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Lie Symmetry Analysis, Particular Solutions and Conservation Laws of Benjiamin Ono Equation

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Abstract: In this paper, by applying the Lie group method and the direct symmetry method, Lie algebras of the Benjiamin Ono equation are obtained, and we find that results of the two methods are same. Based on the Lie algebra, Lie symmetry groups, relationships between new solutions and old solutions, two kinds of ODEs as symmetry reductions are obtained. Making use of the power series method, the exact power series solution of the Benjiamin Ono equation has been derived. We also give the conservation laws of Benjiamin Ono equation by means of Ibragimovs new conservation Theorem.

Keywords: lie symmetry analysis; Benjiamin Ono equation; power series method; particular solutions; conservation laws

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1. Introduction

With the development of science and technology, people pay more and more attention in nonlinear evolution equations (NLPDEs). The search for exact solutions of nonlinear evolution equations becomes an important subject. To solve NLPDEs, a variety of effective methods have been proposed, such as the Jacobi elliptic function method [1], the extended tanh method [2], the Exp-function method [3,4], the Bäcklund transformation [5], the generalized algebraic method [6] and so on. To the best of our knowledge, Lie group analysis is an efficient and direct method for finding exact solutions of nonlinear differential equations [7–10]. Many equations have been studied by this method [11–17]. In this paper, we study the following Benjiamin Ono (BO) equation by means of the classical Lie group. The BO equation is written as

$$u_{tt} + \beta\left(u^2\right)_{xx} + \gamma u_{xxxx} = 0, \tag{1}$$

where β and γ are nonzero constants. The BO equation is a famous nonlinear model for representing the water wave motion with damping structure. Some exact periodic solutions were obtained using the Jacobi elliptic function expansion method in [18]. Wang studied this equation using the Riccati expansion method [19].

We construct the paper as follows: Lie symmetry analysis of BO equation are presented in Section 2. Symmetry reduction and the exact power series solutions for BO equation are obtained in Section 3. The conservation laws are derived in Section 4. Finally, the discussion and conclusions are given in Section 5.

2. Lie Symmetry Analysis of BO Equation

We apply the Lie group approach in this section to consider the BO equation.

2.1. Direct Symmetry

A one-parameter Lie group of infinitesimal transformation:

$$
\begin{aligned}
t^* &= t + \varepsilon\tau(x,t,u) + o(\varepsilon^2),\\
x^* &= x + \varepsilon\varsigma(x,t,u) + o(\varepsilon^2),\\
u^* &= u + \varepsilon\eta(x,t,u) + o(\varepsilon^2),
\end{aligned}
\tag{2}
$$

where ε is a small parameter. The corresponding vector field can be expressed by

$$V = \varsigma(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{3}$$

If the vector field (3) produces a symmetry of (1), so V must satisfy the Lie's symmetry condition

$$pr^{(4)}V(\Delta)\Big|_{\Delta=0} = 0, \tag{4}$$

where $\Delta = u_{tt} + \beta\left(u^2\right)_{xx} + \gamma u_{xxxx} = 0$. Applying the fourth prolongation $pr^{(4)}V$ to (1), then we can get the following symmetry equations

$$\eta^{tt} + 4\beta\eta^{x} + 2\beta u_{xx}\eta + 2\beta u\eta^{xx} + \gamma\eta^{xxxx} = 0, \tag{5}$$

where

$$
\begin{aligned}
&\eta^{t} = D_t(\eta) - u_x D_t(\varsigma) - u_t D_t(\tau),\\
&\eta^{tt} = D_t\left(\eta^{t}\right) - u_{xt} D_t(\varsigma) - u_{tt} D_t(\tau),\\
&\eta^{x} = D_x(\eta) - u_x D_x(\varsigma) - u_t D_x(\tau),\\
&\eta^{xx} = D_x(\eta^{x}) - u_{xt} D_x(\tau) - u_{xx} D_x(\varsigma),\\
&\eta^{xxx} = D_x(\eta^{xx}) - u_{xxt} D_x(\tau) - u_{xxx} D_x(\varsigma),\\
&\eta^{xxxx} = D_x(\eta^{xxx}) - u_{xxxt} D_x(\tau) - u_{xxxx} D_x(\varsigma).
\end{aligned}
\tag{6}
$$

Here, D_i represents a differential operator, which is defined as

$$D_i = \frac{\partial}{\partial x^i} + u_i\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + \ldots,\ i = 1,2 \tag{7}$$

and $\left(x^1, x^2\right) = (t, x)$.

Solving (5) with the help of (6) gives

$$\tau = c_1 t + c_2,\ \varsigma = \frac{c_1}{2}x + c_3,\ \eta = -(c_1 u + c_4), \tag{8}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants.

Therefore, four-dimensional Lie algebras can be obtained

$$V_1 = \frac{\partial}{\partial t},\ V_2 = \frac{\partial}{\partial x},\ V_3 = -\frac{\partial}{\partial u}, V_4 = t\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}. \tag{9}$$

It is easy to check that the vector fields are closed under the Lie bracket (see Table 1).

To get some exact solutions from known problems, we find the Lie symmetry group from the corresponding symmetry and solve the following initial problem to obtain the Lie symmetry group

$$(\tilde{t}, \tilde{x}, \tilde{u})|_{\varepsilon=0} = (t, x, u). \tag{10}$$

Table 1. Commutator of the Lie algebra of (1).

	V_1	V_2	V_3	V_4
V_1	0	0	0	V_1
V_2	0	0	0	$\frac{V_2}{2}$
V_3	0	0	0	V_3
V_4	$-V_1$	$-\frac{V_2}{2}$	$-V_3$	0

Therefore, we can obtain the following Lie symmetry group

$$g : (t, x, u) \to (\widetilde{t}, \widetilde{x}, \widetilde{u}). \tag{11}$$

Solving (10) can get the one-parameter group $g_i(\varepsilon)$ generated by $V_i(i = 1, 2, 3, 4)$

$$\begin{aligned} &g_1 : (t + \varepsilon, x, u), \\ &g_2 : (t, x + \varepsilon, u), \\ &g_3 : (t, x, u + \varepsilon), \\ &g_4 : (te^{\varepsilon}, xe^{\varepsilon/2}, ue^{-\varepsilon}). \end{aligned} \tag{12}$$

The symmetry groups g_2 and g_3 explain the time-and-space-invariance of the equation, and g_1 refers to scaling symmetry. Applying above group g_1, g_2, g_3 and g_4, we can get the Lie symmetry theorem:

Theorem 1. *If $u = f(x, t)$ is a solution of (1) then the expression of the corresponding function solutions*

$$\begin{aligned} g_1(\varepsilon)f(x, t) &= f(t - \varepsilon, x), \\ g_2(\varepsilon)f(x, t) &= f(t, x - \varepsilon), \\ g_2(\varepsilon)f(x, t) &= f(t, x - \varepsilon), \\ g_3(\varepsilon)f(x, t) &= f(t, x,) - \varepsilon, \\ g_4(\varepsilon)f(x, t) &= e^{-\varepsilon} f(te^{-\varepsilon}, xe^{-\varepsilon/2}). \end{aligned} \tag{13}$$

In reference [18], Fu obtained the trigonometric function solution

$$u = \frac{5}{2}k^2\left(1 - 3\csc^2\left(\frac{k}{2}x - 3k^5 t\right)\right). \tag{14}$$

So we can obtain the new exact solution of (1) by applying g_1 and g_3 as follows

$$u = \frac{5}{2}e^{-\frac{2}{5}\varepsilon}k^2\left(1 - 3\csc^2\left(\frac{k}{2}x^{-\frac{1}{5}\varepsilon} - 3k^5 te^{-\varepsilon}\right)\right), \tag{15}$$

and

$$u = \frac{5}{2}k^2\left(1 - 3\csc^2\left(\frac{k}{2}(x - \varepsilon) - 3k^5 t\right)\right). \tag{16}$$

We can obtain many solutions by giving the arbitrary different constants.

Remark 1. *A number of new invariant solutions can be found from the given solutions in [18] for the BO equation.*

2.2. Generalized Symmetry

The symmetry group of (1) is obtained by classical Lie group method in Section 2.1. Next, we will find the symmetries of (1) through the generalized symmetry method. The essence of the generalized symmetry method is to find the symmetry of a known NPDE,

$$F(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt},\ldots)=0, \tag{17}$$

σ satisfies the following equation

$$F'(u)\sigma=0, \tag{18}$$

where (18) reads as

$$F'(u)\sigma=\frac{\partial F}{\partial u}\sigma+\frac{\partial F}{\partial u_t}\sigma_t+\frac{\partial F}{\partial u_x}\sigma_x+\frac{\partial F}{\partial u_{xx}}\sigma_{xx}+\frac{\partial F}{\partial u_{xt}}\sigma_{xt}+\ldots. \tag{19}$$

From (18), the symmetry equation of (1) must satisfy

$$\sigma_{tt}+4\beta\sigma_x+2\beta\sigma u_{xx}+2\beta u\sigma_{xx}+\gamma\sigma_{xxxx}=0. \tag{20}$$

The symmetry of BO equation is

$$\sigma=au_t+bu_x+cu+d, \tag{21}$$

where a,b,c and d are functions of x,t which will be determined later.

Substituting (21) into (20) and using (1) to make the coefficients of the polynomial zero after simplification, we can obtain

$$a=c_1t+c_2,\ b=\frac{c_1}{2}x+c_3,\ c=c_1,\ d=c_4, \tag{22}$$

where c_1,c_2,c_3 and c_4 are arbitrary constants.

Equivalently, (21) can also be written as

$$\sigma=c_1\left(t+\frac{x}{2}+u\right)+c_2u_t+c_3u_x+c_4. \tag{23}$$

The symmetry equivalent vector field can be represented as

$$V=\left(\frac{c_1}{2}x+c_3\right)\frac{\partial}{\partial x}+(c_1t+c_2)\frac{\partial}{\partial t}-(c_1u+c_4)\frac{\partial}{\partial u}. \tag{24}$$

The vector field $V_i(i=1,2,3,4)$ has been obtained in Section 2.1.

3. Symmetry Reduction and Exact Solutions for BO Equation

3.1. Symmetry Reduction

In this section, we present some discussions on (1) based on the symmetries (9). We will discuss the following two situations:

case 1: $V_1+\mu V_2$

From $V_1+\mu V_2=0$, we obtain the invariants and reduced equation as $u=f(\xi)$, where $\xi=x-\mu t$. By substituting it into (1), we arrive at

$$\mu^2f''+2\beta f'^2+2\beta ff''+\gamma f^{(4)}=0, \tag{25}$$

where $f'=df/d\xi$.

case 2: V_4

For this generator V_4, we obtain $u = t^{-1}f(\xi)$, where $\xi = xt^{-\frac{1}{2}}$ is the invariant solution. By substituting it into (1), we get to the following ODE

$$\xi^2 f'' + 7\xi f + 8f + 8\beta f'^2 + 8\beta f f'' - 4\gamma f^{(4)} = 0, \tag{26}$$

where $f' = df/d\xi$.

3.2. Particular Solutions for BO Equation

We use the power series method to research the exact solution of (25) in this section. For (25), we will assume that the series solution is in the form of

$$f(\xi) = \sum_{n=0}^{\infty} q_n \xi^n, \tag{27}$$

where $q_n(0, 1, 2, 3, \ldots)$ are coefficients that will be determined below. We obtain by substituting (27) into (25),

$$\begin{aligned}
&2\mu^2 q_2 + \mu \sum_{n=1}^{\infty} (n+1)(n+2) q_{n+2} \xi^n + 2\beta {q_1}^2 \\
&+ 2\beta \sum_{n=1}^{\infty} \sum_{k=0}^{n} (k+1)(n+1-k) q_{k+1} q_{n+1-k} \xi^n \\
&+ 4\beta q_0 q_2 + 4\beta \sum_{n=1}^{\infty} \sum_{k=1}^{n} (k+1)(n-k) q_{k+1} q_{n+1-k} \xi^n \\
&+ 24\gamma q_4 + \gamma \sum_{n=1}^{\infty} (n+1)(n+2)(n+3)(n+4) q_{n+4} \xi^n = 0.
\end{aligned} \tag{28}$$

For $n = 0$, by comparing all coefficients in (28), we obtain,

$$q_4 = -\frac{\mu^2 q_2 + \beta {q_1}^2 + 2\beta q_0 q_2}{12\gamma}. \tag{29}$$

More generally, for $n \geq 1$, we have

$$\begin{aligned}
q_{n+4} &= \frac{1}{(n+1)(n+2)(n+3)(n+4)\gamma} (\mu(n+1)(n+2) q_{n+2} \\
&+ 2\beta \sum_{k=0}^{n} (k+1)(n+1-k) q_{k+1} q_{n+1-k} + 4 \sum_{k=1}^{n} (k+1)(n-k) q_{k+1} q_{n+1-k})
\end{aligned} \tag{30}$$

from (28).

Clearly, a power series solution for (25) [20–23] can be found using the above process. Hence, this power series solution (27) to (25) is an exact analytic solution.

In fact, the power series solution of (25) can be reached:

$$\begin{aligned}
f(\xi) &= q_0 + q_1\xi + q_2\xi^2 + q_3\xi^3 + q_4\xi^4 + \sum_{n=1}^{\infty} q_{n+4} \omega^n \\
&= q_0 + q_1\xi + q_2\xi^2 + q_3\xi^3 - \frac{\mu^2 q_2 + \beta {q_1}^2 + 2\beta q_0 q_2}{12\gamma} \xi^4 \\
&+ \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)\gamma} (\mu(n+1)(n+2) q_{n+2} \\
&+ 2\beta \sum_{k=0}^{n} (k+1)(n+1-k) q_{k+1} q_{n+1-k} + 4 \sum_{k=1}^{n} (k+1)(n-k) q_{k+1} q_{n+1-k}) \xi^n.
\end{aligned} \tag{31}$$

Thus, the power series solution of BO equation can be obtained from solution (31)

$$
\begin{aligned}
f(\xi) &= q_0 + q_1(x-\mu t) + q_2(x-\mu t)^2 + q_3(x-\mu t)^3 + q_4(x-\mu t)^4 + \sum_{n=1}^{\infty} q_{n+4}(x-\mu t)^n \\
&= q_0 + q_1(x-\mu t) + q_2(x-\mu t)^2 + q_3(x-\mu t)^3 - \frac{\mu^2 q_2 + \beta {q_1}^2 + 2\beta q_0 q_2}{12\gamma}(x-\mu t)^4 \\
&+ \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)\gamma}(\mu(n+1)(n+2)q_{n+2} \\
&+ 2\beta \sum_{k=0}^{n} (k+1)(n+1-k)q_{k+1}q_{n+1-k} + 4\sum_{k=1}^{n} (k+1)\,(n-k)q_{k+1}q_{n+1-k})(x-\mu t)^n,
\end{aligned} \tag{32}
$$

where $q_i(i = 0, \ldots, 4)$ are arbitrary constants. The other terms of the sequence $q_n(n \geq 4)$ can be defined continuously by (30).

Remark 2. *Obviously, (26) can also be solved by the power series method, which has been ommitted here. It is not difficult for us to find that the power series method is a useful method to solve PDEs using the process above. As far as we know, the solutions obtained in this section were not found in other references.*

4. Conservation Laws of BO Equation

The conservation law of NLPDEs [24–27] plays an important role in nonlinear scientific research. The law of conservation is widely used in the development of appropriate numerical methods, mathematical analysis, in especially, uniqueness, existence, stability analysis and so on. We will discuss the conservation law of (1) through using the adjoint equation [28] and symmetry (9) in this section. The adjoint equation of (1) can be written as

$$v_{tt} + 2\beta u v_{xx} + \gamma v_{xxxx} = 0, \tag{33}$$

and the formal Lagrangian for the system form is expressed as

$$L = v(u_{tt} + 2\beta u_{xx} + 2\beta u u_{xx} + \gamma u_{xxxx}). \tag{34}$$

Next, we will recall the "new conservation theorem" proposed by Ibragimov [28].

Theorem 2. *Any Lie point, non-local symmetry and Lie-Bäcklund*

$$V = \varsigma(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u} \tag{35}$$

of (1) supplies a conservation law $D_i\left(T^i\right) = 0$ for the system comprising (1) and its adjoint equation [29,30]. The conservation vector can be derived from the following formula

$$
\begin{aligned}
T^i &= \xi^i L + W^\alpha\left[\frac{\partial L}{\partial {u_i}^\alpha} - D_j\left(\frac{\partial L}{\partial {u_{ij}}^\alpha}\right) + D_j D_k\left(\frac{\partial L}{\partial {u_{ijk}}^\alpha}\right)\right] \\
&+ D_i(W^\alpha)\left[\frac{\partial L}{\partial {u_{ij}}^\alpha} - D_k\left(\frac{\partial L}{\partial {u_{ijk}}^\alpha}\right) + \ldots\right],
\end{aligned} \tag{36}
$$

where $W^\alpha = \eta^\alpha - \xi^i {u_j}^\alpha$ is Lie characteristic function and L is determined by (34).

The operator V yields the conservation law $D_t(T^t) + D_x(T^x) = 0$, where the conserved vector $T = \left(T^t, T^x\right)$ is given by (36) and has the components

$$T^t = \xi^t L + W(-v_t) + W_t(v), \tag{37}$$

$$\begin{aligned}
\mathrm{T}^x &= \xi^x\,\mathrm{L} + \mathrm{W}[4\beta u_x v - D_x(2\beta u v) - D_{xxx}(\gamma v)] \\
&+ D_x(\mathrm{W})[2\beta u v + D_{xx}(\gamma v)] \\
&+ D_{xx}(\mathrm{W})[-D_x(\gamma v)] \\
&+ D_{xxx}(\mathrm{W})(\gamma v).
\end{aligned} \tag{38}$$

Thus, the corresponding parts of the non-local conservation laws of (1) and (33) can be defined using (37) and (38). The operator V of (1) has been obtained in Section 2.1.

We consider the conserved vectors for generators of BO equation. We have the following cases for classical generators:

case 1: $V = \frac{\partial}{\partial t}$. Using this operator, we can get $\mathrm{W} = -u_t$. Thus, the conservation vector of (1) can be obtained as

$$\begin{aligned}
\mathrm{T}^t &= 2\beta v {u_x}^2 + 2\beta v u u_{xx} + \gamma v u_{xxxx} + u_t v_t, \\
\mathrm{T}^x &= -u_t(2\beta u_x v - 2\beta u v_x - \gamma v_{xxx}) - u_{xt}(2\beta u v + \gamma v_{xx}) + \gamma u_{xxt} v_x - \gamma u_{xxxt} v.
\end{aligned} \tag{39}$$

For example, let $v = xt$ here, then

$$\begin{aligned}
{\mathrm{T}_1}^t &= 2\beta x t {u_x}^2 + 2\beta x t u u_{xx} + \gamma x t u_{xxxx} + x u_t, \\
{\mathrm{T}_1}^x &= -u_t(2\beta x t u_x - 2\beta x u) - 2\beta x t u u_{xt} + \gamma t u_{xxt} - \gamma x t u_{xxxt}.
\end{aligned} \tag{40}$$

The following situations are similar, and will not be explained one by one here.

case 2: $V = \frac{\partial}{\partial x}$. Using this operator, we can get $\mathrm{W} = -u_x$. Thus, the conservation vector of (1) can be obtained as

$$\begin{aligned}
\mathrm{T}^t &= u_x v_t - u_{xt} v, \\
\mathrm{T}^x &= v[u_{tt} + \beta(2{u_x}^2 + 2u u_{xx} + \gamma u_{xxxx})] \\
&- u_x(2\beta u_x v - 2\beta u v_x - \gamma v_{xxx}) \\
&- u_{xx}(2\beta u v + \gamma v_{xx}) + \gamma u_{xxx} v_x - \gamma u_{xxxx} v.
\end{aligned} \tag{41}$$

case 3: $V = -\frac{\partial}{\partial u}$. Using this operator, we can get $\mathrm{W} = 1$. Thus, the conservation vector of (1) can be obtained as

$$\begin{aligned}
\mathrm{T}^t &= -v_t, \\
\mathrm{T}^x &= 2\beta u_x v - 2\beta u v_x - \gamma v_{xxx}.
\end{aligned} \tag{42}$$

case 4: $V = t\partial t + \frac{x}{2}\partial x - u\partial u$. Using this operator, $\mathrm{W} = -u - t u_t - \frac{x}{2} u_x$. Thus, the conservation vector of (1) can be obtained as

$$\begin{aligned}
\mathrm{T}^t &= t v[u_{tt} + \beta(2{u_x}^2 + 2u u_{xx}) + \gamma u_{xxxx}] + (u + \frac{x}{2} u_x + t u_x) v_t \\
&- (2u_t + \frac{x}{2} u_{xt} + t u_{tt}) v. \\
\mathrm{T}^x &= \frac{x v}{2}[u_{tt} + \beta(2{u_x}^2 + 2u u_{xx}) + \gamma u_{xxxx}] - (u + \frac{x}{2} u_x + t u_t)(2\beta u_x v - 2\beta u v_x - \gamma v_{xxx}) \\
&- (\frac{3}{2} u_x + \frac{x}{2} u_{xx} - t u_{xt})(2\beta u v + \gamma u_{xx}) + (2u_{xx} - \frac{x}{2} u_{xxx} + t u_{xxt}) \gamma v_x \\
&- (\frac{5}{2} u_{xxx} + \frac{x}{2} u_{xxxx} - t u_{xxxt}) \gamma v.
\end{aligned} \tag{43}$$

This vector can obtain any solution V to the adjoint Equation (33) and provides an unlimited number of conservation laws for the BO equation.

Remark 3. *v is the solution of the adjoint equation. We can then find the solution u of Equation (1) according to v. By taking different special solutions of V, more conservation laws of Equation (1) can be obtained. The conservation laws listed here are trivial.*

Remark 4. *The accuracy of the conservation vector* $\left(T^t, T^x\right)$ *has been checked using Maple software.*

5. Conclusions

In this paper, using the Lie group analysis symmetry of the nonlinear Benjamin–Ono equation, the classical Lie group symmetry and the relationship between the new solution and the old solution, we can solve the new special solution of the BO equation. At the same time, by reducing the original equation, we get the particular solution of the correlation generator. Finally, according to the obtained Lie symmetry generator, we construct the conservation law of the related classical vector field of the equation. These conclusions may help to explain some practical physical problems and provide a theoretical basis and methods for solving practical problems.

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