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Formulation, Solution's Existence, and Stability Analysis for Multi-Term System of Fractional-Order Differential Equations

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Abstract: In the recent past, multi-term fractional equations have been studied using symmetry methods. In some cases, many practical test problems with some symmetries are provided to demonstrate the authenticity and utility of the used techniques. Fractional-order differential equations can be formulated by using two types of differential operators: single-term and multi-term differential operators. Boundary value problems with single- as well as multi-term differential operators have been extensively studied, but several multi-term fractional differential equations still need to be formulated, and examination should be done with symmetry or any other feasible techniques. Therefore, the purpose of the present research work is the formulation and study of a new couple system of multi-term fractional differential equations with delay, as well as supplementation with nonlocal boundary conditions. After model formulation, the existence of a solution and the uniqueness conditions will be developed, utilizing fixed point theory and functional analysis. Moreover, results related to Ulam's and other types of functional stability will be explored, and an example is carried out to illustrate the findings of the work.

Keywords: fractional differential equations; multi-term operators; existence and uniqueness of solution; functional stability; delay term

MSC: 26A33; 34A08; 35B40

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1. Introduction

The study of fractional-order mathematical models, which govern real-world problems appearing in various disciplines of engineering, has been extensively improved by the methods of fractional calculus. The main reason for the importance of the selection of fractional-order differential and integral operators is the nonlocal behavior of these operators. Theoretical and applied aspects of fractional-order differential equations have been extensively explored. Agarwal et al. in [1] used Pettis integral and nonreflexive Banach spaces to establish results related to the existence of Abel's integral equation and Cauchy type problem. A new aspect known as practical stability has been studied by the authors in [2]. This new notion of stability enables us to study the behavior of a couple of solutions when both initial values and intervals are different. Positive solutions for a class of boundary value problem, equipped with p-Laplacian operator and multipoint, are studied with the help of a fixed-point theorem of mixed-type monotone operators [3]. To see the effect of initial conditions on the fractional derivative, a model is formulated in [4], which also discusses the delay differential equation for the existence of a solution.

Besides the theoretical aspects of fractional differential equations, some applications of this emerging field can also be found in economics, ecology, immune systems, pathology, and chaos theory [5–10]. In this connection, various researchers have been inspired by the great popularity of fractional differential equations (FDEs) and used the theory to some extent. Numerical investigation of Whitham–Broer–Kaup equations was carried out by Ali et al. in [11], utilizing Laplace transform together with Adomian decomposition method. For more exploration of the numerical solution of partial and ordinary FDEs, one can refer to [12–15]. More information regarding the theoretical and practical development of FDE can be found in [16–21] and the references therein.

Another important aspect of FDE is the investigation of the well posedness of a model formulated via fractional differential operator. In this connection, the existence and uniqueness of a solution studied by various techniques, such as monotone iterative technique, were discussed in [22]; topological degree theory was the concern in [23]; a method of successive approximation was developed in [24]; fixed point index theory was used in [25], and tools of fixed point theory (FPT) were used in [26]. For more details, see the recent work cited as [27–30].

Initial and boundary values problems (BVP) of ordinary and fractional-order differential equations have many applications in the modeling of biological problems and dynamical systems. Nevertheless, many physical, chemical, and other processes depend not only on boundary points but on the interior of the domain. Therefore, to analyze the underlying problem, we need a novel boundary and initial conditions called the nonlocal conditions. These special types of conditions relate initial and boundary values of the solution to their values in the interior points of the domain [31]. For further details, [32–34] can enrich the reader's understanding regarding nonlocal conditions.

The first FDE supplemented the classical boundary condition with two points and was studied by Benchohra et al. in [35] using results from nonlinear analysis and FPT. The authors first converted the underlying problem to the corresponding integral equation and then used Schaefer's fixed point theorem for the existence of a solution. Furthermore, the solution uniqueness result was also developed by utilizing the Banach contraction principle. To be precise, the model under consideration is given as follows:

$$\begin{aligned} &{}^{c}D^{\alpha_1}\mathscr{V}(t) = \mathscr{E}(t, \mathscr{V}(t)), \quad 0 \le t \le T, \\ &a_0\mathscr{V}(0) + a_1\mathscr{V}(1) = c, \end{aligned}$$

where a_0, a_1, c, and T are arbitrary constants, with the given condition that the sum of a_0 and a_1 must be a non-zero real number together with $T > 0$. The fractional differential operator used in the above-mentioned model is the Caputo operator, which is symbolized as ${}^{c}D^{\alpha_1}$, where α_1 is any real number from the semi-open interval (0,1]. Furthermore, the operator $\mathscr{E}$ is a given continuous function in the corresponding domain. The authors also stated that the problem under consideration takes the form of an initial value problem for $a_0 = 1$ and $a_1 = 0$, while the terminal value problem can be obtained by fixing $a_0 = 0$ and $a_1 = 1$.. An appropriate example was also given to justify the obtained results.

The imposing nonlocal conditions on the unknown function of second-order FDE were investigated in [36]. The author derived the desired result by using Krasnoselskii's fixed point theorem and Banach contraction to demonstrate the existence of a solution to the following model:

$$\begin{aligned} &{}^{c}D^{\alpha_1}\mathscr{V}(t) = \mathscr{H}(t, \mathscr{V}(t)), \quad 0 \le t \le 1, \\ &a_0\mathscr{V}(0) - a_1\mathscr{V}'(0) = a_2\mathscr{V}(\sigma_1), \quad b_0\mathscr{V}(1) + b_1\mathscr{V}'(1) = b_2\mathscr{V}(\sigma_2). \end{aligned}$$

Here, the nonlinear function $\mathscr{H} : [0,1] \times R \mapsto R$ is a given continuous function, while $a_i,\ b_i(i = 1,2,3)$ are taken from the set of all real numbers. The points σ_1 and σ_2 are taken from the interior of the domain, and the considered model uses a Caputo fractional derivative of order α_1, which is mathematically symbolized as ${}^{c}D^{\alpha_1}$, where $\alpha_1 \in (0,2]$. Moreover, the arbitrary constant should not satisfy the following condition: $(a_2 + a_3\sigma_1)(b_1 - b_3) = (b_1 + b_2 - b_3\sigma_2)(a_1 - a_3)$.

Another example is related to geophysical morphodynamics. This type of investigation is associated with the dynamical aspect of coastlines and sandbanks [37]. Based on these applications, researchers recently studied mathematical aspects and different types of stabilities for a coupled system of fractional-order differential equations.

In real-world problems, there exist several phenomena that can be modeled and studied by a coupled system of differential equations, rather than other types of differential equations. Among various other applications, binary fluid convection is an example of fractional calculus in physics [38]. In binary fluid convection, two ideal infinitely long plates are considered as a channel for fluid flow.

In this regard, Ahmad et al. [39] investigated the following coupled system of pantograph differential equations:

$$\begin{aligned}
&{}^cD^{\eta}\mathscr{V}_1(t) = \mathscr{F}(t, \mathscr{V}_1(t), \mathscr{V}_1(\lambda t), \mathscr{V}_2(t)),\\
&{}^cD^{\beta}\mathscr{V}_2(t) = \mathscr{K}(t, \mathscr{V}_2(t), \mathscr{V}_2(\lambda t), \mathscr{V}_1(t)),\ \eta,\ \beta \in (0,1],\ 0 < \lambda < 1, \quad t \in [0,1],\\
&h_1(\mathscr{V}_1) = a_1\mathscr{V}_1(0) - b_1\mathscr{V}_1(\zeta_1) - c_1\mathscr{V}_1(1), \quad h_2(\mathscr{V}_2) = a_2\mathscr{V}_2(0) - b_2\mathscr{V}_2(\zeta_2) - c_2\mathscr{V}_2(1).
\end{aligned} \tag{1}$$

Here, the domain of the unknown function is unit interval [0,1], while the nonlinear functions $\mathscr{F}, \mathscr{K} : [0,1] \times (-\infty,\infty)^{[3]} \mapsto (-\infty,\infty)$ are continuous. The given continuous function $h_1, h_2 : C([0,1], R) \mapsto (-\infty,\infty)$, which is involved in the auxiliary conditions, makes the considered problem non-local, and the arbitrary constants $a_j, b_j (j = 1,2,3)$ will be chosen from the set of real numbers. The aforementioned system of FDEs utilized a Caputo derivative of order η and β, symbolized as ${}^cD^{\eta}$ and ${}^cD^{\beta}$, respectively. The arbitrary constants from the set of real numbers can be chosen such that $a_j - b_j - c_j \neq 0$ for j=1,2. The solution's existence and uniqueness were obtained by using FPT and results from non-linear analysis. Furthermore, after deriving the integral form of the solution, the considered problem Equation (1) was studied with the help of an operator. Banach contraction principle and Shafer's fixed point theorems were considered in developing the desired results. Moreover, the results for Ulam–Hyres (UH) and generalized Ulam–Hyres (GUH) stability together with appropriate examples were also considered.

The study of differential equations equipped with multi-term differential operators of fractional orders has been given very little attention. Exploration of the functional stability of such kinds of multi-term differential operators has been given negligible attention. Therefore, motivated by the existence results discussed above and applications of multi-term FED, we formulate a new model here. This novel model consists of a coupled system of FDEs containing more than one fractional derivative of first order supplemented with non-local boundary conditions. The novel aspects of our newly formulated model are as follows: To the best of our knowledge no contribution exists regarding the formulation of coupled systems of FDEs, equipped with multi-term fractional differential operators. Therefore, the formulation of a coupled system with n-fractional differential multi-term differential operators is the first novel aspect of our work. As mentioned, less attention is given to the functional stability of multi-term FDEs, so our analysis of the four types of UH stability is our second novel aspect. The final novel aspect is the incorporation of nonlocal conditions and proportional type delay. The model under consideration is given as follow:

$$\begin{cases}
\sum_{i=1}^{n} \sigma_i {}^cD^{\alpha_i}\mathscr{V}_1(t) = f_1(t, \mathscr{V}_1(t), \mathscr{V}_1(\lambda_1 t), \mathscr{V}_2(t)), \lambda_1, \lambda_2, t \in [0,1],\\
\sum_{i=1}^{n} \eta_i {}^cD^{\beta_i}\mathscr{V}_2(t) = f_2(t, \mathscr{V}_2(t), \mathscr{V}_2(\lambda_2 t), \mathscr{V}_1(t)), \alpha_i,\ \beta_i \in (0,1],\ \text{for}\ \ i = 1,2,3,\dots,n\\
g_1(\mathscr{V}_1) = a_1\mathscr{V}_1(0) - b_1\mathscr{V}_1(\zeta_1) - c_1\mathscr{V}_1(1), \quad g_2(\mathscr{V}_2) = a_2\mathscr{V}_2(0) - b_2\mathscr{V}_2(\zeta_2) - c_2\mathscr{V}_2(1).
\end{cases} \tag{2}$$

In the considered problem Equation (2), standard Caputo fractional-order derivatives of order $\alpha_i, \beta_i \in (0,1]$, symbolized as ${}^cD^{\alpha_i}$, ${}^cD^{\beta_i}(i = 1,2\dots,n)$, are utilized with the conditions that $\alpha_1 > \alpha_i$ and $\beta_1 > \beta_i$ for i = 2, 3, . . . , n. The auxiliary conditions imposed on the given problem are nonlocal due to the points ζ_1 and ζ_2 being chosen from the interior of the domain. The arbitrary constants $\eta_i,\ \sigma_i,\ a_j,\ b_j,\ c_j,\ (j = 1,2),\ (i = 1,2,\dots,n)$ can be selected from the set of real numbers, with conditions that $\eta_1 \neq 0, \sigma_1 \neq 0$, $a_1 \neq b_1 + c_1$, and $a_2 \neq b_2 + c_2$. The non-linear functions $f_1, f_2 : [0,1] \times R \times R \times R \mapsto R$ and $g_1, g_2 : C([0,1], R) \times R \mapsto R$ are continuous, where $1 \in R$ is any real number greater than zero.

The scope of work for studying the model represented in Equation (2) is given as follows: The tools of FPT and functional analysis will be used to obtain the conditions for solution existence and uniqueness. In this regard, the problem under consideration will be converted into an integral equation and then to a fixed point problem. To achieve our goals, we will impose certain assumptions on the given functions and arbitrary constants. Furthermore, Banach's contraction principle and Krasnoselskii's fixed point theorem will be used in the investigation of uniqueness and at least one solution, respectively. Moreover, the solution of the problem under consideration has interesting behavior under certain assumptions imposed on the given data. In addition to the above, results regarding Ulam's type stability, such as UH, GUH, UH Rassias, and GUH Rassias, will be derived. For verification of the obtained results, we give an appropriate example.

The structure of the rest of the paper is as follows. Basic supporting results will be given in Section 2. The results that rely on Krasnoselskii's fixed point theorem and Banach contraction principle will be provided in Section 3. Stability-related results are explored in Section 4. An elaborating example is provided at the end of the obtained results.

2. Preliminaries

This section of research is devoted to basic results and definitions of FPT and fractional calculus, which will be needed for investigation of the main work.

Definition 1 ([17]). *"The integral of fractional-order α of a function $\mathscr{V}(t) \in L[0,d]$ is denoted by $I^{\alpha}\mathscr{V}(t)$ and defined as*

$$\mathbb{I}^{\alpha}\mathscr{V}(t) = \int_0^t \frac{\mathscr{V}(\chi)}{\Gamma(\alpha)(t-\chi)^{1-\alpha}} d\chi,$$

provided that the right-hand side is pointwise convergent".

Definition 2 ([17]). *"The fractional-order Caputo derivative for a function $\mathscr{V}(t) \in L^1([0,d], R_+)$ on the interval $[0,d]$ is defined as*

$$^{c}D^{\alpha}\mathscr{V}(t) = \int_0^t \frac{\mathscr{V}^{n}(\chi)}{\Gamma(n-\alpha)(t-\chi)^{\alpha+1-n}} d\chi,$$

provided that the right-hand side is pointwise convergent. Where $n = \lceil \alpha \rceil$, and $\lceil \alpha \rceil$ is defined to be the smallest integer equal or greater than α".

Theorem 1 ([17]). *"The solution of FDE*

$$^{c}D^{\alpha}\mathscr{V}(t) = 0, \quad where \quad n-1 < \alpha \leq n,$$

is given by

$$\mathscr{V}(t) = A_1 + A_2 t + A_3 t^2 + A_4 t^3 + \dots\dots\dots A_n t^{n-1},$$

where $A_i \in R$ for $i = 1, 2, \dots, n$".

Lemma 1 ([17]). *"The relation between fractional-order integral and derivative is given as*

$$I^{\alpha}[^{c}D^{\alpha}\mathscr{V}(t)] = A_1 + A_2 t + A_3 t^2 + A_4 t^3 + \dots\dots\dots A_n t^{n-1} + \mathscr{V}(t),$$

where $A_i \in R$ for $i = 1, 2, \dots, n$".

Theorem 2 ([40]). *" Assume that $\mathcal{H}$ is a non-empty, convex, bounded, and closed subset of a Banach space $\mathcal{X}$. Let $\mathscr{J}_1$ and $\mathscr{J}_1$ be two operators, provided that $\mathscr{J}_1\mathscr{U}_1 + \mathscr{J}_2\mathscr{U}_2 \in \mathcal{H}$ whenever $\mathscr{U}, \mathscr{U}_2 \in \mathcal{H}$, $\mathscr{J}_1$ is continuous and compact, and $\mathscr{J}_2$ is a contraction map. Then, we get $\mathscr{U} \in \mathcal{H}$, provided that $\mathscr{U} = \mathscr{J}_1\mathscr{U} + \mathscr{J}_2\mathscr{U}$".*

3. Main Result

This section is devoted to the main results associated with our newly formulated model.

Theorem 3. *For $\mathscr{N}_1, \mathscr{N}_2 \in C(J, R)$, the solution of system of linear MFDDEs,*

$$\begin{cases} \sum_{i=1}^{n} \sigma_i {}^c D^{\alpha_i} \mathscr{V}_1(t) = \mathscr{N}_1(t), \\ \sum_{i=1}^{n} \eta_i {}^c D^{\beta_i} \mathscr{V}_2(t) = \mathscr{N}_2(t), \alpha_i,\ \beta_i \in (0,1],\ for\ \ i = 1,2,3,\dots,n, \quad t \in [0,1], \\ g_1(\mathscr{V}_1) = a_1\mathscr{V}_1(0) - b_1\mathscr{V}_1(\zeta_1) - c_1\mathscr{V}_1(1), \quad g_2(\mathscr{V}_2) = a_2\mathscr{V}_2(0) - b_2\mathscr{V}_2(\zeta_2) - c_2\mathscr{V}_2(1), \end{cases} \tag{3}$$

is given by

$$\begin{aligned} \mathscr{V}_1(t) &= -d_1 g_1(\mathscr{V}_1) - \frac{d_1 b_1}{\sigma_1 \Gamma(\alpha_1)} \int_0^{\zeta_1} (\zeta_1 - \mathscr{U})^{\alpha_1 - 1} \mathscr{N}_1(\mathscr{U}) d\mathscr{U} \\ &+ \sum_{i=2}^{n} \frac{d_1 b_1 \sigma_i}{\sigma_1 \Gamma(\alpha_1 - \alpha_i)} \int_0^{\zeta_1} (\zeta_1 - \mathscr{U})^{\alpha_1 - \alpha_i - 1} \mathscr{V}_1(\mathscr{U}) d\mathscr{U} - \frac{d_1 c_1}{\sigma_1 \Gamma(\alpha_1)} \int_0^1 (1 - \mathscr{U})^{\alpha_1 - 1} \mathscr{N}_1(\mathscr{U}) d\mathscr{U} \\ &+ \sum_{i=2}^{n} \frac{d_1 c_1 \sigma_i}{\sigma_1 \Gamma(\alpha_1 - \alpha_i)} \int_0^1 (1 - \mathscr{U})^{\alpha_1 - \alpha_i - 1} \mathscr{V}_1(\mathscr{U}) d\mathscr{U} + \frac{1}{\sigma_1 \Gamma(\alpha_1)} \int_0^t (t - \mathscr{U})^{\alpha_1 - 1} \mathscr{N}_1(\mathscr{U}) d\mathscr{U} \\ &- \sum_{i=2}^{n} \frac{\sigma_i}{\sigma_1 \Gamma(\alpha_1 - \alpha_i)} \int_0^t (t - \mathscr{U})^{\alpha_1 - \alpha_i - 1} \mathscr{V}_1(\mathscr{U}) d\mathscr{U}, \\ \mathscr{V}_2(t) &= -d_2 g_2(\mathscr{V}_2) - \frac{d_2 b_2}{\eta_1 \Gamma(\beta_1)} \int_0^{\zeta_2} (\zeta_2 - \mathscr{U})^{\beta_1 - 1} \mathscr{N}_2(\mathscr{U}) d\mathscr{U} \\ &+ \sum_{i=2}^{n} \frac{d_2 b_2 \eta_i}{\eta_1 \Gamma(\beta_1 - \beta_i)} \int_0^{\zeta_2} (\zeta_2 - \mathscr{U})^{\beta_1 - \beta_i - 1} \mathscr{V}_2(\mathscr{U}) d\mathscr{U} - \frac{d_2 c_2}{\eta_1 \Gamma(\beta_1)} \int_0^1 (1 - \mathscr{U})^{\beta_1 - 1} \mathscr{N}_2(\mathscr{U}) d\mathscr{U} \\ &+ \sum_{i=2}^{n} \frac{d_2 c_2 \eta_i}{\eta_1 \Gamma(\beta_1 - \beta_i)} \int_0^1 (1 - \mathscr{U})^{\beta_1 - \beta_i - 1} \mathscr{V}_2(\mathscr{U}) d\mathscr{U} + \frac{1}{\eta_1 \Gamma(\beta_1)} \int_0^t (t - \mathscr{U})^{\beta_1 - 1} \mathscr{N}_2(\mathscr{U}) d\mathscr{U} \\ &- \sum_{i=2}^{n} \frac{\eta_i}{\eta_1 \Gamma(\beta_1 - \beta_i)} \int_0^t (t - \mathscr{U})^{\beta_1 - \beta_i - 1} \mathscr{V}_2(\mathscr{U}) d\mathscr{U}, \end{aligned} \tag{4}$$

where $d_1 = \frac{1}{-a_1+b_1+c_1}$, $\quad d_2 = \frac{1}{-a_2+b_2+c_2}$, $\quad a_1 - b_1 - c_1 \neq 0$, *and* $a_2 - b_2 - c_2 \neq 0$.

Proof. Applying fractional-order integral $I^{\alpha_1}, I^{\beta_1}$ on Equation (3), and in view of Lemma (1),

$$\begin{aligned} \mathscr{V}_1(t) &= A_1 + \frac{1}{\sigma_1 \Gamma(\alpha_1)} \int_0^t (t - \mathscr{U})^{\alpha_1 - 1} \mathscr{N}_1(\mathscr{U}) d\mathscr{U} \\ &- \sum_{i=2}^{n} \frac{\sigma_i}{\sigma_1 \Gamma(\alpha_1 - \alpha_i)} \int_0^t (t - \mathscr{U})^{\alpha_1 - \alpha_i - 1} \mathscr{V}_1(\mathscr{U}) d\mathscr{U}, \\ \mathscr{V}_2(t) &= B_1 + \frac{1}{\eta_1 \Gamma(\beta_1)} \int_0^t (t - \mathscr{U})^{\beta_1 - 1} \mathscr{N}_2(\mathscr{U}) d\mathscr{U} \\ &- \sum_{i=2}^{n} \frac{\eta_i}{\eta_1 \Gamma(\beta_1 - \beta_i)} \int_0^t (t - \mathscr{U})^{\beta_1 - \beta_i - 1} \mathscr{V}_2(\mathscr{U}) d\mathscr{U}. \end{aligned} \tag{5}$$

Now, by using the subsidiary condition Equation (3) in Equation (5),

$$
\begin{aligned}
A_1 &= -g_1(\mathscr{V}_1)\frac{1}{-a_1+b_1+c_1} - \frac{1}{-a_1+b_1+c_1}\frac{b_1}{\sigma_1\Gamma(\alpha_1)}\int_0^{\zeta_1}(\zeta_1-\mathscr{U})^{\alpha_1-1}\mathscr{N}_1(\mathscr{U})d\mathscr{U} \\
&+ \frac{1}{-a_1+b_1+c_1}\sum_{i=2}^{n}\frac{b_1\sigma_i}{\sigma_1\Gamma(\alpha_1-\alpha_i)}\int_0^{\zeta_1}(\zeta_1-\mathscr{U})^{\alpha_1-\alpha_i-1}\mathscr{V}_1(\mathscr{U})d\mathscr{U} \\
&- \frac{1}{-a_1+b_1+c_1}\frac{c_1}{\sigma_1\Gamma(\alpha_1)}\int_0^{1}(1-\mathscr{U})^{\alpha_1-1}\mathscr{N}_1(\mathscr{U})d\mathscr{U} \\
&+ \frac{1}{-a_1+b_1+c_1}\sum_{i=2}^{n}\frac{c_1\sigma_i}{\sigma_1\Gamma(\alpha_1-\alpha_i)}\int_0^{1}(1-\mathscr{U})^{\alpha_1-\alpha_i-1}\mathscr{V}_1(\mathscr{U})d\mathscr{U}, \\
B_1 &= -g_2(\mathscr{V}_2)\frac{1}{-a_2+b_2+c_2} - \frac{1}{-a_2+b_2+c_2}\frac{b_2}{\eta_1\Gamma(\beta_1)}\int_0^{\zeta_2}(\zeta_2-\mathscr{U})^{\beta_1-1}\mathscr{N}_2(\mathscr{U})d\mathscr{U} \\
&+ \frac{1}{-a_2+b_2+c_2}\sum_{i=2}^{n}\frac{b_2\eta_i}{\eta_1\Gamma(\beta_1-\beta_i)}\int_0^{\zeta_2}(\zeta_2-\mathscr{U})^{\beta_1-\beta_i-1}\mathscr{V}_2(\mathscr{U})d\mathscr{U} \\
&- \frac{1}{-a_2+b_2+c_2}\frac{c_2}{\eta_1\Gamma(\beta_1)}\int_0^{1}(1-\mathscr{U})^{\beta_1-1}\mathscr{N}_2(\mathscr{U})d\mathscr{U} \\
&+ \frac{1}{-a_2+b_2+c_2}\sum_{i=2}^{n}\frac{c_2\eta_i}{\eta_1\Gamma(\beta_1-\beta_i)}\int_0^{1}(1-\mathscr{U})^{\beta_1-\beta_i-1}\mathscr{V}_2(\mathscr{U})d\mathscr{U}.
\end{aligned} \tag{6}
$$

One can obtain the desired solution Equation (4) by using Equation (6) in (5) and assuming that $d_1 = \frac{1}{-a_1+b_1+c_1} \neq 0,\ d_2 = \frac{1}{-a_2+b_2+c_2} \neq 0.$ □

Corollary 1. *In view of Theorem (3), the solution of the given system of MFDDEs*

$$
\begin{aligned}
&\sum_{i=1}^{n}\sigma_i{}^{c}D^{\alpha_i}\mathscr{V}_1(t) = f_1(t,\mathscr{V}_1(t),\mathscr{V}_1(\lambda_1 t),\mathscr{V}_2(t)),\ \lambda_1,\ \lambda_2 \in [0,1], \\
&\sum_{i=1}^{n}\eta_i{}^{c}D^{\beta_i}\mathscr{V}_2(t) = f_2(t,\mathscr{V}_2(t),\mathscr{V}_2(\lambda_2 t),\mathscr{V}_1(t)),\ \alpha_i,\ \beta_i \in (0,1],\ \textit{for}\ \ i = 1,2,3,\ldots,n,\ t\in[0,1], \\
&g_1(\mathscr{V}_1) = a_1\mathscr{V}_1(0) - b_1\mathscr{V}_1(\zeta_1) - c_1\mathscr{V}_1(1), \quad g_2(\mathscr{V}_2) = a_2\mathscr{V}_2(0) - b_2\mathscr{V}_2(\zeta_2) - c_2\mathscr{V}_2(1),
\end{aligned}
$$

is given by

$$
\begin{aligned}
\mathscr{V}_1(t) &= -d_1 g_1(\mathscr{V}_1) - \frac{d_1 b_1}{\sigma_1\Gamma(\alpha_1)}\int_0^{\zeta_1}(\zeta_1-\mathscr{U})^{\alpha_1-1}f_1(\mathscr{U},\mathscr{V}_1(\mathscr{U}),\mathscr{V}_1(\lambda_1\mathscr{U}),\mathscr{V}_2(\mathscr{U}))d\mathscr{U} \\
&+ \sum_{i=2}^{n}\frac{d_1 b_1\sigma_i}{\sigma_1\Gamma(\alpha_1-\alpha_i)}\int_0^{\zeta_1}(\zeta_1-\mathscr{U})^{\alpha_1-\alpha_i-1}\mathscr{V}_1(\mathscr{U})d\mathscr{U} \\
&- \frac{d_1 c_1}{\sigma_1\Gamma(\alpha_1)}\int_0^{1}(1-\mathscr{U})^{\alpha_1-1}f_1(\mathscr{U},\mathscr{V}_1(\mathscr{U}),\mathscr{V}_1(\lambda_1\mathscr{U}),\mathscr{V}_2(\mathscr{U}))d\mathscr{U} \\
&+ \sum_{i=2}^{n}\frac{d_1 c_1\sigma_i}{\sigma_1\Gamma(\alpha_1-\alpha_i)}\int_0^{1}(1-\mathscr{U})^{\alpha_1-\alpha_i-1}\mathscr{V}_1(\mathscr{U})d\mathscr{U} \\
&+ \frac{1}{\sigma_1\Gamma(\alpha_1)}\int_0^{t}(t-\mathscr{U})^{\alpha_1-1}f_1(\mathscr{U},\mathscr{V}_1(\mathscr{U}),\mathscr{V}_1(\lambda_1\mathscr{U}),\mathscr{V}_2(\mathscr{U}))d\mathscr{U} \\
&- \sum_{i=2}^{n}\frac{\sigma_i}{\sigma_1\Gamma(\alpha_1-\alpha_i)}\int_0^{t}(t-\mathscr{U})^{\alpha_1-\alpha_i-1}\mathscr{V}_1(\mathscr{U})d\mathscr{U},
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{V}_2(t) = & -d_2 g_2(\mathscr{V}_2) - \frac{d_2 b_2}{\eta_1 \Gamma(\beta_1)} \int_0^{\zeta_2} (\zeta_2 - \mathscr{U})^{\beta_1 - 1} f_2(\mathscr{U}, \mathscr{V}_2(\mathscr{U}), \mathscr{V}_2(\lambda_2 \mathscr{U}), \mathscr{V}_1(\mathscr{U})) d\mathscr{U} \\
& + \sum_{i=2}^{n} \frac{d_2 b_2 \eta_i}{\eta_1 \Gamma(\beta_1 - \beta_i)} \int_0^{\zeta_2} (\zeta_2 - \mathscr{U})^{\beta_1 - \beta_i - 1} \mathscr{V}_2(\mathscr{U}) d\mathscr{U} \\
& - \frac{d_2 c_2}{\eta_1 \Gamma(\beta_1)} \int_0^{1} (1 - \mathscr{U})^{\beta_1 - 1} f_2(\mathscr{U}, \mathscr{V}_2(\mathscr{U}), \mathscr{V}_2(\lambda_2 \mathscr{U}), \mathscr{V}_1(\mathscr{U})) d\mathscr{U} \\
& + \sum_{i=2}^{n} \frac{d_2 c_2 \eta_i}{\eta_1 \Gamma(\beta_1 - \beta_i)} \int_0^{1} (1 - \mathscr{U})^{\beta_1 - \beta_i - 1} \mathscr{V}_2(\mathscr{U}) d\mathscr{U} \\
& + \frac{1}{\eta_1 \Gamma(\beta_1)} \int_0^{t} (t - \mathscr{U})^{\beta_1 - 1} f_2(\mathscr{U}, \mathscr{V}_2(\mathscr{U}), \mathscr{V}_2(\lambda_2 \mathscr{U}), \mathscr{V}_1(\mathscr{U})) d\mathscr{U} \\
& - \sum_{i=2}^{n} \frac{\eta_i}{\eta_1 \Gamma(\beta_1 - \beta_i)} \int_0^{t} (t - \mathscr{U})^{\beta_1 - \beta_i - 1} \mathscr{V}_2(\mathscr{U}) d\mathscr{U}.
\end{aligned}
$$

where $d_1 = \frac{1}{-a_1 + b_1 + c_1}$, $\quad d_2 = \frac{1}{-a_2 + b_2 + c_2}$, $\quad a_1 \neq b_1 + c_1$, *and* $a_2 \neq b_2 + c_2$.

Assumptions and Existence Results

There are several approaches for studying the existence of a solution and stability. These include iterative methods, topological degree theory, fixed point theory, and so on. In each approach, we need to impose some auxiliary information in the form of hypotheses. Here, we use operator theory, fixed point theorems, and some results from functional analysis. Therefore, the required solution of the underlying MFDDEs Equation (2) will basically be expressed in the form of an operator equation and will also provide some subsidiary assumptions for the proposed model.

Let X = C (J,R) denote the subspace of all continuous functions defined on interval J, and let us formulate an operator $\mathcal{T} : X \times X \mapsto X \times X$ by the following:

$\mathcal{T}\Big(\mathscr{V}_1(t), \mathscr{V}_2(t)\Big) = \Big(\mathcal{T}_1(\mathscr{V}_1(t), \mathscr{V}_2(t)), \mathcal{T}_2(\mathscr{V}_1(t), \mathscr{V}_2(t))\Big)$, where $\Big(X \times X, ||(\mathscr{V}_1, \mathscr{V}_2)||\Big)$ is Banach space with norm given as $||(\mathscr{V}_1, \mathscr{V}_2)|| = \sup_{t \in J}\{|\mathscr{V}_1(t)|\} + \sup_{t \in J}\{|\mathscr{V}_2(t)|\} = ||\mathscr{V}_1|| + ||\mathscr{V}_2||$ for every $\mathscr{V}_1, \mathscr{V}_2 \in X$ and

$$
\begin{aligned}
\mathcal{T}_1(\mathscr{V}_1(t), \mathscr{V}_2(t)) = & -d_1 g_1(\mathscr{V}_1) + \sum_{i=2}^{n} \frac{d_1 b_1 \sigma_i}{\sigma_1 \Gamma(\alpha_1 - \alpha_i)} \int_0^{\zeta_1} (\zeta_1 - \mathscr{U})^{\alpha_1 - \alpha_i - 1} \mathscr{V}_1(\mathscr{U}) d\mathscr{U} \\
& - \frac{d_1 b_1}{\sigma_1 \Gamma(\alpha_1)} \int_0^{\zeta_1} (\zeta_1 - \mathscr{U})^{\alpha_1 - 1} f_1(\mathscr{U}, \mathscr{V}_1(\mathscr{U}), \mathscr{V}_1(\lambda_1 \mathscr{U}), \mathscr{V}_2(\mathscr{U})) d\mathscr{U} \\
& - \frac{d_1 c_1}{\sigma_1 \Gamma(\alpha_1)} \int_0^{1} (1 - \mathscr{U})^{\alpha_1 - 1} f_1(\mathscr{U}, \mathscr{V}_1(\mathscr{U}), \mathscr{V}_1(\lambda_1 \mathscr{U}), \mathscr{V}_2(\mathscr{U})) d\mathscr{U} \\
& + \sum_{i=2}^{n} \frac{d_1 c_1 \sigma_i}{\sigma_1 \Gamma(\alpha_1 - \alpha_i)} \int_0^{1} (1 - \mathscr{U})^{\alpha_1 - \alpha_i - 1} \mathscr{V}_1(\mathscr{U}) d\mathscr{U} \\
& + \frac{1}{\sigma_1 \Gamma(\alpha_1)} \int_0^{t} (t - \mathscr{U})^{\alpha_1 - 1} f_1(\mathscr{U}, \mathscr{V}_1(\mathscr{U}), \mathscr{V}_1(\lambda_1 \mathscr{U}), \mathscr{V}_2(\mathscr{U})) d\mathscr{U} \\
& - \sum_{i=2}^{n} \frac{\sigma_i}{\sigma_1 \Gamma(\alpha_1 - \alpha_i)} \int_0^{t} (t - \mathscr{U})^{\alpha_1 - \alpha_i - 1} \mathscr{V}_1(\mathscr{U}) d\mathscr{U},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T}_2\big(\mathscr{V}_1(t),\mathscr{V}_2(t)\big) = & -d_2 g_2(\mathscr{V}_2) + \sum_{i=2}^{n} \frac{d_2 b_2 \eta_i}{\eta_1 \Gamma(\beta_1-\beta_i)} \int_0^{\zeta_2} (\zeta_2-\mathscr{U})^{\beta_1-\beta_i-1} \mathscr{V}_2(\mathscr{U}) d\mathscr{U} \\
& - \frac{d_2 b_2}{\eta_1 \Gamma(\beta_1)} \int_0^{\zeta_2} (\zeta_2-\mathscr{U})^{\beta_1-1} f_2(\mathscr{U},\mathscr{V}_2(\mathscr{U}),\mathscr{V}_2(\lambda_2\mathscr{U}),\mathscr{V}_1(\mathscr{U})) d\mathscr{U} \\
& - \frac{d_2 c_2}{\eta_1 \Gamma(\beta_1)} \int_0^{1} (1-\mathscr{U})^{\beta_1-1} f_2(\mathscr{U},\mathscr{V}_2(\mathscr{U}),\mathscr{V}_2(\lambda_2\mathscr{U}),\mathscr{V}_1(\mathscr{U})) d\mathscr{U} \\
& + \sum_{i=2}^{n} \frac{d_2 c_2 \eta_i}{\eta_1 \Gamma(\beta_1-\beta_i)} \int_0^{1} (1-\mathscr{U})^{\beta_1-\beta_i-1} \mathscr{V}_2(\mathscr{U}) d\mathscr{U} \\
& + \frac{1}{\eta_1 \Gamma(\beta_1)} \int_0^{t} (t-\mathscr{U})^{\beta_1-1} f_2(\mathscr{U},\mathscr{V}_2(\mathscr{U}),\mathscr{V}_2(\lambda_2\mathscr{U}),\mathscr{V}_1(\mathscr{U})) d\mathscr{U} \\
& - \sum_{i=2}^{n} \frac{\eta_i}{\eta_1 \Gamma(\beta_1-\beta_i)} \int_0^{t} (t-\mathscr{U})^{\beta_1-\beta_i-1} \mathscr{V}_2(\mathscr{U}) d\mathscr{U}.
\end{aligned}
$$

We consider the following assumptions, which will be needed for further progress in this work.

H1. *For $\mathscr{V}_1, \mathscr{V}_1^* \in X$, there exist $\mathscr{L}_{f_1}, \mathscr{L}_{f_1^\lambda},\ \mathscr{L}_{f_1^c} \geq 0$, such that,*

$$
\begin{aligned}
& |f_1(t,\mathscr{V}_1(t),\mathscr{V}_1(\lambda_1 t),\mathscr{V}_2(t)) - f_1(t,\mathscr{V}_1^*(t),\mathscr{V}_1^*(\lambda_1 t),\mathscr{V}_2^*(t))| \\
& \quad \leq \mathscr{L}_{f_1}||\mathscr{V}_1-\mathscr{V}_1^*|| + \mathscr{L}_{f_1^\lambda}||\mathscr{V}_1(\lambda t)-\mathscr{V}_1^*(\lambda_1 t)|| + \mathscr{L}_{f_1^c}||\mathscr{V}_2-\mathscr{V}_2^*||.
\end{aligned}
$$

H2. *For $\mathscr{V}_1, \mathscr{V}_1^* \in X$, there exist $\mathscr{L}_{f_1}, \mathscr{L}_{f_1^\lambda},\ \mathscr{L}_{f_1^c} \geq 0$, such that*

$$
\begin{aligned}
& |f_2(t,\mathscr{V}_2(t),\mathscr{V}_2(\lambda_2 t),\mathscr{V}_1(t)) - f_2(t,\mathscr{V}_2^*(t),\mathscr{V}_2^*(\lambda_2 t),\mathscr{V}_1^*(t))| \\
& \quad \leq \mathscr{L}_{f_2}||\mathscr{V}_2-\mathscr{V}_2^*|| + \mathscr{L}_{f_2^\lambda}||\mathscr{V}_2(\lambda_2 t)-\mathscr{V}_2^*(\lambda_2 t)|| + \mathscr{L}_{f_2^c}||\mathscr{V}_1-\mathscr{V}_1^*||.
\end{aligned}
$$

H3. *For $\mathscr{V}_j, \mathscr{V}_j^* (j=1,2) \in X$, there exist $\mathscr{L}_{g_j} (j=1,2) > 0$, such that*

$$
|g_j(\mathscr{V}_j(t)) - g_j(\mathscr{V}_j^*(t))| \leq \mathscr{L}_{g_j}||\mathscr{V}_j-\mathscr{V}_j^*||.
$$

H4. *For any $\mathscr{V}_1 \in X$, there exist $\mathscr{H}_{f_1^a}, \mathscr{H}_{f_1^b}.\mathscr{H}_{f_1^c} : C(J,R^+)$, such that*

$$
|f_1(t,\mathscr{V}_1(t),\mathscr{V}_1(\lambda_1 t)),\mathscr{V}_2| \leq \mathscr{H}_{f_1^a}(t) + \mathscr{H}_{f_1^b}(t)|\mathscr{V}_1| + \mathscr{H}_{f_1^c}(t)|\mathscr{V}_2|.
$$

H5. *For any $\mathscr{V}_2 \in X$, there exist $\mathscr{H}_{f_2^a}, \mathscr{H}_{f_2^b}.\mathscr{H}_{f_2^c} : C(J,R^+)$, such that*

$$
|f_2(t,\mathscr{V}_2(t),\mathscr{V}_2(\lambda_2 t)),\mathscr{V}_1| \leq \mathscr{H}_{f_2^a}(t) + \mathscr{H}_{f_2^b}(t)|\mathscr{V}_1| + \mathscr{H}_{f_2^c}(t)|\mathscr{V}_2|.
$$

H6. *For any $\mathscr{V}_2 \in X$, there exist $\mathscr{H}_{g_j} (j=1,2) \in C(J,R^+)$, such that*

$$
|g_j(\mathscr{V}_j)| \leq \mathscr{H}_{g_j}(t).
$$

Remark 1. *To move the calculations from tedious operations, the following notations will be used*

$$\mathcal{L}_1 = |d_1|\mathcal{L}_{g_1} + \sum_{i=2}^{n}\left(|d_1||b_1|\zeta_1^{\alpha_1-\alpha_i} + |d_1|c_1| + 1\right)\frac{|\sigma_i|}{|\sigma_1|\Gamma(\alpha_1-\alpha_i+1)} \tag{7}$$

$$+\left(|d_1||b_1|\zeta_1^{\alpha_1} + |d_1|c_1| + 1\right)\frac{(\mathcal{L}_{f_1} + \mathcal{L}_{f_1^{\lambda}+\mathcal{L}_{f_1^{c}}})}{|\sigma_1|\Gamma(\alpha_1+1)}, \tag{8}$$

$$\mathcal{L}_2 = |d_2|\mathcal{L}_{g_2} + \sum_{i=2}^{n}\left(|d_2||b_2|\zeta_2^{\beta_1-\beta_i} + |d_2|c_2| + 1\right)\frac{|\eta_i|}{|\eta_1|\Gamma(\beta_1-\beta_i+1)} \tag{9}$$

$$+\left(|d_2||b_2|\zeta_2^{\beta_1} + |d_2|c_2| + 1\right)\frac{(\mathcal{L}_{f_2} + \mathcal{L}_{f_2^{\lambda}+\mathcal{L}_{f_2^{c}}})}{|\eta_1|\Gamma(\beta_1+1)}, \tag{10}$$

$$\mathcal{L}_{1_a} = \sum_{i=2}^{n}\frac{|\sigma_i|}{|\sigma_1|\Gamma(\alpha_1-\alpha_i+1)} + \frac{\mathcal{L}_{f_1} + \mathcal{L}_{f_1^{\lambda}+\mathcal{L}_{f_1^{c}}}}{|\sigma_1|\Gamma(\alpha_1+1)}, \tag{11}$$

$$\mathcal{L}_{2_a} = \sum_{i=2}^{n}\frac{|\eta_i|}{|\eta_1|\Gamma(\beta_1-\beta_i+1)} + \frac{\mathcal{L}_{f_2} + \mathcal{L}_{f_2^{\lambda}+\mathcal{L}_{f_2^{c}}}}{|\eta_1|\Gamma(\beta_1+1)}, \tag{12}$$

$$\mathcal{R}_1 > \frac{|d_1|||\mathscr{H}_{g_1}(t)|| + \frac{\left[||\mathscr{H}_{f_1^a}(t)||\right]}{|\sigma_1|\Gamma(\alpha_1+1)}(|d_1||b_1|\sigma_1^{\alpha_1} + |d_1||c_1| + 1)}{1 - \frac{\left[||\mathscr{H}_{f_1^b}(t)|| + ||\mathscr{H}_{f_1^c}(t)||\right]}{|\sigma_1|\Gamma(\alpha_1+1)}(|d_1||b_1|\sigma_1^{\alpha_1} + |d_1||c_1| + 1) - \sum_{i=2}^{n}\frac{|\sigma_i|(|d_1||b_1|\sigma_1^{\alpha_1-\alpha_i} + |d_1||c_1|1)}{|\sigma_1|\Gamma(\alpha_1-\alpha_i+1)}}, \tag{13}$$

$$\mathcal{R}_2 > \frac{|d_2|||\mathscr{H}_{g_2}(t)|| + \frac{\left[||\mathscr{H}_{f_2^a}(t)||\right]}{|\eta_1|\Gamma(\beta_1+1)}(|d_2||b_2|\zeta_1^{\beta_1} + |d_2||c_2| + 1)}{1 - \frac{\left[||\mathscr{H}_{f_2^b}(t)|| + ||\mathscr{H}_{f_2^c}(t)||\right]}{|\eta_1|\Gamma(\beta_1+1)}(|d_2||b_2|\zeta_1^{\beta_1} + |d_2||c_2| + 1) - \sum_{i=2}^{n}\frac{|\eta_i|(|d_2||b_2|\zeta_1^{\beta_1-\beta_i} + |d_2||c_1|1)}{|\eta_1|\Gamma(\beta_1-\beta_i+1)}}. \tag{14}$$

Theorem 4. *Consider that (**H1**)–(**H3**) holds and $\mathcal{L} < 1$; then, the operator $\mathcal{T}$ has at most one fixed point, where $\mathcal{L} = \max\{\mathcal{L}_1, \mathcal{L}_2\}$ and $\mathcal{L}_1, \mathcal{L}_2$ are defined by Equations (7) and (9), respectively.*

Proof. Consider that $\mathscr{V}_j(t), \mathscr{V}_j^*(t) (j = 1, 2) \in X$,

$$\begin{aligned}
&\Big|\mathcal{T}_1\big(\mathscr{V}_1(t), \mathscr{V}_2(t)\big) - \mathcal{T}_1\big(\mathscr{V}_1^*(t), \mathscr{V}_2^*(t)\big)\Big| \leq |d_1||g_1(\mathscr{V}_1) - g_1(\mathscr{V}_1^*)| \\
&+ \frac{|d_1||b_1|}{|\sigma_1|\Gamma(\alpha_1)} \int_0^{\zeta_1} (\zeta_1 - \mathscr{U})^{\alpha_1 - 1} \Big| f_1(\mathscr{U}, \mathscr{V}_1(\mathscr{U}), \mathscr{V}_1(\lambda_1 \mathscr{U}), \mathscr{V}_2(\mathscr{U})) \\
&- f_1(\mathscr{U}, \mathscr{V}_1^*(\mathscr{U}), \mathscr{V}_1^*(\lambda_1 \mathscr{U}), \mathscr{V}_2^*(\mathscr{U}))\Big| d\mathscr{U} \\
&+ \sum_{i=2}^{n} \frac{|d_1||b_1||\sigma_i|}{|\sigma_1|\Gamma(\alpha_1 - \alpha_i)} \int_0^{\zeta_1} (\zeta_1 - \mathscr{U})^{\alpha_1 - \alpha_i - 1} \Big| \mathscr{V}_1(\mathscr{U}) - \mathscr{V}_1^*(\mathscr{U}) \Big| d\mathscr{U} \\
&+ \frac{|d_1||c_1|}{|\sigma_1|\Gamma(\alpha_1)} \int_0^{1} (1 - \mathscr{U})^{\alpha_1 - 1} \Big| f_1(\mathscr{U}, \mathscr{V}_1(\mathscr{U}), \mathscr{V}_1(\lambda_1 \mathscr{U}), \mathscr{V}_2(\mathscr{U})) \\
&- f_1(\mathscr{U}, \mathscr{V}_1^*(\mathscr{U}), \mathscr{V}_1^*(\lambda_1 \mathscr{U}), \mathscr{V}_2^*(\mathscr{U}))\Big| d\mathscr{U} \\
&+ \sum_{i=2}^{n} \frac{|d_1||c_1||\sigma_i|}{|\sigma_1|\Gamma(\alpha_1 - \alpha_i)} \int_0^{1} (1 - \mathscr{U})^{\alpha_1 - \alpha_i - 1} \Big| \mathscr{V}_1(\mathscr{U}) - \mathscr{V}_1^*(\mathscr{U}) \Big| d\mathscr{U} \\
&+ \frac{1}{|\sigma_1|\Gamma(\alpha_1)} \int_0^{t} (t - \mathscr{U})^{\alpha_1 - 1} \Big| f_1(\mathscr{U}, \mathscr{V}_1(\mathscr{U}), \mathscr{V}_1(\lambda_1 \mathscr{U}), \mathscr{V}_2(\mathscr{U})) \\
&- f_1(\mathscr{U}, \mathscr{V}_1^*(\mathscr{U}), \mathscr{V}_1^*(\lambda_1 \mathscr{U}), \mathscr{V}_2^*(\mathscr{U}))\Big| d\mathscr{U} \\
&+ \sum_{i=2}^{n} \frac{|\sigma_i|}{|\sigma_1|\Gamma(\alpha_1 - \alpha_i)} \int_0^{t} (t - \mathscr{U})^{\alpha_1 - \alpha_i - 1} \Big| \mathscr{V}_1(\mathscr{U}) - \mathscr{V}_1^*(\mathscr{U}) \Big| d\mathscr{U}.
\end{aligned} \tag{15}$$

Now, by making use of (**H1**) and (**H3**) in the inequality Equation (15), we obtain the following:

$$\begin{aligned}
&\Big|\mathcal{T}_1\big(\mathscr{V}_1(t), \mathscr{V}_2(t)\big) - \mathcal{T}_1\big(\mathscr{V}_1^*(t), \mathscr{V}_2^*(t)\big)\Big| \leq |d_1||g_1(\mathscr{V}_1) - g_1(\mathscr{V}_1^*)| \\
&\quad + \frac{|d_1||b_1|\big[(\mathscr{L}_{f_1} + \mathscr{L}_{f_1^\lambda})||\mathscr{V}_1 - \mathscr{V}_1^*|| + \mathscr{L}_{f_1^c}||\mathscr{V}_2 - \mathscr{V}_2^*||\big]}{|\sigma_1|\Gamma(\alpha_1)} \int_0^{\zeta_1} (\zeta_1 - \mathscr{U})^{\alpha_1 - 1} d\mathscr{U} \\
&\quad + \sum_{i=2}^{n} \frac{|d_1||b_1||\sigma_i||\mathscr{V}_1 - \mathscr{V}_1^*|}{|\sigma_1|\Gamma(\alpha_1 - \alpha_i)} \int_0^{\zeta_1} (\zeta_1 - \mathscr{U})^{\alpha_1 - \alpha_i - 1} d\mathscr{U} \\
&\quad + \frac{|d_1||c_1|\big[(\mathscr{L}_{f_1} + \mathscr{L}_{f_1^\lambda})||\mathscr{V}_1 - \mathscr{V}_1^*|| + \mathscr{L}_{f_1^c}||\mathscr{V}_2 - \mathscr{V}_2^*||\big]}{|\sigma_1|\Gamma(\alpha_1)} \int_0^{1} (1 - \mathscr{U})^{\alpha_1 - 1} d\mathscr{U} \\
&\quad + \sum_{i=2}^{n} \frac{|d_1||c_1||\sigma_i||\mathscr{V}_1 - \mathscr{V}_1^*|}{|\sigma_1|\Gamma(\alpha_1 - \alpha_i)} \int_0^{1} (1 - \mathscr{U})^{\alpha_1 - \alpha_i - 1} d\mathscr{U} \\
&\quad + \frac{\big[(\mathscr{L}_{f_1} + \mathscr{L}_{f_1^\lambda})||\mathscr{V}_1 - \mathscr{V}_1^*|| + \mathscr{L}_{f_1^c}||\mathscr{V}_2 - \mathscr{V}_2^*||\big]}{|\sigma_1|\Gamma(\alpha_1)} \int_0^{t} (t - \mathscr{U})^{\alpha_1 - 1} d\mathscr{U} \\
&\quad + \sum_{i=2}^{n} \frac{|\sigma_i||\mathscr{V}_1 - \mathscr{V}_1^*|}{|\sigma_1|\Gamma(\alpha_1 - \alpha_i)} \int_0^{t} (t - \mathscr{U})^{\alpha_1 - \alpha_i - 1} d\mathscr{U}.
\end{aligned} \tag{16}$$

Evaluating the integral involved in Equation (16) and using Equation (7), we get

$$\Big\|\mathcal{T}_1\big(\mathscr{V}_1(t), \mathscr{V}_2(t)\big) - \mathcal{T}_1\big(\mathscr{V}_1^*(t), \mathscr{V}_2^*(t)\big)\Big\| \leq \mathscr{L}_1||(\mathscr{V}_1, \mathscr{V}_2) - (\mathscr{V}_1^*, \mathscr{V}_2^*)||. \tag{17}$$

By similar calculations, one can infer the following expression:

$$\Big\|\mathcal{T}_2\big(\mathscr{V}_1(t), \mathscr{V}_2(t)\big) - \mathcal{T}_2\big(\mathscr{V}_1^*(t), \mathscr{V}_2^*(t)\big)\Big\| \leq \mathscr{L}_2||(\mathscr{V}_1, \mathscr{V}_2) - (\mathscr{V}_1^*, \mathscr{V}_2^*)||. \tag{18}$$

Now, from Equations (17) and (18), we obtain

$$
\begin{aligned}
\left|\left|\mathcal{T}\left(\mathscr{V}_1(t),\mathscr{V}_2(t)\right)-\mathcal{T}\left(\mathscr{V}_1^*(t),\mathscr{V}_2^*(t)\right)\right|\right| &= \left|\left|\mathcal{T}_1\left(\mathscr{V}_1(t),\mathscr{V}_2(t)\right)-\mathcal{T}_1\left(\mathscr{V}_1^*(t),\mathscr{V}_2^*(t)\right)\right|\right| \\
&+\left|\left|\mathcal{T}_2\left(\mathscr{V}_1(t),\mathscr{V}_2(t)\right)-\mathcal{T}_2\left(\mathscr{V}_1^*(t),\mathscr{V}_2^*(t)\right)\right|\right|, \\
&\leq \mathscr{L}_1||(\mathscr{V}_1,\mathscr{V}_2)-(\mathscr{V}_1^*,\mathscr{V}_2^*)|| \\
&+\mathscr{L}_2||(\mathscr{V}_1,\mathscr{V}_2)-(\mathscr{V}_1^*,\mathscr{V}_2^*)||, \\
&\leq \mathscr{L}||(\mathscr{V}_1,\mathscr{V}_2)-(\mathscr{V}_1^*,\mathscr{V}_2^*)||.
\end{aligned}
$$

Therefore, the use of the Banach contraction principle implies that the operator $\mathcal{T}$ possesses a fixed point. Consequently, our proposed model Equation (2) has a solution. Furthermore, this solution is unique. □

Theorem 5. *The considered model Equation (2) possesses at least one solution if (**H1**) and (**H1**) and (**H4**)–(**H6**) hold together with $\mathscr{L}_a < 1$, where $\mathscr{L}_a = \max(\mathscr{L}_{1_a}, \mathscr{L}_{2_a})$ and $\mathscr{L}_{1_a}, \mathscr{L}_{2_a}$ are defined by Equations (11) and (12), respectively.*

Proof. In order to prove the existence of at least one solution, we define the operators $\mathcal{F},\mathcal{G} : X\times X \mapsto X\times X$ given by $\mathcal{F}\Big(\mathscr{V}_1(t),\mathscr{V}_2(t)\Big) = \Big(\mathcal{F}_1\big(\mathscr{V}_1(t),\mathscr{V}_2(t)\big),\mathcal{F}_2\big(\mathscr{V}_1(t),\mathscr{V}_2(t)\big)\Big)$ and $\mathcal{G}\Big(\mathscr{V}_1(t),\mathscr{V}_2(t)\Big) = \Big(\mathcal{G}_1\big(\mathscr{V}_1(t),\mathscr{V}_2(t)\big),\mathcal{G}_2\big(\mathscr{V}_1(t),\mathscr{V}_2(t)\big)\Big)$, where

$$
\begin{aligned}
\mathcal{F}_1\big(\mathscr{V}_1(t),\mathscr{V}_2(t)\big) &= -d_1 g_1(\mathscr{V}_1)+\sum_{i=2}^{n}\frac{d_1 b_1\sigma_i}{\sigma_1\Gamma(\alpha_1-\alpha_i)}\int_0^{\zeta_1}(\zeta_1-\mathscr{U})^{\alpha_1-\alpha_i-1}\mathscr{V}_1(\mathscr{U})d\mathscr{U} \\
&-\frac{d_1 b_1}{\sigma_1\Gamma(\alpha_1)}\int_0^{\zeta_1}(\zeta_1-\mathscr{U})^{\alpha_1-1}f_1(\mathscr{U},\mathscr{V}_1(\mathscr{U}),\mathscr{V}_1(\lambda_1\mathscr{U}),\mathscr{V}_2(\mathscr{U}))d\mathscr{U} \\
&-\frac{d_1 c_1}{\sigma_1\Gamma(\alpha_1)}\int_0^{1}(1-\mathscr{U})^{\alpha_1-1}f_1(\mathscr{U},\mathscr{V}_1(\mathscr{U}),\mathscr{V}_1(\lambda_1\mathscr{U}),\mathscr{V}_2(\mathscr{U}))d\mathscr{U} \\
&+\sum_{i=2}^{n}\frac{d_1 c_1\sigma_i}{\sigma_1\Gamma(\alpha_1-\alpha_i)}\int_0^{1}(1-\mathscr{U})^{\alpha_1-\alpha_i-1}\mathscr{V}_1(\mathscr{U})d\mathscr{U},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_2\big(\mathscr{V}_1(t),\mathscr{V}_2(t)\big)(t) &= -d_2 g_2(\mathscr{V}_2)+\sum_{i=2}^{n}\frac{d_2 b_2\eta_i}{\eta_1\Gamma(\beta_1-\beta_i)}\int_0^{\zeta_2}(\zeta_2-\mathscr{U})^{\beta_1-\beta_i-1}\mathscr{V}_2(\mathscr{U})d\mathscr{U} \\
&-\frac{d_2 b_2}{\eta_1\Gamma(\beta_1)}\int_0^{\zeta_2}(\zeta_2-\mathscr{U})^{\beta_1-1}f_2(\mathscr{U},\mathscr{V}_2(\mathscr{U}),\mathscr{V}_2(\lambda_2\mathscr{U}),\mathscr{V}_1(\mathscr{U}))d\mathscr{U} \\
&-\frac{d_2 c_2}{\eta_1\Gamma(\beta_1)}\int_0^{1}(1-\mathscr{U})^{\beta_1-1}f_2(\mathscr{U},\mathscr{V}_2(\mathscr{U}),\mathscr{V}_2(\lambda_2\mathscr{U}),\mathscr{V}_1(\mathscr{U}))d\mathscr{U} \\
&+\sum_{i=2}^{n}\frac{d_2 c_2\eta_i}{\eta_1\Gamma(\beta_1-\beta_i)}\int_0^{1}(1-\mathscr{U})^{\beta_1-\beta_i-1}\mathscr{V}_2(\mathscr{U})d\mathscr{U}, \\
\mathcal{G}_1\big(\mathscr{V}_1(t),\mathscr{V}_2(t)\big) &= \frac{1}{\sigma_1\Gamma(\alpha_1)}\int_0^{t}(t-\mathscr{U})^{\alpha_1-1}f_1(\mathscr{U},\mathscr{V}_1(\mathscr{U}),\mathscr{V}_1(\lambda_1\mathscr{U}),\mathscr{V}_2(\mathscr{U}))d\mathscr{U} \\
&-\sum_{i=2}^{n}\frac{\sigma_i}{\sigma_1\Gamma(\alpha_1-\alpha_i)}\int_0^{t}(t-\mathscr{U})^{\alpha_1-\alpha_i-1}\mathscr{V}_1(\mathscr{U})d\mathscr{U}, \\
\mathcal{G}_2\big(\mathscr{V}_1(t),\mathscr{V}_2(t)\big) &= \frac{1}{\eta_1\Gamma(\beta_1)}\int_0^{t}(t-\mathscr{U})^{\beta_1-1}f_2(\mathscr{U},\mathscr{V}_2(\mathscr{U}),\mathscr{V}_2(\lambda_2\mathscr{U}),\mathscr{V}_1(\mathscr{U}))d\mathscr{U} \\
&-\sum_{i=2}^{n}\frac{\eta_i}{\eta_1\Gamma(\beta_1-\beta_i)}\int_0^{t}(t-\mathscr{U})^{\beta_1-\beta_i-1}\mathscr{V}_2(\mathscr{U})d\mathscr{U}.
\end{aligned}
$$

Moreover, we construct a ball $\mathcal{H} = \{(\mathscr{V}_1, \mathscr{V}_2) \in X \times X : ||(\mathscr{V}_1, \mathscr{V}_2)|| \leq \mathcal{R}\}$, with positive radius $\mathcal{R} > \max(\mathcal{R}_1, \mathcal{R}_2)$, where $\mathcal{R}_1$ and $\mathcal{R}_2$ are defined by Equations (13) and (14). For simplicity, we divide the proof into various steps.

Step 1: We claim in this step that $\mathcal{F}\big(\mathscr{V}_1(t), \mathscr{V}_2(t)\big) + \mathcal{G}\big(\mathscr{V}_1^*(t), \mathscr{V}_2^*(t)\big) \in H \subset X \times X$ for every $\big(\mathscr{V}_1(t), \mathscr{V}_2(t)\big), \big(\mathscr{V}_1^*(t), \mathscr{V}_2^*(t)\big) \in H$. To do this, we proceed as follows:

$$\begin{aligned}
\Big|\mathcal{F}_1\big(\mathscr{V}_1(t), \mathscr{V}_2(t)\big) + \mathcal{G}_1\big(\mathscr{V}_1^*(t), \mathscr{V}_2^*(t)\big)\Big| &\leq |d_1| g_1(\mathscr{V}_1)| \\
&+ \frac{|d_1||b_1|}{|\sigma_1|\Gamma\alpha_1)} \int_0^{\zeta_1} (\zeta_1 - \mathscr{U})^{\alpha_1 - 1} \big|f_1(\mathscr{U}, \mathscr{V}_1(\mathscr{U}), \mathscr{V}_1(\lambda_1 \mathscr{U}), \mathscr{V}_2(\mathscr{U}))\big| d\mathscr{U} \\
&+ \sum_{i=2}^{n} \frac{|d_1||b_1||\sigma_i|}{|\sigma_1|\Gamma(\alpha_1 - \alpha_i)} \int_0^{\zeta_1} (\zeta_1 - \mathscr{U})^{\alpha_1 - \alpha_i - 1} \big|\mathscr{V}_1(\mathscr{U})\big| d\mathscr{U} \\
&+ \frac{|d_1||c_1|}{|\sigma_1|\Gamma(\alpha_1)} \int_0^{1} (1 - \mathscr{U})^{\alpha_1 - 1} \big|f_1(\mathscr{U}, \mathscr{V}_1(\mathscr{U}), \mathscr{V}_1(\lambda_1 \mathscr{U}), \mathscr{V}_2(\mathscr{U}))\big| d\mathscr{U} \\
&+ \sum_{i=2}^{n} \frac{|d_1||c_1||\sigma_i|}{|\sigma_1|\Gamma(\alpha_1 - \alpha_i)} \int_0^{1} (1 - \mathscr{U})^{\alpha_1 - \alpha_i - 1} \big|\mathscr{V}_1(\mathscr{U})\big| d\mathscr{U} \\
&+ \frac{1}{|\sigma_1|\Gamma(\alpha_1)} \int_0^{t} (t - \mathscr{U})^{\alpha_1 - 1} \big|f_1(\mathscr{U}, \mathscr{V}_1(\mathscr{U}), \mathscr{V}_1(\lambda_1 \mathscr{U}), \mathscr{V}_2(\mathscr{U}))\big| d\mathscr{U} \\
&+ \sum_{i=2}^{n} \frac{1|\sigma_i|}{|\sigma_1|\Gamma(\alpha_1 - \alpha_i)} \int_0^{t} (t - \mathscr{U})^{\alpha_1 - \alpha_i - 1} \big|\mathscr{V}_1(\mathscr{U})\big| d\mathscr{U}.
\end{aligned} \tag{19}$$

Now, by making use of (**H4**) and (**H6**) in the inequality Equation (19), we get

$$\begin{aligned}
\Big|\Big|\mathcal{F}_1\big(\mathscr{V}_1(t), \mathscr{V}_2(t)\big) + \mathcal{G}_1\big(\mathscr{V}_1^*(t), \mathscr{V}_2^*(t)\big)\Big|\Big| &\leq |d_1|||\mathscr{H}_{g_1}(t)|| \\
+ \frac{\Big[||\mathscr{H}_{f_1^a}(t)|| + ||\mathscr{H}_{f_1^b}(t)||||\mathscr{V}_1|| + ||\mathscr{H}_{f_1^c}(t)||||\mathscr{V}_2||\Big]}{|\sigma_1|\Gamma(\alpha_1 + 1)} &(|d_1||b_1|\zeta_1^{\alpha_1} + |d_1||c_1| + 1) \\
+ \sum_{i=2}^{n} \frac{|\sigma_i|||\mathscr{V}_1||}{|\sigma_1|\Gamma(\alpha_1 - \alpha_i + 1)} &(|d_1||b_1|\zeta_1^{\alpha_1 - \alpha_i} + |d_1||c_1| + 1).
\end{aligned}$$

or

$$\Big|\Big|\mathcal{F}_1\big(\mathscr{V}_1(t), \mathscr{V}_2(t)\big) + \mathcal{G}_1\big(\mathscr{V}_1^*(t), \mathscr{V}_2^*(t)\big)\Big|\Big| < \mathcal{R}_1. \tag{20}$$

Similarly, one can obtain the following inequality for $\mathcal{F}_2$:

$$\Big|\Big|\mathcal{F}_2\big(\mathscr{V}_1(t), \mathscr{V}_2(t)\big) + \mathcal{G}_2\big(\mathscr{V}_1^*(t), \mathscr{V}_2^*(t)\big)\Big|\Big| < \mathcal{R}_2. \tag{21}$$

Hence, from Equations (20) and (21), one can infer that $\mathcal{F}\big(\mathscr{V}_1(t), \mathscr{V}_2(t)\big) + \mathcal{G}\big(\mathscr{V}_1^*(t), \mathscr{V}_2^*(t)\big) \in H$.

Step 2: In this step, we claim that $\mathcal{F}$ is uniformly bounded. For verification, we proceed as follows:

Let $\mathscr{V}_1, \mathscr{V}_2 \in X$. Then, we have

$$\begin{aligned}\Big|\mathcal{F}_1(\mathscr{V}_1(t),\mathscr{V}_2(t))\Big| \leq{}& |d_1|g_1(\mathscr{V}_1)| + \frac{d_1|b_1|}{|\sigma_1|\Gamma(\alpha_1)}\int_0^{\zeta_1}(\zeta_1-\mathscr{U})^{\alpha_1-1}|f_1(\mathscr{U},\mathscr{V}_1(\mathscr{U}),\mathscr{V}_1(\lambda_1\mathscr{U}),\mathscr{V}_2(\mathscr{U}))|d\mathscr{U}\\ &+\sum_{i=2}^{n}\frac{|d_1||b_1||\sigma_i|}{|\sigma_1|\Gamma(\alpha_1-\alpha_i)}\int_0^{\zeta_1}(\zeta_1-\mathscr{U})^{\alpha_1-\alpha_i-1}|\mathscr{V}_1(\mathscr{U})|d\mathscr{U}\\ &+\frac{|d_1||c_1|}{|\sigma_1|\Gamma(\alpha_1)}\int_0^1(1-\mathscr{U})^{\alpha_1-1}|f_1(\mathscr{U},\mathscr{V}_1(\mathscr{U}),\mathscr{V}_1(\lambda_1\mathscr{U}),\mathscr{V}_2(\mathscr{U}))|d\mathscr{U}\\ &+\sum_{i=2}^{n}\frac{|d_1||c_1||\sigma_i|}{|\sigma_1|\Gamma(\alpha_1-\alpha_i)}\int_0^1(1-\mathscr{U})^{\alpha_1-\alpha_i-1}|\mathscr{V}_1(\mathscr{U})|d\mathscr{U}.\end{aligned} \tag{22}$$

Now, by making use of (**H4**) and (**H6**) in the inequality Equation (22), we get

$$\begin{aligned}\Big|\Big|\mathcal{F}_1(\mathscr{V}_1(t),\mathscr{V}_2(t))\Big|\Big| \leq{}& |d_1|||\mathscr{H}_{g_1}(t)||\\ &+\frac{|d_1|\Big[||\mathscr{H}_{f_1^a}(t)||+||\mathscr{H}_{f_1^b}(t)||||\mathscr{V}_1||+||\mathscr{H}_{f_1^c}(t)||||\mathscr{V}_2||\Big]}{|\sigma_1|\Gamma(\alpha_1+1)}(|b_1|\zeta_1^{\alpha_1}+|c_1|)\\ &+\sum_{i=2}^{n}\frac{|d_1||\sigma_i|||\mathscr{V}_1||}{|\sigma_1|\Gamma(\alpha_1-\alpha_i+1)}(|b_1|\zeta_1^{\alpha_1-\alpha_i}+|c_1|).\end{aligned} \tag{23}$$

Similarly, one can obtain the following inequality for $\mathcal{F}_2$:

$$\begin{aligned}\Big|\Big|\mathcal{F}_2(\mathscr{V}_1(t),\mathscr{V}_2(t))\Big|\Big| \leq{}& |d_2|||\mathscr{H}_{g_2}(t)||\\ &+\frac{|d_2|\Big[||\mathscr{H}_{f_2^a}(t)||+||\mathscr{H}_{f_2^b}(t)||||\mathscr{V}_2||+||\mathscr{H}_{f_2^c}(t)||||\mathscr{V}_1||\Big]}{|\eta_1|\Gamma(\beta_1+1)}(|b_2|\zeta_1^{\beta_1}+|c_2|)\\ &+\sum_{i=2}^{n}\frac{|d_2||\eta_i|||\mathscr{V}_2||}{|\eta_1|\Gamma(\beta_1-\beta_i+1)}(|b_2|\zeta_1^{\beta_1-\beta_i}+|c_2|).\end{aligned} \tag{24}$$

Therefore, from Equations (23) and (24), one can get that $\mathcal{F}$ is uniformly bounded.

Step 3: Now, we claim that $\mathcal{F}$ is continuous. For the proof, consider a sequence $(\mathscr{V}_{1_n},\mathscr{V}_{2_n})\in\mathcal{H}$, which converges to $(\mathscr{V}_1,\mathscr{V}_2)$. Now, we need to prove the relation that $<\mathcal{F}(\mathscr{V}_{1_n},\mathscr{V}_{2_n})>$ converges to $<\mathcal{F}(\mathscr{V}_1,\mathscr{V}_2)>$, as $n\mapsto\infty$. Consider the following:

$$\begin{aligned}&\Big|\mathcal{F}_1(\mathscr{V}_{1_n}(t),\mathscr{V}_{2_n}(t))-\mathcal{F}_1(\mathscr{V}_1(t),\mathscr{V}_2(t))\Big| \leq |d_1||g_1(\mathscr{V}_{1_n})-g_1(\mathscr{V}_1)|\\ &+\frac{|d_1||b_1|}{|\sigma_1|\Gamma(\alpha_1)}\int_0^{\zeta_1}(\zeta_1-\mathscr{U})^{\alpha_1-1}\Big|f_1(\mathscr{U},\mathscr{V}_{1_n}(t\mathscr{U},\mathscr{V}_{1_n}(\lambda_1\mathscr{U}),\mathscr{V}_{2_n}(\mathscr{U}))\\ &-f_1(\mathscr{U},\mathscr{V}_1(t\mathscr{U},\mathscr{V}_1(\lambda_1\mathscr{U}),\mathscr{V}_2^*(\mathscr{U}))\Big|d\mathscr{U}\\ &+\sum_{i=2}^{n}\frac{|d_1||b_1||\sigma_i|}{|\sigma_1|\Gamma(\alpha_1-\alpha_i)}\int_0^{\zeta_1}(\zeta_1-\mathscr{U})^{\alpha_1-\alpha_i-1}\Big|\mathscr{V}_{1_n}(\mathscr{U})-\mathscr{V}_1(\mathscr{U})\Big|d\mathscr{U}\\ &+\frac{|d_1||c_1|}{|\sigma_1|\Gamma(\alpha_1)}\int_0^1(1-\mathscr{U})^{\alpha_1-1}\Big|f_1(\mathscr{U},\mathscr{V}_{1_n}(t\mathscr{U},\mathscr{V}_{1_n}(\lambda_1\mathscr{U}),\mathscr{V}_{2_n}(\mathscr{U}))\\ &-f_1(\mathscr{U},\mathscr{V}_1(t\mathscr{U},\mathscr{V}_1(\lambda_1\mathscr{U}),\mathscr{V}_2^*(\mathscr{U}))\Big|d\mathscr{U}\\ &+\sum_{i=2}^{n}\frac{|d_1||c_1||\sigma_i|}{|\sigma_1|\Gamma(\alpha_1-\alpha_i)}\int_0^1(1-\mathscr{U})^{\alpha_1-\alpha_i-1}\Big|\mathscr{V}_1(\mathscr{U})-\mathscr{V}_1(\mathscr{U})\Big|d\mathscr{U}.\end{aligned}$$

Hence, by Lebsegue's dominated convergent theorem, we have $||\mathcal{F}_1(\mathscr{V}_{1_n},\mathscr{V}_{2_n})-\mathcal{F}_1(\mathscr{V}_1(t),\mathscr{V}_2(t))||\mapsto 0$, as $n\mapsto\infty$.

In the same way, the operator $\mathcal{F}_2$ can be proved continuous. Furthermore, from the continuity of $\mathcal{F}_1$ and $\mathcal{F}_2$, we infer that $\mathcal{F}$ is continuous.

Step 4: In this step, it will be proved that bounded sets will map to equicontinuous sets under $\mathcal{F}$. Consider for $t_1, t_2 \in J$, $t_1 < t_2$ and $(\mathscr{V}_1, \mathscr{V}_2) \in \mathcal{H} \subset X \times X$,

$$\left|\left|\mathcal{F}_1(\mathscr{V}_1(t_2), \mathscr{V}_2(t_2)) - \mathcal{F}_1(\mathscr{V}_1(t_1), \mathscr{V}_2(t_1))\right|\right| \leq |d_1||g_1(\mathscr{V}_1(t_2)) - g_1(\mathscr{V}_1(t_1))|.$$

Clearly, as $t_1 \mapsto t_2$, we have $||\mathcal{F}_1(\mathscr{V}_1(t_2), \mathscr{V}_2(t_2)) - \mathcal{F}_1(\mathscr{V}_1(t_1), \mathscr{V}_2(t_1))|| = 0$. By a similar procedure, it can be shown that bounded sets map to the set of equicontinous functions under the operator $\mathcal{F}_2$. Moreover, the aforementioned result also holds for $\mathcal{F}$.

Step 5: The contraction $\mathcal{G}$ can be derived in a similar manner as for Theorem 4.

Thus, all requirements of Krasnoselskii's fixed point theorem are fulfilled; therefore, the model of fractional-order differential equations has at least one solution. □

4. Stability Analysis

The results carried out in this part of paper are specific to the functional stability of the model under consideration in Equation (2). Four types of functional-stability-related results will be explored in the context of fractional calculus. Initially, definitions of these kinds of stability results will be provided. Later on, based on some auxiliary hypotheses, results related to the existence of functional stability will be explored.

Definition 3. *The Ulam–Hyres stability for the model Equation (2) can be achieved if one can find* $\mathcal{B} = \max(\mathcal{B}_1, \mathcal{B}_2)(constant) > 0$ *such that for each solution* $(\mathscr{V}_1, \mathscr{V}_2) \in X \times X$ *of the following differential inequality and* $\epsilon = \max(\epsilon_1, \epsilon_2) > 0$,

$$\begin{cases} \left|\sum_{i=1}^{n} \sigma_i{}^c D^{\alpha_i} \mathscr{V}_1(t) - f_1(t, \mathscr{V}_1(t), \mathscr{V}_1(\lambda_1 t), \mathscr{V}_2(t))\right| \leq \epsilon_1, & t \in [0,1], \\ \left|\sum_{i=1}^{n} \eta_i{}^c D^{\beta_i} \mathscr{V}_2(t) - f_2(t, \mathscr{V}_2(t), \mathscr{V}_2(\lambda_2 t), \mathscr{V}_1(t))\right| \leq \epsilon_1, & t \in [0,1]. \end{cases} \tag{25}$$

and a unique solution $(\mathscr{V}_1^*, \mathscr{V}_2^*) \in X \times X$ *of the given problem Equation (2) such that* $|(\mathscr{V}_1, \mathscr{V}_2) - (\mathscr{V}_1^*, \mathscr{V}_2^*)| \leq \mathcal{B}\epsilon$, *while the solution satisfies the definition of generalized Ulam–Hyers (GUH) stability, if a positive function* $\mathcal{K} : (0, \infty) \mapsto (0, \infty)$ *can be found with the condition* $\mathcal{K}(0) = 0$ *such that* $|(\mathscr{V}_1, \mathscr{V}_2) - (\mathscr{V}_1^*, \mathscr{V}_2^*)| \leq \mathcal{B}\mathcal{K}(t)$.

Definition 4. *The problem Equation (2)'s solution satisfies the UH Rassias stability criteria, with regard to a continuous function* $\chi = \max(\chi_1, \chi_2) \in X \times X$ *if we have* $\mathcal{B}^* = \max(\mathcal{B}_1^*, \mathcal{B}_2^*)$*(positive constant)* > 0*, and* $\epsilon = \max(\epsilon_1, \epsilon_2) > 0$*, for each solution* $(\mathscr{V}_1, \mathscr{V}_2) \in X \times X$ *of the following differential inequality:*

$$\begin{cases} \left|\sum_{i=1}^{n} \sigma_i{}^c D^{\alpha_i} \mathscr{V}_1(t) - f_1(t, \mathscr{V}_1(t), \mathscr{V}_1(\lambda_1 t), \mathscr{V}_2(t))\right| \leq \chi_1(t)\epsilon_1, & t \in [0,1], \\ \left|\sum_{i=1}^{n} \eta_i{}^c D^{\beta_i} \mathscr{V}_2(t) - f_2(t, \mathscr{V}_2(t), \mathscr{V}_2(\lambda_2 t), \mathscr{V}_1(t))\right| \leq \chi_2(t)\epsilon_2, & t \in [0,1]. \end{cases} \tag{26}$$

and a unique solution $(\mathscr{V}_1^*, \mathscr{V}_2^*) \in X \times X$ *for the given problem Equation (2) such that* $|(\mathscr{V}_1, \mathscr{V}_2) - (\mathscr{V}_1^*, \mathscr{V}_2^*)| \leq \mathcal{B}^*\chi(t)\epsilon$.

Definition 5. *The solution of the considered problem is GUH Rassias stable, with regard to continuous function* $\chi = \max(\chi_1, \chi_2) \in X \times X$ *if we have* $\mathcal{B}^* = \max(\mathcal{B}_1^*, \mathcal{B}_2^*)$*(positive constant)* > 0*, and for each solution* $(\mathscr{V}_1, \mathscr{V}_2) \in X \times X$ *of the following differential inequality*

$$\begin{cases} \left| \sum_{i=1}^{n} \sigma_i {}^c D^{\alpha_i} \mathscr{V}_1(t) - f_1(t, \mathscr{V}_1(t), \mathscr{V}_1(\lambda_1 t), \mathscr{V}_2(t)) \right| \leq \chi_1(t), \quad t \in [0,1], \\ \left| \sum_{i=1}^{n} \eta_i {}^c D^{\beta_i} \mathscr{V}_2(t) - f_2(t, \mathscr{V}_2(t), \mathscr{V}_2(\lambda_2 t), \mathscr{V}_1(t)) \right| \leq \chi_2(t), \quad t \in [0,1]. \end{cases} \tag{27}$$

and a unique solution $(\mathscr{V}_1^*, \mathscr{V}_2^*) \in X \times X$ *of the given problem Equation (2) such that* $|(\mathscr{V}_1, \mathscr{V}_2) - (\mathscr{V}_1^*, \mathscr{V}_2^*)| \leq \mathcal{B}^* \chi(t)$.

Remark 2. *The solution of the inequality Equation (25) is* $(\mathscr{V}_1, \mathscr{V}_2) \in X \times X$*, iff one can find functions,* $\xi_1, \xi_2 \in X$ *such that*

$$(i) \quad |\xi_1(t)| \leq \epsilon_1, \ |\xi_2(t)| \leq \epsilon_2, \ t \in J,$$

and

$$(ii) \quad \begin{cases} \sum_{i=1}^{n} \sigma_i {}^c D^{\alpha_i} \mathscr{V}_1(t) = f_1(t, \mathscr{V}_1(t), \mathscr{V}_1(\lambda_1 t), \mathscr{V}_2(t)) + \xi_1(t), \quad t \in [0,1], \\ \sum_{i=1}^{n} \eta_i {}^c D^{\beta_i} \mathscr{V}_2(t) = f_2(t, \mathscr{V}_2(t), \mathscr{V}_2(\lambda_2 t), \mathscr{V}_1(t)) + \xi_2(t), \quad t \in [0,1]. \end{cases}$$

Remark 3. *The solution of the inequality Equation (26) is* $(\mathscr{V}_1, \mathscr{V}_2) \in X \times X$*, iff one can find functions* $\xi_1, \xi_2 \in X$ *such that*

$$(i) \quad |\xi_1(t)| \leq \chi_1(t)\epsilon_1, \ |\xi_2(t)| \leq \chi_2(t)\epsilon_v, \ t \in J,$$

and

$$(ii) \quad \begin{cases} \sum_{i=1}^{n} \sigma_i {}^c D^{\alpha_i} \mathscr{V}_1(t) = f_1(t, \mathscr{V}_1(t), \mathscr{V}_1(\lambda_1 t), \mathscr{V}_2(t)) + \xi_1(t), \quad t \in [0,1], \\ \sum_{i=1}^{n} \eta_i {}^c D^{\beta_i} \mathscr{V}_2(t) = f_2(t, \mathscr{V}_2(t), \mathscr{V}_2(\lambda_2 t), \mathscr{V}_1(t)) + \xi_2(t), \quad t \in [0,1]. \end{cases}$$

Remark 4. *The solution of the inequality Equation (27) is* $(\mathscr{V}_1, \mathscr{V}_2) \in X \times X$*, iff one can find functions,* $\xi_1, \xi_2 \in X$ *such that*

$$(i) \quad |\xi_1(t)| \leq \chi_1(t), \ |\xi_2(t)| \leq \chi_2(t), \ t \in J,$$

and

$$(ii) \quad \begin{cases} \sum_{i=1}^{n} \sigma_i {}^c D^{\alpha_i} \mathscr{V}_1(t) = f_1(t, \mathscr{V}_1(t), \mathscr{V}_1(\lambda_1 t), \mathscr{V}_2(t)) + \xi_1(t), \quad t \in [0,1], \\ \sum_{i=1}^{n} \eta_i {}^c D^{\beta_i} \mathscr{V}_2(t) = f_2(t, \mathscr{V}_2(t), \mathscr{V}_2(\lambda_2 t), \mathscr{V}_1(t)) + \xi_2(t), \quad t \in [0,1]. \end{cases}$$

Lemma 2. *Consider that* $(\mathscr{V}_1, \mathscr{V}_2) \in X \times X$ *is a solution of system of MFDDEs*

$$\begin{cases} \sum_{i=1}^{n} \sigma_i {}^c D^{\alpha_i} \mathscr{V}_1(t) = f_1(t, \mathscr{V}_1(t), \mathscr{V}_1(\lambda_1 t), \mathscr{V}_2(t)) + \xi_1(t), \ \lambda_1, \lambda_2, \ t \in [0,1], \\ \sum_{i=1}^{n} \eta_i {}^c D^{\beta_i} \mathscr{V}_2(t) = f_2(t, \mathscr{V}_2(t), \mathscr{V}_2(\lambda_2 t), \mathscr{V}_1(t)) + \xi_2(t), \ \alpha_i, \beta_i \in (0,1], \ \text{for } i = 1,2,3,\ldots,n, \\ g_1(\mathscr{V}_1) = a_1 \mathscr{V}_1(0) - b_1 \mathscr{V}_1(\zeta_1) - c_1 \mathscr{V}_1(1), \quad g_2(\mathscr{V}_2) = a_2 \mathscr{V}_2(0) - b_2 \mathscr{V}_2(\zeta_2) - c_2 \mathscr{V}_2(1). \end{cases} \tag{28}$$

that satisfies the following relations:

$$\left|\left|\mathscr{V}_1(t) - \mathcal{T}_1(\mathscr{V}_1, \mathscr{V}_2)\right|\right| \leq \frac{\left(|d_1||b_1|\zeta_1^{\alpha_1} + |d_1||c_1| + 1\right)\epsilon_1}{|\sigma_1|\Gamma(\alpha_1 + 1)},$$
$$\left|\left|\mathscr{V}_2(t) - \mathcal{T}_2(\mathscr{V}_1, \mathscr{V}_2)\right|\right| \leq \frac{\left(|d_2||b_2|\zeta_2^{\beta_1} + |d_2||c_2| + 1\right)\epsilon_2}{|\eta_1|\Gamma(\beta_1 + 1)}.$$

Proof. Suppose $(\mathscr{V}_1, \mathscr{V}_2) \in X \times X$ to be the solution of the problem Equation (28). Then, by corollary, (1), we have

$$\begin{aligned}
\mathscr{V}_1(t) = & -d_1 g_1(\mathscr{V}_1) - \frac{d_1 b_1}{\sigma_1 \Gamma(\alpha_1)} \int_0^{\zeta_1} (\zeta_1 - \mathscr{U})^{\alpha_1 - 1} f_1(\mathscr{U}, \mathscr{V}_1(\mathscr{U}), \mathscr{V}_1(\lambda_1 \mathscr{U}), \mathscr{V}_2(\mathscr{U})) d\mathscr{U} \\
& + \sum_{i=2}^{n} \frac{d_1 b_1 \sigma_i}{\sigma_1 \Gamma(\alpha_1 - \alpha_i)} \int_0^{\zeta_1} (\zeta_1 - \mathscr{U})^{\alpha_1 - \alpha_i - 1} \mathscr{V}_1(\mathscr{U}) d\mathscr{U} \\
& - \frac{d_1 c_1}{\sigma_1 \Gamma(\alpha_1)} \int_0^1 (1 - \mathscr{U})^{\alpha_1 - 1} f_1(\mathscr{U}, \mathscr{V}_1(\mathscr{U}), \mathscr{V}_1(\lambda_1 \mathscr{U}), \mathscr{V}_2(\mathscr{U})) d\mathscr{U} \\
& + \sum_{i=2}^{n} \frac{d_1 c_1 \sigma_i}{\sigma_1 \Gamma(\alpha_1 - \alpha_i)} \int_0^1 (1 - \mathscr{U})^{\alpha_1 - \alpha_i - 1} \mathscr{V}_1(\mathscr{U}) d\mathscr{U} \\
& + \frac{1}{\sigma_1 \Gamma(\alpha_1)} \int_0^t (t - \mathscr{U})^{\alpha_1 - 1} f_1(\mathscr{U}, \mathscr{V}_1(\mathscr{U}), \mathscr{V}_1(\lambda_1 \mathscr{U}), \mathscr{V}_2(\mathscr{U})) d\mathscr{U} \\
& - \sum_{i=2}^{n} \frac{\sigma_i}{\sigma_1 \Gamma(\alpha_1 - \alpha_i)} \int_0^t (t - \mathscr{U})^{\alpha_1 - \alpha_i - 1} \mathscr{V}_1(\mathscr{U}) d\mathscr{U} - \frac{d_1 b_1}{\sigma_1 \Gamma(\alpha_1)} \int_0^{\zeta_1} (\zeta_1 - \mathscr{U})^{\alpha_1 - 1} \xi_1(\mathscr{U}) d\mathscr{U} \\
& - \frac{d_1 c_1}{\sigma_1 \Gamma(\alpha_1)} \int_0^1 (1 - \mathscr{U})^{\alpha_1 - 1} \xi_1(\mathscr{U}) d\mathscr{U} + \frac{1}{\sigma_1 \Gamma(\alpha_1)} \int_0^t (t - \mathscr{U})^{\alpha_1 - 1} \xi_1(\mathscr{U}) d\mathscr{U},
\end{aligned} \tag{29}$$

$$\begin{aligned}
\mathscr{V}_2(t) = & -d_2 g_2(\mathscr{V}_2) - \frac{d_2 b_2}{\eta_1 \Gamma(\beta_1)} \int_0^{\zeta_2} (\zeta_2 - \mathscr{U})^{\beta_1 - 1} f_2(\mathscr{U}, \mathscr{V}_2(\mathscr{U}), \mathscr{V}_2(\lambda_2 \mathscr{U}), \mathscr{V}_1(\mathscr{U})) d\mathscr{U} \\
& + \sum_{i=2}^{n} \frac{d_2 b_2 \eta_i}{\eta_1 \Gamma(\beta_1 - \beta_i)} \int_0^{\zeta_2} (\zeta_2 - \mathscr{U})^{\beta_1 - \beta_i - 1} \mathscr{V}_2(\mathscr{U}) d\mathscr{U} \\
& - \frac{d_2 c_2}{\eta_1 \Gamma(\beta_1)} \int_0^1 (1 - \mathscr{U})^{\beta_1 - 1} f_2(\mathscr{U}, \mathscr{V}_2(\mathscr{U}), \mathscr{V}_2(\lambda_2 \mathscr{U}), \mathscr{V}_1(\mathscr{U})) d\mathscr{U} \\
& + \sum_{i=2}^{n} \frac{d_2 c_2 \eta_i}{\eta_1 \Gamma(\beta_1 - \beta_i)} \int_0^1 (1 - \mathscr{U})^{\beta_1 - \beta_i - 1} \mathscr{V}_2(\mathscr{U}) d\mathscr{U} \\
& + \frac{1}{\eta_1 \Gamma(\beta_1)} \int_0^t (t - \mathscr{U})^{\beta_1 - 1} f_2(\mathscr{U}, \mathscr{V}_2(\mathscr{U}), \mathscr{V}_2(\lambda_2 \mathscr{U}), \mathscr{V}_1(\mathscr{U})) d\mathscr{U} \\
& - \sum_{i=2}^{n} \frac{\eta_i}{\eta_1 \Gamma(\beta_1 - \beta_i)} \int_0^t (t - \mathscr{U})^{\beta_1 - \beta_i - 1} \mathscr{V}_2(\mathscr{U}) d\mathscr{U} - \frac{d_2 b_2}{\eta_1 \Gamma(\beta_1)} \int_0^{\zeta_2} (\zeta_2 - \mathscr{U})^{\beta_1 - 1} \xi_2(\mathscr{U}) d\mathscr{U} \\
& - \frac{d_2 c_2}{\eta_1 \Gamma(\beta_1)} \int_0^1 (1 - \mathscr{U})^{\beta_1 - 1} \xi_2(\mathscr{U}) d\mathscr{U} + \frac{1}{\eta_1 \Gamma(\beta_1)} \int_0^t (t - \mathscr{U})^{\beta_1 - 1} \xi_2(\mathscr{U}) d\mathscr{U}.
\end{aligned} \tag{30}$$

For simplicity, let us introduce $\mathcal{T}_1(\mathscr{V}_1,\mathscr{V}_2)$ and $\mathcal{T}_2(\mathscr{V}_1,\mathscr{V}_2)$ in Equations (29) and (30). Then, we obtain

$$\begin{aligned}
\mathscr{V}_1(t)-\mathcal{T}_1(\mathscr{V}_1,\mathscr{V}_2) = &-\frac{d_1b_1}{\sigma_1\Gamma(\alpha_1)}\int_0^{\zeta_1}(\zeta_1-\mathscr{U})^{\alpha_1-1}+\xi_1(\mathscr{U})d\mathscr{U}\\
&-\frac{d_1c_1}{\sigma_1\Gamma(\alpha_1)}\int_0^{1}(1-\mathscr{U})^{\alpha_1-1}\xi_1(\mathscr{U})d\mathscr{U}\\
&+\frac{1}{\sigma_1\Gamma(\alpha_1)}\int_0^{t}(t-\mathscr{U})^{\alpha_1-1}\xi_1(\mathscr{U})d\mathscr{U},\\
\mathscr{V}_2(t)-\mathcal{T}_2(\mathscr{V}_1,\mathscr{V}_2) = &-\frac{d_2b_2}{\eta_1\Gamma(\beta_1)}\int_0^{\zeta_2}(\zeta_2-\mathscr{U})^{\beta_1-1}\xi_2(\mathscr{U})d\mathscr{U}\\
&-\frac{d_2c_2}{\eta_1\Gamma(\beta_1)}\int_0^{1}(1-\mathscr{U})^{\beta_1-1}\xi_2(\mathscr{U})d\mathscr{U}\\
&+\frac{1}{\eta_1\Gamma(\beta_1)}\int_0^{t}(t-\mathscr{U})^{\beta_1-1}\xi_2(\mathscr{U})d\mathscr{U}.
\end{aligned}$$

By taking the absolute and using Remark (2), we have

$$\begin{aligned}
\Big|\Big|\mathscr{V}_1(t)-\mathcal{T}_1(\mathscr{V}_1,\mathscr{V}_2)\Big|\Big| &\le \frac{(|d_1||b_1|\zeta_1^{\alpha_1}+|d_1||c_1|+1)\epsilon_1}{|\sigma_1|\Gamma(\alpha_1+1)},\\
\Big|\Big|\mathscr{V}_2(t)-\mathcal{T}_2(\mathscr{V}_1,\mathscr{V}_2)\Big|\Big| &\le \frac{(|d_2||b_2|\zeta_2^{\beta_1}+|d_2||c_2|+1)\epsilon_2}{|\eta_1|\Gamma(\beta_1+1)}.
\end{aligned}$$

This proves the required result. □

Theorem 6. *Under the assumptions (**H1**)–(**H3**), the fractional-order model Equation (2) is UH stable and GUH stable if $\mathscr{L}<1$, where $\mathscr{L}=\max(\mathscr{L}_1,\mathscr{L}_2)$ and $\mathscr{L}_1,\mathscr{L}_2$ are defined by Equations (7) and (9), respectively.*

Proof. Given any solution $(\mathscr{V}_1,\mathscr{V}_2)\in\times X$, and the unique solution $(\mathscr{V}_1^*,\mathscr{V}_2^*)$ of the given problem Equation (2), the following holds:

$$\begin{aligned}
||(\mathscr{V}_1,\mathscr{V}_2)-(\mathscr{V}_1^*,\mathscr{V}_2^*)|| &= ||(\mathscr{V}_1,\mathscr{V}_2)-\mathcal{T}(\mathscr{V}_1^*,\mathscr{V}_2^*)||,\\
&\le ||\mathscr{V}_1-\mathcal{T}_1(\mathscr{V}_1,\mathscr{V}_2)||+||\mathcal{T}_1(\mathscr{V}_1^*,\mathscr{V}_2^*)-\mathcal{T}_1(\mathscr{V}_1,\mathscr{V}_2)||+||\mathscr{V}_2-\mathcal{T}_2(\mathscr{V}_1,\mathscr{V}_2)||\\
&+||\mathcal{T}_2(\mathscr{V}_1^*,\mathscr{V}_2^*)-\mathcal{T}_2(\mathscr{V}_1,\mathscr{V}_2)||.
\end{aligned}$$

Using Theorem 4 and Lemma 2, we have

$$\begin{aligned}
||(\mathscr{V}_1,\mathscr{V}_2)-(\mathscr{V}_1^*,\mathscr{V}_2^*)|| \le\le &\frac{(|d_1||b_1|\zeta_1^{\alpha_1}+|d_1||c_1|+1)\epsilon_1}{|\sigma_1|\Gamma(\alpha_1+1)}+\frac{(|d_2||b_2|\zeta_2^{\beta_1}+|d_2||c_2|+1)\epsilon_2}{|\eta_1|\Gamma(\beta_1+1)}\\
&+\mathscr{L}||(\mathscr{V}_1,\mathscr{V}_2)-(\mathscr{V}_1^*,\mathscr{V}_2^*)||,\\
&\le\Bigg[\frac{(|d_1||b_1|\zeta_1^{\alpha_1}+|d_1||c_1|+1)\epsilon_1}{|\sigma_1|\Gamma(\alpha_1+1)}\\
&+\frac{(|d_2||b_2|\zeta_2^{\beta_1}+|d_2||c_2|+1)\epsilon_2}{|\eta_1|\Gamma(\beta_1+1)}\Bigg]\frac{1}{1-\mathscr{L}},\\
&\le\mathcal{B}\epsilon.
\end{aligned}$$

Let $\mathcal{B} = \max\left[\frac{\left(|d_1||b_1|\zeta_1^{\alpha_1}+|d_1||c_1|+1\right)}{|\sigma_1|\Gamma(\alpha_1+1)(1-\mathcal{L})}, \frac{\left(|d_2||b_2|\zeta_2^{\beta_1}+|d_2||c_2|+1\right)}{|\eta_1|\Gamma(\beta_1+1)(1-\mathcal{L})}\right]$; then, the solution of the considered problem Equation (2) is UH stable. Furthermore, we can set $\mathcal{K}(\epsilon) = \epsilon$,; then, the considered problem Equation (2) is GUH stable. □

To prove the next stability result, we need the following assumption, given as **(H7)**. For any non-decreasing function $\chi_1, \chi_2 \in X$, there exist the positive constants $\mathcal{G}_1$ and $\mathcal{G}_2$, such that

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-\mathcal{U})^{\alpha_1-1}\chi_1(\mathcal{U})d\mathcal{U} \le \mathcal{G}_1\chi_1(t),$$
$$\frac{1}{\Gamma(\beta)}\int_0^t (t-\mathcal{U})^{\beta_1-1}\chi_2(\mathcal{U})d\mathcal{U} \le \mathcal{G}_2\chi_2(t).$$

Lemma 3. *Consider that $(\mathcal{V}_1, \mathcal{V}_2) \in X \times X$ is a solution of a system of MFDDEs,*

$$\begin{cases} \sum_{i=1}^{n} \sigma_i{}^c D^{\alpha_i}\mathcal{V}_1(t) = f_1(t, \mathcal{V}_1(t), \mathcal{V}_1(\lambda_1 t), \mathcal{V}_2(t)) + \xi_1(t), \ \lambda_1, \lambda_2, \ t \in [0,1], \\ \sum_{i=1}^{n} \eta_i{}^c D^{\beta_i}\mathcal{V}_2(t) = f_2(t, \mathcal{V}_2(t), \mathcal{V}_2(\lambda_2 t), \mathcal{V}_1(t)) + \xi_2(t), \alpha_i, \beta_i \in (0,1], \, for\, i = 1,2,3,\ldots,n, \\ g_1(\mathcal{V}_1) = a_1\mathcal{V}_1(0) - b_1\mathcal{V}_1(\zeta_1) - c_1\mathcal{V}_1(1), \quad g_2(\mathcal{V}_2) = a_2\mathcal{V}_2(0) - b_2\mathcal{V}_2(\zeta_2) - c_2\mathcal{V}_2(1). \end{cases} \tag{31}$$

that satisfies the following relations,

$$\left|\left|\mathcal{V}_1(t) - \mathcal{T}_1(\mathcal{V}_1, \mathcal{V}_2)\right|\right| \le \frac{(|d_1||b_1| + |d_1||c_1| + 1)\mathcal{G}_1\chi_1(t)\epsilon_1}{|\sigma_1|\Gamma(\alpha_1+1)},$$
$$\left|\left|\mathcal{V}_2(t) - \mathcal{T}_2(\mathcal{V}_1, \mathcal{V}_2)\right|\right| \le \frac{(|d_2||b_2| + |d_2||c_2| + 1)\mathcal{G}_2\chi_2(t)\epsilon_2}{|\eta_1|\Gamma(\beta_1+1)}.$$

Proof. Suppose a solution $(\mathcal{V}_1, \mathcal{V}_2) \in X \times X$ of the problem Equation (31). Then, by Corollary (1), we have

$$\begin{aligned} \mathcal{V}_1(t) - \mathcal{T}_1(\mathcal{V}_1, \mathcal{V}_2) =& -\frac{d_1 b_1}{\sigma_1\Gamma(\alpha_1)}\int_0^{\zeta_1}(\zeta_1-\mathcal{U})^{\alpha_1-1} + \xi_1(\mathcal{U})d\mathcal{U} \\ &- \frac{d_1 c_1}{\sigma_1\Gamma(\alpha_1)}\int_0^1 (1-\mathcal{U})^{\alpha_1-1}\xi_1(\mathcal{U})d\mathcal{U} \\ &+ \frac{1}{\sigma_1\Gamma(\alpha_1)}\int_0^t (t-\mathcal{U})^{\alpha_1-1}\xi_1(\mathcal{U})d\mathcal{U}, \\ \mathcal{V}_2(t) - \mathcal{T}_2(\mathcal{V}_1, \mathcal{V}_2) =& -\frac{d_2 b_2}{\eta_1\Gamma(\beta_1)}\int_0^{\zeta_2}(\zeta_2-\mathcal{U})^{\beta_1-1}\xi_2(\mathcal{U})d\mathcal{U} \\ &- \frac{d_2 c_2}{\eta_1\Gamma(\beta_1)}\int_0^1 (1-\mathcal{U})^{\beta_1-1}\xi_2(\mathcal{U})d\mathcal{U} \\ &+ \frac{1}{\eta_1\Gamma(\beta_1)}\int_0^t (t-\mathcal{U})^{\beta_1-1}\xi_2(\mathcal{U})d\mathcal{U}. \end{aligned}$$

By taking the absolute and using Remark (3), we have

$$\left|\left|\mathcal{V}_1(t) - \mathcal{T}_1(\mathcal{V}_1, \mathcal{V}_2)\right|\right| \le \frac{(|d_1||b_1| + |d_1||c_1| + 1)\mathcal{G}_1\chi_1(t)\epsilon_1}{|\sigma_1|\Gamma(\alpha_1+1)},$$
$$\left|\left|\mathcal{V}_2(t) - \mathcal{T}_2(\mathcal{V}_1, \mathcal{V}_2)\right|\right| \le \frac{(|d_2||b_2| + |d_2||c_2| + 1)\mathcal{G}_2\chi_2(t)\epsilon_2}{|\eta_1|\Gamma(\beta_1+1)}.$$

This proves the required result. □

Theorem 7. *If the assumptions (**H1**)–(**H3**) and (**H7**) are supposed, then the problem Equation (2) is UHR stable and GUHR stable, if $\mathscr{L} < 1$, where $\mathscr{L} = \max(\mathscr{L}_1, \mathscr{L}_2)$ and $\mathscr{L}_1, \mathscr{L}_2$ are given by Equations (7) and (9), respectively.*

Proof. Suppose a given general solution $\mathscr{U} \in CJ, R)$, as well as a unique solution $\mathscr{U}^*$, for the model Equation (2). Then,

$$||(\mathscr{V}_1, \mathscr{V}_2) - (\mathscr{V}_1^*, \mathscr{V}_2^*)|| \leq ||\mathscr{V}_1 - \mathcal{T}_1(\mathscr{V}_1, \mathscr{V}_2)|| + ||\mathcal{T}_1(\mathscr{V}_1^*, \mathscr{V}_2^*) - \mathcal{T}_1(\mathscr{V}_1, \mathscr{V}_2)|| + ||\mathscr{V}_2 - \mathcal{T}_2(\mathscr{V}_1, \mathscr{V}_2)|| + ||\mathcal{T}_2(\mathscr{V}_1^*, \mathscr{V}_2^*) - \mathcal{T}_2(\mathscr{V}_1, \mathscr{V}_2)||.$$

Using Theorem (4) and Lemma (3), we have

$$\begin{aligned} ||(\mathscr{V}_1, \mathscr{V}_2) - (\mathscr{V}_1^*, \mathscr{V}_2^*)|| \leq & \Bigg[\frac{\big(|d_1||b_1| + |d_1||c_1| + 1\big)\mathscr{G}_1\chi_1(t)\epsilon_1}{|\sigma_1|\Gamma(\alpha_1+1)} \\ & + \frac{\big(|d_2||b_2| + |d_2||c_2| + 1\big)\mathscr{G}_2\chi_2(t)\epsilon_2}{|\eta_1|\Gamma(\beta_1+1)}\Bigg]\frac{1}{1-\mathscr{L}}, \\ \leq & \mathcal{B}^*\chi(t)\epsilon. \end{aligned}$$

Let $\mathcal{B}^* = \max\left[\frac{\big(|d_1||b_1|+|d_1||c_1|+1\big)\mathscr{G}_1}{|\sigma_1|\Gamma(\alpha_1+1)(1-\mathscr{L})}, \frac{\big(|d_2||b_2|+|d_2||c_2|+1\big)\mathscr{G}_2}{|\eta_1|\Gamma(\beta_1+1)(1-\mathscr{L})}\right]$; then, the solution of the considered problem Equation (2) is UHR stable. Furthermore, set $\epsilon = 1$,; then, the proposed problem Equation (2) is GUHR stable. □

To see the applications of the obtained results, an illustrative example is provided in the next part.

Example 1. *Taking some specific functions in the model Equation (2), we construct the following MFDDEs:*

$$\begin{cases} \sum_{i=1}^{100} \frac{14}{3^i} {}^cD^{\frac{3}{5^i}} \mathscr{V}_1(t) = \frac{e^{-\sin(t)}}{14}\mathscr{V}_1(t) + \frac{\log(1+\sin(t))}{71}\mathscr{V}_1(\frac{t}{5}) - \frac{coh(e^t)}{34}\mathscr{V}_2(t) + \frac{\cosh(t^3)}{9e^t - t^2}, \\ \sum_{i=1}^{100} \frac{33}{8^i} {}^cD^{\frac{3}{7^i}} \mathscr{V}_2(t) = \frac{\tan(x)}{45}\mathscr{V}_2(t) + \frac{cos^{-1}(t-0.5))}{13}\mathscr{V}_2(\frac{t}{7}) - \frac{coh(t^2+1)}{53}\mathscr{V}_1(t) + \frac{\cosh(t^3)}{9e^t - t^2}, \\ \frac{3\mathscr{V}_1}{18+|\mathscr{V}_1|} = \frac{1}{14}\mathscr{V}_1(0) - \mathscr{V}_1(\zeta_1) - \frac{2}{50}\mathscr{V}_1(1),\ t \in [0,1] \\ \frac{e^t\mathscr{V}_2}{12(t^2+5)+|\mathscr{V}_2|} = \frac{2}{15}\mathscr{V}_2(0) - \frac{3}{4}\mathscr{V}_2(\zeta_2) - \frac{1}{100}\mathscr{V}_2(1). \end{cases} \tag{32}$$

Here, $n = 100$, $\alpha_i = \frac{3}{5^i}$, $\beta_i = \frac{3}{7^i}$, $a_1 = \frac{1}{14}$, $a_2 = \frac{2}{15}$, $b_1 = 1; b_2 = \frac{3}{4}$, $c_1 = \frac{2}{50}$, $c_2 = \frac{1}{100}$, $\sigma_i = \frac{14}{3^i}$,
$\eta_i = \frac{33}{8^i}$, $\zeta_1 = \frac{1}{24}$, $\zeta_2 = \frac{2}{31}$, $\lambda_1 = \frac{1}{5}$, $\lambda_2 = \frac{1}{7}$,

$$\begin{aligned} f_1(t, \mathscr{V}_1(t), \mathscr{V}_1(\lambda_1 t), \mathscr{V}_2(t)) &= \frac{e^{-\sin(t)}}{14}\mathscr{V}_1(t) + \frac{\log(1+\sin(t))}{71}\mathscr{V}_1(\frac{t}{5}) - \frac{coh(e^t)}{34}\mathscr{V}_2(t) + \frac{\cosh(t^3)}{9e^t - t^2}, \\ f_2(t, \mathscr{V}_2(t), \mathscr{V}_2(\lambda_2 t), \mathscr{V}_1(t)) &= \frac{\tan(x)}{45}\mathscr{V}_2(t) + \frac{cos^{-1}(t-0.5))}{13}\mathscr{V}_2(\frac{t}{7}) - \frac{coh(t^2+1)}{53}\mathscr{V}_1(t) \\ &\quad + \frac{\cosh(t^3)}{9e^t - t^2}, \\ g_1(\mathscr{V}_1) = \frac{3\mathscr{V}_1}{18+|\mathscr{V}_1|}, \quad \textit{and} \quad & g_2(\mathscr{V}_2) = \frac{e^t\mathscr{V}_2}{12(t^2+5)+|\mathscr{V}_2|}. \end{aligned} \tag{33}$$

Now, the reader can easily obtain from Equation (33) that $\mathscr{L}_{f_1} = \frac{1}{14}$, $\mathscr{L}_{f_1^\lambda} = \frac{\log(1+\sin(1))}{71}$, $\mathscr{L}_{f_1^c} = \frac{\cosh(e)}{34}$, $\mathscr{L}_{f_2} = \frac{\tan(1)}{45}$, $\mathscr{L}_{f_2^\lambda} = 1.0471975512$; $\mathscr{L}_{f_2^c} = \frac{\cosh(2)}{53}$, $\mathscr{L}_{g_1} = \frac{3}{18}$, $\mathscr{L}_{g_2} = \frac{e}{65}$, *consequently* $\mathscr{L} = 0.9539 < 1$. *Therefore, by Theorem (4), the problem Equation (32) has a unique solution. Set* $\chi_1(t) = t + 1$, *and* $\chi_2(t) = 2t^2 + 3 \; \forall t \in [0, 1]$. *Then, we can get the following:*

$$\frac{1}{\Gamma(\frac{3}{5})} \int_0^t (t - \mathscr{U})^{\frac{3}{5}-1} (\mathscr{U} + 1) d\mathscr{U} \leq 1.12(t + 1),$$
$$\frac{1}{\Gamma(\frac{3}{5})} \int_0^t (t - \mathscr{U})^{\frac{3}{5}-1} (2\mathscr{U}^2 + 3) d\mathscr{U} \leq 1.12(t + 1) \leq 1.13(2t^2 + 3)$$

*As a condition, (**H7**) is satisfied with* $\mathscr{G}_1 = 1.12$ *and* $\mathscr{G}_2 = 1.13$. *Therefore, all requirements of Theorem (7) are fulfilled, so this leads to the fact that the solution of the problem Equation (2) is UHR stable with respect to a continuous function* $\chi(t) = 2t^2 + 3$ *together with constant* $\mathcal{B}^* = 13.1184$. *Furthermore, the solution of the aforementioned problem is GUHR stable. In addition to this, the problem Equation (2) is UH and GUH stable because of Theorem (6).*

5. Conclusions

Findings of the research are described here. We derived some adequate results regarding the existence and functional stability of a class of nonlocal BVP of coupled systems of MFDEs with proportional type delay term. We have used the results from functional analysis and FPT to derive the conditions for the solution's existence and uniqueness. Furthermore, results related to UH stabilities were also investigated. The results reached in this manuscript generalize some results from the existing literature. In addition, the outcomes have been illustrated through a proper example. Moreover, the special case of the considered problem Equation (2) can be obtained by fixing the parameters, $\eta_i = \sigma_i = 0$, for $i = 2, 3, \ldots, n$), which was investigated in [39]. As future work, we recommend studying the present problem, as well as other similar expected problems, using symmetry methods.

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