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On Codazzi Couplings on the Metric $\left(E^4 = I\right)-$Manifolds

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Abstract: Let $\mathcal{M}^k$ be a metric $\left(E^4 = I\right)-$manifold equipped with electromagnetic-type structure E, a pseudo-Riemannian metric g and a nondegenerate $2-$form $\hat{\omega}$. The paper deals with Codazzi couplings of an affine connection ∇ with E, g and $\hat{\omega}$. We present some results concerning the relationship of these Codazzi couplings. In addition, we construct the connection between Codazzi couplings and $e - (E^4 = I)$ Kaehler manifolds.

Keywords: electromagnetic-type structure; Codazzi couplings; conjugate connection; statistical structure

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1. Introduction

A metric $\left(E^4 = I\right)-$manifold is a $k-$dimensional pseudo-Riemannian manifold $\mathcal{M}^k$ which consists of a $(1,1)$ tensor field E and a pseudo-Riemannian metric g satisfying the following:

(a) $E^4 = I$, whose characteristic polynomial is $(\lambda - 1)^{\beta_1}(\lambda + 1)^{\beta_2}\left(\lambda^2 + 1\right)^s$ with $\beta_1 + \beta_2 + 2s = k$;
(b) $g(EK, L) + g(K, EL) = 0$, then g is necessarily pseudo-Riemannian and $\beta_1 = \beta_2$.

A $\left(E^4 = I\right)-$structure combines an almost-product structure and an almost-complex structure. In addition, it is a generalization of the electromagnetic tensor field. The condition (b) is the condition that the pseudo-Riemannian metric g is an aem (adapted in the electromagnetic sense metric). In addition, this condition generalizes in a sense of that of Mishra [1] and Hlavaty [2]. For g being an aem, a metric $\left(E^4 = I\right)-$manifold will be called $e - \left(E^4 = I\right)$ Kaehler manifold if E is parallel relative to the Levi–Civita connection ∇^g of g $(\nabla^g E = 0)$ [3].

Let the triple $(\mathcal{M}^k, E, g)$ be a metric $\left(E^4 = I\right)-$manifold. Then, fundamental $2-$form $\hat{\omega}$ can be defined by the formula:

$$\hat{\omega}(K, L) = g(EK, L) = -g(K, EL) = -\hat{\omega}(L, K).$$

In the present paper, we will take manifold as a smooth $k-$manifold and use the character E, g and $\hat{\omega}$ for the electromagnetic-type structure, the pseudo-Riemannian metric g and the $2-$form, respectively. Furthermore, the quadruple $(\mathcal{M}^k, E, g, \hat{\omega})$ is denoted as $e - (E^4 = I)$ Kaehler manifold. It is easy to say that the following satisfy:

(i) if E is an electromagnetic-type structure, $\widehat{E} = E^{-1} = E^3$ is an electromagnetic-type structure, which we will call an $\widehat{E}$ conjugate electromagnetic-type structure.
(ii) if E is an electromagnetic-type structure, $P = E^2$ is an almost-product structure.

Note that substitution $K = \widehat{E}K$ and $L = \widehat{E}L$ in $g(EK, L) + g(K, EL) = 0$ immediately gives that $g\left(\widehat{E}K, L\right) + g\left(K, \widehat{E}L\right) = 0$. Moreover, $\nabla^g E = 0$ if and only if $\nabla^g \widehat{E} = 0$. Hence, $(\mathcal{M}^k, \widehat{E}, g)$ is another $e - (E^4 = I)$ Kaehler manifold.

The $e-(E^4=I)$ Kaehler manifolds were firstly constructed by Gadea and Amilibia in [3]. In this paper, the authors showed many of the results obtained for Kaehler manifolds in this more general context. In particular, the Riemannian curvature tensor R satisfies an identity, involving E, in addition to the usual symmetries. As in the Kaehler case, this is used to show that R is determined by its values on quadruplets (K, EK, K, EK), $K \in$ Sec $T(\mathcal{M}^k)$. This, in turn, leads to an analogue of the notion of constant holomorphic sectional curvature. The authors show that such manifolds are locally products of Kaehler manifolds and ones in which $E^2 = I$. They give models for the latter in which $\mathcal{M}^k$ is the tangent bundle to a sphere.

In [4], Fei and Zhang studied the interaction of Codazzi couplings with para-Kaehler geometry. The authors obtain the structural results that the Kleinian group acts on an arbitrary affine connection by $g-$conjugation, $\hat{\omega}-$conjugation, and $\mathfrak{L}-$gauge transformation, where g is the pseudo-Riemannian metric, $\hat{\omega}$ is a non-degenerate $2-$form and $\mathfrak{L}$ is the tangent bundle isomorphism on smooth manifolds. They established the relationship of Codazzi couplings of a torsion-free connection with a compatible triple. They also showed the compatibility of a pair of connections with Kaehler and para-Kaehler structures, which generalizes special Kaehler geometry (where the connection is curvature-free) to Codazzi-Kaehler geometry (where the connection need not be curvature-free). Later, Gezer and Cakicioglu [5] obtained some new results concerning with Codazzi pairs on the anti-Hermitian context by using different arguments. The paper aims to study Codazzi couplings on the metric $\left(E^4=I\right)-$manifold $(\mathcal{M}^k, E, g, \hat{\omega})$. The analogous case with almost Hermitian case was worked out earlier by Fei and Zhang [4].

2. Conjugate Connection and Codazzi Coupling

Let $(\mathcal{M}^k, E, g, \hat{\omega})$ be a metric $\left(E^4=I\right)-$manifold and ∇ be an affine connection. Next, we define, respectively, the conjugate connections of ∇ according to g, $\hat{\omega}$ and E by the equations [6]:

$$Mg(K,L) = g(\nabla_M K, L) + g(K, \nabla^*_M L),$$
$$M\hat{\omega}(K,L) = \hat{\omega}(\nabla_M K, L) + \hat{\omega}(K, \nabla^\dagger_M L)$$

and

$$\nabla^E(K,L) = \widehat{E}(\nabla_K EL)$$

for all vector fields K, L, M on $\mathcal{M}^k$. These connections are called a $g-$conjugate connection, $\hat{\omega}-$conjugate connection and $E-$conjugate connection, respectively. Note that both $g-$conjugate connection and $\hat{\omega}-$conjugate connection satisfy $(\nabla^*)^* = \nabla$ and $\left(\nabla^\dagger\right)^\dagger = \nabla$. It is clear that $\nabla g = 0$ if and only if ∇^* (or $\nabla^\dagger$) coincides with ∇ . For conjugate connections, we also refer to [7–9].

Considering the pair (∇, g), the $(0,3)-$tensor fields $\mathfrak{C}$ and Γ are constructed, respectively, by

$$\mathfrak{C}(K,L,M) := (\nabla_M g)(K,L)$$

and

$$\Gamma(K,L,M) := -(\nabla^\dagger_M \hat{\omega})(K,L) = (\nabla_M \hat{\omega})(K,L),$$

where the tensor field $\mathfrak{C}$ is referred to as the cubic form associated to the pair (∇, g) and Γ is in analogous to the cubic form $\mathfrak{C}$ [4].

The curvature tensor field R of an affine connection ∇ is defined by, for all vector fields K, L, M,

$$R(K,L)M = \nabla_K \nabla_L M - \nabla_L \nabla_K M - \nabla_{[K,L]} M$$

and its $(0,4)-$curvature tensor field is as follows:

$$R(K,L,M,N) = g(R(K,L)M, N).$$

For the curvature tensor fields of ∇, ∇^* and ∇^E, the theorem is given below.

Theorem 1. *Let $(\mathcal{M}^k, E, g, \hat{\omega})$ be a metric $\left(E^4 = I\right)-$manifold. ∇^* and ∇^E assign, respectively, $g-$conjugation and $E-$conjugation of an affine connection ∇ on $\mathcal{M}^k$. The relationship between the $(0,4)-$curvature tensor fields R, R^* and R^E of ∇, ∇^* and ∇^E is as follows:*

$$R(K, L, EM, N) = -R^*(K, L, N, EM) = R^E(K, L, M, EN)$$

for all vector fields K, L, M on $\mathcal{M}^k$.

Proof. It suffices to prove it only on one basis, because the relation is linear in the arguments K, L, N and M. Thus, we suppose that $K, L, N, M \in \{\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^k}\}$ and take computational advantage of the following vanishing Lie brackets

$$[K, L] = [L, N] = [N, M] = 0.$$

From here, it is obtained that

$$\begin{aligned} KL\hat{\omega}(M, N) &= K(Lg(EM, N)) \\ &= K(g(\nabla_L(EM), N) + g(EM, \nabla^*_L N)) \\ &= K(g(\nabla_L(EM), N)) + K(g(EM, \nabla^*_L N)) \\ &= g(\nabla_K \nabla_L(EM), N) + g(\nabla_L(EM), \nabla^*_K N) \\ &\quad + g(\nabla_K(EM), \nabla^*_L N) + g(EM,\ \nabla^*_K \nabla^*_L N) \end{aligned}$$

and similarly

$$\begin{aligned} LK\hat{\omega}(M, N) &= L(Kg(EM, N)) \\ &= g(\nabla_L \nabla_K(EM), N) + g(\nabla_K(EM), \nabla^*_L N) \\ &\quad + g(\nabla_L(EM), \nabla^*_K N) + g(EN, \nabla^*_L \nabla^*_K N). \end{aligned}$$

When we subtract the above equations from each other, we find

$$\begin{aligned} 0 &= [K, L]\hat{\omega}(M, N) = KL\hat{\omega}(M, N) - LK\hat{\omega}(M, N) \\ 0 &= g(\nabla_K \nabla_L(EM) - \nabla_L \nabla_K(EM), N) \\ &\quad + g(EM,\ \nabla^*_K \nabla^*_L N - \nabla^*_L \nabla^*_K N) \\ &\quad - g(\nabla_L \nabla_K(EM), N) - g(\nabla_K(EM), \nabla^*_L N) \\ &\quad - g(\nabla_L(EM), \nabla^*_K N) - g(EM, \nabla^*_L \nabla^*_K N) \\ 0 &= g(R(K, L)EM, N) + g(R^*(K, L)N, EM) \\ 0 &= R(K, L, EM, N) + R^*(K, L, N, EM) \end{aligned}$$

$$\Rightarrow R(K, L, EM, N) = -R^*(K, L, N, EM)$$

and similarly

$$\begin{aligned} & g(\nabla_K \nabla_L(EM) - \nabla_L \nabla_K(EM), N) + g(EM,\ \nabla^*_K \nabla^*_L N - \nabla^*_L \nabla^*_K N) = 0 \\ & \hat{\omega}\left(\widehat{E}(\nabla_K \nabla_L(EM) - \nabla_L \nabla_K(EM)), N\right) + \hat{\omega}(M, \nabla^*_K \nabla^*_L N - \nabla^*_L \nabla^*_K N) = 0 \\ & \hat{\omega}\left(\widehat{E}\nabla_K E\left(\widehat{E}\nabla_L EM\right) - \widehat{E}\nabla_L E\left(\widehat{E}\nabla_K EM\right), N\right) + \hat{\omega}(M, \nabla^*_K \nabla^*_L N - \nabla^*_L \nabla^*_K N) = 0 \\ & \hat{\omega}\left(\widehat{E}\nabla_K E\left(\nabla^E_L M\right) - \widehat{E}\nabla_L E\left(\nabla^E_K M\right), N\right) + \hat{\omega}(M, \nabla^*_K \nabla^*_L N - \nabla^*_L \nabla^*_K N) = 0 \\ & \hat{\omega}\left(\nabla^E_K \nabla^E_L M - \nabla^E_L \nabla^E_K M, N\right) + \hat{\omega}(M, \nabla^*_K \nabla^*_L N - \nabla^*_L \nabla^*_K N) = 0 \\ & -g\left(\nabla^E_K \nabla^E_L M - \nabla^E_L \nabla^E_K M, EN\right) + g(\nabla^*_K \nabla^*_L N - \nabla^*_L \nabla^*_K N,\ EM) = 0 \\ & -R^E(K, L, M, EN) + R^*(K, L, N, EM) = 0 \\ & \Rightarrow R^E(K, L, M, EN) = R^*(K, L, N, EM). \end{aligned}$$

Thus, it is obtained that $R(K, L, EM, N) = -R^*(K, L, N, EM) = R^E(K, L, M, EN)$ which completes the proof. $\square$

Given an arbitrary affine connection ∇ on a pseudo-Riemannian manifold $(\mathcal{M}^k, g)$, for any $(1,1)-$tensor field E and a symmetric bilinear form ρ on $\mathcal{M}^k$, we call (∇, E) and (∇, ρ), respectively, Codazzi-coupled, if their covariant derivative (∇E) and $(\nabla \rho)$, respectively, is (totally) symmetric in K, L, M [6]:

$$(\nabla_M E)K = (\nabla_K E)M, \, (\nabla_M \rho)(K, L) = (\nabla_K \rho)(M, L).$$

Next, we search characterization of Codazzi couplings of an affine connection ∇ on $\mathcal{M}^k$ with a pseudo-Riemannian metric g and an electromagnetic-type structure E. We give the following proposition, which is analogous to the result given in [4] for a Hermitian setting.

Proposition 1. *Let ∇ be an affine connection on the metric $\left(E^4 = I\right)-$manifold $(\mathcal{M}^k, E, g, \hat{\omega})$. If $(\nabla, \hat{\omega})$ is Codazzi-coupled, the following are provided:*

(i) $\Gamma(K, L, M) = (\nabla_M \hat{\omega})(K, L)$ *is not totally symetric;*
(ii) $\Gamma(K, L, M) = \Gamma(K, M, L) \Leftrightarrow \nabla^\dagger$ *and* ∇ *have equal torsions;*
(iii) ∇^* *and* ∇ *have equal torsions if and only if* (∇^*, E) *is Codazzi-coupled.*

Proof. (i) Since $\hat{\omega}$ is skew-symmetric, $\Gamma(K, L, M) = -\Gamma(L, K, M)$. Therefore, Γ is not totally symetric.

(ii)

$$\begin{aligned}
&\Gamma(K, L, M) - \Gamma(K, M, L) = (\nabla_M \hat{\omega})(K, L) - (\nabla_L \hat{\omega})(K, M) \\
&= Z\hat{\omega}(K, L) - \hat{\omega}(\nabla_M K, L) - \hat{\omega}(K, \nabla_M L) \\
&- L\hat{\omega}(K, M) + \hat{\omega}(\nabla_L K, M) + \hat{\omega}(K, \nabla_L M) \\
&= \hat{\omega}(\nabla_M K, L) + \hat{\omega}\left(K, \nabla^\dagger_M L\right) - \hat{\omega}(\nabla_M K, L) - \hat{\omega}(K, \nabla_M L) \\
&- \hat{\omega}(\nabla_L K, M) - \hat{\omega}\left(K, \nabla^\dagger_L M\right) + \hat{\omega}(\nabla_L K, M) + \hat{\omega}(K, \nabla_L M) \\
&= \hat{\omega}\left(K, \left(\nabla^\dagger - \nabla\right)_M L\right) - \hat{\omega}(K, \left(\nabla^\dagger - \nabla\right)_L M) \\
&= \hat{\omega}\left(K, \left(\nabla^\dagger - \nabla\right)_M L - \left(\nabla^\dagger - \nabla\right)_L M\right) \\
&= \hat{\omega}\left(K, T^{\nabla^\dagger}(M, L) - T^\nabla(M, L)\right) = 0 \Leftrightarrow T^{\nabla^\dagger} = T^\nabla,
\end{aligned}$$

where

$$\begin{aligned}
T^{\nabla^\dagger}(M, L) &= \nabla^\dagger_M L - \nabla^\dagger_L M - [M, L] \\
T^\nabla(M, L) &= \nabla_M L - \nabla_L M - [M, L].
\end{aligned}$$

(iii) From the covariant derivative, we have

$$
\begin{aligned}
&(\nabla_M\hat{\omega})(K,L) = (\nabla_K\hat{\omega})(M,L)\\
&\hat{\omega}(K,L) - \hat{\omega}(\nabla_M K,L) - \hat{\omega}(K,\nabla_M L) = K\hat{\omega}(M,L) - \hat{\omega}(\nabla_K M,L) - \hat{\omega}(M,\ \nabla_K L)\\
&Mg(EK,L) - g(E\nabla_M K,L) - g(EK,\nabla_M L) = Kg(EM,L) - g(E\nabla_K M,L) - g(EM,\nabla_K L)\\
&g(\nabla^*_M(EK),L) - g(E\nabla_M K,L) = g(\nabla^*_K(EM),L) - g(E\nabla_K M,L)\\
&\hat{\omega}\left(\widehat{E}\nabla^*_M(EK),L\right) - \hat{\omega}(\nabla_M K,L) = \hat{\omega}\left(\widehat{E}\nabla^*_K(EM),L\right) - \hat{\omega}(\nabla_K M,L)\\
&\hat{\omega}\left(\widehat{E}\{\nabla^*_M(EK) - \nabla^*_K(EM)\},L\right) = \hat{\omega}(\nabla_M K - \nabla_K M,L)\\
&E^{-1}\{\nabla^*_M(EK) - \nabla^*_K(EM)\} = \nabla_M K - \nabla_K M\\
&\widehat{E}\{(\nabla^*_M E)K + E(\nabla^*_M K) - (\nabla^*_K E)M - E(\nabla^*_K M)\} = \nabla_M K - \nabla_K M\\
&\widehat{E}\{(\nabla^*_M E)K - (\nabla^*_K E)M\} + \widehat{E}\{\ E(\nabla^*_M K - \nabla^*_K M)\} = \nabla_M K - \nabla_K M\\
&\widehat{E}\{(\nabla^*_M E)K - (\nabla^*_K E)M\} + ((\nabla^*_M K - \nabla^*_K M) - [M,K]) = \nabla_M K - \nabla_K M - [M,K]\\
&\widehat{E}\{(\nabla^*_M E)K - (\nabla^*_K E)M\} = T^{\nabla}(M,K) - T^{\nabla^*}(M,K),
\end{aligned}
$$

where

$$T^{\nabla^*}(M,L) = \nabla^*_M L - \nabla^*_L M - [M,L].$$

Therefore, $T^{\nabla^*}(M,K) = T^{\nabla}(M,K) \Leftrightarrow \left(\nabla^*_M E\right)K = (\nabla^*_K E)M$, that is, (∇^*, E) is Codazzi-coupled. □

Proposition 2. *Let ∇ be an affine connection on the metric $\left(E^4 = I\right)$−manifold $(\mathcal{M}^k, E, g, \hat{\omega})$. The followings are equivalent:*

(i) (∇, E) *is Codazzi-coupled;*
(ii) ∇ *and* ∇^E *have equal torsions;*
(iii) $(\nabla^E, \widehat{E})$ *is Codazzi-coupled,*

where $\widehat{E}$ is a conjugate electromagnetic-type structure on $(\mathcal{M}^k, E, g, \hat{\omega})$.

Proof. Let (∇, E) be Codazzi-coupled. Using $T^{\nabla}(K,L) = \nabla_K L - \nabla_L K - [K,L]$ and $T^{\nabla^E}(K,L) = \nabla^E_K L - \nabla^E_L K - [K,L]$, we yield

$$
\begin{aligned}
&T^{\nabla^E}(K,L) - T^{\nabla}(K,L) = \nabla^E_K L - \nabla_K L - \nabla^E_L K + \nabla_L K\\
&= \widehat{E}(\nabla_K EL) - \nabla_K L - \widehat{E}(\nabla_L EK) + \nabla_L K\\
&= \widehat{E}[(\nabla_K E)L + E(\nabla_K L)] - \nabla_K L - \widehat{E}[(\nabla_L E)K + E(\nabla_L K)] + \nabla_L K\\
&= \widehat{E}(\nabla_K E)L + \nabla_K L - \nabla_K L - \widehat{E}(\nabla_L E)K - \nabla_L K + \nabla_L K\\
&= \widehat{E}[(\nabla_K E)L - (\nabla L E)K] = 0.
\end{aligned}
$$

On the other hand, it is straightforward to obtain

$$
\begin{aligned}
&\left(\nabla^E_K \widehat{E}\right)L - \left(\nabla^E_L \widehat{E}\right)K = \nabla^E_K\left(\widehat{E}L\right) - \widehat{E}\left(\nabla^E_K L\right) - \nabla^E_L\left(\widehat{E}K\right) + \widehat{E}\left(\nabla^E_L K\right)\\
&= \widehat{E}(\nabla_K L) - \widehat{E}\left(\widehat{E}\nabla_K EL\right) - \widehat{E}(\nabla_L K) + \widehat{E}\left(\widehat{E}\nabla_L EK\right)\\
&= \widehat{E}(\nabla_K L) - \widehat{E}^2[(\nabla_K E)L + E(\nabla_K L)] - \widehat{E}(\nabla_L K) + \widehat{E}^2[(\nabla_L E)K + E(\nabla_L K)]\\
&= \widehat{E}(\nabla_K L) - P(\nabla_K E)L - \widehat{E}(\nabla_K L) - \widehat{E}(\nabla_L K) + P(\nabla_L E)K + \widehat{E}(\nabla_L K)\\
&= P[(\nabla_L E)K - (\nabla_K E)L],
\end{aligned}
$$

which gives to us: $\left(\nabla^E_K \widehat{E}\right)L - \left(\nabla^E_L \widehat{E}\right)K = 0 \Leftrightarrow (\nabla_L E)K - (\nabla_K E)L = 0$. Hence, the proof is completed. □

Proposition 3. *Let ∇ be an affine connection on the metric $\left(E^4 = I\right)-$manifold $(\mathcal{M}^k, E, g, \hat{\omega})$. If (∇, E) and (∇, P) are Codazzi-coupled, (∇^E, E) is Codazzi-coupled, where $P = E^2$.*

Proof. Standard calculations give

$$\begin{aligned}
&\left(\nabla^E_K E\right)K - \left(\nabla^E_L E\right)K = \left(\nabla^E_K EL\right) - E\left(\nabla^E_K L\right) - \left(\nabla^E_L EK\right) + E\left(\nabla^E_L K\right) \\
&= E^{-1}(\nabla_K PL) - E\left(\widehat{E}\nabla_K EL\right) - \widehat{E}(\nabla_L PK) + E\left(\widehat{E}\nabla_L EK\right) \\
&= \widehat{E}(\nabla_K PL) - \nabla_K EL - \widehat{E}(\nabla_L PK) + \nabla_L EK \\
&= \widehat{E}[(\nabla_K P)L + P\nabla_K L] - (\nabla_K E)L - E\nabla_K L \\
&-\widehat{E}[(\nabla_L P)K + P\nabla_L K] + (\nabla_L E)K + E\nabla_L K \\
&= \widehat{E}(\nabla_K P)L + E\nabla_K L - (\nabla_K E)L - E\nabla_K L \\
&-\widehat{E}(\nabla_L P)K - E\nabla_L K + (\nabla_L E)K + E\nabla_L K \\
&= \widehat{E}[(\nabla_K P)L - (\nabla_L P)K] + (\nabla_L E)K - (\nabla_K E)L.
\end{aligned}$$

Thus, the result is given. □

Proposition 4. *Let ∇ be an affine connection on the metric $\left(E^4 = I\right)-$manifold $(\mathcal{M}^k, E, g, \hat{\omega})$. In that case, the following are equivalent:*

(i) $\hat{\omega}\left(T^{\nabla}(K, L), M\right) = \hat{\omega}\left(T^{\nabla^\dagger}(K, L), M\right) + \left(\nabla^\dagger \hat{\omega}\right)(K, L, M) - \left(\nabla^\dagger \hat{\omega}\right)(L, K, M)$;

(ii) $\nabla \hat{\omega}$ *is symmetric if and only if* $\nabla^\dagger \hat{\omega}$ *is symmetric.*

Proof. (i) From the definition of $\hat{\omega}-$conjugation, it follows that

$$\begin{aligned}
&\hat{\omega}\left(T^{\nabla}(K, L), M\right) = \hat{\omega}(\nabla_K L - \nabla_L K - [K, L], M) \\
&= \hat{\omega}(\nabla_K L, M) - \hat{\omega}(\nabla_L K, M) - \hat{\omega}([K, L], M) \\
&= K\hat{\omega}(L, M) - \hat{\omega}\left(L, \nabla^\dagger_K M\right) - L\hat{\omega}(K, M) + \hat{\omega}\left(K, \nabla^\dagger_L M\right) \\
&-\hat{\omega}\left(\nabla^\dagger_K L - \nabla^\dagger_L K - T^{\nabla^\dagger}(K, L), M\right) \\
&= K\hat{\omega}(L, M) - \hat{\omega}\left(L, \nabla^\dagger_K M\right) - L\hat{\omega}(K, M) + \hat{\omega}\left(K, \nabla^\dagger_L M\right) \\
&-\hat{\omega}\left(\nabla^\dagger_K L, M\right) + \hat{\omega}\left(\nabla^\dagger_L K, M\right) + \hat{\omega}\left(T^{\nabla^\dagger}(K, L), M\right) \\
&= \left(\nabla^\dagger_K \hat{\omega}\right)(L, M) - \left(\nabla^\dagger_L \hat{\omega}\right)(K, M) + \hat{\omega}\left(T^{\nabla^\dagger}(K, L), M\right).
\end{aligned}$$

(ii) Similarly, we obtain

$$\begin{aligned}
&\left(\nabla^\dagger \hat{\omega}\right)(K, L, M) = \left(\nabla^\dagger_K \hat{\omega}\right)(L, M) \\
&= K\hat{\omega}(L, M) - \hat{\omega}\left(\nabla^\dagger_K L, M\right) - \hat{\omega}\left(L, \nabla^\dagger_K M\right) \\
&= K\hat{\omega}(L, M) - K\hat{\omega}(L, M) + \hat{\omega}(L,\ \nabla_K M) - K\hat{\omega}(L, M) + \hat{\omega}(\nabla_K L, M) \\
&= -\ K\hat{\omega}(L, M) + \hat{\omega}(L,\ \nabla_K M) + \hat{\omega}(\nabla_K L, M) \\
&= -(\nabla_K \hat{\omega})(L, M) = -(\nabla \hat{\omega})(K, L, M).
\end{aligned}$$

□

Proposition 5. *Let ∇ be an affine connection on the metric $\left(E^4 = I\right)-$manifold $(\mathcal{M}^k, E, g, \hat{\omega})$. In that case,*

(i) $\hat{\omega}(R(K, L, M), N) = -\hat{\omega}\left(M, R^\dagger(K, L, N)\right) = \hat{\omega}\left(R^\dagger(K, L, N), M\right)$;

(ii) *If $\nabla^\dagger$ is flat, ∇ is flat, too,*
where $\nabla^\dagger$ denotes $\hat{\omega}-$conjugation of ∇ on M^k and R and $R^\dagger$ are, respectively, the curvature tensor fields of ∇ and $\nabla^\dagger$.

Proof.

$$\begin{aligned}
&\hat{\omega}\Big(R^{\nabla}(K,L,M),N\Big) = \hat{\omega}\Big(\nabla_K\nabla_L M - \nabla_L\nabla_K M - \nabla_{[K,L]}M, N\Big) \\
&= \hat{\omega}(\nabla_K\nabla_L M, N) - \hat{\omega}(\nabla_L\nabla_K M, N) - \hat{\omega}\Big(\nabla_{[K,L]}M, N\Big) \\
&= K\hat{\omega}(\nabla_L M, N) - \hat{\omega}\Big(\nabla_L M, \nabla^\dagger_K N\Big) - L\hat{\omega}(\nabla_K M, N) + \hat{\omega}\Big(\nabla_K M, \nabla^\dagger_L N\Big) \\
&-[K,L]\hat{\omega}(M,N) + \hat{\omega}\Big(M, \nabla^\dagger_{[K,L]}N\Big).
\end{aligned}$$

On the other hand, using $[K,L]\hat{\omega}(M,N) = KL\hat{\omega}(M,N) - LK\hat{\omega}(M,N)$, we have

$$\begin{aligned}
KL\hat{\omega}(M,N) &= K(L\hat{\omega}(M,N)) \\
&= K(\hat{\omega}(\nabla_L M, N) + \hat{\omega}\Big(M, \nabla^\dagger_L N\Big) \\
&= K(\hat{\omega}(\nabla_L M, N)) + K\Big(\hat{\omega}\Big(M, \nabla^\dagger_L N\Big)\Big) \\
&= \hat{\omega}(\nabla_K\nabla_L M, N) + \hat{\omega}\Big(\nabla_L M, \nabla^\dagger_K N\Big) \\
&+ \hat{\omega}\Big(\nabla_K M,\ \nabla^\dagger_L N\Big) + \hat{\omega}\Big(M,\ \nabla^\dagger_K\nabla^\dagger_L N\Big)
\end{aligned}$$

and similarly

$$\begin{aligned}
LK\hat{\omega}(M,N) &= \hat{\omega}(\nabla_L\nabla_K M, N) + \hat{\omega}\Big(\nabla_K M, \nabla^\dagger_L N\Big) \\
&+ \hat{\omega}\Big(\nabla_L M,\ \nabla^\dagger_K N\Big) + \hat{\omega}\Big(M,\ \nabla^\dagger_L\nabla^\dagger_K N\Big).
\end{aligned}$$

Thus

$$\begin{aligned}
&LK\hat{\omega}(M,N) - KL\hat{\omega}(M,N) = \hat{\omega}(\nabla_L\nabla_K M, N) + \hat{\omega}\Big(\nabla_K M, \nabla^\dagger_L N\Big) \\
&+\hat{\omega}\Big(\nabla_L M,\ \nabla^\dagger_K N\Big) + \hat{\omega}\Big(M, \nabla^\dagger_L\nabla^\dagger_K N\Big) \\
&-\hat{\omega}(\nabla_K\nabla_L M, N) - \hat{\omega}\Big(\nabla_L M, \nabla^\dagger_K N\Big) \\
&-\hat{\omega}\Big(\nabla_K M,\ \nabla^\dagger_L N\Big) - \hat{\omega}\Big(M,\ \nabla^\dagger_K\nabla^\dagger_L N\Big) \\
&= -\hat{\omega}(\nabla_K\nabla_L M - \nabla_L\nabla_K M, N) - \hat{\omega}\Big(M,\ \nabla^\dagger_K\nabla^\dagger_L N - \nabla^\dagger_L\nabla^\dagger_K N\Big).
\end{aligned}$$

From the last equation, we obtain

$$\begin{aligned}
&\hat{\omega}\Big(R^{\nabla}(K,L,M),N\Big) = \hat{\omega}(\nabla_K\nabla_L M, N) + \hat{\omega}\Big(\nabla_L M,\ \nabla^\dagger_K N\Big) \\
&-\hat{\omega}\Big(\nabla_L M, \nabla^\dagger_K N\Big) - \hat{\omega}(\nabla_L\nabla_K M, N) - \hat{\omega}\Big(\nabla_K M, \nabla^\dagger_L N\Big) + \hat{\omega}\Big(\nabla_K M,\ \nabla^\dagger_L N\Big) \\
&-\hat{\omega}(\nabla_K\nabla_L M - \nabla_L\nabla_K M, N) \\
&-\hat{\omega}\Big(M, \nabla^\dagger_K\nabla^\dagger_L N - \nabla^\dagger_L\nabla^\dagger_K N\Big) + \hat{\omega}\Big(M, \nabla^\dagger_{[K,L]}N\Big) \\
&= \hat{\omega}(\nabla_K\nabla_L M - \nabla_L\nabla_K M, N) - \hat{\omega}(\nabla_K\nabla_L M - \nabla_L\nabla_K M, N) \\
&-\hat{\omega}\Big(M, \nabla^\dagger_K\nabla^\dagger_L N - \nabla^\dagger_L\nabla^\dagger_K N - \nabla^\dagger_{[K,L]}N\Big) \\
&= -\hat{\omega}\Big(M, R^{\nabla^\dagger}(K,L,N)\Big) = \hat{\omega}\Big(R^{\nabla^\dagger}(K,L,N), M\Big).
\end{aligned}$$

□

Proposition 6. *Let ∇ be an affine connection on the metric $\left(E^4 = I\right)-$manifold $(M^k, E, g, \hat{\omega})$. If $\nabla\hat{\omega}$ is symmetric and (∇, E) is Codazzi-coupled, $(\nabla^\dagger, E)$ is so.*

Proof.

$$
\begin{aligned}
&\hat{\omega}\left(\left(\nabla_K^\dagger E\right)L - \left(\nabla_L^\dagger E\right)K, M\right) = \hat{\omega}\left(\left(\nabla_K^\dagger E\right)L, M\right) - \hat{\omega}\left(\left(\nabla_L^\dagger E\right)K, M\right)\\
&= \hat{\omega}\left(\nabla_K^\dagger EL - E\nabla_K^\dagger L, M\right) - \hat{\omega}\left(\nabla_L^\dagger EK - E\nabla_L^\dagger K, M\right)\\
&= \hat{\omega}\left(\nabla_K^\dagger EL, M\right) - \hat{\omega}\left(E\nabla_K^\dagger L, M\right) - \hat{\omega}\left(\nabla_L^\dagger EK, M\right) + \hat{\omega}\left(E\nabla_L^\dagger K, M\right)\\
&= -\hat{\omega}\left(M, \nabla_K^\dagger EL\right) - \hat{\omega}\left(E\nabla_K^\dagger L, M\right) + \hat{\omega}\left(M, \nabla_L^\dagger EK\right) + \hat{\omega}\left(E\nabla_L^\dagger K, M\right)\\
&= -K\hat{\omega}(M, EL) + \hat{\omega}(\nabla_K M, EL) + L\hat{\omega}(M, EK)\\
&-\hat{\omega}(\nabla_L M, EK) - \hat{\omega}\left(E\nabla_K^\dagger L, M\right) + \hat{\omega}\left(E\nabla_L^\dagger K, M\right)\\
&= K\hat{\omega}(EL, M) - \hat{\omega}(EL, \nabla_K M) - L\hat{\omega}(EK, M)\\
&+\hat{\omega}(EK,\ \nabla_L M) - \hat{\omega}\left(E\left(\nabla_K^\dagger L - \nabla_L^\dagger K\right), M\right)\\
&= K\hat{\omega}(EL, M) - \hat{\omega}(EL, \nabla_K M) - L\hat{\omega}(EK, M)\\
&+\hat{\omega}(EK,\ \nabla_L M) - \hat{\omega}\left(E\left(T^{\nabla^\dagger}(K, L) + [K, L]\right), M\right).
\end{aligned}
$$

From $T^{\nabla^\dagger} = T^{\nabla}$, we obtain

$$
\begin{aligned}
&= K\hat{\omega}(EL, M) - \hat{\omega}(EL, \nabla_K M) - L\hat{\omega}(EK, M)\\
&+\hat{\omega}(EK,\ \nabla_L M) - \hat{\omega}(E(\nabla_K L - \nabla_L K), M)\\
&= (\nabla_K\hat{\omega})(EL, M) + \hat{\omega}(\nabla_K EL, M) + \hat{\omega}(EL, \nabla_K M)\\
&-\hat{\omega}(EL, \nabla_K M) - (\nabla_L\hat{\omega})(EK, M)\\
&-\hat{\omega}(\nabla_L EK, M) - \hat{\omega}(EK, \nabla_L M) + \hat{\omega}(EK, \nabla_L M)\\
&-\hat{\omega}(E\nabla_K L, M) + \hat{\omega}(E\nabla_L K, M)\\
&= (\nabla_K\hat{\omega})(EL, M) + \hat{\omega}(\nabla_K EL, M) - (\nabla_L\hat{\omega})(EK, M)\\
&-\hat{\omega}(\nabla_L EK, M) - \hat{\omega}(E\nabla_K L, M) + \hat{\omega}(E\nabla_L K, M)\\
&= (\nabla_K\hat{\omega})(EL, M) + \hat{\omega}((\nabla_K E)L, M) + \hat{\omega}(E\nabla_K L, M) - (\nabla_L\hat{\omega})(EK, M)\\
&-\hat{\omega}((\nabla_L E)K, M) - \hat{\omega}(E\nabla_L K, M) - \hat{\omega}(E\nabla_K L, M) + \hat{\omega}(E\nabla_L K, M)\\
&= (\nabla_K\hat{\omega})(EL, M) - (\nabla_L\hat{\omega})(EK, M) + \hat{\omega}((\nabla_K E)L - (\nabla_L E)K, M).
\end{aligned}
$$

Hence, the proof is completed. □

Recall that a structure is integrable if $N_E = 0$, where N_E is Nijenhuis tensor. In that case, the integrability of the electromagnetic-type structure E is equivalent to $N_E = 0$:

$$N_E(K, L) = [EK, EL] - E[EK, L] - E[K, EL] + E^2[K, L].$$

Proposition 7. *Let ∇ be an affine connection on the metric $\left(E^4 = I\right)-$manifold $(\mathcal{M}^k, E, g, \hat{\omega})$. In the case that (∇, E) is Codazzi-coupled, (∇, P) is Codazzi-coupled if and only if E is integrable, where $P = E^2$.*

Proof. From the condition that (∇, E) being Codazzi-coupled, we have

$$\begin{aligned} N_E(K,L) &= [EK,EL] - E[K,EL] - E[EK,L] + E^2[K,L] \\ &= \nabla_{EK}EL - \nabla_{EL}EK - E(\nabla_K EL - \nabla_{EL}K) \\ &\quad -E(\nabla_{EK}L - \nabla_L EK) + P(\nabla_K L - \nabla_L K) \\ &= ((\nabla_{EK}E)L + E(\nabla_{EK})L) - ((\nabla_{EL}E)K + E(\nabla_{EL})K) \\ &\quad -E((\nabla_K E)L + E\nabla_K L - \nabla_{EL}K) \\ &\quad -E(\nabla_{EK}L - (\nabla_L E)K - E(\nabla_L K)) + P(\nabla_K L - \nabla_L K) \\ &= (\nabla_{EK}E)L + E(\nabla_{EK})L - (\nabla_{EL}E)K - E(\nabla_{EL})K - E(\nabla_K E)L \\ &\quad -P\nabla_K L + E\nabla_{EL}K - E\nabla_{EK}L + E(\nabla_L E)K \\ &\quad +P\nabla_L K + P\nabla_K L - P\nabla_L K \\ &= (\nabla_{EK}E)L - (\nabla_{EL}E)K - E(\nabla_K E)L + E(\nabla_L E)K \\ &= (\nabla_L E)EK - (\nabla_K E)EL + E((\nabla_L E)K - (\nabla_K E)L) \\ &= (\nabla_L P)K - E(\nabla_L E)K - (\nabla_K P)L + E(\nabla_K E)L \\ &\quad +E((\nabla_L E)K - (\nabla_K E)L) \\ &= (\nabla_L P)K - (\nabla_K P)L. \end{aligned}$$

From this, we can say that (∇, P) is Codazzi-coupled if and only if E is integrable. □

3. (Codazzi) $e - (E^4 = I)$ Kaehler Manifold

Next, we search the Codazzi couplings with respect to the torsion-free connection ∇: Codazzi coupling of ∇ with E, Codazzi coupling of ∇ with g, and $\nabla\hat{\omega} = 0$ (that is, ∇ is an almost-symplectic connection). By means of these Codazzi couplings, we plan to approach (Codazzi) $e - (E^4 = I)$ Kaehler manifold.

Let $(\mathcal{M}^k, g)$ be a (pseudo-)Riemannian manifold with the torsion-free connection ∇. If (∇, g) is Codazzi-coupled, the manifold $\mathcal{M}^k$ with a statistical structure (∇, g) is named a statistical manifold. This type of manifold was first described by Lauritzen [10]. Statistical manifolds have been extensively researched in affine differential geometry [8,10] and have an important role plays in information geometry. The following theorem is analogue to the theorem given by Fei and Zhang [4] for a Hermitian setting.

Theorem 2. *Let ∇ be an affine connection on the metric $\left(E^4 = I\right)-$manifold $(\mathcal{M}^k, E, g, \hat{\omega})$. Assuming that*

(i) *(∇, g) is Codazzi-coupled;*
(ii) *(∇, J) and (∇, P) are Codazzi-coupled, where $P = J^2$.*

Then, (M^k, g, E) is a (Codazzi) $e - (E^4 = I)$ Kaehler manifold.

Proof. We shall prove that E is integrable and $\hat{\omega}$ is closed.

From Proposition 7, we have that E is integrable if (∇, E) and (∇, P) are Codazzi-coupled to a torsion-free connection ∇. Therefore, we only shall prove that $d\hat{\omega} = 0$. We obtain

$$\begin{aligned} &(\nabla_M\hat{\omega})(K,L) = M(\hat{\omega}(K,L)) - \hat{\omega}(\nabla_M K, L) - \hat{\omega}(K, \nabla_M L) \\ &= M(g(EK,L)) - g(E\nabla_M K, L) - g(EK, \nabla_M L) \\ &= (\nabla_M g)(EK,L) + g((\nabla_M E)K, L) \\ &= \mathfrak{C}(EK,L,M) + g((\nabla_M E)K, L). \end{aligned} \tag{1}$$

Using $(\nabla_M g)(EK,L) = Mg(EK,L) - g(\nabla_M EK, L) - g(EK,\ \nabla_M L)$, we obtain

$$= Mg(EK,L) - g((\nabla_M E)K,L) - g(E\nabla_M K, L) - g(EK,\ \nabla_M L)).$$

Similarly

$$\begin{aligned}(\nabla_K\hat{\omega})(L,M) &= K(\hat{\omega}(L,M)) - \hat{\omega}(\nabla_K L,M) - \hat{\omega}(L,\nabla_K M)\\ &= \mathfrak{C}(EL,M,K) + g((\nabla_K E)L,M)\\ &= Kg(EL,M) - g((\nabla_K E)L,M) - g(E\nabla_K L,M) - g(EL,\ \nabla_K M))\end{aligned} \tag{2}$$

and

$$\begin{aligned}(\nabla_L\hat{\omega})(M,K) &= L(\hat{\omega}(M,K)) - \hat{\omega}(\nabla_L M,K) - \hat{\omega}(M,\nabla_L K)\\ &= \mathfrak{C}(EM,K,L) + g((\nabla_L E)M,K)\\ &= Lg(EM,K) - g((\nabla_L E)M,K) - g(E\nabla_L M,K) - g(EM,\ \nabla_L K)).\end{aligned} \tag{3}$$

We use (1), (2) and (3) in the following equation

$$d\hat{\omega}(K,L,M) = (\nabla_K\hat{\omega})(L,M) + (\nabla_L\hat{\omega})(M,K) + (\nabla_M\hat{\omega})(K,L)$$

(see also [4]). In addition, we find

$$\begin{aligned}d\hat{\omega}(K,L,M) &= (\nabla_K\hat{\omega})(L,M) + (\nabla_L\hat{\omega})(M,K) + (\nabla_M\hat{\omega})(K,L)\\ &= \mathfrak{C}(EK,L,M) + \mathfrak{C}(EL,M,K) + \mathfrak{C}(EM,K,L)\\ &\quad + g((\nabla_M E)K,L) + g((\nabla_K E)L,M) + g((\nabla_L E)M,K)\end{aligned}$$

and similarly

$$\begin{aligned}d\hat{\omega}(M,L,K) &= (\nabla_M\hat{\omega})(L,K) + (\nabla_L\hat{\omega})(K,M) + (\nabla_K\hat{\omega})(M,L)\\ &= \mathfrak{C}(EM,L,K) + \mathfrak{C}(EL,K,M) + \mathfrak{C}(EK,M,L)\\ &\quad + g((\nabla_K E)M,L) + g((\nabla_M E)L,K) + g((\nabla_L E)K,M).\end{aligned}$$

Moreover, we know that (∇, g) is Codazzi-coupled, then $\mathfrak{C}$ is totally symetric [4]. Using (∇, E) being Codazzi-coupled and $d\hat{\omega}$ being totally skew-symetric, we obtain

$$\begin{aligned}d\hat{\omega}(K,L,M) - d\hat{\omega}(M,L,K) &= 0\\ -2d\hat{\omega}(K,L,M) = 0 \Rightarrow d\hat{\omega}(K,L,M) &= 0.\end{aligned}$$

This gives a result. □

4. $E-$Parallel Affine Connections

Let ∇ be an affine connection and E be an electromagnetic-type structure. If

$$\nabla_K EL = E\nabla_K L$$

is satisfied for any vector fields K, L on $\mathcal{M}^k$, ∇ is named an $E-$parallel affine connection on $\mathcal{M}^k$.

Proposition 8. *Let ∇ be an affine connection on the metric $(E^4 = I)-$manifold $\left(\mathcal{M}^k, E, g, \hat{\omega}\right)$. ∇^* and $\nabla^\dagger$ assign, respectively, $g-$conjugation and $\hat{\omega}-$conjugation of ∇ on $\mathcal{M}^k$. In that case,*

(i) *∇^* is $E-$parallel if and only if ∇ is so.*

(ii) *$\nabla^\dagger$ is $E-$parallel if and only if ∇ is so.*

Proof. (i) From the definitions of $g-$conjugation and $E-$parallel, it is obtained that

$$\begin{aligned}
&\hat{\omega}(\nabla^*_K EL - E\nabla^*_K L, M) = \hat{\omega}(\nabla^*_K EL, M) - \hat{\omega}(E\nabla^*_K L, M) \\
&= g(E\nabla^*_K EL, M) - g(E(E\nabla^*_K L), M) = -g(\nabla^*_K EL, EM) + g(E\nabla^*_K L, EM) \\
&= -g(EM, \nabla^*_K EL) - g\left(E^2 M,\ \nabla^*_K L\right) \\
&= -Kg(EM, EL) + g(\nabla_K EM, EL) - Kg\left(E^2 M, L\right) + g\left(\nabla_K E^2 M, L\right) \\
&= -Kg(EM, EL) + g(\nabla_K EM, EL) + Kg(EM, EL) + g\left(\nabla_K E^2 M, L\right) \\
&= g(EL,\ \nabla_K EM) + g((\nabla_K E)EM, L) + g(E\nabla_K EM, L) \\
&= g(EL, \nabla_K EM) + g((\nabla_K E)EM, L) - g(\nabla_K EM, EL) \\
&= g((\nabla_K E)EM, L).
\end{aligned}$$

Thus, $\nabla^*_K EL = E\nabla^*_K L$ if and only if $\nabla_K\ EM = E\nabla_K\ M$.

(ii) Firstly, we have

$$\hat{\omega}(EK, L) = g(E(EK), L) = -g(EK, EL) = -\hat{\omega}(K, EL).$$

From the definitions of $\hat{\omega}-$conjugation and the above equation, we obtain

$$\begin{aligned}
&\hat{\omega}\left(\nabla^\dagger_K EL - E\nabla^\dagger_K L, M\right) = \hat{\omega}\left(\nabla^\dagger_K EL, M\right) - \hat{\omega}\left(E\nabla^\dagger_K L, M\right) \\
&= -\hat{\omega}\left(M,\ \nabla^\dagger_K EL\right) + \hat{\omega}\left(M, E\nabla^\dagger_K L\right) = -\hat{\omega}\left(M,\ \nabla^\dagger_K EL\right) - \hat{\omega}\left(EM, \nabla^\dagger_K L\right) \\
&= -K\hat{\omega}(M, EL) + \hat{\omega}(\nabla_K M, EL) - K\hat{\omega}(EM, L) + \hat{\omega}(\nabla_K EM, L) \\
&= -K\hat{\omega}(M, EL) + \hat{\omega}(\nabla_K M, EL) + K\hat{\omega}(M, EL) + \hat{\omega}(\nabla_K EM, L) \\
&= -\hat{\omega}(EL, \nabla_K M) - \hat{\omega}(L, \nabla_K EM) = \hat{\omega}(L, E\nabla_K M) - \hat{\omega}(L, \nabla_K EM),
\end{aligned}$$

which completes the proof. □

Proposition 9. *Let ∇ be an $E-$parallel affine connection on the metric $\left(E^4 = I\right)-$manifold $\left(\mathcal{M}^k, E, g, \hat{\omega}\right)$. ∇^* and $\nabla^\dagger$ assign respectively $g-$conjugation and $\hat{\omega}-$conjugation of ∇ on $\mathcal{M}^k$. The followings are provided:*

(i) $\nabla^\dagger = \nabla^*$;

(ii) $(\nabla, \hat{\omega})$ *is a Codazzi-coupled if and only if* (∇, g) *is so.*

Proof. (i) From the definitions of $g-$conjugation, $\hat{\omega}-$conjugation and $E-$parallel, we obtain

$$\begin{aligned}
&M\hat{\omega}(K, L) = \hat{\omega}(\nabla_M K, L) + \hat{\omega}\left(K, \nabla^\dagger_M L\right) \\
&Mg(EK, L) = g(E\nabla_M K, L) + g\left(EK,\ \nabla^\dagger_M L\right) \\
&Mg(EK, L) - g(E\nabla_M K, L) = g\left(EK,\ \nabla^\dagger_M L\right) \\
&Mg(EK, L) - g(\nabla_M EK, L) = g\left(EK,\ \nabla^\dagger_M L\right) \\
&g(EK, \nabla^*_M L) = g\left(EK,\ \nabla^\dagger_M L\right) \Leftrightarrow \nabla^*_M L = \nabla^\dagger_M L \Leftrightarrow \nabla^* = \nabla^\dagger.
\end{aligned}$$

(ii) From the Codazzi equation, we have

$$\begin{aligned}
(\nabla_M \hat{\omega})(K, L) &= (\nabla_K \hat{\omega})(M, L) \\
M\hat{\omega}(K, L) - \hat{\omega}(\nabla_M K, L) - \hat{\omega}(K, \nabla_M L) &= K\hat{\omega}(M, L) - \hat{\omega}(\nabla_K M, L) - \hat{\omega}(M, \nabla_K L) \\
Mg(EK, L) - g(E\nabla_M K, L) - g(EK, \nabla_M L) &= Kg(EM, L) - g(E\nabla_K M, L) - g(EM, \nabla_K L)
\end{aligned}$$

$$
\begin{aligned}
& -Mg(K,EL)+g(\nabla_M K,EL)+g(M,E\nabla_M L) \\
= & -Kg(M,EL)+g(\nabla_K M,EL)+g(M,E\nabla_K L)-Mg(K,EL) \\
& +g(\nabla_M K,EL)+g(K,\nabla_M EL) \\
= & -Kg(M,EL)+g(\nabla_K M,EL)+g(M,\nabla_K EL) \\
& (\nabla_M g)(K,EL)=(\nabla_K g)(M,EL),
\end{aligned}
$$

which gives the proof. □

To give results about statistical structures, for a moment, we shall assume a torsion-free $E-$parallel affine connection ∇.

Theorem 3. *Let ∇ be a torsion-free $E-$parallel affine connection on the metric $\left(E^4=I\right)-$manifold $\left(\mathcal{M}^k,E,g,\hat{\omega}\right)$, by ∇^* and $\nabla^\dagger$ we assign respectively $g-$conjugation and $\hat{\omega}-$conjugation of ∇ on $\mathcal{M}^k$. If $(\nabla,\hat{\omega})$ is a Codazzi-coupled, the followings are provided:*

(i) (∇^*,g) *is a statistical structure;*
(ii) $(\nabla^\dagger,g)$ *is a statistical structure.*

Proof. The proof is obtained from Proposition 9 and Proposition 2.10 in [4]. □

Theorem 4. *Let ∇ be a torsion-free $E-$parallel affine connection on the metric $\left(E^4=I\right)-$manifold $\left(\mathcal{M}^k,E,g,\hat{\omega}\right)$. ∇^* assigns the $g-$conjugation of ∇ on $\mathcal{M}^k$. In the case that $(\nabla,\hat{\omega})$ being a Codazzi coupled, $(\nabla^*,\hat{\omega})$ is a Codazzi-coupled if and only if $(\nabla_{EM}g)(K,L)=(\nabla_M g)(EK,L)$.*

Proof. Let ∇ be a torsion-free $E-$parallel affine connection and $(\nabla,\hat{\omega})$ be a Codazzi coupled. Then, from Proposition 9, we also have that (∇,g) is Codazzi-coupled. Moreover, from Proposition 8, we can say that ∇^* is $E-$parallel. Using all the above, we obtain

$$
\begin{aligned}
& (\nabla^*_M\hat{\omega})(K,L)-(\nabla^*_K\hat{\omega})(M,L)=M\hat{\omega}(K,L)-\hat{\omega}(\nabla^*_M K,L)-\hat{\omega}(K,\nabla^*_M L) \\
& -K\hat{\omega}(M,L)+\hat{\omega}(\nabla^*_K M,L)+\hat{\omega}(M,\nabla^*_K L) \\
& =Mg(EK,L)-g(E\nabla^*_M K,L)-g(EK,\nabla^*_M L) \\
& -Kg(EM,L)+g(E\nabla^*_K M,L)+g(EM,\nabla^*_K L) \\
& =g(\nabla^*_M EK,L)+g(EK,\nabla_M L)-g(E\nabla^*_M K,L)-g(EK,\nabla^*_M L) \\
& -g(\nabla^*_K EM,L)-g(EM,\nabla_K L)+g(E\nabla^*_K M,L)+g(EM,\nabla^*_K L) \\
& =g((\nabla^*_M E)K,L)+g(E\nabla^*_M K,L)+g(EK,\nabla_M L)-g(E\nabla^*_M K,L)-g(EK,\nabla^*_M L) \\
& -g((\nabla^*_K E)M,L)-g(E\nabla^*_K M,L)-g(EM,\nabla_K L)+g(E\nabla^*_K M,L)+g(EM,\nabla^*_K L) \\
& =g(EK,\nabla_M L)-g(EK,\nabla^*_M L)-g(EM,\nabla_K L)+g(EM,\nabla^*_K L) \\
& =g(EK,\nabla_M L)-Mg(EK,L)+g(\nabla_M EK,L)-g(EM,\nabla_K L) \\
& +Kg(EM,L)-g(\nabla_K EM,L)=-(\nabla_M g)(EK,L)+(\nabla_K g)(EM,L) \\
& =-(\nabla_M g)(EK,L)+(\nabla_{EM}g)(K,L).
\end{aligned}
$$

□

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