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New Estimates for Hermite-Hadamard Inequality in Quantum Calculus via (α, m) Convexity

Peng Xu [1], Saad Ihsan Butt [2,*], Qurat Ul Ain [2] and Hüseyin Budak [3]

[1] School of Computer Science of Information Technology, Qiannan Normal University for Nationalities, Duyun 558000, China; gdxupeng@gzhu.edu.cn
[2] Department of Mathematics, COMSATS University Islamabad, Lahore Campus, Lahore 54000, Pakistan; quratulain4566@gmail.com
[3] Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce 81620, Turkey; hsyn.budak@gmail.com
* Correspondence: saadihsanbutt@gmail.com

Abstract: This study provokes the existence of quantum Hermite-Hadamard inequalities under the concept of q-integral. We analyse and illustrate a new identity for the differentiable function mappings whose second derivatives in absolute value are (α, m) convex. Some basic inequalities such as Hölder's and Power mean have been used to obtain new bounds and it has been determined that the main findings are generalizations of many results that exist in the literature. We make links between our findings and a number of well-known discoveries in the literature. The conclusion in this study unify and generalise previous findings on Hermite-Hadamard inequalities.

Keywords: quantum calculus; Hermite-Hadamard inequalities; (α, m) convexity

MSC: 26A33; 26D15; 26E60

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Citation: Xu, P.; Ihsan Butt, S.; Ain, Q.U.; Budak, H. New Estimates for Hermite-Hadamard Inequality in Quantum Calculus via (α, m) Convexity. *Symmetry* **2022**, *14*, 1394. https://doi.org/10.3390/sym14071394

Academic Editor: Palle E.T. Jorgensen

Received: 19 June 2022
Accepted: 5 July 2022
Published: 6 July 2022

1. Introduction

When there is no limit in calculus, it is referred as q-calculus. Euler is the inventor of q-parameter and also the creator of q-calculus. Jackson began his work in a symmetrical manner in the nineteenth century and presented q-definite integrals. Q-calculus is used in a wide range of subjects, including mathematics, number theory, hyper geometry and physics. One can see in [1–4] and references therein. In q-calculus, we substitute classical derivative with difference operator, allowing you to work with sets of non-differentiable functions. Quantum difference operators are of tremendous importance because of their applications in a variety of mathematical disciplines, including orthogonal polynomials, basic hypergeometric functions, combinatorics, mechanics and the theory of relativity. Many essential concepts of quantum calculus are covered in Kac and Cheung's book [5]. These ideas help us to develop new inequalities, which can be useful in the discovery of new boundaries.

Integral inequalities is historically viewed as a classical field of research. From classical to modern applications, inequalities have been used in mathematical analysis. In 1934, Polya and Hardy introduced classical work on inequalities. Integral inequalities plays vital role in differential equation theory. Many researchers have studied integral inequalities in classical calculus along with their applications (see [6–9]) . Because the value of mathematical inequalities was well established in past, inequalities such as Hermite-Hadamard, Popoviciu's, Steffensen-Grüss, Jensen, Hardy and Cauchy-Schwarz performed an essential role in the theory of classical calculus and q-calculus [10–14].

In convexity theory, Hermite-Hadamard is one of the most well known inequality, which was developed by Hermite and Hadamard (see also [15], [16] p. 137). Convexity is very simple and natural concept to solve many problems of mathematics. Convexity is growing area of research that has applications in complex analysis, number theory and many other fields. Convexity also has a significant impact on people's lives with numerous uses in industry, medicine and business. Convex functions are studied by researchers in a variety of fields and are defined as:

Definition 1 ([6]). *If $\mathfrak{g} : [\theta_1, \theta_2] \subset \Re \to \Re$ is convex, then for every $\varkappa,\ y \in [\theta_1, \theta_2]$ and every $\kappa \in [0,1]$, we have:*

$$\mathfrak{g}(\kappa y + (1-\kappa)\varkappa) \leq \kappa \mathfrak{g}(y) + (1-\kappa)\mathfrak{g}(\varkappa).$$

Definition 2 ([17]). *If $\mathfrak{g} : [0, \theta_2) \to \Re$ is called (α, m) convex, then following inequality holds*

$$\mathfrak{g}(\kappa\varkappa + m(1-\kappa)y) \leq \kappa^{\alpha}\mathfrak{g}(\varkappa) + m(1-\kappa^{\alpha})\mathfrak{g}(y),$$

holds $\forall \varkappa, y \in [0, \theta_2]$. $\kappa \in [0,1]$, $(\alpha, m) \in [0,1]^2$ and $m \in (0,1]$.

Convexity has a geometrical interpretation with various applications. In accordance with these inequalities: if $\mathfrak{g} : I \to \Re$ is a convex function on I over the real numbers and $\theta_1, \theta_2 \in I$ with $\theta_1 < \theta_2$, then

$$\mathfrak{g}\left(\frac{\theta_1+\theta_2}{2}\right) \leq \frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2} \mathfrak{g}(\varkappa)\, d\varkappa \leq \frac{\mathfrak{g}(\theta_1)+\mathfrak{g}(\theta_2)}{2}. \tag{1}$$

If $\mathfrak{g}$ is a concave function, both sides of inequality are in reversed manner. We can see that Hermite-Hadamard inequality come from Jensen's inequality. Over the last few years, Hermite-Hadamard inequalities for convex functions have gotten a lot of attention, and as a result, there have been a number of refinements and generalizations.

The goal of this paper is to use the newly developed concept of q^{θ_2}-integral to investigate H-H inequality for (α, m) convex functions. We also analyse how our outcomes compare to similar outcomes in the literature.

2. Description of q-Calculus

We will consider as $q \in (0,1)$ throughout the whole article. In this part, we set up the notation given below (see Ref. [5]):

$$[n]_q = \frac{1-q^n}{1-q}.$$

Jackson integral [3] of $\mathfrak{g}$ was described by Jackson from 0 to θ_2 as follows:

$$\int_0^{\theta_2} \mathfrak{g}(\varkappa)\ d_q\varkappa \ = (1-q)\theta_2 \sum_{n=0}^{\infty} q^n \mathfrak{g}(\theta_2 q^n), \tag{2}$$

provided that the sum converges absolutely.

The Jackson integral [3] of a function $\mathfrak{g}$ over the interval $[\theta_1, \theta_2]$ is as follows:

$$\int_{\theta_1}^{\theta_2} \mathfrak{g}(\varkappa)\ d_q\varkappa \ = \int_0^{\theta_2} \mathfrak{g}(\varkappa)\ d_q\varkappa \ - \int_0^{\theta_1} \mathfrak{g}(\varkappa)\ d_q\varkappa \ .$$

Definition 3 ([18]). *Let $\mathfrak{g} : [\theta_1, \theta_2] \underset{cont.}{\longrightarrow} \Re$. The q_{θ_1}-derivative of f at $\varkappa \in [\theta_1, \theta_2]$ is defined as:*

$${}_{\theta_1}D_q\mathfrak{g}(\varkappa) \ = \frac{\mathfrak{g}(\varkappa) - \mathfrak{g}(q\varkappa + (1-q)\theta_1)}{(1-q)(\varkappa-\theta_1)},\ \varkappa \neq \theta_1. \tag{3}$$

Since $\mathfrak{g} : [\theta_1, \theta_2] \underset{cont.}{\longrightarrow} \Re$, *we can define*

$$ {}_{\theta_1}D_q\mathfrak{g}(\theta_1) = \lim_{\varkappa \to \theta_1} {}_{\theta_1}D_q\mathfrak{g}(\varkappa) $$

The function $\mathfrak{g}$ *is said to be* q_{θ_1}*-differentiable on* $[\theta_1, \theta_2]$ *if* ${}_{\theta_1}D_q\mathfrak{g}(\varkappa)$ *exists* $\forall \varkappa \in [\theta_1, \theta_2]$. *If we take* $\theta_1 = 0$ *in (3), then we have* ${}_0D_q\mathfrak{g}(\varkappa) = D_q\mathfrak{g}(\varkappa)$, *where* $D_q\mathfrak{g}(\varkappa)$ *is a known q-derivative of* $\mathfrak{g}$ *at* $\varkappa \in [0, \theta_2]$ *in (see Ref. [5]) given as:*

$$ D_q\mathfrak{g}(\varkappa) = \frac{\mathfrak{g}(\varkappa) - \mathfrak{g}(q\varkappa)}{(1-q)\varkappa}, \ \varkappa \neq 0. $$

Definition 4 ([19])**.** *Let* $\mathfrak{g} : [\theta_1, \theta_2] \underset{cont.}{\longrightarrow} \Re$. *The* q^{θ_2}*-derivative of* $\mathfrak{g}$ *at* $\varkappa \in [\theta_1, \theta_2]$ *is given as:*

$$ {}^{\theta_2}D_q\mathfrak{g}(\varkappa) = \frac{\mathfrak{g}(q\varkappa + (1-q)\theta_2) - \mathfrak{g}(\varkappa)}{(1-q)(\theta_2 - \varkappa)}, \ \varkappa \neq \theta_2. $$

Definition 5. *Let* $\mathfrak{g} : [\theta_1, \theta_2] \underset{cont.}{\longrightarrow} \Re$. *The second* q^{θ_2}*-derivative of* $\mathfrak{g}$ *at* $\varkappa \in [\theta_1, \theta_2]$ *is given as:*

$$ \begin{aligned} & {}^{\theta_2}D_q^2\mathfrak{g}(\varkappa) \\ = & {}^{\theta_2}D_q\left({}^{\theta_2}D_q\mathfrak{g}(\varkappa)\right) \\ = & \frac{\mathfrak{g}(q^2\varkappa + (1-q^2)\theta_2) - ([2]_q)\mathfrak{g}(q\varkappa + (1-q)\theta_2) + q\mathfrak{g}(\varkappa)}{(1-q)^2 q(\theta_2 - \varkappa)^2}. \end{aligned} $$

Definition 6 ([18])**.** *If* $\mathfrak{g} : [\theta_1, \theta_2] \underset{cont.}{\longrightarrow} \Re$. *Then, the* q_{θ_1}*-definite integral on* $[\theta_1, \theta_2]$ *is defined as:*

$$ \int_{\theta_1}^{\theta_2} \mathfrak{g}(\varkappa) \ {}_{\theta_1}d_q\varkappa = (1-q)(\theta_2 - \theta_1)\sum_{n=0}^{\infty} q^n \mathfrak{g}(q^n\theta_2 + (1-q^n)\theta_1) = (\theta_2 - \theta_1)\int_0^1 \mathfrak{g}((1-\kappa)\theta_1 + \kappa\theta_2) \ d_q\kappa \ . $$

In [20], researchers presented the q_{θ_1}-Hermite-Hadamard inequalities for generalized convex function in q-calculus:

Theorem 1. *Let* $\mathfrak{g} : [\theta_1, \theta_2] \to \Re$ *is a convex differentiable function on* $[\theta_1, \theta_2]$, *we have*

$$ \mathfrak{g}\left(\frac{q\theta_1 + \theta_2}{1+q}\right) \leq \frac{1}{\theta_2 - \theta_1}\int_{\theta_1}^{\theta_2} \mathfrak{g}(\varkappa) \ {}_{\theta_1}d_q\varkappa \leq \frac{q\mathfrak{g}(\theta_1) + \mathfrak{g}(\theta_2)}{1+q}. \tag{4} $$

For the both sides of the inequality (4), the authors defined specific boundaries in [20,21]. In [19], Bermudo et al. proposed the following definitions and derived corresponding Hermite-Hadamard inequalities.

Definition 7 ([19])**.** *Let* $\mathfrak{g} : [\theta_1, \theta_2] \underset{cont.}{\longrightarrow} \Re$, *then* q^{θ_2}*-definite integral on* $[\theta_1, \theta_2]$ *is given as:*

$$ \int_{\theta_1}^{\theta_2} \mathfrak{g}(\varkappa) \ {}^{\theta_2}d_q\varkappa = (1-q)(\theta_2 - \theta_1)\sum_{n=0}^{\infty} q^n \mathfrak{g}(q^n\theta_1 + (1-q^n)\theta_2) = (\theta_2 - \theta_1)\int_0^1 \mathfrak{g}(\kappa\theta_1 + (1-\kappa)\theta_2) \ d_q\kappa $$

Theorem 2 ([19])**.** *If* $\mathfrak{g} : [\theta_1, \theta_2] \to \Re$ *is convex and differentiable function on* $[\theta_1, \theta_2]$, *then q-Hermite-Hadamard inequalities are given as follows:*

$$\mathfrak{g}\left(\frac{\theta_1+q\theta_2}{[2]_q}\right)\leq\frac{1}{\theta_2-\theta_1}\int\limits_{\theta_1}^{\theta_2}\mathfrak{g}(\varkappa)\ {}^{\theta_2}d_q\varkappa\ \leq\frac{\mathfrak{g}(\theta_1)+q\mathfrak{g}(\theta_2)}{[2]_q},\tag{5}$$

where $0<q<1$.

The following inequalities can be obtain from Theorems 1 and 2.

Corollary 1 ([19]). *With the assumptions of Theorem 2, we have*

$$\mathfrak{g}\left(\frac{q\theta_1+\theta_2}{1+q}\right)+\mathfrak{g}\left(\frac{\theta_1+q\theta_2}{1+q}\right)\leq\frac{1}{\theta_2-\theta_1}\left\{\int\limits_{\theta_1}^{\theta_2}\mathfrak{g}(\varkappa)\ {}_{\theta_1}d_q\varkappa\ +\int\limits_{\theta_1}^{\theta_2}\mathfrak{g}(\varkappa)\ {}^{\theta_2}d_q\varkappa\right\}\leq\mathfrak{g}(\theta_1)+\mathfrak{g}(\theta_2)\tag{6}$$

and

$$\mathfrak{g}\left(\frac{\theta_1+\theta_2}{2}\right)\leq\frac{1}{2(\theta_2-\theta_1)}\left\{\int\limits_{\theta_1}^{\theta_2}\mathfrak{g}(\varkappa)\ {}_{\theta_1}d_q\varkappa\ +\int\limits_{\theta_1}^{\theta_2}\mathfrak{g}(\varkappa)\ {}^{\theta_2}d_q\varkappa\right\}\leq\frac{\mathfrak{g}(\theta_1)+\mathfrak{g}(\theta_2)}{2}.\tag{7}$$

Theorem 3 (Hölder's inequality, Ref. [22] p. 604). *Let* $\varkappa>0$, $\wp_1>1$. *If* $\frac{1}{\wp_1}+\frac{1}{\wp_2}=1$. *Then*

$$\int_0^{\varkappa}|\mathfrak{g}(\varkappa)g(\varkappa)|d_q\varkappa\leq\left(\int_0^{\varkappa}|\mathfrak{g}(\varkappa)|^{\wp_1}d_q\varkappa\right)^{\frac{1}{\wp_1}}\left(\int_0^{\varkappa}|g(\varkappa)|^{\wp_2}d_q\varkappa\right)^{\frac{1}{\wp_2}}.$$

In recent years, many papers have been devoted to inequalities for quantum integrals. For some of them, one can refer to [23–30].

3. Main Results

Now, we present some novel Hermite-Hadamard inequalities with the concept of quantum integral.

Lemma 1. *Let* $\mathfrak{g}:[\theta_1,\theta_2]\subset\Re\to\Re$ *is a twice* q^{θ_2}*-differentiable function on* (θ_1,θ_2) *such that* ${}^{\theta_2}D_q^2\mathfrak{g}\in C[\theta_1,\theta_2]$ *and integrable on* $[\theta_1,\theta_2]$, *we have:*

$$\begin{aligned}&\frac{\mathfrak{g}(\theta_1)+q\mathfrak{g}(m\theta_2)}{[2]_q}-\frac{1}{(m\theta_2-\theta_1)}\int\limits_{\theta_1}^{m\theta_2}\mathfrak{g}(\varkappa)\ {}^{m\theta_2}d_q\varkappa\\&=\frac{q^2(m\theta_2-\theta_1)^2}{[2]_q}\int\limits_0^1\kappa(1-q\kappa)\ {}^{\theta_2}D_q^2\mathfrak{g}(\kappa\theta_1+m(1-\kappa)\theta_2)d_q\kappa.\end{aligned}\tag{8}$$

Proof. By using Definition 5, we have

$$\begin{aligned}&{}^{\theta_2}D_q^2\mathfrak{g}(\kappa\theta_1+m(1-\kappa)\theta_2)\\=\ &{}^{\theta_2}D_q\left({}^{\theta_2}D_q(\mathfrak{g}(\kappa\theta_1+m(1-\kappa)\theta_2))\right)\\=\ &\frac{\mathfrak{g}(q^2\kappa\theta_1+m(1-\kappa q^2)\theta_2)-(1+q)\mathfrak{g}(q\kappa\theta_1+m(1-q\kappa)\theta_2)+q\mathfrak{g}(\kappa\theta_1+m(1-\kappa)\theta_2)}{(1-q)^2q(m\theta_2-\theta_1)^2\kappa^2}.\end{aligned}$$

Also,

$$\int_0^1\kappa(1-q\kappa)\ {}^{\theta_2}D_q^2\mathfrak{g}(\kappa\theta_1+m(1-\kappa)\theta_2)\ d_q\kappa\tag{9}$$

$$
\begin{aligned}
= & \int_0^1 \frac{\mathfrak{g}\left(q^2\kappa\theta_1+m\left(1-\kappa q^2\right)\theta_2\right)-(1+q)\mathfrak{g}(q\kappa\theta_1+m(1-q\kappa)\theta_2)+q\mathfrak{g}(\kappa\theta_1+m(1-\kappa)\theta_2)}{(1-q)^2q(m\theta_2-\theta_1)^2\kappa}d_q\kappa \\
& -\int_0^1 q\left[\frac{\mathfrak{g}\left(q^2\kappa\theta_1+m\left(1-\kappa q^2\right)\theta_2\right)-(1+q)\mathfrak{g}(q\kappa\theta_1+m(1-q\kappa)\theta_2)+q\mathfrak{g}(\kappa\theta_1+m(1-\kappa)\theta_2)}{(1-q)^2q(m\theta_2-\theta_1)^2}\right]d_q\kappa
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_0^1 \frac{\mathfrak{g}\left(q^2\kappa\theta_1+m\left(1-\kappa q^2\right)\theta_2\right)-(1+q)\mathfrak{g}(q\kappa\theta_1+m(1-q\kappa)\theta_2)+q\mathfrak{g}(\kappa\theta_1+m(1-\kappa)\theta_2)}{(1-q)^2q(m\theta_2-\theta_1)^2\kappa}d_q\kappa \\
= & (1-q)\sum_{n=0}^{\infty}\frac{\mathfrak{g}\left(q^{n+2}\theta_1+m\left(1-q^{n+2}\right)\theta_2\right)}{(1-q)^2q(m\theta_2-\theta_1)^2}-(1-q)(1+q)\sum_{n=0}^{\infty}\frac{\mathfrak{g}\left(q^{n+1}\theta_1+m\left(1-q^{n+1}\right)\theta_2\right)}{(1-q)^2q(m\theta_2-\theta_1)^2} \\
& +q(1-q)\sum_{n=0}^{\infty}\frac{\mathfrak{g}(q^n\theta_1+m(1-q^n)\theta_2)}{(1-q)^2q(m\theta_2-\theta_1)^2} \\
= & \sum_{n=0}^{\infty}\frac{\mathfrak{g}\left(q^{n+2}\theta_1+m\left(1-q^{n+2}\right)\theta_2\right)}{(1-q)q(m\theta_2-\theta_1)^2}-\sum_{n=0}^{\infty}\frac{\mathfrak{g}\left(q^{n+1}\theta_1+m\left(1-q^{n+1}\right)\theta_2\right)}{(1-q)q(m\theta_2-\theta_1)^2} \\
& -q\left[\sum_{n=0}^{\infty}\frac{\mathfrak{g}\left(q^{n+1}\theta_1+m\left(1-q^{n+1}\right)\theta_2\right)}{(1-q)q(m\theta_2-\theta_1)^2}-\sum_{n=0}^{\infty}\frac{\mathfrak{g}(q^n\theta_1+m(1-q^n)\theta_2)}{(1-q)q(m\theta_2-\theta_1)^2}\right] \\
= & \frac{\mathfrak{g}(m\theta_2)-\mathfrak{g}(q\theta_1+m(1-q)\theta_2)}{(1-q)q(m\theta_2-\theta_1)^2}-q\left[\frac{\mathfrak{g}(m\theta_2)-\mathfrak{g}(\theta_1)}{(1-q)q(m\theta_2-\theta_1)^2}\right].
\end{aligned} \tag{10}
$$

From (2) and Definition 7,

$$
\begin{aligned}
& \int_0^1 q\left[\frac{\mathfrak{g}\left(q^2\kappa\theta_1+m\left(1-\kappa q^2\right)\theta_2\right)-(1+q)\mathfrak{g}(q\kappa\theta_1+m(1-q\kappa)\theta_2)+q\mathfrak{g}(\kappa\theta_1+m(1-\kappa)\theta_2)}{(1-q)^2q(m\theta_2-\theta_1)^2}\right]d_q\kappa \\
= & q\left[(1-q)(m\theta_2-\theta_1)\sum_{n=0}^{\infty}\frac{q^{n+2}\mathfrak{g}\left(q^{n+2}\theta_1+m\left(1-q^{n+2}\right)\theta_2\right)}{(1-q)^2q^3(m\theta_2-\theta_1)^3}\right. \\
- & (1-q)(1+q)(m\theta_2-\theta_1)\sum_{n=0}^{\infty}\frac{q^{n+1}\mathfrak{g}\left(q^{n+1}\theta_1+m\left(1-q^{n+1}\right)\theta_2\right)}{(1-q)^2q^2(m\theta_2-\theta_1)^3} \\
& \left.+q(1-q)(m\theta_2-\theta_1)\sum_{n=0}^{\infty}\frac{q^n\mathfrak{g}(q^n\theta_1+m(1-q^n)\theta_2)}{(1-q)^2q(m\theta_2-\theta_1)^3}\right] \\
= & q\left[\frac{1}{(1-q)^2q^3(m\theta_2-\theta_1)^3}\right. \\
& \left(\int_{\theta_1}^{m\theta_2}\mathfrak{g}(\varkappa)\,{}^{m\theta_2}d_q\varkappa-(1-q)(\theta_2-\theta_1)\mathfrak{g}(\theta_1)-(1-q)(m\theta_2-\theta_1)q\mathfrak{g}(q\theta_1+m(1-q)\theta_2)\right) \\
- & \frac{1+q}{(1-q)^2q^2(m\theta_2-\theta_1)^3}\left(\int_{\theta_1}^{m\theta_2}\mathfrak{g}(\varkappa)\,{}^{m\theta_2}d_q\varkappa-(1-q)(1+q)(m\theta_2-\theta_1)\mathfrak{g}(\theta_1)\right) \\
+ & \left.\frac{1}{(1-q)^2(m\theta_2-\theta_1)^3}\int_{\theta_1}^{m\theta_2}\mathfrak{g}(\varkappa)\,{}^{m\theta_2}d_q\varkappa\right] \\
= & \frac{1+q}{(m\theta_2-\theta_1)^3q^2}\int_{\theta_1}^{m\theta_2}\mathfrak{g}(\varkappa)\,{}^{m\theta_2}d_q\varkappa+\frac{q^2+q-1}{(1-q)q^2(m\theta_2-\theta_1)^2}\mathfrak{g}(\theta_1)-\frac{\mathfrak{g}(q\theta_1+m(1-q)\theta_2)}{(1-q)q(m\theta_2-\theta_1)^2}
\end{aligned} \tag{11}
$$

Using (10) and (11) in (9), we have

$$
\int_0^1 \kappa(1-q\kappa)\,{}^{\theta_2}D_q^2\mathfrak{g}(\kappa\theta_1+m(1-\kappa)\theta_2)\,d_q\kappa \tag{12}
$$

$$= \frac{\mathfrak{g}(m\theta_2) - \mathfrak{g}(q\theta_1 + m(1-q)\theta_2)}{(1-q)q(m\theta_2 - \theta_1)^2} - q\left[\frac{\mathfrak{g}(m\theta_2) - \mathfrak{g}(\theta_1)}{(1-q)q(m\theta_2 - \theta_1)^2}\right]$$
$$- \frac{1+q}{(m\theta_2 - \theta_1)^3 q^2}\int_{\theta_1}^{m\theta_2} \mathfrak{g}(\varkappa)\ {}^{m\theta_2}d_q\varkappa - \frac{q^2+q-1}{(1-q)q^2(m\theta_2-\theta_1)^2}\mathfrak{g}(\theta_1) + \frac{\mathfrak{g}(q\theta_1 + m(1-q)\theta_2)}{(1-q)q(m\theta_2 - \theta_1)^2}$$
$$= \frac{\mathfrak{g}(\theta_1) + q\mathfrak{g}(m\theta_2)}{(m\theta_2 - \theta_1)^2 q^2} - \frac{1+q}{(m\theta_2 - \theta_1)^3 q^2}\int_{\theta_1}^{m\theta_2} \mathfrak{g}(\varkappa)\ {}^{m\theta_2}d_q\varkappa.$$

Multiplying both sides of (12) by $\frac{(m\theta_2-\theta_1)^2 q^2}{1+q}$, we get required identity. □

Remark 1. *By putting $m = 1$ and taking limit $q \to 1^-$ in Lemma 1, we get*

$$\frac{\mathfrak{g}(\theta_1) + \mathfrak{g}(\theta_2)}{2} - \frac{1}{\theta_2 - \theta_1}\int_{\theta_1}^{\theta_2} \mathfrak{g}(\varkappa)\, d\varkappa = \frac{(\theta_2 - \theta_1)^2}{2}\int_0^1 \kappa(1-\kappa)\ \mathfrak{g}''(\kappa\theta_1 + (1-\kappa)\theta_2)\, d\kappa,$$

which is given in [31].

Theorem 4. *If $\mathfrak{g} : [\theta_1, \theta_2] \subset \Re \to \Re$ is a twice q^{θ_2}-differentiable function on (θ_1, θ_2) such that ${}^{\theta_2}D_q^2\mathfrak{g} \in C[\theta_1, \theta_2]$ and integrable on $[\theta_1, \theta_2]$, then we have following inequality, provided that $\left|{}^{\theta_2}D_q^2\mathfrak{g}\right|$ is (α, m) convex on $[\theta_1, \theta_2]$*

$$\left|\frac{\mathfrak{g}(\theta_1) + q\mathfrak{g}(m\theta_2)}{[2]_q} - \frac{1}{m\theta_2 - \theta_1}\int_{\theta_1}^{m\theta_2} \mathfrak{g}(\varkappa)\ {}^{m\theta_2}d_q\varkappa\right| \tag{13}$$
$$\leq \frac{q^2(m\theta_2 - \theta_1)^2}{1+q}\left[\frac{[\alpha+3]_q - q[\alpha+2]_q}{[\alpha+3]_q[\alpha+2]_q}\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_1)\right|\right.$$
$$\left. + \ m\left(\frac{1}{[3]_q[2]_q} - \frac{[\alpha+3]_q - q[\alpha+2]_q}{[\alpha+3]_q[\alpha+2]_q}\right)\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_2)\right|\right].$$

Proof. Taking modulus on Lemma 1 and then using (α, m) convexity of $\left|{}^{\theta_2}D_q^2\mathfrak{g}\right|$, we obtain following

$$\left|\frac{\mathfrak{g}(\theta_1) + q\mathfrak{g}(m\theta_2)}{1+q} - \frac{1}{m\theta_2 - \theta_1}\int_{\theta_1}^{m\theta_2} \mathfrak{g}(\varkappa)\ {}^{m\theta_2}d_q\varkappa\right|$$
$$\leq \frac{q^2(m\theta_2 - \theta_1)^2}{1+q}\int_0^1 (\kappa(1-q\kappa))\left|{}^{\theta_2}D_q^2\mathfrak{g}(\kappa\theta_1 + m(1-\kappa)\theta_2)\right| d_q\kappa$$
$$\leq \frac{q^2(m\theta_2 - \theta_1)^2}{1+q}\int_0^1 (\kappa(1-q\kappa))\left[\kappa^\alpha\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_1)\right| + m(1-\kappa^\alpha)\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_2)\right|\right] d_q\kappa$$
$$= \frac{q^2(m\theta_2 - \theta_1)^2}{1+q}\left[\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_1)\right|\int_0^1 \kappa(\kappa^\alpha(1-q\kappa))d_q\kappa + m\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_2)\right|\int_0^1 (1-\kappa^\alpha)(\kappa(1-q\kappa))d_q\kappa\right]$$
$$= \frac{q^2(m\theta_2 - \theta_1)^2}{1+q}\left[\frac{[\alpha+3]_q - q[\alpha+2]_q}{[\alpha+3]_q[\alpha+2]_q}\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_1)\right|\right.$$
$$\left. + \ m\left(\frac{1}{[3]_q[2]_q} - \frac{[\alpha+3]_q - q[\alpha+2]_q}{[\alpha+3]_q[\alpha+2]_q}\right)\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_2)\right|\right].$$

Hence the theorem is proved. □

Remark 2. *By taking limit as $q \to 1^-$ and $\alpha = m = 1$ in Theorem 4, we get following Trapezoidal inequality:*

$$\left|\frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2}\mathfrak{g}(\varkappa)d\varkappa - \frac{\mathfrak{g}(\theta_1)+\mathfrak{g}(\theta_2)}{2}\right| \leq \frac{(\theta_2-\theta_1)^2}{12}\left[\frac{|\mathfrak{g}''(\theta_1)|+|\mathfrak{g}''(\theta_2)|}{2}\right],$$

which is given by Sarikaya and Aktan in [32], Proposition 2.

Example 1. *Let consider the convex function $\mathfrak{g}:[0,1]\to\mathbb{R}$ defined by $\mathfrak{g}(\varkappa)=\varkappa^3$ and let $m=\frac{1}{2}$ and $\alpha = 1$. Under these assumptions, we have*

$$\begin{aligned}\int_{\theta_1}^{m\theta_2}\mathfrak{g}(\varkappa)\ {}^{m\theta_2}d_q\varkappa &= \int_0^{\frac{1}{2}} x^3\ {}^{\frac{1}{2}}d_q\varkappa \\ &= \frac{1-q}{2}\sum_{n=0}^{\infty}q^n(1-q^n)^3 \\ &= \frac{1}{16}\left[1-\frac{3}{[2]_q}+\frac{3}{[3]_q}-\frac{1}{[4]_q}\right].\end{aligned}$$

Then the left hand side of the inequality (13) reduces to

$$\left|\frac{\mathfrak{g}(\theta_1)+q\mathfrak{g}(m\theta_2)}{1+q} - \frac{1}{m\theta_2-\theta_1}\int_{\theta_1}^{m\theta_2}\mathfrak{g}(\varkappa)\ {}^{m\theta_2}d_q\varkappa\right|$$
$$\left|\frac{q}{8[2]_q}-\frac{1}{8}\left[1-\frac{3}{[2]_q}+\frac{3}{[3]_q}-\frac{1}{[4]_q}\right]\right|.$$

On the other hand, by Definition 5, we get

$${}^{\theta_2}D_q^2\mathfrak{g}(\varkappa) = {}^{1}D_q^2\varkappa^3 = [2]_q[3]_q\varkappa + [2]_q\left(3-[3]_q\right).$$

Hence, we have

$$\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_1)\right| = \left|{}^{1}D_q^2\mathfrak{g}(0)\right| = [2]_q\left(3-[3]_q\right)$$

and

$$|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_2)| = |{}^{1}D_q^2\mathfrak{g}(1)| = 3[2]_q.$$

Therefore, the right hand side of the inequality (13) reduces to

$$\begin{aligned} &\frac{q^2(m\theta_2-\theta_1)^2}{1+q}\left[\frac{[\alpha+3]_q-q[\alpha+2]_q}{[\alpha+3]_q[\alpha+2]_q}\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_1)\right|\right. \\ &\left.+m\left(\frac{1}{[3]_q[2]_q}-\frac{[\alpha+3]_q-q[\alpha+2]_q}{[\alpha+3]_q[\alpha+2]_q}\right)|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_2)|\right] \\ =\ &\frac{q^2}{4(1+q)}\left[\left(\frac{1}{[3]_q}-\frac{q}{[4]_q}\right)[2]_q\left(3-[3]_q\right)\right. \\ &\left.+\frac{1}{2}\left(\frac{1}{[3]_q[2]_q}-\frac{1}{[3]_q}+\frac{q}{[4]_q}\right)3[2]_q\right] \\ =\ &\frac{q^2(2-q+2q^2)}{4[3]_q[4]_q}\end{aligned}$$

By the inequality (13), we have the inequality

$$\left|\frac{q}{8[2]_q} - \frac{1}{8}\left[1 - \frac{3}{[2]_q} + \frac{3}{[3]_q} - \frac{1}{[4]_q}\right]\right| \leq \frac{q^2(2-q+2q^2)}{4[3]_q[4]_q}. \tag{14}$$

One can see the validity of the inequality (14) in Figure 1.

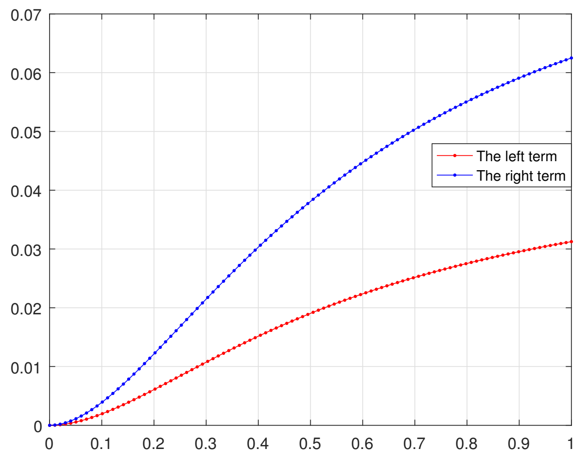

Figure 1. An example to the inequality (13).

Theorem 5. *Let* $\mathfrak{g} : [\theta_1, \theta_2] \subset \Re \to \Re$ *is a twice* q^{θ_2}*-differentiable function on* (θ_1, θ_2) *and* ${}^{\theta_2}D_q^2\mathfrak{g} \in C[\theta_1, \theta_2]$ *and integrable on* $[\theta_1, \theta_2]$*. If* $\left|{}^{\theta_2}D_q^2\mathfrak{g}\right|^{\wp_1}$*,* $\wp_1 \geq 1$*, is* (α, m) *convex on* $[\theta_1, \theta_2]$*, we have the following inequality:*

$$\begin{aligned}
&\left|\frac{\mathfrak{g}(\theta_1) + q\mathfrak{g}(m\theta_2)}{[2]_q} - \frac{1}{m\theta_2 - \theta_1}\int_{\theta_1}^{m\theta_2} \mathfrak{g}(\varkappa)\ {}^{m\theta_2}d_q\varkappa\right| \\
\leq\ & \frac{q^2(m\theta_2 - \theta_1)^2}{([2]_q)^{2-\frac{1}{\wp_1}}([3]_q)^{\frac{1}{\wp_1}}}\left(\frac{[\alpha+3]_q - q[\alpha+2]_q}{[\alpha+3]_q[\alpha+2]_q}\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_1)\right|^{\wp_1}\right. \\
+\ & \left. m\left(\frac{1}{[3]_q[2]_q} - \frac{[\alpha+3]_q - q[\alpha+2]_q}{[\alpha+3]_q[\alpha+2]_q}\right)|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_2)|^{\wp_1}\right)^{\frac{1}{\wp_1}}.
\end{aligned}$$

Proof. By applying modulus on Lemma 1 and applying Power mean inequality, we get

$$\begin{aligned}
&\left|\frac{\mathfrak{g}(\theta_1) + q\mathfrak{g}(m\theta_2)}{1+q} - \frac{1}{m\theta_2 - \theta_1}\int_{\theta_1}^{m\theta_2} \mathfrak{g}(\varkappa)\ {}^{m\theta_2}d_q\varkappa\right| \\
\leq\ & \frac{q^2(m\theta_2 - \theta_1)^2}{1+q}\int_0^1 (\kappa(1-q\kappa))\left|{}^{\theta_2}D_q^2\mathfrak{g}(\kappa\theta_1 + m(1-\kappa)\theta_2)\right| d_q\kappa \\
\leq\ & \frac{q^2(m\theta_2 - \theta_1)^2}{1+q}\left(\int_0^1 (\kappa(1-q\kappa))d_q\kappa\right)^{1-\frac{1}{\wp_1}}\left(\int_0^1 (\kappa(1-q\kappa))\left|{}^{\theta_2}D_q^2\mathfrak{g}(\kappa\theta_1 + m(1-\kappa)\theta_2)\right|^{\wp_1} d_q\kappa\right)^{\frac{1}{\wp_1}}.
\end{aligned}$$

Applying (α, m) convexity of $\left|{}^{\theta_2}D_q^2\mathfrak{g}\right|^{\wp_1}$, we have

$$\left|\frac{\mathfrak{g}(\theta_1)+q\mathfrak{g}(m\theta_2)}{1+q}-\frac{1}{m\theta_2-\theta_1}\int_{\theta_1}^{m\theta_2}\mathfrak{g}(\varkappa)\,{}^{m\theta_2}d_q\varkappa\right|$$
$$\leq \frac{q^2(m\theta_2-\theta_1)^2}{1+q}\left(\int_0^1(\kappa(1-q\kappa))d_q\kappa\right)^{1-\frac{1}{\wp_1}}$$
$$\times\left(\int_0^1(\kappa(1-q\kappa))\left[\kappa^\alpha\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_1)\right|^{\wp_1}+m(1-\kappa^\alpha)\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_2)\right|^{\wp_1}\right]d_q\kappa\right)^{\frac{1}{\wp_1}}$$
$$= \frac{q^2(m\theta_2-\theta_1)^2}{1+q}\left(\int_0^1(\kappa(1-q\kappa))d_q\kappa\right)^{1-\frac{1}{\wp_1}}$$
$$\times\left(\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_1)\right|^{\wp_1}\int_0^1\kappa^\alpha(\kappa(1-q\kappa))d_q\kappa+m\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_2)\right|^{\wp_1}\int_0^1(1-\kappa^\alpha)(\kappa(1-q\kappa))d_q\kappa\right)^{\frac{1}{\wp_1}}$$
$$= \frac{q^2(m\theta_2-\theta_1)^2}{1+q}\left(\frac{1}{[2]_q[3]_q}\right)^{1-\frac{1}{\wp_1}}$$
$$\times\left(\frac{[\alpha+3]_q-q[\alpha+2]_q}{[\alpha+3]_q[\alpha+2]_q}\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_1)\right|^{\wp_1}+m\left(\frac{1}{[3]_q[2]_q}-\frac{[\alpha+3]_q-q[\alpha+2]_q}{[\alpha+3]_q[\alpha+2]_q}\right)|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_2)|^{\wp_1}\right)^{\frac{1}{\wp_1}}.$$

Hence we get required results. □

Remark 3. *By taking $m=\alpha=1$ and then taking $q\to1^-$ in Theorem 5, we get*

$$\left|\frac{\mathfrak{g}(\theta_1)+\mathfrak{g}(\theta_2)}{2}-\frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2}\mathfrak{g}(\varkappa)d\varkappa\right|\leq\frac{(\theta_2-\theta_1)^2}{12.2^{\frac{1}{\wp_1}}}\left(\left|\mathfrak{g}''(\theta_1)\right|^{\wp_1}+\left|\mathfrak{g}''(\theta_2)\right|^{\wp_1}\right)^{\frac{1}{\wp_1}}$$

which is given by Ali et al. in [33].

Theorem 6. *Let $\mathfrak{g}:[\theta_1,\theta_2]\subset\Re\to\Re$ is a twice q^{θ_2}-differentiable function on (θ_1,θ_2) and ${}^{\theta_2}D_q^2\mathfrak{g}\in C[\theta_1,\theta_2]$ and integrable on $[\theta_1,\theta_2]$. If $\left|{}^{\theta_2}D_q^2\mathfrak{g}\right|^{\wp_1}$ is (α,m) convex on $[\theta_1,\theta_2]$, for some $\wp_1>1$ and $\frac{1}{\wp_2}+\frac{1}{\wp_1}=1$, then we have,*

$$\left|\frac{\mathfrak{g}(\theta_1)+q\mathfrak{g}(m\theta_2)}{1+q}-\frac{1}{m\theta_2-\theta_1}\int_{\theta_1}^{m\theta_2}\mathfrak{g}(\varkappa)\,{}^{m\theta_2}d_q\varkappa\right|$$
$$\leq\frac{q^2(m\theta_2-\theta_1)^2}{1+q}(u_1)^{\frac{1}{\wp_2}}\left(\frac{\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_1)\right|^{\wp_1}+m([\alpha+1]_q-1)\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_2)\right|^{\wp_1}}{[\alpha+1]_q}\right)^{\frac{1}{\wp_1}},\tag{15}$$

where $u_1=(1-q)\sum_{n=0}^{\infty}(q^n)^{\wp_2+1}(1-q^{n+1})^{\wp_2}$.

Proof. Take modulus on Lemma 1 and then, applying well-known Hölder's inequality, we get

$$\left|\frac{\mathfrak{g}(\theta_1)+q\mathfrak{g}(m\theta_2)}{1+q}-\frac{1}{m\theta_2-\theta_1}\int_{\theta_1}^{m\theta_2}\mathfrak{g}(\varkappa)\,{}^{m\theta_2}d_q\varkappa\right|$$

$$\leq \frac{q^2(m\theta_2-\theta_1)^2}{1+q}\int_0^1(\kappa(1-q\kappa))\left|{}^{\theta_2}D_q^2\mathfrak{g}(\kappa\theta_1+m(1-\kappa)\theta_2)\right|d_q\kappa$$

$$\leq \frac{q^2(m\theta_2-\theta_1)^2}{1+q}\left(\int_0^1(\kappa(1-q\kappa))^{\wp_2}d_q\kappa\right)^{\frac{1}{\wp_2}}\left(\int_0^1\left|{}^{\theta_2}D_q^2\mathfrak{g}(\kappa\theta_1+m(1-\kappa)\theta_2)\right|^{\wp_1}d_q\kappa\right)^{\frac{1}{\wp_1}}.$$

Since $\left|{}^{\theta_2}D_q^2\mathfrak{g}\right|^{\wp_1}$ is (α,m) convex, we have

$$\left|\frac{\mathfrak{g}(\theta_1)+q\mathfrak{g}(m\theta_2)}{1+q}-\frac{1}{m\theta_2-\theta_1}\int_{\theta_1}^{m\theta_2}\mathfrak{g}(\varkappa)\,{}^{m\theta_2}d_q\varkappa\right|$$

$$\leq \frac{q^2(m\theta_2-\theta_1)^2}{1+q}\left(\int_0^1(\kappa(1-q\kappa))^{\wp_2}d_q\kappa\right)^{\frac{1}{\wp_2}}$$

$$\times\left(\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_1)\right|^{\wp_1}\int_0^1\kappa^\alpha\,d_q\kappa+m\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_2)\right|^{\wp_1}\int_0^1(1-\kappa^\alpha)d_q\kappa\right)^{\frac{1}{\wp_1}}$$

$$=\frac{q^2(m\theta_2-\theta_1)^2}{1+q}(u_1)^{\frac{1}{\wp_2}}\left(\frac{\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_1)\right|^{\wp_1}+m([\alpha+1]_q-1)\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_2)\right|^{\wp_1}}{[\alpha+1]_q}\right)^{\frac{1}{\wp_1}}.$$

Using the fact that

$$u_1=\int_0^1(\kappa(1-q\kappa))^{\wp_2}d_q\kappa=(1-q)\sum_{n=0}^{\infty}(q^n)^{\wp_2+1}\left(1-q^{n+1}\right)^{\wp_2},$$

the required result can be obtained. □

Remark 4. *By taking $m=\alpha=1$ and $q\to 1^-$ in Theorem 6, we get*

$$u_1=\int_0^1(\kappa(1-\kappa))^{\wp_2}d\kappa=B(\wp_2+1,\wp_2+1),$$

where $B(\varkappa,y)$ is Euler Beta function.

Inequality (15) reduces in following inequality

$$\left|\frac{\mathfrak{g}(\theta_1)+\mathfrak{g}(\theta_2)}{2}-\frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2}\mathfrak{g}(\varkappa)d\varkappa\right|\leq\frac{(\theta_2-\theta_1)^2}{2}(B(\wp_2+1,\wp_2+1))^{\frac{1}{\wp_2}}\left(\frac{|\mathfrak{g}''(\theta_1)|^{\wp_1}+|\mathfrak{g}''(\theta_2)|^{\wp_1}}{2}\right)^{\frac{1}{\wp_1}}$$

which is given by Ali et al. in [33].

Theorem 7. *By using the assumptions of Theorem 6, following inequality holds*

$$\left|\frac{\mathfrak{g}(\theta_1)+q\mathfrak{g}(m\theta_2)}{1+q}-\frac{1}{m\theta_2-\theta_1}\int_{\theta_1}^{m\theta_2}\mathfrak{g}(\varkappa)\,{}^{m\theta_2}d_q\varkappa\right| \tag{16}$$

$$\leq\frac{q^2(m\theta_2-\theta_1)^2}{1+q}\left(\frac{1}{[\wp_2+1]_q}\right)^{\frac{1}{\wp_2}}\left(u_2\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_1)\right|^{\wp_1}+m\,u_3\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_2)\right|^{\wp_1}\right)^{\frac{1}{\wp_1}},$$

where

$$u_2 = (1-q)\sum_{n=0}^{\infty} q^{n(\alpha+1)}\left(1-q^{\alpha n+1}\right)^{\wp_1} \text{ and } u_3 = (1-q)\sum_{n=0}^{\infty} q^{n}(1-q^{n\alpha}\alpha)\left(1-q^{\alpha n+1}\right)^{\wp_1}.$$

Proof. Applying modulus in Lemma 1 and also using well-known Hölder's inequality, we get

$$\begin{aligned}
&\left|\frac{\mathfrak{g}(\theta_1)+q\mathfrak{g}(m\theta_2)}{1+q} - \frac{1}{m\theta_2-\theta_1}\int_{\theta_1}^{m\theta_2}\mathfrak{g}(\varkappa)\ {}^{m\theta_2}d_q\varkappa\right| \\
\le\ & \frac{q^2(m\theta_2-\theta_1)^2}{1+q}\int_0^1 (\kappa(1-q\kappa))\left|{}^{\theta_2}D_q^2\mathfrak{g}(\kappa\theta_1+m(1-\kappa)\theta_2)\right| d_q\kappa \\
\le\ & \frac{q^2(m\theta_2-\theta_1)^2}{1+q}\left(\int_0^1 \kappa^{\wp_2}d_q\kappa\right)^{\frac{1}{\wp_2}}\left(\int_0^1 (1-q\kappa)^{\wp_1}\left|{}^{\theta_2}D_q^2\mathfrak{g}(\kappa\theta_1+m(1-\kappa)\theta_2)\right|^{\wp_1} d_q\kappa\right)^{\frac{1}{\wp_1}}.
\end{aligned}$$

As $\left|{}^{\theta_2}D_q^2\mathfrak{g}\right|^{\wp_1}$ is (α, m) convex, we have

$$\begin{aligned}
&\left|\frac{\mathfrak{g}(\theta_1)+q\mathfrak{g}(m\theta_2)}{1+q} - \frac{1}{m\theta_2-\theta_1}\int_{\theta_1}^{m\theta_2}\mathfrak{g}(\varkappa)\ {}^{m\theta_2}d_q\varkappa\right| \\
\le\ & \frac{q^2(m\theta_2-\theta_1)^2}{1+q}\left(\int_0^1 \kappa^{\wp_2}d_q\kappa\right)^{\frac{1}{\wp_2}} \\
& \times\left(\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_1)\right|^{\wp_1}\int_0^1 (1-q\kappa^\alpha)^{\wp_1}\kappa^\alpha\ d_q\kappa + m\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_2)\right|^{\wp_1}\int_0^1 (1-q\kappa^\alpha)^{\wp_1}(1-\kappa^\alpha)d_q\kappa\right)^{\frac{1}{\wp_1}} \\
=\ & \frac{q^2(m\theta_2-\theta_1)^2}{1+q}\left(\frac{1}{[\wp_2+1]_q}\right)^{\frac{1}{\wp_2}}\left(u_2\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_1)\right|^{\wp_1} + m\ u_3\left|{}^{\theta_2}D_q^2\mathfrak{g}(\theta_2)\right|^{\wp_1}\right)^{\frac{1}{\wp_1}}.
\end{aligned}$$

One can easily see that

$$u_2 = \int_0^1 (1-q\kappa^\alpha)^{\wp_1}\kappa^\alpha\ d_q\kappa = (1-q)\sum_{n=0}^{\infty} q^{n(\alpha+1)}\left(1-q^{(n\alpha+1)}\right)^{\wp_1}$$

and

$$u_3 = \int_0^1 (1-q\kappa^\alpha)^{\wp_1}(1-\kappa^\alpha)d_q\kappa = (1-q)\sum_{n=0}^{\infty} q^{n}(1-q^{\alpha n})\left(1-q^{\alpha n+1}\right)^{\wp_1}.$$

We get the required results. □

Remark 5. *By taking $m = \alpha = 1$ and $q \to 1^-$ in Theorem 7, we have*

$$u_2 = \int_0^1 \kappa(1-\kappa)^{\wp_1}d_q\kappa = \frac{1}{(\wp_1+1)(\wp_1+2)}$$

and

$$u_3 = \int_0^1 (1-\kappa)^{\wp_1}(1-\kappa)d\kappa = \frac{1}{\wp_1+2}.$$

Moreover, the inequality (16) reduces to the following inequality

$$\left| \frac{\mathfrak{g}(\theta_1)+\mathfrak{g}(\theta_2)}{2} - \frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2} \mathfrak{g}(\varkappa)d\varkappa \right|$$
$$\leq \frac{(\theta_2-\theta_1)^2}{2(\wp_1+1)}\left(\frac{1}{\wp_1+2}\right)^{\frac{1}{\wp_1}}\left((\wp_1+2)|\mathfrak{g}''(\theta_1)|^{\wp_1}+|\mathfrak{g}''(\theta_2)|^{\wp_1}\right)^{\frac{1}{\wp_1}}.$$

4. Conclusions

The main findings of our study are designed to prove quantum Hermite-Hadamard inequalities utilizing the idea of convex function to get improved outcomes. Furthermore, we demonstrated that the newly discovered inequalities are strong generalizations of similar findings in the literature. Adopting the novel approach, we extended the study of Hermite-Hadamard type integral inequalities using Power-mean and Hölder's integral inequalities. It is interesting to extend such findings for other convexities. We presume that our newly announced concept will be the focus of much research in this fascinating field of inequalities and analysis.

Author Contributions: Conceptualization, S.I.B.; writing—original draft preparation Q.U.A. and S.I.B.; writing—review and editing, H.B. and P.X.; methodology, S.I.B. and H.B.; validation, H.B.; investigation, S.I.B. and Q.U.A.; resources, Q.U.A.; data curation, P.X.; supervision, H.B.; formal analysis, S.I.B.; visualization, H.B. All authors have read and agreed to the published version of the manuscript.

Funding: This work was funded in part by the National Natural Science Foundation of China (grant no. 62002079).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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