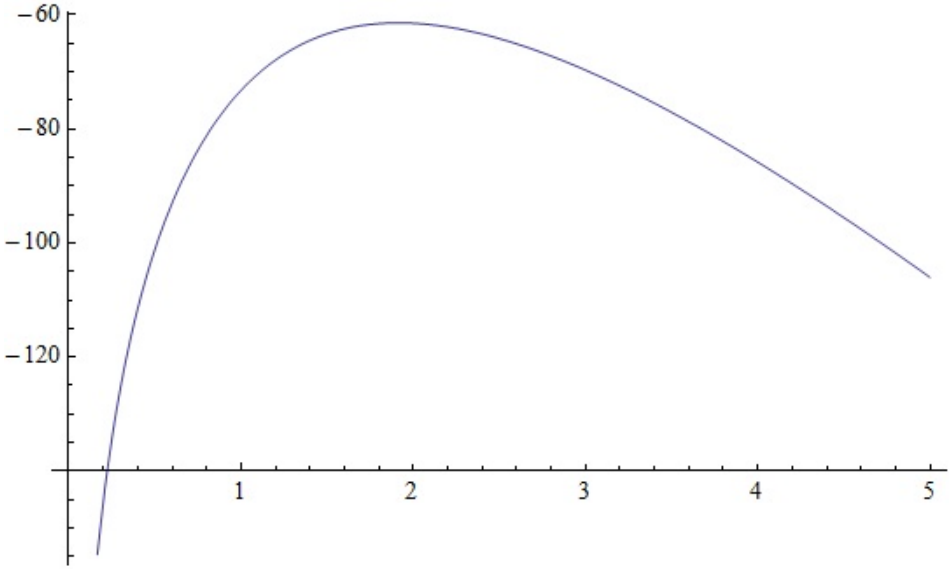

Figure 1. Profile log-likelihood function of θ.

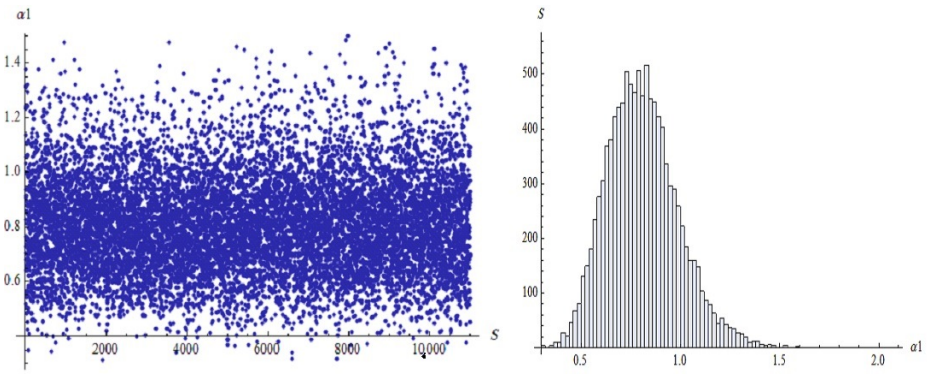

Figure 2. Simulated numbers and corresponding histogram generated under MCMC.

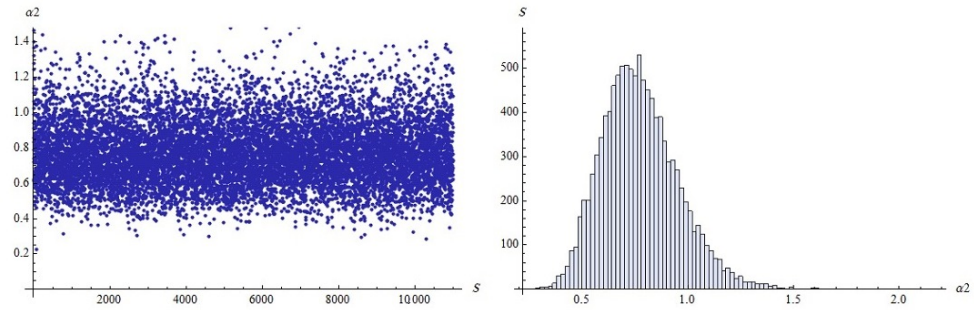

Figure 3. Simulated numbers and corresponding histogram generated under MCMC.

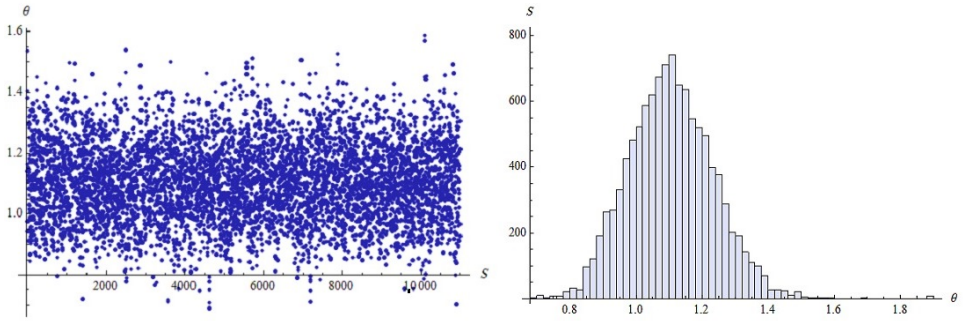

Figure 4. Simulated numbers and corresponding histogram generated under MCMC.

Table 6. The point and the corresponding 95% interval estimate.

Pa.	ML	Bayes	95% A.C.I.	Lenth	95% C.I.	Lenth
α_1	1.5607	0.9695	(0.7588, 2.3626)	1.6038	(1.1172, 1.7971)	0.6799
α_2	1.4957	1.1230	(0.7172, 2.2742)	1.5571	(1.0846, 1.6138)	0.5291
θ	1.9123	1.5013	(1.4658, 2.3588)	0.8930	(1.3123, 1.5896)	0.2773

6.2. Example 2: Simulated Data

A data set was generated from a Burr XII distribution using the following algorithms.

1. We fixed $\underline{\eta} = \{\alpha_1, \alpha_2, \theta\} = \{1.0, 1.3, 0.5\}$, and the hyper parameters were selected to satisfy $E(\eta_i) \simeq \frac{a_i}{b_i}$.
2. For the given $n = 50$, $m = 25$ and $\tau = 5.5$, we generated a type I HSC random sample from a Burr XII distribution with parameters $\alpha_1 + \alpha_2$ and $\theta \in \{0.0008, 0.0012, 0.0016, 0.0024, 0.0025, 0.0038, 0.0057, 0.0069, 0.0103, 0.0105, 0.012, 0.0134, 0.0172, 0.0237, 0.0312, 0.0336, 0.0507, 0.0589, 0.1098, 0.1295, 0.1577, 0.1675, 0.1732, 0.1959, 0.2141\}$.
3. From this data, we noticed that $r = m = 25$.
4. The number of censored failure causes $n_3 = 3$ were generated from the Bernouli distribution with a probability $p = 0.1$ and a sample size of 25.
5. The two observed causes $n_1 = 12$ and $n_2 = 10$ were generated from the binomail distribution with parameter $(r - n_3)$ and the probability of success $\frac{\alpha_1}{\alpha_1+\alpha_2}$ and $\frac{\alpha_2}{\alpha_1+\alpha_2}$, respectively.
6: The fixed point method was used to compute the MLE with an initial value of 0.52 taken from the profile log-likelihood function (17) depicted in Figure 5.
7. The simulated number and the corresponding histogram generated under MCMC methods are presented by Figures 6–8.
8. Point estimates, 95% ML intervals and a Bayes estimate are given in Table 7.
9. Under the estimated values of the parameters in Table 7, the survival probability when $t = 0.05$ for MLE is given by S_{1-MLE} = 0.822363, $S2_{2-MLE}$ =0.84961 and, for a Bayes estimate, S_{1-B} = 0.836762, $S2_{2-B}$ = 0.836863.

Table 7. Point and 95% interval for the ML and Bayes estimate.

Exact	ML	Bayes	95% A.C.I.	Length	95% C.I.	Length
$\alpha_1 = 1.0$	1.0368	1.0311	(0.3994, 1.6741)	1.2748	(0.8247, 1.8636)	1.0390
$\alpha = 1.3$	0.8640	1.0304	(0.2896, 1.4383)	1.1488	(0.7965, 2.0385)	1.2420
$\theta = 0.5$	0.5248	0.5567	(0.3559, 0.6936)	0.3377	(0.2862, 0.526)	0.2398

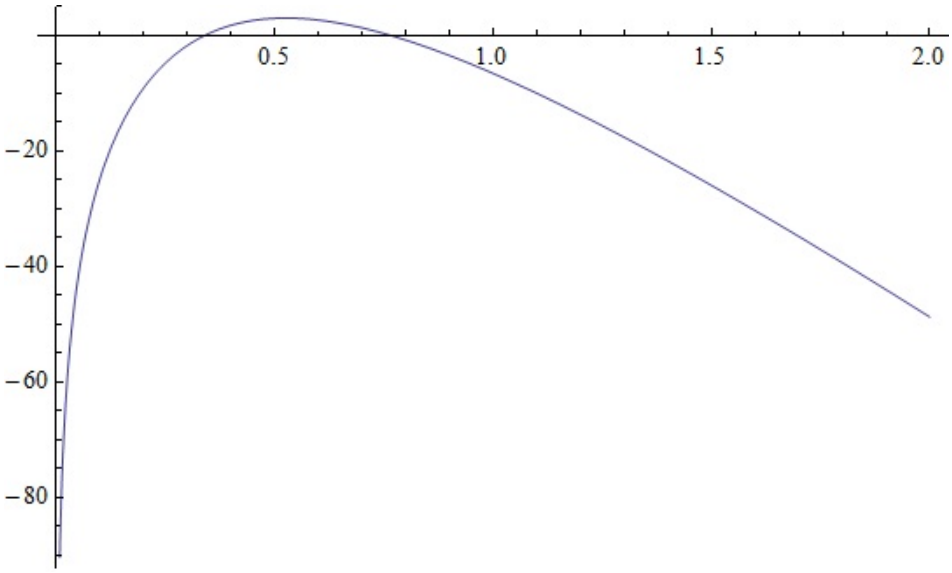

Figure 5. Profile log-likelihood function of θ.

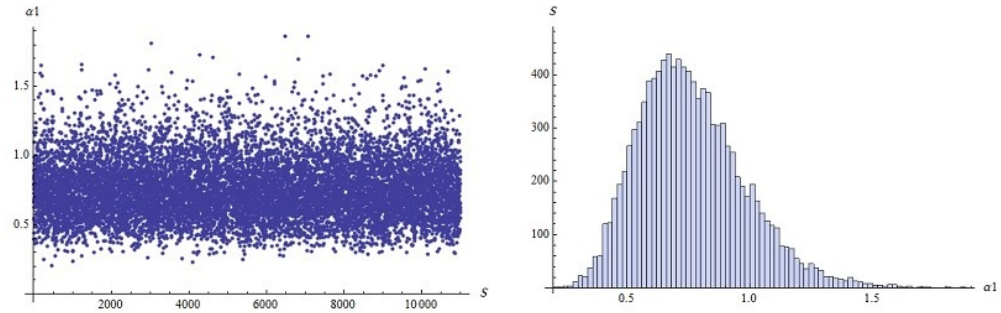

Figure 6. Simulated numbers and corresponding histogram generated under MCMC.

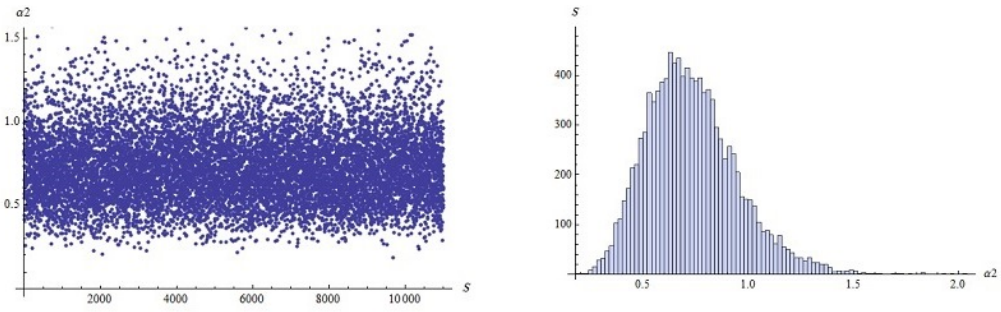

Figure 7. Simulated numbers and corresponding histogram generated under MCMC.

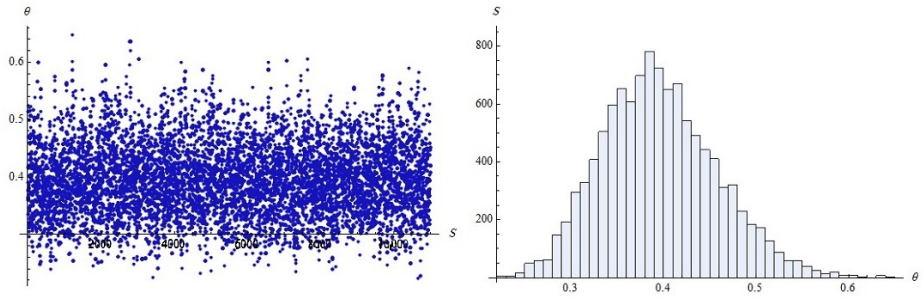

Figure 8. Simulated numbers and corresponding histogram generated under MCMC.

7. Conclusions

In a life testing experiment, failure may occur due to different causes. In this paper, we considered partially observed independent causes of failure with a Burr XII life time distribution. The competing risk model parameters are estimated by two different methods of estimation, namely the MLE and Bayes methods, under a symmetric squared error loss function. In addition to a simulation study, a real data set is used to show the practicality of the proposed method. In addition, the interval estimates are listed for simulation as well as for real data sets. Finally, the MCMC approach with an importance sampling step is considered to compute the Bayes estimates. The results obtained from the real data and Monte Carlo simulation study in Tables 4–7 suggested that the proposed model under a type I HSC is capable of measuring competing disease risks. Small values of masking probability are preferred over the large values. The affected sample size m and ideal test time τ are more crucial than the sample size n. Estimation based on the Bayes method leads to better results than the MLE method. The results of the MSEs and interval length decrease for larger m and τ. In a life testing experiment, failure may occur due to different causes. In this paper, we considered partially observed independent causes of failure with a Burr XII life time distribution. The competing risk model parameters we re estimated by two different methods of estimation, namely the MLE and Bayes methods. In addition to a simulation study, a real data set is used to show the practicality of the proposed method. In addition, the interval estimates are listed for the simulation as well as for the real data set. Finally, the MCMC approach with an importance sampling step is considered to compute the Bayes estimates. The results obtained from the real data and Monte Carlo simulation study in Tables 4–7 suggested that

1 The proposed model under the type I HSC is capable of measuring competing disease risks;
2 Small values of masking probability are preferred over large values;
3 The affected sample size m and ideal test time τ are more crucial than the sample size n;
4 Estimation based on Bayes method leads to better results than the MLE;
5 The results of the MSEs and interval length decrease for larger m and τ.

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