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Complexity Analysis of E-Bayesian Estimation Under Type-II Censoring with Application to Organ Transplant Blood Data

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Abstract: The E-Bayesian estimation approach has been presented for estimating the parameter and/or reliability characteristics of various models. Several investigations in the literature have considered this method under the assumption that just one parameter is unknown. So, based on Type-II censoring, this study proposes for the first time an effort to use the E-Bayesian estimation approach to estimate the full model parameters as well as certain related functions such as the reliability and hazard rate functions. To illustrate this purpose, we apply the proposed technique to the two-parameter generalized inverted exponential distribution which can be considered to be one of the most flexible asymmetrical probability distributions. Moreover, the E-Bayesian method, maximum likelihood, and Bayesian estimation approaches are also considered for comparison purposes. Under the assumption of independent gamma priors, the Bayes and E-Bayes estimators are developed using the symmetrical squared error loss function. Due to the complex form of the joint posterior density, two approximation techniques, namely the Lindley and Markov chain Monte Carlo methods, are considered to carry out the Bayes and E-Bayes estimates and also to construct the associate credible intervals. Monte Carlo simulations are performed to assess the performance of the proposed estimators. To demonstrate the applicability of the proposed methods in real phenomenon, one real data set is analyzed and it shows that the proposed method is effective and easy to operate in a real-life scenario.

Keywords: E-Bayesian estimation; reliability function; hazard rate function; generalized inverted exponential distribution; Markov chain Monte Carlo techniques

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1. Introduction

The exponential distribution is the most popular distribution for lifetime data analysis because of the simplicity of its probability density function (PDF). In spite of its popularity, it has serious limitations in modeling data because of its constant failure rate. Many authors proposed generalizations of the exponential distribution by adding a new parameter(s) or mixing it with other well-known distributions to solve this limitation. For instance, consider the exponentiated exponential distribution proposed by [1], alpha power exponential distribution by [2], and Marshall–Olkin alpha power exponential distribution by [3]. Another modification to the exponential distribution has been done by using its inverted version. To formally define this, let the random variable Y follow an exponential distribution, then $X = Y^{-1}$ follows an inverted exponential distribution. Reference [4] used the maximum likelihood method to estimate the parameter and the reliability function (RF) of the inverted exponential distribution using complete samples. By including an exponentiated parameter, Reference [5] introduced a generalized form of the inverted exponential distribution. The new distribution was called the generalized inverted exponential distribution (GIED). They have constructed various statistical properties of this distribution and observed that the hazard rate function (HRF) of the GIED can be increasing or decreasing shaped. The random variable X is said to follow the GIED with scale parameter δ and shape parameter θ if its PDF is given by

$$f(x;\delta,\theta) = \delta\theta x^{-2} e^{-\delta/x}\left(1 - e^{-\delta/x}\right)^{\theta-1}, x > 0, \delta, \theta > 0. \tag{1}$$

The corresponding RF and HRF at mission time $t > 0$ are given, respectively, by

$$R(t;\delta,\theta) = \left(1 - e^{-\delta/t}\right)^{\theta}, t > 0, \delta, \theta > 0 \tag{2}$$

and

$$h(t;\delta,\theta) = \delta\theta t^{-2}\left(e^{\delta/t} - 1\right)^{-1}, t > 0, \delta, \theta > 0. \tag{3}$$

The GIED is a specific case of the exponentiated inverse Weibull distribution. Reference [6] developed some of the GIED's distributional properties in relation to this topic. Reference [7] used the maximum likelihood method to investigate the reliability of a multi-component stress-strength for the GIED. Reference [8] obtained the reliability estimation of the GIED under progressive Type-II censoring. Reference [9] used the Bayesian perspective to estimate the GIED parameters under progressive Type-II censoring. Reference [10] studied the maximum likelihood and Bayesian estimations of the GIED by considering hybrid censoring. Recently, Reference [11] obtained the E-Bayesian estimations of the shape parameter and some lifetime parameters of the GIED. See [12–14] for more details about GIED.

Reference [15] introduced an estimation method, named Expected-Bayesian (E-Bayesian) method, to estimate the failure rate of the exponential distribution. Using this approach, the E-Bayesian estimator (EBE) is given by taking the expectation of the usual Bayesian estimator (BE) over the hyper-parameters. Recently, many authors considered the E-Bayesian estimation method to estimate the unknown parameters of some distributions. Reference [15] introduced the E-Bayesian estimation method and estimated the exponential distribution parameter under Type-I censored data. Reference [16] studied the E-Bayesian estimation of the Burr-XII distribution using Type-II censored data. The E-Bayesian estimation for the geometric model based on record statistics was investigated in [17]. Reference [18] studied the E-Bayesian estimation of the exponentiated parameter using the LINEX loss function. Reference [19] discussed E-Bayesian estimation for the simple step-stress model based on Type-II censored data. Reference [20] obtained the E-Bayesian estimation of Burr Type-XII distribution using adaptive progressive Type-II censored data. Reference [21] studied the E-Bayesian estimations for the exponential distribution based on record data. Reference [11] considered the E-Bayesian estimations for some lifetime parameters of the GIED using Type-II censoring. For more studies about E-Bayesian estimation, one may refer to [22–25].

All of the previous research focused on E-Bayesian estimation problems for just distributions with one parameter such as exponential or geometric distributions. When studying distributions with more than one parameter, on the other hand, they always assumed that only one of these parameters is unknown to avoid analytical issues and computing complexity. Although this approach is simple, it raises a significant concern regarding the other unestimated parameters. To the best of our knowledge, the E-Bayesian estimation with the assumption that all model parameters are unknown has not yet been studied. For this reason, the main objectives of the present paper are as follows:

1. To derive the frequentist and BEs of the unknown parameters, RF and HRF for the GIED when the lifetime data are collected under Type-II censored sampling.
2. To construct the asymptotic confidence intervals (ACIs) and Bayesian credible intervals (BCIs) of the unknown parameters, RF and HRF.
3. To propose a simple approach to acquire the EBEs of the unknown parameters, RF and HRF as well as the corresponding E-Bayesian credible intervals (E-BCIs).

4. To compare the performance of the different proposed estimators using a simulation study by considering some criteria including root mean squared errors (RMSEs), relative absolute biases (RABs), and confidence lengths (CLs).
5. To show the usefulness and applicability of the proposed estimators by analyzing an organ transplant blood data set.

Under the assumption of independent gamma priors, the BEs and EBEs are developed using squared error loss (SEL). The SEL function is the generally employed symmetric loss function in which the overestimation and underestimation are treated equally. Since the Bayesian and E-Bayesian estimators cannot be obtained in closed forms, therefore we propose to use two procedures in order to approximate them, namely Lindley's approximation and Markov chain Monte Carlo (MCMC) methods.

The rest of this paper is organized as follows. The maximum likelihood method is considered in Section 2 to estimate the parameters RF and HRF. Section 3 is devoted to the Bayesian estimation using Lindley's approximation and MCMC technique. The E-Bayesian estimations are considered in Section 4. In Section 5 we discuss the elicitation procedure used to determine the hyper-parameter values. Various interval estimations are considered in Section 6. A simulation study is performed in Section 7. One real data set is analyzed in Section 8. Finally, the paper is concluded in Section 9.

2. Maximum Likelihood Estimation

Suppose that n independent items taken from a population with PDF and RF given by (1) and (2), respectively, are placed on a test. Let $x_{1:m}, \ldots, x_{m:m}$ be an observed Type-II censored sample of size $m < n$ from the GIED. Then from (1) and (2), we can write the likelihood function, ignoring the constant term, as follows

$$\begin{aligned} L(\delta,\theta) = {} & (\delta\theta)^m \prod_{i=1}^m x_i^{-2} \exp\left\{-\left[\sum_{i=1}^m (\delta/x_i) + \log(\xi(x_i;\delta))\right]\right\} \\ & \times \exp\left\{-\theta\left[-\sum_{i=1}^m \log(\xi(x_i;\delta)) - (n-m)\log(\xi(x_m;\delta))\right]\right\}, \end{aligned} \tag{4}$$

where $x_i = x_{i:m}$, $\xi(x_i;\delta) = 1 - e^{-\delta/x_i}$, and $\xi(x_m;\delta) = 1 - e^{-\delta/x_m}$. To obtain the maximum likelihood estimates (MLEs) of δ and θ, we first obtain the natural logarithm of (4) up to proportional as

$$\log L(\delta,\theta) \propto m\log(\delta\theta) - \sum_{i=1}^m (\delta/x_i) + (\theta-1)\sum_{i=1}^m \log(\xi(x_i;\delta)) + \theta(n-m)\log(\xi(x_m;\delta)). \tag{5}$$

The MLEs of δ and θ, denoted by $\hat{\delta}_{ML}$ and $\hat{\theta}_{ML}$, are the solution of the following two normal equations

$$\frac{m}{\delta} - \sum_{i=1}^m x_i^{-1} + (\theta-1)\sum_{i=1}^m \frac{x_i^{-1}e^{-\delta/x_i}}{\xi(x_i;\delta)} + \theta(n-m)\frac{x_m^{-1}e^{-\frac{\delta}{x_m}}}{\xi(x_m;\delta)} = 0, \tag{6}$$

and

$$\frac{m}{\theta} + \sum_{i=1}^m \log(\xi(x_i;\delta)) + (n-m)\log(\xi(x_m;\delta)) = 0. \tag{7}$$

From Equation (7) and for constant δ, we can obtain the $\hat{\theta}_{ML}$ as a function in δ as

$$\hat{\theta}_{ML}(\delta) = -m\left[\sum_{i=1}^m \log(\xi(x_i;\delta)) + (n-m)\log(\xi(x_m;\delta))\right]^{-1}. \tag{8}$$

Substituting (8) in (6), the MLE of δ is the solution of the following nonlinear equation

$$\frac{m}{\delta} - \sum_{i=1}^m x_i^{-1} + (\hat{\theta}_{ML}(\delta)-1)\sum_{i=1}^m \frac{x_i^{-1}e^{-\delta/x_i}}{\xi(x_i;\delta)} + \hat{\theta}_{ML}(\delta)(n-m)\frac{x_m^{-1}e^{-\frac{\delta}{x_m}}}{\xi(x_m;\delta)} = 0. \tag{9}$$

Any numerical technique can be used to obtain $\hat{\delta}_{ML}$ from (9). Once $\hat{\delta}_{ML}$ is obtained the $\hat{\theta}_{ML}$ can be obtained directly from (8). Using the invariance property of the MLEs, the MLEs of RF and HRF can be obtained by replacing δ and θ in (2) and (3) by $\hat{\delta}_{ML}$ and $\hat{\theta}_{ML}$, respectively. Thus

$$\hat{R}_{ML}(t) = (1 - e^{-\hat{\delta}_{ML}/t})^{\hat{\theta}_{ML}}$$

and

$$\hat{h}_{ML}(t) = \hat{\delta}_{ML}\hat{\theta}_{ML}t^{-2}(e^{\hat{\delta}_{ML}/t} - 1)^{-1}.$$

For more details about the reliability characteristics estimations, one may refer to [26–29].

3. Bayesian Estimation

This section discusses the Bayesian estimation of the parameters δ and θ and the reliability indices under the SEL function. It is interesting to note that a joint conjugate prior for the parameters does not exist when the two parameters are unknown; for further information, see [9]. As a result, we investigate the BEs using independent gamma priors for both δ and θ with the joint prior distribution as below.

$$g(\delta, \theta) \propto \delta^{\tau_1 - 1}\theta^{\tau_2 - 1}e^{-(\nu_1\delta + \nu_2\theta)}, \tau_j, \nu_j > 0, j = 1, 2. \tag{10}$$

Here, we choose to use gamma priors due to its mathematical flexibility and its ability to cover a wide variety of prior beliefs of the experimenter. The joint posterior distribution of δ and θ can be derived from (4) and (9) as follows

$$\begin{aligned} g(\delta, \theta|\underline{x}) = {} & A^{-1}\delta^{m+\tau_1-1}\theta^{m+\tau_2-1}\exp\left\{-\left[\textstyle\sum_{i=1}^{m}(\delta/x_i) + \log(\xi(x_i;\delta))\right] - \nu_1\delta\right\} \\ & \times \exp\left\{-\theta\left[-\textstyle\sum_{i=1}^{m}\log(\xi(x_i;\delta)) - (n-m)\log(\xi(x_m;\delta)) + \nu_2\right]\right\}, \end{aligned} \tag{11}$$

where $\underline{x} = (x_1, \ldots, x_m)$ and A is the normalized constant. From (11), the BE under SEL function is given by the posterior mean. However, due to the complex form of the joint posterior distribution in (11), the BEs of the unknown parameters are obtained in the form of a ratio of two-dimensional integrals for which a closed-form solution is not easily tractable due to implicit mathematical expressions. Because of that, we propose to use two approaches, namely Lindley and MCMC methods to approximate the BEs from (11) as in the following subsections.

3.1. Bayesian Estimation Using Lindley Approximation

Following the approach proposed by [30], we can obtain approximate explicit BEs containing no integrals. Consider the ratio of integral in the form

$$\Lambda(X) = \frac{\int_{(\delta,\theta)} \varphi(\delta,\theta)\exp[\ell(\delta,\theta) + \psi(\delta,\theta)]d(\delta,\theta)}{\int_{(\delta,\theta)} \exp[\ell(\delta,\theta) + \psi(\delta,\theta)]d(\delta,\theta)} \tag{12}$$

where $\varphi(\delta,\theta)$ is any function in δ and θ, $\ell(\delta,\theta) = \log L(\delta,\theta)$ and $\psi(\delta,\theta) = \log g(\delta,\theta)$. Applying Lindley's approximation, this ratio of integrals defined in (12) can be rewritten as

$$\begin{aligned} \Lambda(X) &= \varphi(\delta,\theta) + 0.5\left[(\varphi_{\delta\delta} + 2\varphi_\delta\psi_\delta)\sigma_{\delta\delta} + (\varphi_{\theta\delta} + 2\varphi_\theta\psi_\delta)\sigma_{\theta\delta} + (\varphi_{\delta\theta} + 2\varphi_\delta\psi_\theta)\sigma_{\delta\theta}\right. \\ &+ \left.(\varphi_{\theta\theta} + 2\varphi_\theta\psi_\theta)\sigma_{\theta\theta}\right] + 0.5\left[(\varphi_\delta\sigma_{\delta\delta} + \varphi_\theta\sigma_{\delta\theta})(\ell_{\delta\delta\delta}\sigma_{\delta\delta} + \ell_{\delta\theta\delta}\sigma_{\delta\theta} + \ell_{\theta\delta\delta}\sigma_{\theta\delta}\right. \\ &+ \left.\ell_{\theta\theta\delta}\sigma_{\theta\theta}) + (\varphi_\delta\sigma_{\theta\delta} + \varphi_\theta\sigma_{\theta\theta})(\ell_{\theta\delta\delta}\sigma_{\delta\delta} + \ell_{\delta\theta\theta}\sigma_{\delta\theta} + \ell_{\theta\delta\theta}\sigma_{\theta\delta} + \ell_{\theta\theta\theta}\sigma_{\theta\theta})\right], \end{aligned} \tag{13}$$

where φ_δ, φ_θ, $\varphi_{\delta\theta}$, $\varphi_{\theta\delta}$, $\varphi_{\delta\delta}$, and $\varphi_{\theta\theta}$ are the derivatives of $\varphi(\delta,\theta)$. All terms in (13) are evaluated at the MLEs $\hat{\delta} = \hat{\delta}_{ML}$ and $\hat{\theta} = \hat{\theta}_{ML}$ of δ and θ, respectively. The other quantities in (13) are obtained as

$$\ell_{\delta\delta} = -\frac{m}{\delta^2} - (\theta - 1)\sum_{i=1}^{m}\frac{e^{-\delta/x_i}}{x_i^2\xi^2(x_i;\delta)} - \theta(n-m)\frac{e^{-\delta/x_m}}{x_m^2\xi^2(x_m;\delta)},$$

$$\ell_{\theta\theta} = -\frac{m}{\theta^2},\ \ell_{\delta\theta} = \ell_{\theta\delta} = \sum_{i=1}^{m} \frac{e^{-\delta/x_i}}{x_i\xi(x_i;\delta)} + (n-m)\frac{e^{-\delta/x_m}}{x_m\xi(x_m;\delta)},$$

$$\ell_{\delta\delta\delta} = \frac{2m}{\delta^3} + (\theta-1)\sum_{i=1}^{m} \frac{e^{-\delta/x_i}(1+e^{-\delta/x_i})}{x_i^3\xi^3(x_i;\delta)} + \theta(n-m)\frac{e^{-\delta/x_m}(1+e^{-\delta/x_m})}{x_m^3\xi^3(x_m;\delta)},$$

$$\ell_{\theta\theta\theta} = \frac{2m}{\theta^3},\ \ell_{\theta\delta\delta} = \ell_{\delta\delta\theta} = \ell_{\delta\theta\delta} = -\sum_{i=1}^{m} \frac{e^{-\delta/x_i}}{x_i^2\xi^2 x_i;\delta)} - (n-m)\frac{e^{-\delta/x_m}}{x_m^2\xi^2(x_m;\delta)},$$

and

$$\ell_{\delta\theta\theta} = \ell_{\theta\theta\delta} = \ell_{\theta\delta\theta} = 0.$$

Additionally, $\sigma_{\delta\delta}, \sigma_{\theta\theta}$, and $\sigma_{\delta\theta} = \sigma_{\theta\delta}$ are obtained as

$$\sigma_{\delta\delta} = \frac{-\ell_{\theta\theta}}{\ell_{\delta\delta}\ell_{\theta\theta} - \ell_{\delta\theta}^2},\ \sigma_{\theta\theta} = \frac{-\ell_{\delta\delta}}{\ell_{\delta\delta}\ell_{\theta\theta} - \ell_{\delta\theta}^2} \text{ and } \sigma_{\delta\theta} = \frac{\ell_{\delta\theta}}{\ell_{\delta\delta}\ell_{\theta\theta} - \ell_{\delta\theta}^2}.$$

In addition, from the prior distribution in (10) we have

$$\psi(\delta,\theta) = \log[g(\delta,\theta)] = (\tau_1-1)\log(\delta) + (\tau_2-1)\log(\theta) - (\nu_1\delta + \nu_2\theta).$$

Thus

$$\psi_\delta = \frac{\tau_1-1}{\delta} - \nu_1 \text{ and } \psi_\theta = \frac{\tau_2-1}{\theta} - \nu_2. \tag{14}$$

Now, when $\varphi(\delta,\theta) = \delta$, hence $\varphi_\delta = 1$ and $\varphi_\theta = \varphi_{\delta\delta} = \varphi_{\theta\theta} = \varphi_{\delta\theta} = \varphi_{\theta\delta} = 0$. Then we can obtain the BE of δ as follows

$$\tilde{\delta}_{BL} = \hat{\delta} + \psi_\delta\sigma_{\delta\delta} + \psi_\theta\sigma_{\delta\theta} + \frac{1}{2}B_1, \tag{15}$$

where $B_1 = \sigma_{\delta\delta}^2\ell_{\delta\delta\delta} + 3\sigma_{\delta\delta}\sigma_{\delta\theta}\ell_{\delta\delta\theta} + \sigma_{\theta\delta}\sigma_{\theta\theta}\ell_{\theta\theta\theta}$. Similarly, for $\varphi(\delta,\theta) = \theta$ and $\varphi_\theta = 1$, $\varphi_\delta = \varphi_{\delta\delta} = \varphi_{\theta\theta} = \varphi_{\delta\theta} = \varphi_{\theta\delta} = 0$, we can obtain the BE of θ as follows

$$\tilde{\theta}_{BL} = \hat{\theta} + \psi_\delta\sigma_{\theta\delta} + \psi_\theta\sigma_{\theta\theta} + \frac{1}{2}B_2, \tag{16}$$

where $B_2 = \sigma_{\delta\delta}\sigma_{\theta\theta}\ell_{\delta\delta\delta} + 2\sigma_{\delta\theta}^2\ell_{\delta\delta\theta} + \sigma_{\theta\theta}\sigma_{\delta\delta}\ell_{\theta\delta\delta} + \sigma_{\theta\theta}^2\ell_{\theta\theta\theta}$.

Using the same approach we can obtain the BE of the RF by considering $\varphi(\delta,\theta) = [\xi(t;\delta)]^\theta$ and in this case we have the following quantities

$$\varphi_\delta = \theta t^{-1}e^{-\delta/t}[\xi(t;\delta)]^{\theta-1},\ \varphi_\theta = [\xi(t;\delta)]^\theta\log(\xi(t;\delta)),\ \varphi_{\theta\theta} = [\xi(t;\delta)]^\theta[\log(\xi(t;\delta))]^2,$$

$$\varphi_{\delta\delta} = \theta t^{-2}e^{-\delta/t}[\xi(t;\delta)]^{\theta-1}[(\theta-1)e^{-\delta/t} - 1]$$

and

$$\varphi_{\delta\theta} = \varphi_{\theta\delta} = t^{-1}e^{-\delta/t}[\xi(t;\delta)]^{\theta-1}[1 + \theta\log(\xi(t;\delta))].$$

Thus, the BE of the RF can be obtained as

$$\tilde{R}_{BL}(t) = \hat{R}(t) + B_3 + \frac{1}{2}B_4, \tag{17}$$

where $\hat{R}_{ML} = \hat{R}_{ML}(t)$, $\xi(t;\delta) = 1 - e^{-\delta/t},\ t > 0$, $B_3 = \varphi_\delta\psi_\delta\sigma_{\delta\delta} + \varphi_\theta\psi_\delta\sigma_{\theta\delta} + \varphi_\delta\psi_\theta\sigma_{\delta\theta} + \varphi_\theta\psi_\theta\sigma_{\theta\theta}$ and

$$\begin{aligned} B_4 = {} & \varphi_{\delta\delta}\sigma_{\delta\delta} + 2\varphi_{\delta\theta}\sigma_{\delta\theta} + \varphi_{\theta\theta}\sigma_{\theta\theta} + (\varphi_\delta\sigma_{\delta\delta} + \varphi_\theta\sigma_{\delta\theta})(\ell_{\delta\delta\delta}\sigma_{\delta\delta} + \ell_{\delta\theta\delta}\sigma_{\delta\theta} + \ell_{\theta\delta\delta}\sigma_{\theta\delta}) \\ & + (\varphi_\delta\sigma_{\theta\delta} + \varphi_\theta\sigma_{\theta\theta})(\ell_{\theta\delta\delta}\sigma_{\delta\delta} + \ell_{\theta\theta\theta}\sigma_{\theta\theta}). \end{aligned}$$

Let $\varphi(\delta, \theta) = \delta\theta t^{-2}(e^{\delta/t} - 1)^{-1}$, then we have

$$\varphi_\delta = \theta t^{-3}(e^{\delta/t} - 1)^{-2}[(t-\delta)e^{\delta/t} - t],\ \varphi_\theta = \delta t^{-2}(e^{\delta/t} - 1)^{-1},\ \varphi_{\theta\theta} = 0,$$

$$\varphi_{\delta\delta} = \theta t^{-4} e^{\delta/t}(e^{\delta/t} - 1)^{-3}[\delta(1 + e^{\delta/t}) - 2t(e^{\delta/t} - 1)],$$

and

$$\varphi_{\delta\theta} = \varphi_{\theta\delta} = t^{-3}(e^{\delta/t} - 1)^{-2}[e^{\delta/t}(t-\delta) - t].$$

Therefore, the BE of the HRF can be obtained as

$$\tilde{h}_{BL}(t) \quad = \quad \hat{h}(t) + B_5 + \frac{1}{2}B_6, \tag{18}$$

where $\hat{h}_{ML} = \hat{h}_{ML}(t)$,

$$B_5 = \varphi_\delta \psi_\delta \sigma_{\delta\delta} + \varphi_\theta \psi_\delta \sigma_{\theta\delta} + \varphi_\delta \psi_\theta \sigma_{\delta\theta} + \varphi_\theta \psi_\theta \sigma_{\theta\theta}$$

and

$$\begin{aligned} B_6 = {} & \varphi_{\delta\delta}\sigma_{\delta\delta} + 2\varphi_{\delta\theta}\sigma_{\delta\theta} + (\varphi_\delta\sigma_{\delta\delta} + \varphi_\theta\sigma_{\delta\theta})(\ell_{\delta\delta\delta}\sigma_{\delta\delta} + \ell_{\delta\theta\delta}\sigma_{\delta\theta} + \ell_{\theta\delta\delta}\sigma_{\theta\delta}) \\ & + (\varphi_\delta\sigma_{\theta\delta} + \varphi_\theta\sigma_{\theta\theta})(\ell_{\theta\delta\delta}\sigma_{\delta\delta} + \ell_{\theta\theta\theta}\sigma_{\theta\theta}). \end{aligned}$$

3.2. Bayesian Estimation Using MCMC

In this subsection, we propose to use the MCMC method to obtain the BEs of δ and θ as well as $R(t)$ and $h(t)$. The MCMC method can be used to simulate samples from (11) and in turn to obtain the BEs. To generate samples from (11), we need to sample successively from a target distribution. Therefore, to implement the MCMC procedure, we revise the posterior distribution in (11) as follows

$$g(\delta, \theta|\underline{x}) \quad \propto \quad g(\delta|\theta, \underline{x})\, g(\theta|\delta, \underline{x}), \tag{19}$$

where $g(\delta|\theta, \underline{x})$ is the conditional distribution of δ given θ and data and can be written in the form

$$\begin{aligned} g(\delta|\theta, \underline{x}) \propto \delta^{m+\tau_1-1} & \exp\Big\{-\Big[\textstyle\sum_{i=1}^{m}(\delta/x_i) + \log(\xi(x_i;\delta))\Big] - \nu_1\delta\Big\} \\ & \times \exp\Big\{-\theta\Big[-\textstyle\sum_{i=1}^{m}\log(\xi(x_i;\delta)) - (n-m)\log(\xi(x_m;\delta))\Big]\Big\} \end{aligned} \tag{20}$$

and $g(\theta|\delta, \underline{x})$ is the conditional distribution of θ given δ and data in the form

$$g(\theta|\delta, \underline{x}) \propto \theta^{m+\tau_2-1} \exp\Big\{-\theta\Big[-\textstyle\sum_{i=1}^{m}\log(\xi(x_i;\delta)) - (n-m)\log(\xi(x_m;\delta)) + \nu_2\Big]\Big\}. \tag{21}$$

It can be seen from (21) that $g(\theta|\delta, \underline{x})$ is gamma density, therefore, samples of θ can be easily generated by considering any gamma generating routine. On the other hand, the conditional distribution of δ given by (20) cannot be reduced to any well-known distributions. Therefore, following [31], we have employed a hybrid strategy combining the Metropolis–Hasting (M–H) algorithm with the Gibbs algorithm for obtaining the samples from the posterior distributions to develop the BEs and construct the associated credible intervals. One can also consider the Hamiltonian Monte Carlo method, instead of the Gibbs sampling, which uses an approximate Hamiltonian dynamics simulation based on numerical integration which is then corrected by performing a Metropolis acceptance step, for details see [32]. The hybrid M–H steps (for updating δ) within Gibbs steps (for updating θ) are carried out using the following steps:

Step 1: Set $j = 1$

Step 2: Generate $\delta^{(j)}$ from (20) using M-H steps with $N(\hat{\delta}_{ML}, \sigma_{\delta\delta})$ as a proposal distribution.

Step 3: Generate $\theta^{(j)}$ from (21).

Step 4: Set $j = j + 1$.

Step 5: Redo steps 2–4, M times, and obtain $(\delta^{(j)}, \theta^{(j)})$, $j = 1, 2, \ldots, M$.

Step 6: Get the BEs of δ, θ, $R(t)$ and $h(t)$ at distinct time $t > 0$ under SEL function.

To remove the affection of an initial guess value and also to guarantee the sampling convergence, the first simulated varieties, say M_0, are discarded in the beginning of the analysis implementation (burn-in period) and the remaining samples can be further utilized to carried out the BEs. Then, for sufficiently large M, the drawn Bayes MCMC samples of the unknown parameters δ, θ, $R(t)$ and $h(t)$ as in $\vartheta^{(j)} = (\delta^{(j)}, \theta^{(j)}, R^{(j)}(t), h^{(j)}(t))$, $j = M_0 + 1$, $M_0 + 2, \ldots, M$ can be used to develop the Bayesian inference. Thus, the approximate BEs of ϑ based on SEL function is given by

$$\tilde{\vartheta}_{BM} = \sum_{j=M_0+1}^{M} \vartheta^{(j)} / (M - M_0)\ .$$

4. E-Bayesian Estimation

In this section, we obtain the EBEs of $\delta, \theta, R(t)$, and $h(t)$. Reference [15] proposed the E-Bayesian estimation to estimate the failure rate of the exponential distribution using SEL function. He also investigated some properties of the E-Bayesian estimation. In the classical Bayesian estimation, the values of the hyper-parameters are considered to be constants and determined through experience or arbitrary by the researcher. On the other hand, the E-Bayesian estimation method assumes that these hyper-parameters are random variables and have prior distributions. Therefore, the main advantage of the E-Bayesian estimation over the classical Bayesian estimation is that it takes the expectation of the usual BE over the hyper-parameters to take into account all possible values of the hyper-parameters. Let $\tilde{\mu}_B(a, b)$ be the BE of the unknown parameter μ, where a and b are the hyper-parameters. Then, the EBE of the parameter μ can be obtained as follows

$$\hat{\mu}_{EB} = \int\int_D \tilde{\mu}_B(a, b)\, \pi(a, b) da\, db,$$

where D is the domain of a and b, and $\pi(a, b)$ is the prior distributions of the hyper-parameters a and b. According to [15], the prior distributions of the hyper-parameters should be selected to guarantee that the prior distributions are decreasing functions in the parameters. In our case, we obtain the first derivative of the prior distribution of the parameter δ as

$$\frac{dg(\delta)}{d\delta} = \frac{\nu_1^{\tau_1}}{\Gamma(\tau_1)} \delta^{\tau_1 - 2} e^{-\nu_1 \delta} [(\tau_1 - 1) - \nu_1 \delta]. \tag{22}$$

Similarly, the first derivative of the prior distribution of the parameter θ is given by

$$\frac{dg(\theta)}{d\theta} = \frac{\nu_2^{\tau_2}}{\Gamma(\tau_2)} \theta^{\tau_2 - 2} e^{-\nu_2 \theta} [(\tau_2 - 1) - \nu_2 \theta], \tag{23}$$

where $g(\delta)$ and $g(\theta)$ are the prior distributions of the parameters δ and θ, respectively. From (22) and (23), we can observe that when $0 < \tau_1 < 1, \nu_1 > 0$ and $0 < \tau_2 < 1, \nu_2 > 0$, the derivatives $\frac{dg(\delta)}{d\delta}$ and $\frac{dg(\theta)}{d\theta}$ are less than zero, therefore, $g(\delta)$ and $g(\theta)$ are decreasing functions in δ and θ, respectively. Based on these results, we obtain the EBEs by considering three different prior distributions for the hyper-parameters to show the influence of these distributions on the results of E-Bayesian estimations. For the parameter δ, we consider using the following three prior distributions for the hyper-parameters τ_1 and ν_1

$$\left.\begin{array}{lll}\phi_1(\tau_1,\nu_1)=\frac{\tau_1^{s_1-1}(1-\tau_1)^{s_2-1}}{c_1B(s_1,s_2)}, & 0<\tau_1<1, & 0<\nu_1<c_1\\ \phi_2(\tau_1,\nu_1)=\frac{2\nu_1\tau_1^{s_1-1}(1-\tau_1)^{s_2-1}}{c_1^2B(s_1,s_2)}, & 0<\tau_1<1, & 0<\nu_1<c_1\\ \phi_3(\tau_1,\nu_1)=\frac{3\nu_1^2\tau_1^{s_1-1}(1-\tau_1)^{s_2-1}}{c_1^3B(s_1,s_2)}, & 0<\tau_1<1, & 0<\nu_1<c_1\end{array}\right\}, \tag{24}$$

where $B(.,.)$ is the beta function. Additionally, for the parameter θ, the following three prior distributions for the hyper-parameters τ_2 and ν_2 are considered

$$\left.\begin{array}{lll}\pi_1(\tau_2,\nu_2)=\frac{\tau_2^{z_1-1}(1-\tau_2)^{z_2-1}}{c_2B(z_1,z_2)}, & 0<\tau_2<1, & 0<\nu_2<c_2\\ \pi_2(\tau_2,\nu_2)=\frac{2\nu_2\tau_2^{z_1-1}(1-\tau_2)^{z_2-1}}{c_2^2B(z_1,z_2)}, & 0<\tau_2<1, & 0<\nu_2<c_2\\ \pi_3(\tau_2,\nu_2)=\frac{3\nu_2^2\tau_2^{z_1-1}(1-\tau_2)^{z_2-1}}{c_2^3B(z_1,z_2)}, & 0<\tau_2<1, & 0<\nu_2<c_2\end{array}\right\}. \tag{25}$$

For more details about the E-Bayesian estimation, one can refer to [16,19,20,33,34]. The EBEs of δ and θ are, respectively, obtained as follows

$$\hat{\delta}_{EB}=\int\int_D \tilde{\delta}_B(\tau_1,\nu_1)\,\phi_k(\tau_1,\nu_1)\,d\tau_1\,d\nu_1,\ k=1,2,3, \tag{26}$$

and

$$\hat{\theta}_{EB}=\int\int_D \tilde{\theta}_B(\tau_2,\nu_2)\,\pi_k(\tau_2,\nu_2)\,d\tau_2\,d\nu_2,\ k=1,2,3, \tag{27}$$

where $\tilde{\delta}_B(\tau_1,\nu_1)$ and $\tilde{\theta}_B(\tau_2,\nu_2)$ are the BEs of δ and θ, respectively. Using the same approach, we can obtain the EBEs of the RF and HRF. Since the BEs $\tilde{\delta}_B(\tau_1,\nu_1)$ and $\tilde{\theta}_B(\tau_2,\nu_2)$ have no closed form, then it is not easy to obtain the EBEs in (26) and (27). To overcome this problem, we propose to use the Lindley approximation and MCMC methods in the next subsections to obtain the EBEs of δ, θ, $R(t)$ and $h(t)$.

4.1. E-Bayesian Estimation Using Lindley Approximation

To obtain the EBEs, we need to obtain the expected values of ψ_δ and ψ_θ given by (14) over the prior distributions of the hyper-parameters given by (24) and (25), respectively. Then, using (14) and (24) we have the following results:

(1) For the prior distribution $\phi_1(\tau_1,\nu_1)$

$$\begin{aligned}\psi_{\delta.1} &= \int_0^{c_1}\int_0^1 \psi_\delta\,\phi_1(\tau_1,\nu_1)\,d\tau_1\,d\nu_1\\ &= \int_0^{c_1}\int_0^1\left(\frac{\tau_1-1}{\delta}-\nu_1\right)\frac{\tau_1^{s_1-1}(1-\tau_1)^{s_2-1}}{c_1B(s_1,s_2)}\,d\tau_1\,d\nu_1\\ &= \frac{1}{\delta}\left(\frac{s_1}{s_1+s_2}-\frac{c_1\delta}{2}-1\right).\end{aligned} \tag{28}$$

(2) For the prior distribution $\phi_2(\tau_1,\nu_1)$

$$\begin{aligned}\psi_{\delta.2} &= \int_0^{c_1}\int_0^1 \psi_\delta\,\phi_2(\tau_1,\nu_1)\,d\tau_1\,d\nu_1\\ &= \int_0^{c_1}\int_0^1\left(\frac{\tau_1-1}{\delta}-\nu_1\right)\frac{2\nu_1\tau_1^{s_1-1}(1-\tau_1)^{s_2-1}}{c_1^2B(s_1,s_2)}\,d\tau_1\,d\nu_1\\ &= \frac{1}{\delta}\left(\frac{s_1}{s_1+s_2}-\frac{2c_1\delta}{3}-1\right).\end{aligned} \tag{29}$$

(3) For the prior distribution $\phi_3(\tau_1, \nu_1)$

$$\begin{aligned}
\psi_{\delta.3} &= \int_0^{c_1}\int_0^1 \psi_\delta\, \phi_3(\tau_1, \nu_1)\, d\tau_1\, d\nu_1 \\
&= \int_0^{c_1}\int_0^1 \left(\frac{\tau_1 - 1}{\delta} - \nu_1\right) \frac{3\nu_1^2 \tau_1^{s_1-1}(1-\tau_1)^{s_2-1}}{c_1^3 B(s_1, s_2)}\, d\tau_1\, d\nu_1 \\
&= \frac{1}{\delta}\left(\frac{s_1}{s_1+s_2} - \frac{3c_1\delta}{4} - 1\right). \qquad (30)
\end{aligned}$$

Similarly, from (14) and (25) we can obtain the following results:
(1) For the prior distribution $\pi_1(\tau_2, \nu_2)$

$$\begin{aligned}
\psi_{\theta.1} &= \int_0^{c_2}\int_0^1 \psi_\theta\, \pi_1(\tau_2, \nu_2)\, d\tau_2\, d\nu_2 \\
&= \int_0^{c_2}\int_0^1 \left(\frac{\tau_2 - 1}{\theta} - \nu_2\right) \frac{\tau_2^{z_1-1}(1-\tau_2)^{z_2-1}}{c_2 B(z_1, z_2)}\, d\tau_2\, d\nu_2 \\
&= \frac{1}{\theta}\left(\frac{z_1}{z_1+z_2} - \frac{c_2\theta}{2} - 1\right). \qquad (31)
\end{aligned}$$

(2) For the prior distribution $\pi_2(\tau_2, \nu_2)$

$$\begin{aligned}
\psi_{\theta.2} &= \int_0^{c_2}\int_0^1 \psi_\theta\, \pi_2(\tau_2, \nu_2)\, d\tau_2\, d\nu_2 \\
&= \int_0^{c_2}\int_0^1 \left(\frac{\tau_2 - 1}{\theta} - \nu_2\right) \frac{2\nu_2 \tau_2^{z_1-1}(1-\tau_2)^{z_2-1}}{c_2^2 B(z_1, z_2)}\, d\tau_2\, d\nu_2 \\
&= \frac{1}{\theta}\left(\frac{z_1}{z_1+z_2} - \frac{2c_2\theta}{3} - 1\right). \qquad (32)
\end{aligned}$$

(3) For the prior distribution $\pi_3(\tau_2, \nu_2)$

$$\begin{aligned}
\psi_{\theta.3} &= \int_0^{c_2}\int_0^1 \psi_\theta\, \pi_3(\tau_2, \nu_2)\, d\tau_2\, d\nu_2 \\
&= \int_0^{1}\int_0^{c_2} \left(\frac{\tau_2 - 1}{\theta} - \nu_2\right) \frac{3\nu_2^2 \tau_2^{z_1-1}(1-\tau_2)^{z_2-1}}{c_2^3 B(z_1, z_2)}\, d\tau_2\, d\nu_2 \\
&= \frac{1}{\theta}\left(\frac{z_1}{z_1+z_2} - \frac{3c_2\theta}{4} - 1\right). \qquad (33)
\end{aligned}$$

Now, the different EBEs of the parameter δ can be obtained using Lindely approximation by using (15) and the results in (28)–(33) as follows

$$\tilde{\delta}_{EBL.k} = \hat{\delta} + \psi_{\delta.k}\sigma_{\delta\delta} + \psi_{\theta.k}\sigma_{\delta\theta} + \frac{1}{2}B_1,\ k = 1, 2, 3. \qquad (34)$$

From (16) and the results in (28)–(32), we can obtain the EBEs of θ as

$$\tilde{\theta}_{EBL.k} = \hat{\theta} + \psi_{\delta.k}\sigma_{\theta\delta} + \psi_{\theta.k}\sigma_{\theta\theta} + \frac{1}{2}B_2,\ k = 1, 2, 3. \qquad (35)$$

Similarly, we can obtain the EBEs of the RF from (17) and (28)–(32) as follows

$$\tilde{R}_{EBL.k}(t) = \hat{R}(t) + B_{3.k} + \frac{1}{2}B_4,\ k = 1, 2, 3, \qquad (36)$$

where $B_{3.k} = \varphi_\delta \psi_{\delta.k} \sigma_{\delta\delta} + \varphi_\theta \psi_{\delta.k} \sigma_{\theta\delta} + \varphi_\delta \psi_{\theta.k} \sigma_{\delta\theta} + \varphi_\theta \psi_{\theta.k} \sigma_{\theta\theta}$. Additionally, the EBEs of the HRF can be obtained from (18) and (28)–(32) as follow

$$\tilde{h}_{EBL.k}(t) \quad = \quad \hat{h}(t) + B_{5.k} + \frac{1}{2}B_6, \; k = 1,2,3, \tag{37}$$

where $B_{5.k} = \varphi_\delta \psi_{\delta.k} \sigma_{\delta\delta} + \varphi_\theta \psi_{\delta.k} \sigma_{\theta\delta} + \varphi_\delta \psi_{\theta.k} \sigma_{\delta\theta} + \varphi_\theta \psi_{\theta.k} \sigma_{\theta\theta}$.

4.2. E-Bayesian Estimation Using MCMC

Here, the MCMC method is used to obtain the EBEs of $\delta, \theta, R(t)$, and $h(t)$. The MCMC method is used to generate samples of δ and θ based on the full conditional distributions given by (20) and (21) and the prior distributions of the hyper-parameters in (24) and (25). The EBEs are obtained according to the following steps

Step 1: Set the initial values of δ and θ as $(\delta^{(0)}, \theta^{(0)}) = (\hat{\delta}, \hat{\theta})$.

Step 2: Determine the values of c_1, c_2, s_1, s_2, z_1 and z_2.

Step 3: Set $j = 1$

Step 4: Generate $\tau_1^{(j)}$ and $\nu_1^{(j)}$ from (24).

Step 5: Generate $\tau_2^{(j)}$ and $\nu_2^{(j)}$ from (25).

Step 6: Generate $\delta^{(j)}$ from (20) using M-H steps with $N(\hat{\delta}, \sigma_{\delta\delta})$.

Step 7: Generate $\theta^{(j)}$ from (21).

Step 8: Set $j = j + 1$.

Step 9: Redo steps 3–8, M times, and obtain $(\delta^{(j)}, \theta^{(j)}), j = 1, 2, \ldots, M$.

Step 10: Get the EBEs of ϑ, where $\vartheta = (\delta, \theta, R(t), h(t))$, under SEL function as

$$\tilde{\vartheta}_{EBM.k} = \textstyle\sum_{j=M_0+1}^{M} \vartheta_k^{(j)} / (M - M_0), \; k = 1,2,3,$$

where M_0 is burn-in.$\vartheta^{(j)} = (\delta^{(j)}, \theta^{(j)}, R^{(j)}(t), h^{(j)}(t)), \; j = M_0 + 1, M_0 + 2, \ldots, M$

5. Hyper-Parameter Value Selection

In the Bayesian paradigm, the elicitation procedure used to determine the hyper-parameter value, when an informative prior of the density parameter is taken into account, is the main issue. This problem has been discussed in the literature [35,36]. Moreover, the value of hyper-parameters for the unknown parameter under interest is made by assuming two independent information namely prior mean and prior variance of the model parameters under consideration. In this regard, we propose the following steps of the past samples algorithm to select the values of hyper-parameters (τ_1, ν_1) and (τ_2, ν_2) of δ and θ, respectively, as

Step 1: Set the parameter value of δ and θ.

Step 2: Set $j = 1$

Step 3: Draw a random sample of size n from GIED.

Step 4: Compute the MLEs $\hat{\delta}$ and $\hat{\theta}$ of δ and θ, respectively.

Step 5: Set $j = 1$.

Step 6: Repeat steps 2–5 M times to get $\hat{\delta}^{(j)}$ and $\hat{\theta}^{(j)}$ for $j = 1, 2, \ldots, M$.

Step 7: Equating the mean and the variance of $\hat{\delta}^{(j)}$ and $\hat{\theta}^{(j)}$ for $j = 1, 2, \ldots, M$ to the mean and variance of the corresponding gamma density priors, respectively, as

$$\frac{1}{M} \textstyle\sum_{j=1}^{M} \hat{\delta}^{(j)} = \frac{\tau_1}{\nu_1} \text{ and } \frac{1}{M-1} \textstyle\sum_{j=1}^{M} \left(\hat{\delta}^{(j)} - \frac{1}{M} \textstyle\sum_{j=1}^{M} \hat{\delta}^{(j)} \right)^2 = \frac{\tau_1}{{\nu_1}^2}, \tag{38}$$

and
$$\frac{1}{M}\sum_{j=1}^{M}\hat{\theta}^{(j)}=\frac{\tau_2}{\nu_2}\text{ and }\frac{1}{M-1}\sum_{j=1}^{M}\left(\hat{\theta}^{(j)}-\frac{1}{M}\sum_{j=1}^{M}\hat{\theta}^{(j)}\right)^2=\frac{\tau_2}{{\nu_2}^2}. \tag{39}$$

Step 8: Solving (38), the estimated hyper-parameter values $\widehat{\tau}_1$ and $\widehat{\nu}_1$ of τ_1 and ν_1 for δ can obtained respectively by

$$\widehat{\tau}_1=\frac{\left(\frac{1}{M}\sum_{j=1}^{M}\hat{\delta}^{(j)}\right)^2}{\frac{1}{M-1}\sum_{j=1}^{M}\left(\hat{\delta}^{(j)}-M^{-1}\sum_{j=1}^{M}\hat{\delta}^{(j)}\right)^2},\text{ and }\widehat{\nu}_1=\frac{\frac{1}{M}\sum_{j=1}^{M}\hat{\delta}^{(j)}}{\frac{1}{M-1}\sum_{j=1}^{M}\left(\hat{\delta}^{(j)}-M^{-1}\sum_{j=1}^{M}\hat{\delta}^{(j)}\right)^2}.$$

Step 9: Solving (39), the estimated hyper-parameter values $\widehat{\tau}_2$ and $\widehat{\nu}_2$ of τ_2 and ν_2 for θ can obtained respectively by

$$\widehat{\tau}_2=\frac{\left(\frac{1}{M}\sum_{j=1}^{M}\hat{\theta}^{(j)}\right)^2}{\frac{1}{M-1}\sum_{j=1}^{M}\left(\hat{\theta}^{(j)}-M^{-1}\sum_{j=1}^{M}\hat{\theta}^{(j)}\right)^2},\text{ and }\widehat{\nu}_2=\frac{\frac{1}{M}\sum_{j=1}^{M}\hat{\theta}^{(j)}}{\frac{1}{M-1}\sum_{j=1}^{M}\left(\hat{\theta}^{(j)}-M^{-1}\sum_{j=1}^{M}\hat{\theta}^{(j)}\right)^2}.$$

Similarly, one can use the same above steps of the past samples algorithm to determine the value of priors s_i and $z_i, i=1,2$, respectively.

6. Interval Estimation

In this section, we propose to use the asymptotic properties of the MLEs of δ and θ, or any function of them such as $R(t)$ and $h(t)$ in order to construct associated confidence intervals. Further, using the MCMC simulated varieties of the BEs and EBEs for δ, θ, $R(t)$, and $h(t)$, the associated credible intervals are constructed.

6.1. Asymptotic Confidence Intervals

The $100(1-\kappa)\%$ ACIs for ϑ, where $\vartheta=(\delta,\theta,R(t),h(t))$, can be estimated using the observed Fisher information matrix $\mathbf{I}^{-1}(\cdot)$, which is defined as the inverse of the matrix of second partial derivatives of (4) with respect to δ and θ locally at their MLEs $\hat{\delta}$ and $\hat{\theta}$, see [37], as
$$\mathbf{I}^{-1}(\hat{\delta},\hat{\theta})\cong\begin{bmatrix}-\ell_{\delta\delta} & -\ell_{\delta\theta}\\ -\ell_{\theta\delta} & -\ell_{\theta\theta}\end{bmatrix}^{-1}=\begin{bmatrix}\hat{\sigma}_{\hat{\delta}\hat{\delta}} & \hat{\sigma}_{\hat{\delta}\hat{\theta}}\\ \hat{\sigma}_{\hat{\theta}\hat{\delta}} & \hat{\sigma}_{\hat{\theta}\hat{\theta}}\end{bmatrix}. \tag{40}$$

The second partial derivatives $\ell_{ij},\ i,j=1,2$, as in (40) are previously reported in Section 3.1. Under some mild regularity conditions, the MLEs $\hat{\delta}$ and $\hat{\theta}$ are approximately distributed as normal distribution $\hat{\delta}\sim N(\delta,\sigma_{\delta\delta})$ and $\hat{\theta}\sim N(\theta,\sigma_{\theta\theta})$, respectively. To construct the $100(1-\kappa)\%$ ACIs of the RF $R(t)$ and HRF $h(t)$, we need to approximate the variance estimate of them.

Thus, the delta method which is a general approach to obtain the approximate estimates of the variance associated with the MLEs of $R(t)$ and $h(t)$ is used for this purpose. However, according to delta method based on (40), the variance of $\hat{R}(t)$ and $\hat{h}(t)$ obtained at their MLEs $\hat{\delta}$ and $\hat{\theta}$ can be approximated, respectively by

$$\hat{\sigma}^2_{\hat{R}}=[\nabla\hat{R}]^{\top}\mathbf{I}^{-1}(\hat{\delta},\hat{\theta})[\nabla\hat{R}]|_{(\delta,\theta)=(\hat{\delta},\hat{\theta})}$$

and
$$\hat{\sigma}^2_{\hat{h}}=[\nabla\hat{h}]^{\top}\mathbf{I}^{-1}(\hat{\delta},\hat{\theta})[\nabla\hat{h}]|_{(\delta,\theta)=(\hat{\delta},\hat{\theta})},$$

where $\nabla\hat{R}$ and $\nabla\hat{h}$ are the gradients of $R(t)$ and $h(t)$, respectively, with respect to δ and θ, i.e.,
$$[\nabla R]^{\top}=[\partial R(t)/\partial\alpha\ ,\partial R(t)/\partial\lambda\ ],$$

and

$$[\nabla h]^{\top} = [\partial h(t)/\partial \alpha \ , \partial h(t)/\partial \lambda \ ],$$

Hence, $100(1-\kappa)\%$ two-sided ACI for any function of δ and θ obtained based on their MLEs $\hat{\vartheta}$, where $\hat{\vartheta} = (\hat{\delta}, \hat{\theta}, \hat{R}(t), \hat{h}(t))$, are given by

$$\hat{\vartheta} \mp z_{\kappa/2}\sqrt{\hat{\sigma}^2_{\hat{\vartheta}}}$$

where $z_{\kappa/2}$ is the $(\kappa/2)$-th upper percentile of the standard normal distribution.

6.2. Bayes and E-Bayes Credible Intervals

To construct the corresponding Bayesian and E-Bayesian credible intervals of any function of the unknown model parameters δ and θ, the associated MCMC simulated varieties obtained in Sections 3.2 and 4.2 are used, respectively. To construct the BCIs of the parameters δ and θ as well as $R(t)$ and $h(t)$, order the simulated samples of Bayesian MCMC estimates $\vartheta^{(j)}$ for $j = 1, 2, \ldots, M$ after burn-in M_0 as $\vartheta_{(M_0+1)}, \vartheta_{(M_0+2)}, \ldots, \vartheta_{(M)}$. Hence, the $100(1-\kappa)\%$ two-sided BCIs of ϑ is given by

$$\left(\vartheta_{(M-M_0)(\kappa/2)}, \vartheta_{(M-M_0)(1-(\kappa/2))}\right).$$

Similarly, using the simulated samples of E-Bayesian MCMC estimates $\vartheta^{(j)}$ for $j = 1, 2, \ldots, M$ after burn-in M_0, the $100(1-\kappa)\%$ two-sided E-BCIs for δ, θ, $R(t)$ and $h(t)$ can be easily constructed.

7. Simulation Study

Various Monte Carlo simulations are applied to analyze the behavior of the proposed estimators produced in the previous sections, including the MLEs, Bayesian (Lindley's approximation and MCMC), and E-Bayesian (Lindley's approximation and MCMC), as well as the corresponding asymptotic/credible intervals. By considering two sets of parameter values, $(\delta, \theta) = (1, 1)$ and $(\delta, \theta) = (3, 3)$, we simulate a large 1000 Type-II censored samples for different combinations of n (number of total test units) and m (effective sample size) such as $n = 40$, 60, and 80, m is taken as failure proportion such as $m/n = 50$ and 80% for each n. The RF and HRF are evaluated when $(\delta, \theta) = (1, 1)$ at the distinct time $t = 0.25$, hence the corresponding actual values become 0.9816844 and 0.2985178, respectively. Similarly, when $(\delta, \theta) = (3, 3)$ for given time $t = 0.75$, the actual values of $R(t)$ and $h(t)$ are 0.9460533 and 0.2985178, respectively.

In the Bayesian paradigm, the choice of the hyper-parameter values is the main issue. For this propose, to assign values for the hyper-parameters $\tau_i, \nu_i,\ i = 1, 2$, of the gamma prior, we propose to use the procedure of past samples data described in Section 5. In this regard, we generate 1000 complete samples of size 30 (say) from the GIED as past samples for each plausible values of the unknown model parameters δ and θ. Consequently, the values of $\tau_i, \nu_i,\ i = 1, 2$, are $(\tau_1, \tau_2, \nu_1, \nu_2) = (12.326, 15.090, 10.990, 13.710)$ for $(\delta, \theta) = (1, 1)$ and $(\tau_1, \tau_2, \nu_1, \nu_2) = (24.926, 7.9651, 7.7901, 2.3014)$ for $(\delta, \theta) = (3, 3)$. In addition, to carry out the influence of the various prior parameters on the desired EBEs of δ and θ, we generate τ_1 and ν_1 for the parameter δ and τ_2 and ν_2 for the parameter θ from beta and uniform densities in (24) and (25), respectively. Based on the past samples and for fixed $c_i = 0.5$ the values of $s_i, z_i,\ i = 1, 2$, which are used to calculate the desired EBEs are $(s_1, s_2, z_1, z_2) = (8.5420, 7.3215, 2.7534, 1.9968)$. Using the hybrid strategy combining M–H with the Gibbs algorithm proposed in Sections 3.2 and 4.2, we generate 12,000 MCMC samples and discard the first 2000 samples from the generated sequence to remove the affection of the selection of the start value. Here, the initial values for running the MCMC sampler algorithm was taken to be the MLEs.

For each estimation issue, the average point estimates of the unknown parameters δ and θ as well as $R(t)$ and $h(t)$ with their RMSEs and RABs as well as the average CLs (ACLs) of the interval estimates are calculated using the following formulas

$$\overline{\hat{\vartheta}_\eta} = \frac{1}{Q}\sum\nolimits_{i=1}^{Q} \hat{\vartheta}_\eta^{(i)},\ \eta = 1,2,3,4, \tag{41}$$

$$\text{RMSE}(\hat{\vartheta}_\eta) = \sqrt{\frac{1}{Q}\sum\nolimits_{i=1}^{Q}\left(\hat{\vartheta}_\eta^{(i)} - \vartheta_\eta\right)^2},\ \eta = 1,2,3,4, \tag{42}$$

$$\text{RAB}(\hat{\vartheta}_\eta) = \frac{1}{Q}\sum\nolimits_{i=1}^{Q}\frac{\left|\hat{\vartheta}_\eta^{(i)} - \vartheta_\eta\right|}{\vartheta_\eta},\ \eta = 1,2,3,4, \tag{43}$$

and

$$\text{ACL}(\hat{\vartheta}_\eta) = \frac{1}{Q}\sum\nolimits_{i=1}^{Q}\left(\hat{\vartheta}_{\eta_U}^{(i)} - \hat{\vartheta}_{\eta_L}^{(i)}\right),\ \eta = 1,2,3,4, \tag{44}$$

where Q is the number of replicates, $\hat{\vartheta}$ is the MLE, BE, or EBE of the parameter ϑ and $(\hat{\vartheta}_{\eta_L}, \hat{\vartheta}_{\eta_U})$ denote the $100(1-\gamma)\%$ confidence interval bounds, where $\vartheta_1 = \delta$, $\vartheta_2 = \theta$, $\vartheta_3 = R(t)$ and $\vartheta_4 = h(t)$. The average estimates of δ, θ, $R(t)$ and $h(t)$ with their RMSEs and RABs are calculated and reported in Tables 1–4. Uniformly, each column in the Tables 1–4 for each m contains three values correspond to the AE, RMSE, and RAB summarized as the first, second, and third values, respectively. In addition, the ACLs of 95% asymptotic/credible intervals of δ, θ, $R(t)$, and $h(t)$ are listed in Tables 5 and 6. All numerical computations are performed using R statistical programming language software version 4.0.4 by mainly two useful recommended packages; namely the "CODA" package which used for carrying out the computations of MCMC by [38] and the "maxLik" package to maximize the likelihood function proposed by [39].

From Tables 1–6, it can be observed that the proposed estimators of the unknown parameters for the GIED are very good in terms of RMSEs, RABs, and ACLs criteria. As n and m increase, the RMSEs, RABs, and ACLs decrease as expected. So, to get better estimation results, one may tend to increase the effective sample size. Additionally, it can be seen that as the failure percentage m/n increases, the point and interval estimates become even better. Comparing the performance of the proposed estimation methods, simulation results showed that the EBEs for δ and $R(t)$ perform better than other methods on the basis of minimum RMSEs and RABs. On the other hand, the BEs for θ and $h(t)$ have the smallest RMSEs and RABs compared with other estimates in most of the cases. In addition, the credible intervals are performed better than the ACIs due to the gamma prior information in terms of shortest ACLs. Further, it is also noted the Bayesian and E-Bayesian approaches using the hybrid M–H with the Gibbs algorithm sampler have performed better than Lindley's approximation procedure with respect to both RMSEs and RABs. Comparing the performance of the EBEs based on the three different prior PDFs, it can be seen that the RMSEs and RABs of the unknown model parameters δ and θ as well as $R(t)$ and $h(t)$ are greater based on prior distribution 1 than those based on the other prior distributions.

Table 1. The AEs (first row), RMSEs (second row), and RABs (third row) of δ.

(δ,θ)	n	m	MLE	BE		EBE					
				Lindley	MCMC	Lindley			MCMC		
						Prior 1	Prior 2	Prior 3	Prior 1	Prior 2	Prior 3
(1,1)	40	20	1.1195	1.0655	1.1091	1.2909	1.2929	1.2754	1.0353	1.0349	1.0346
			0.3359	0.8849	0.1848	0.4885	0.4870	0.4511	0.0432	0.0413	0.0373
			0.2449	0.5270	0.1457	0.3550	0.3582	0.3398	0.0366	0.0360	0.0346
		32	1.0864	1.0945	1.0602	1.1349	1.1352	1.1236	0.9889	0.9845	0.9804
			0.2555	0.2513	0.1446	0.3336	0.3082	0.2995	0.0276	0.0253	0.0246

Table 1. *Cont.*

(δ, θ)	n	m	MLE	BE		EBE					
				Lindley	MCMC	Lindley			MCMC		
						Prior 1	Prior 2	Prior 3	Prior 1	Prior 2	Prior 3
			0.1962	0.1792	0.1129	0.2332	0.2292	0.2246	0.0231	0.0191	0.0199
	60	30	1.0846	1.1507	1.0762	1.1986	1.1925	1.1887	1.0244	1.0241	1.0183
			0.2473	0.3115	0.1580	0.3520	0.3354	0.3244	0.0370	0.0315	0.0259
			0.1872	0.2375	0.1225	0.2625	0.2546	0.2469	0.0302	0.0277	0.0205
		48	1.0523	1.0891	1.0284	1.0864	1.0760	1.0570	1.0149	1.0121	0.9999
			0.2070	0.1577	0.1195	0.2357	0.2237	0.2124	0.0343	0.0260	0.0133
			0.1572	0.1163	0.0933	0.1775	0.1691	0.1639	0.0292	0.0211	0.0105
	80	40	1.0615	1.1383	1.0625	1.1689	1.1649	1.1557	1.0078	1.0060	1.0174
			0.2263	0.2123	0.1423	0.3004	0.2950	0.2836	0.0372	0.0329	0.0272
			0.1719	0.1633	0.1109	0.2230	0.2205	0.2153	0.0313	0.0278	0.0230
		64	1.0401	1.0777	1.0389	1.0626	1.0554	1.0521	1.0227	1.0186	1.0100
			0.1734	0.1334	0.1148	0.1924	0.1887	0.1830	0.0267	0.0209	0.0125
			0.1317	0.1036	0.0897	0.1447	0.1464	0.1414	0.0227	0.0187	0.0103
$(3,3)$	40	20	3.2145	2.9732	3.0465	3.7890	3.6391	3.5856	3.0097	3.0081	3.0025
			0.6869	1.8437	0.5090	1.2100	1.0123	0.9299	0.0293	0.0244	0.0216
			0.1764	0.3442	0.1329	0.2969	0.2535	0.2413	0.0083	0.0064	0.0054
		32	3.1491	3.0591	3.1632	3.2092	3.1506	3.1247	2.9995	2.9959	3.0109
			0.6223	1.2058	0.4951	0.6105	0.5489	0.5296	0.0387	0.0223	0.0211
			0.1583	0.2170	0.1277	0.1559	0.1403	0.1397	0.0115	0.0065	0.0054
	60	30	3.1672	3.3550	3.0339	3.9009	3.8185	3.7457	3.0123	3.0152	3.0029
			0.6004	1.3644	0.4601	1.2556	1.1694	1.0769	0.0179	0.0174	0.0159
			0.1522	0.2917	0.1202	0.3181	0.2953	0.2755	0.0047	0.0052	0.0043
		48	3.1233	3.1677	3.0336	3.1506	3.1392	3.0962	2.9883	2.9995	2.9953
			0.5181	0.4353	0.3899	0.4987	0.4723	0.4412	0.0162	0.0152	0.0077
			0.1368	0.0989	0.1022	0.1298	0.1190	0.1167	0.0042	0.0042	0.0021
	80	40	3.1536	3.6011	3.0362	3.8030	3.7162	3.6524	2.9836	2.9883	2.9958
			0.5431	1.1031	0.4133	1.1158	1.0335	0.9558	0.0177	0.0164	0.0115
			0.1401	0.2925	0.1078	0.2855	0.2615	0.2443	0.0055	0.0046	0.0034
		64	3.0989	3.2017	3.0676	3.1378	3.1077	3.0782	2.9834	3.0048	2.9971
			0.4282	0.2576	0.3590	0.4349	0.4033	0.3905	0.0184	0.0126	0.0090
			0.1104	0.0753	0.0938	0.1114	0.1051	0.1018	0.0055	0.0034	0.0025

Table 2. The AEs (first row), RMSEs (second row), and RABs (third row) of θ.

(δ, θ)	n	m	MLE	BE		EBE					
				Lindley	MCMC	Lindley			MCMC		
						Prior 1	Prior 2	Prior 3	Prior 1	Prior 2	Prior 3
$(1,1)$	40	20	1.2447	3.5536	1.0715	4.2877	4.2837	4.2701	1.0919	1.0928	1.0847
			0.6271	4.3123	0.0747	3.4352	3.4134	3.4077	0.2499	0.2480	0.2437
			0.4054	3.4991	0.0715	3.2877	3.2837	3.2701	0.1876	0.1849	0.1819
		32	1.1219	3.1617	1.0239	3.3064	3.3084	3.2868	1.0289	1.0238	1.1926
			0.3388	2.3146	0.0301	2.4449	2.4231	2.3947	0.1888	0.1852	0.1840
			0.2434	2.1689	0.0251	2.3064	2.3084	2.2868	0.1453	0.1427	0.1420
	60	30	1.1741	4.0378	1.0348	4.1653	4.1387	4.1031	1.0738	1.0720	1.0588
			0.4560	3.2420	0.0444	3.2280	3.1965	3.1546	0.2217	0.2135	0.2043
			0.3059	3.1179	0.0368	3.1653	3.1387	3.1031	0.1675	0.1626	0.1545
		48	1.0748	3.1598	0.9901	3.2145	3.1709	3.1601	1.0396	1.0354	1.0263
			0.2512	2.2328	0.0271	2.2993	2.2366	2.2317	0.1603	0.1572	0.1537
			0.1805	2.1598	0.0222	2.2145	2.1709	2.1601	0.1245	0.1212	0.1194
	80	40	1.1268	4.0632	1.0234	4.2453	4.2217	4.2179	1.0445	1.0437	1.0502
			0.3750	3.1498	0.0381	3.2842	3.2570	3.2525	0.1810	0.1794	0.1776
			0.2557	3.0896	0.0331	3.2453	3.2217	3.2179	0.1381	0.1380	0.1365
		64	1.0581	3.1554	1.0157	3.1573	3.1106	3.1052	1.0406	1.0320	1.0265

Table 2. *Cont.*

(δ,θ)	n	m	MLE	BE Lindley	BE MCMC	EBE Lindley Prior 1	EBE Lindley Prior 2	EBE Lindley Prior 3	EBE MCMC Prior 1	EBE MCMC Prior 2	EBE MCMC Prior 3
			0.2138	2.2110	0.0272	2.2053	2.1530	2.1449	0.1374	0.1351	0.1316
			0.1594	2.1554	0.0223	2.1573	2.1106	2.1052	0.1066	0.1045	0.1028
$(3,3)$	40	20	3.6916	2.9921	2.8953	7.7877	7.2529	6.7721	3.1168	3.0733	3.0487
			1.7495	9.6710	0.1075	8.2725	6.9536	5.6102	0.6392	0.6163	0.5983
			0.3935	1.4207	0.0349	1.5959	1.4176	1.2574	0.1645	0.1588	0.1541
		32	3.4354	3.6815	3.0195	5.0940	4.9258	4.8271	3.0742	3.0556	3.0504
			1.4172	6.2592	0.0314	2.7488	2.5578	2.3036	0.5572	0.5426	0.5370
			0.3098	0.9418	0.0087	0.6980	0.6419	0.6090	0.1417	0.1401	0.1401
	60	30	3.6010	4.3264	2.9314	7.8495	7.6118	7.3998	3.0996	3.0707	3.0383
			1.6977	8.3728	0.0908	7.0702	6.5662	6.4343	0.5823	0.5595	0.5396
			0.3715	1.3336	0.0261	1.6165	1.5373	1.4666	0.1495	0.1455	0.1413
		48	3.3708	4.4653	2.9461	4.7892	4.7168	4.6264	3.0328	3.0415	3.0138
			1.1613	2.5480	0.0607	2.0855	2.0169	1.8728	0.4450	0.4448	0.4274
			0.2674	0.6217	0.0186	0.5964	0.5723	0.5421	0.1153	0.1166	0.1128
	80	40	3.6235	5.5127	2.9491	7.4986	7.1715	7.0092	3.0266	3.0093	3.0297
			1.6710	6.8034	0.0552	6.3800	5.3552	5.2343	0.4760	0.4760	0.4721
			0.3713	1.4079	0.0170	1.4995	1.3905	1.3364	0.1237	0.1256	0.1231
		64	3.2478	4.6542	2.9888	4.6661	4.6283	4.5695	3.0098	3.0416	3.0128
			0.8558	1.7518	0.0320	1.8733	1.8237	1.7680	0.3749	0.3790	0.3729
			0.2067	0.5629	0.0084	0.5554	0.5428	0.5232	0.0987	0.0994	0.0981

Table 3. The AEs (first row), RMSEs (second row), and RABs (third row) of $R(t)$.

(δ,θ)	n	m	MLE	BE Lindley	BE MCMC	EBE Lindley Prior 1	EBE Lindley Prior 2	EBE Lindley Prior 3	EBE MCMC Prior 1	EBE MCMC Prior 2	EBE MCMC Prior 3
$(1,1)$	40	20	0.9816	0.9416	0.9847	0.9629	0.9647	0.9653	0.9826	0.9826	0.9827
			0.0142	0.0401	0.0038	0.0274	0.0264	0.0255	0.0040	0.0039	0.0038
			0.0110	0.0408	0.0034	0.0218	0.0203	0.0197	0.0033	0.0032	0.0031
		32	0.9817	0.9648	0.9923	0.9737	0.9751	0.9754	0.9803	0.9800	0.9798
			0.0127	0.0169	0.0025	0.0165	0.0158	0.0155	0.0041	0.0041	0.0042
			0.0101	0.0172	0.0021	0.0129	0.0122	0.0118	0.0032	0.0032	0.0032
	60	30	0.9820	0.9577	0.9828	0.9679	0.9692	0.9698	0.9821	0.9822	0.9820
			0.0110	0.0240	0.0029	0.0210	0.0196	0.0189	0.0037	0.0035	0.0034
			0.0089	0.0244	0.0024	0.0165	0.0153	0.0147	0.0030	0.0028	0.0027
		48	0.9811	0.9693	0.9803	0.9762	0.9748	0.9759	0.9820	0.9819	0.9812
			0.0108	0.0124	0.0031	0.0132	0.0130	0.0131	0.0032	0.0030	0.0029
			0.0087	0.0126	0.0024	0.0101	0.0099	0.0101	0.0026	0.0024	0.0023
	80	40	0.9814	0.9614	0.9822	0.9697	0.9705	0.9705	0.9814	0.9813	0.9820
			0.0105	0.0203	0.0028	0.0181	0.0174	0.0173	0.0036	0.0035	0.0031
			0.0085	0.0207	0.0023	0.0143	0.0137	0.0136	0.0029	0.0028	0.0025
		64	0.9814	0.9727	0.9821	0.9774	0.9777	0.9779	0.9826	0.9825	0.9819
			0.0093	0.0090	0.0023	0.0111	0.0110	0.0108	0.0025	0.0024	0.0023
			0.0073	0.0091	0.0019	0.0087	0.0087	0.0083	0.0021	0.0020	0.0019
$(3,3)$	40	20	0.9470	0.8917	0.9373	0.8940	0.9065	0.9127	0.9448	0.9454	0.9454
			0.0262	0.0543	0.0134	0.0680	0.0573	0.0526	0.0109	0.0106	0.0104
			0.0223	0.0574	0.0111	0.0586	0.0488	0.0432	0.0090	0.0088	0.0085
		32	0.9457	0.8978	0.9447	0.9339	0.9425	0.9459	0.9447	0.9448	0.9460
			0.0263	0.0482	0.0081	0.0351	0.0314	0.0308	0.0098	0.0096	0.0092
			0.0220	0.0510	0.0067	0.0285	0.0264	0.0260	0.0080	0.0078	0.0077
	60	30	0.9472	0.8787	0.9403	0.8778	0.8879	0.8919	0.9452	0.9460	0.9456
			0.0222	0.0674	0.0107	0.0786	0.0686	0.0648	0.0099	0.0095	0.0094

Table 3. *Cont.*

(δ,θ)	n	m	MLE	BE		EBE					
				Lindley	MCMC	Lindley			MCMC		
						Prior 1	Prior 2	Prior 3	Prior 1	Prior 2	Prior 3
			0.0186	0.0712	0.0087	0.0726	0.0628	0.0586	0.0081	0.0079	0.0078
		48	0.9461	0.9158	0.9415	0.9373	0.9437	0.9452	0.9447	0.9453	0.9455
			0.0226	0.0302	0.0087	0.0272	0.0236	0.0235	0.0080	0.0078	0.0075
			0.0192	0.0320	0.0071	0.0224	0.0197	0.0199	0.0065	0.0065	0.0063
	80	40	0.9471	0.8658	0.9417	0.8828	0.8880	0.8902	0.9444	0.9451	0.9453
			0.0196	0.0802	0.0089	0.0708	0.0658	0.0637	0.0086	0.0085	0.0083
			0.0164	0.0848	0.0072	0.0671	0.0617	0.0594	0.0070	0.0071	0.0069
		64	0.9469	0.9232	0.9441	0.9405	0.9439	0.9452	0.9447	0.9457	0.9456
			0.0185	0.0229	0.0067	0.0214	0.0200	0.0198	0.0068	0.0066	0.0065
			0.0157	0.0242	0.0055	0.0179	0.0167	0.0166	0.0056	0.0055	0.0054

Table 4. The AEs (first row), RMSEs (second row), and RABs (third row) of $h(t)$.

(δ,θ)	n	m	MLE	BE		EBE					
				Lindley	MCMC	Lindley			MCMC		
						Prior 1	Prior 2	Prior 3	Prior 1	Prior 2	Prior 3
$(1,1)$	40	20	0.2799	0.2284	0.2655	0.2426	0.2412	0.2419	0.2925	0.2932	0.2911
			0.1561	0.0701	0.0505	0.1649	0.1682	0.1630	0.0637	0.0632	0.0624
			0.4246	0.2349	0.1418	0.4630	0.4725	0.4563	0.1692	0.1673	0.1647
		32	0.2830	0.2738	0.2942	0.3005	0.2952	0.2995	0.3181	0.3207	0.3232
			0.1452	0.0247	0.0391	0.1618	0.1591	0.1573	0.0631	0.0633	0.0638
			0.3956	0.0828	0.1051	0.4376	0.4257	0.4267	0.1608	0.1605	0.1616
	60	30	0.2824	0.2389	0.2891	0.2525	0.2526	0.2562	0.2975	0.2972	0.2988
			0.1266	0.0596	0.0425	0.1407	0.1366	0.1334	0.0589	0.0565	0.0558
			0.3467	0.1995	0.1152	0.3948	0.3837	0.3710	0.1548	0.1494	0.1456
		48	0.2938	0.2785	0.3170	0.2976	0.3040	0.3140	0.2970	0.2981	0.3065
			0.1260	0.0201	0.0447	0.1351	0.1347	0.1342	0.0487	0.0468	0.0466
			0.3436	0.0672	0.1166	0.3604	0.3575	0.3571	0.1300	0.1232	0.1212
	80	40	0.2908	0.2480	0.2956	0.2538	0.2573	0.2608	0.3048	0.3062	0.2972
			0.1188	0.0505	0.0405	0.1241	0.1224	0.1219	0.0544	0.0541	0.0492
			0.3241	0.1691	0.1085	0.3400	0.3398	0.3371	0.1422	0.1415	0.1309
		64	0.2930	0.2795	0.2958	0.2988	0.3038	0.3069	0.2898	0.2909	0.2971
			0.1077	0.0191	0.0347	0.1164	0.1163	0.1159	0.0387	0.0383	0.0374
			0.2865	0.0638	0.0929	0.3112	0.3185	0.3121	0.1050	0.1028	0.0996
$(3,3)$	40	20	0.2882	0.2954	0.3373	0.2347	0.2474	0.2496	0.3070	0.3033	0.3026
			0.1036	0.0031	0.0682	0.1266	0.1124	0.1097	0.0623	0.0607	0.0593
			0.2776	0.0103	0.1756	0.3528	0.3114	0.3047	0.1619	0.1579	0.1538
		32	0.2918	0.2919	0.3085	0.2913	0.2959	0.2987	0.3061	0.3053	0.3001
			0.1026	0.0066	0.0468	0.0966	0.0955	0.0953	0.0556	0.0544	0.0526
			0.2761	0.0224	0.1221	0.2602	0.2549	0.2589	0.1423	0.1410	0.1400
	60	30	0.2889	0.2630	0.3238	0.2206	0.2278	0.2336	0.3046	0.3008	0.3014
			0.0841	0.0355	0.0551	0.1232	0.1153	0.1106	0.0567	0.0544	0.0535
			0.2260	0.1190	0.1414	0.3387	0.3161	0.3037	0.1473	0.1433	0.1410
		48	0.2932	0.2872	0.3190	0.2937	0.2918	0.2978	0.3054	0.3028	0.3014
			0.0872	0.0113	0.0458	0.0835	0.0815	0.0797	0.0452	0.0443	0.0428
			0.2369	0.0380	0.1181	0.2254	0.2198	0.2137	0.1169	0.1166	0.1132
	80	40	0.2925	0.2506	0.3183	0.2290	0.2382	0.2452	0.3063	0.3030	0.3028
			0.0747	0.0479	0.0475	0.1090	0.1038	0.0987	0.0487	0.0481	0.0473
			0.1997	0.1605	0.1224	0.2966	0.2825	0.2705	0.1260	0.1268	0.1236
		64	0.2915	0.2799	0.3088	0.2887	0.2939	0.2972	0.3046	0.3012	0.3007
			0.0731	0.0186	0.0370	0.0700	0.0691	0.0682	0.0385	0.0374	0.0373
			0.1968	0.0623	0.0971	0.1864	0.1873	0.1830	0.1007	0.0990	0.0983

Table 5. The ACLs for 95% ACI/HPD credible interval estimates of δ and θ.

(δ,θ)	n	m	δ					θ				
			ACI	BCI	E-BCI			ACI	BCI	E-BCI		
					Prior 1	Prior 2	Prior 3			Prior 1	Prior 2	Prior 3
(1,1)	40	20	2.0044	0.0843	0.0840	0.0906	0.0481	1.1463	0.5833	0.9092	0.9037	0.8850
	32	1.1446	0.0702	0.0887	0.0762	0.0549	0.9350	0.5130	0.7191	0.7269	0.7016	
	60	30	1.5032	0.0926	0.0925	0.0875	0.0717	0.9131	0.5503	0.8247	0.7846	0.7614
		48	0.8830	0.0902	0.0967	0.0734	0.0545	0.7451	0.4552	0.5996	0.5976	0.5928
	80	40	1.2299	0.1085	0.1190	0.1087	0.0760	0.7773	0.5028	0.6792	0.6651	0.6610
		64	0.7488	0.0915	0.0471	0.0359	0.0285	0.6398	0.4186	0.5104	0.5153	0.5096
(3,3)	40	20	5.8344	0.0816	0.0984	0.0765	0.0787	2.4699	2.0037	2.4030	2.3875	2.3395
		32	4.6630	0.0955	0.1171	0.0665	0.0686	2.2726	1.8446	2.1491	2.1066	2.0428
	60	30	5.4633	0.1789	0.0448	0.0312	0.0551	2.1863	1.8007	2.2672	2.1761	2.0848
		48	3.6915	0.1058	0.0424	0.0548	0.0235	1.8400	1.5243	1.7089	1.7117	1.6512
	80	40	5.4671	0.0789	0.0238	0.0376	0.0364	2.0351	1.6101	1.8594	1.8458	1.8376
		64	3.0330	0.1031	0.0263	0.0372	0.0316	1.5857	1.3843	1.4412	1.4631	1.4613

Table 6. The ACLs for 95% ACI/HPD credible interval estimates of $R(t)$ and $h(t)$.

(δ,θ)	n	m	δ					θ				
			ACI	BCI	E-BCI			ACI	BCI	E-BCI		
					Prior 1	Prior 2	Prior 3			Prior 1	Prior 2	Prior 3
(1,1)	40	20	0.0530	0.0091	0.0148	0.0146	0.0142	0.5925	0.1509	0.2429	0.2412	0.2391
		32	0.0497	0.0096	0.0151	0.0146	0.0142	0.5766	0.1524	0.2326	0.2306	0.2262
	60	30	0.0439	0.0104	0.0142	0.0134	0.0132	0.5025	0.1628	0.2282	0.2197	0.2168
		48	0.0423	0.0107	0.0126	0.0117	0.0110	0.4892	0.1585	0.1900	0.1834	0.1772
	80	40	0.0393	0.0104	0.0140	0.0137	0.0118	0.4431	0.1556	0.2117	0.2069	0.1891
		64	0.0367	0.0089	0.0090	0.0088	0.0090	0.4284	0.1346	0.1468	0.1461	0.1488
(3,3)	40	20	0.1006	0.0396	0.0413	0.0414	0.0408	0.3905	0.2208	0.2363	0.2352	0.2338
		32	0.1003	0.0317	0.0381	0.0372	0.0353	0.3876	0.1805	0.2142	0.2107	0.2011
	60	30	0.0854	0.0354	0.0388	0.0370	0.0366	0.3253	0.1927	0.2229	0.2120	0.2068
		48	0.0823	0.0289	0.0305	0.0297	0.0291	0.3207	0.1607	0.1732	0.1694	0.1651
	80	40	0.0761	0.0298	0.0329	0.0326	0.0325	0.2871	0.1677	0.1864	0.1847	0.1859
		64	0.0714	0.0248	0.0259	0.0255	0.0256	0.2786	0.1383	0.1470	0.1451	0.1458

8. Real-Life Data Illustration

To examine the applicability of the proposed methodologies to a real phenomenon, we shall use a real-life data set given by [40]. This data set represents $n = 56$ blood samples from organ transplant recipients and assays an aliquot of each sample by a standard approved method of high-performance liquid chromatography. Reference [41] stated that the Lindley distribution can be considered an adequate model to fit this data set. The ordered blood samples are reported in Table 7. Before further proceeding to discuss the proposed estimators, one question arises about whether the given data set fit the GIED or not. Thus, we use the MLEs to obtain the Kolmogorov–Smirnov (K–S) distance and the associated p-value for the complete blood data set. The MLEs of model parameters δ and θ are 325.37 and 3.2132, respectively. The K–S distance is 0.083 with p-value 0.836. This result indicates that the GIED is fitting the complete blood data set quite well.

Table 7. Blood samples from organ transplant recipients.

35, 71, 77, 87, 93, 99, 104, 109, 109, 112, 118, 118, 125, 127, 129, 130, 148, 151, 153, 156, 159, 159, 162, 166, 185, 198, 203, 206, 221, 227, 241, 244, 245, 254, 266, 271, 275, 280, 285, 318, 327, 336, 339, 340, 346, 350, 370, 402, 428, 440, 498, 521, 556, 578, 653, 980.

Moreover, to prove the existence and uniqueness of the MLEs, we propose to provide the contour plot of the log-likelihood function using the complete blood data set as displayed in Figure 1. The maximum of the log-likelihood function is denoted by point x in the innermost contour. The coordinates of x-point provide the MLEs of δ and θ which are becomes $\hat{\delta} \simeq 325.37$ and $\hat{\theta} \simeq 3.2132$. Further, it shows that the MLEs exist and are also unique.

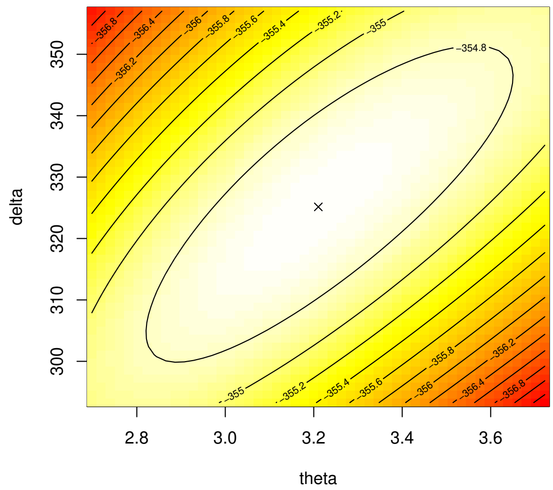

Figure 1. Contour plot of the log-likelihood function δ and θ for blood data set.

Using the data set listed in Table 7, we generate Type-II censored sample with $m = 25$, and then the MLEs, BEs, and EBEs of the unknown parameters δ and θ as well as the $R(t)$ and $h(t)$ at distinct mission time $t = 100$, are provided. To develop the BEs (using Lindley and MCMC methods), we have assumed that the prior information about the model parameters is not available, then the non-informative priors, i.e., $\tau_i, \nu_i = 0,\ i = 1,2$, will be used. Further, to develop the EBEs (using Lindley and MCMC methods), two different choices of $c_i, s_i, z_i,\ i = 1,2$ are taken as 0.5 and 1.5 which are namely Prior 1 and Prior 2, respectively. Using the MCMC algorithms proposed in Sections 3.2 and 4.2, when the start values of δ and θ are taken to be their MLEs $\hat{\delta}$ and $\hat{\theta}$, we generate 20,000 MCMC samples and then first 5000 iterations have been discarded as a burn-in. Hence, using the generated 15,000 samples, the MLEs, BEs, and EBEs of δ, θ, $R(t)$, and $h(t)$ are computed and presented in Table 8. It is clear that the EBEs for δ, θ, $R(t)$, and $h(t)$ are near to corresponding BEs and MLEs under both Priors 1 and 2. Additionally, the different interval estimates of δ, θ, $R(t)$, and $h(t)$ are calculated and listed in Table 9. From Table 9, it can be seen that the E-BCIs based on Priors 1 and 2 perform better than the other interval estimates.

To judge how quickly the MCMC converges and to assess the bias of simulated estimates at each iteration of any unknown parameter, the trace plot (or time-series diagram) is an excellent way for this purpose. Hence, to monitored the convergence of the generated 15,000 EBEs as an example, trace plots of the posterior distributions of δ, θ, $R(t)$, and $h(t)$ under Priors 1 and 2 are plotted in Figure 2. In each trace plot, the sample mean (EBE) is displayed with a horizontal solid line (—), further, lower and upper bounds of 95% credible intervals are displayed as dashed (- - -) horizontal lines. It indicates that the MCMC procedure converges well and it also shows that discarding the first 5000 samples as burn-in is an appropriate size to erase the effect of the initial values. Furthermore, the marginal posterior density estimates under Priors 1 and 2 of δ, θ, $R(t)$, and $h(t)$ using the Gaussian kernel with their histograms based on the MCMC chain values are represented in Figure 3. Similarly, in each histogram plot, the sample mean (EBE) of any unknown parameter is displayed as a vertical dash-dotted line (:). It is evident from the estimates that the generated posteriors of the unknown model parameters δ, θ as well as the reliability characteristics $R(t)$ and $h(t)$ are fairly symmetric for both Priors 1 and 2. Moreover, some vital properties such as mean, median, mode, standard deviation (SD), standard error (SE), and skewness (Sk) for MCMC posterior distributions of the unknown parameters δ, θ, $R(t)$, and $h(t)$ after bun-in are computed and listed in Table 10.

Table 8. The MLEs, BEs, and EBEs of δ, θ, $R(t)$, and $h(t)$ from blood data set.

Parameter	MLE	BE		EBE			
		Lindley	MCMC	Lindley		MCMC	
				Prior 1	Prior 2	Prior 1	Prior 2
δ	287.60	611.61	276.72	606.09	597.36	287.59	287.60
θ	2.4777	25.917	2.3069	25.837	25.716	2.4549	2.3649
$R(100)$	0.8661	0.2760	0.8610	0.2872	0.3045	0.8673	0.8718
$h(100)$	0.0043	0.0004	0.0043	0.0005	0.0005	0.0042	0.0041

Table 9. The 95% ACIs, BCIs, and E-BCIs of δ, θ, $R(t)$, and $h(t)$ from blood data set.

Parameter	ACI	BCI	E-BCI	
			Prior 1	Prior 2
δ	(280.22,294.97)	(270.59,282.82)	(287.59,287.61)	(287.59,287.61)
θ	(1.5041,3.4513)	(2.2141,2.4011)	(2.4548,2.4551)	(2.3648,2.3650)
$R(100)$	(0.8170,0.9152)	(0.8580,0.8639)	(0.8672,0.8673)	(0.8717,0.8718)
$h(100)$	(0.0026,0.0059)	(0.0042,0.0043)	(0.0041,0.0042)	(0.0040,0.0041)

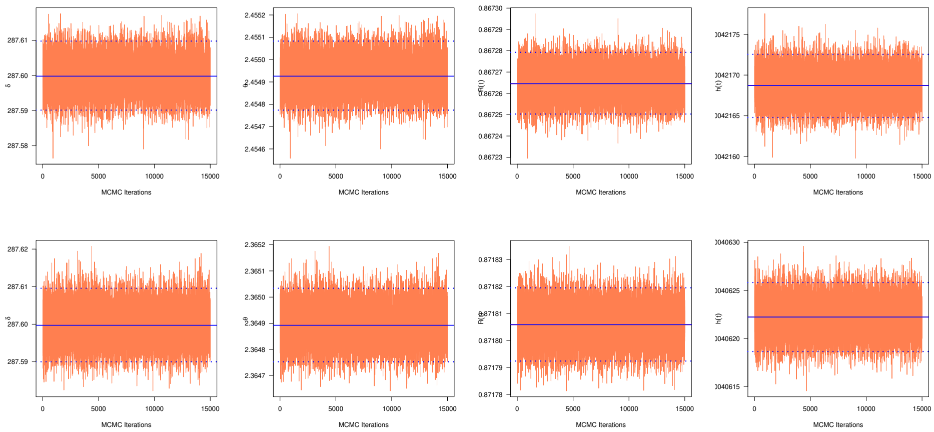

Figure 2. MCMC trace plots of EBEs for Prior 1 (upper panel) and Prior 2 (lower panel) of δ, θ, $R(t)$, and $h(t)$.

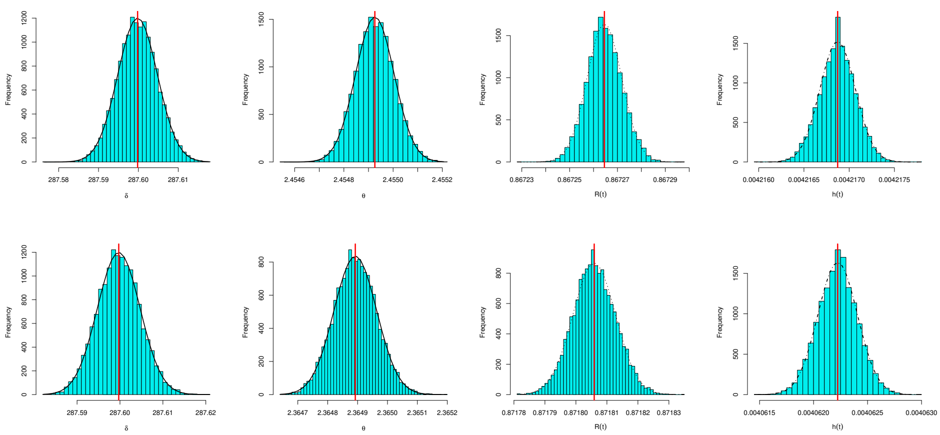

Figure 3. Histogram and kernel density estimates of EBEs for Prior 1 (**upper panel**) and Prior 2 (**lower panel**) of δ, θ, $R(t)$, and $h(t)$.

Table 10. Some posterior characteristics of E-Bayesian MCMC outputs from blood data set.

Prior	Parameter	Mean	Median	Mode	SD	SE	Sk
1	δ	287.599	287.599	287.593	4.988×10^{-3}	4.072×10^{-5}	0.02918
	θ	2.45493	2.45493	2.45482	7.850×10^{-5}	6.410×10^{-7}	0.02913
	$R(100)$	0.86726	0.86726	0.86726	7.299×10^{-6}	5.959×10^{-8}	0.05423
	$h(100)$	0.00422	0.00422	0.00422	1.963×10^{-7}	1.603×10^{-9}	−0.04048
2	δ	287.600	287.600	287.591	4.994×10^{-3}	4.077×10^{-5}	0.04724
	θ	2.36489	2.36489	2.36476	7.164×10^{-5}	5.849×10^{-7}	0.04740
	$R(100)$	0.87181	0.87181	0.87180	6.911×10^{-6}	5.642×10^{-8}	0.04185
	$h(100)$	0.00406	0.00406	0.00406	1.829×10^{-7}	1.494×10^{-9}	−0.01283

9. Conclusions

This paper is primarily related to the E-Bayesian analysis of unknown parameters of multi-parameter population models. The problem of estimating two unknown parameters, reliability, and hazard rate functions of the generalized inverted exponential distribution are discussed based on Type-II censored data. Firstly, the maximum likelihood and Bayesian estimation methods are considered for this purpose. The asymptotic confidence intervals for the unknown parameters as well as Bayesian credible intervals are also obtained. Secondly, for the first time, the E-Bayesian estimation method is adopted to estimate the two unknown parameters as well as the reliability and hazard functions. The E-Bayesian credible intervals for these quantities are also investigated. The Bayesian and E-Bayesian estimations are approximated using Lindley's approximation and MCMC technique based on squared error loss function under the assumption of independent gamma priors. We also proposed an algorithm to determine the values of the hyperparameters using past samples in Bayesian and E-Bayesian procedures. A Monte Carlo simulation is conducted to compare the performance of the different point and interval methods. A practical example using a real-life data set is discussed to demonstrate how the applicability of the proposed methods in real phenomena. The simulation and real data analysis outcomes showed that the proposed procedure to acquire the full model parameters using the E-Bayesian estimation method provides satisfactory estimates and acceptable credible intervals. In future work, one can consider the same inferential methods presented in this paper to other lifetime models such as Weibull and gamma distributions and other sampling schemes such as the progressive Type-II censoring scheme. We hope that the results and methodology discussed in this paper will be beneficial to data analysts and reliability practitioners.

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