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A Note on the Lagrangian of Linear 3-Uniform Hypergraphs

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Abstract: Lots of symmetric properties are well-explored and analyzed in extremal graph theory, such as the well-known symmetrization operation in the Turán problem and the high symmetric in the extremal graphs. This paper is devoted to studying the Lagrangian of hypergraphs, which connects to a very symmetric function—the Lagrangian function. Given an r-uniform hypergraph F, the Lagrangian density $\pi_\lambda(F)$ is the limit supremum of $r!\lambda(G)$ over all F-free G, where $\lambda(G)$ is the Lagrangian of G. An r-uniform hypergraph F is called λ-perfect if $\pi_\lambda(F)$ equals $r!\lambda(K^r_{v(F)-1})$. Yan and Peng conjectured that: for integer $r \geq 3$, there exists $n_0(r)$ such that if G and H are two λ-perfect r-graphs with $|V(G)|$ and $|V(H)|$ no less than $n_0(r)$, then the disjoint union of G and H is λ-perfect. Let S_t denote a 3-uniform hypergraph with t edges $\{e_1, \dots, e_t\}$ satisfying that $e_i \cap e_j = \{v\}$ for all $1 \leq i < j \leq t$. In this paper, we show that the conjecture holds for $G = S_2$ and $H = S_t$ for all $t \geq 62$. Moreover, our result yields a class of Turán densities of 3-uniform hypergraphs. In our proof, we use some new techniques to study Lagrangian density problems; using a result of Sidorenko to find subgraphs, and a result of Talbot to upper bound the Lagrangian of a hypergraph.

Keywords: hypergraph Lagrangian; Lagrangian density; Turán density

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1. Introduction

Symmetry is a major characteristic of mathematical beauty, and it is found in many branches of mathematics. A number of symmetric properties are widely studied, analyzed and applied in graph theory. For example, a well-known symmetrization operation is widely applied in the study of the Turán type problems in extremal graph theory, and the Lagrangian of hypergraphs concerned in this paper connects to a very symmetric function–the Lagrangian function (see [1,2] for surveys).

For a finite set X and an integer $r > 0$, let $X^{(r)} = \{A \subseteq X : |A| = r\}$. An *$r$-uniform hypergraph* (*$r$-graph*) G on the vertex set $V(G)$, is a subset of $V(G)^{(r)}$. We simply denote G as the edge set of G. We call a 2-graph a simple graph. Denote $\overline{G} = V(G)^{(r)} \setminus G$. Let $S \subseteq V(G)$, denote $G - S = \{e \in G : e \cap S = \emptyset\}$ and $G[S] = \{e \in G : e \subseteq S\}$. An edge $e = \{v_1, v_2, \dots, v_r\}$ will be simply denoted by $v_1 v_2 \dots v_r$. Let $K^r_n = X^{(r)}$ denote the complete r-graph on vertex set X with $|X| = n$. Denote $[n] = \{1, 2, \dots, n\}$. Given an r-graph G on vertex set $[n]$, define the *Lagrangian function* of G as

$$w(G, \boldsymbol{x}) = \sum_{e \in G} \prod_{i \in e} x_i,$$

where $\boldsymbol{x} \in [0, \infty)^n$. $w(G, \boldsymbol{x})$ can be interpreted as the density of a blow-up of G divide $r!$. Define the *Lagrangian* of G as $\lambda(G) = \max_{\boldsymbol{x} \in \Delta} w(G, \boldsymbol{x})$, where

$$\Delta = \left\{ \boldsymbol{x} \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1, x_i \geq 0 \right\}.$$

We call a weighting $\boldsymbol{x}$ a *feasible weighting* if $\boldsymbol{x} \in \Delta$. We call a feasible weighting $\boldsymbol{x}$ *optimal* if $w(G, \boldsymbol{x}) = \lambda(G)$. Given r-graphs G and F, G is said to be F-free if G contains no copy of F. The *Lagrangian density* $\pi_\lambda(F)$ of F is defined to be

$$\pi_\lambda(F) = r! \sup\{\lambda(G) : G \text{ is } F\text{-free}\}.$$

The idea of continuous optimization is widely used not only in mathematics, but also in other disciplines (see [3,4] and so on). The hypergraph Lagrangian method, first introduced by Zykov [5] in 1949, is such a continuous optimization method that is helpful to solve the extremal problems. One of the earliest applications of the hypergraph Lagrangian method was applied by Motzkin and Straus [6] in 1965 to establish the connection (See Theorem 1) between the Lagrangian of a simple graph and its maximum clique number. A surprising application is that in the 1980's, Frankl and Rödl [7] used it to disprove a famous conjecture of Erdős. For more developments of the Lagrangian theory of hypergraphs see [8,9]. Actually, determining the Lagrangian density of general r-graphs when $r \geq 3$ is interesting in itself. However, there are very few known results in Lagrangian density problems. We list some of the relevant results known so far as follows.

Since K^r_{n-1} is F-free for an n-vertex r-graph F, then

$$\pi_\lambda(F) \geq r! \lambda(K^r_{n-1}).$$

We call an n-vertex r-graph F λ-perfect if $\pi_\lambda(F) = \lambda(K^r_{n-1})$ (We will show the value of $\lambda(K^r_n)$ in Fact 4).

Motzkin and Straus [6] showed that every 2-graph K^2_t is λ-perfect. Next, we turn our attention to the hypergraphs. Let T be a tree or a forest that satisfies Erdős and Sós' conjecture, and let F be an r-graph obtained by joining $r-2$ fixed vertices into every edge of T. Sidorenko [10] proved that F is λ-perfect for large tree. Let H^r be the r-graph with edge set $\{v_1 \dots v_{r-1} v_r, v_1 \dots v_{r-1} v_{r+1}\}$. Sidorenko [9] showed that H^r is λ-perfect for $r = 3$ and 4. Let M^r_t be the r-uniform matching with t pairwise disjoint edges. Let S_t denote a 3-uniform hypergraph with t edges $\{e_1, \dots, e_t\}$ satisfying that $e_i \cap e_j = \{v\}$ for all $1 \leq i < j \leq t$. Hefetz and Keevash [11] showed that M^3_2 is λ-perfect. The authors [12] proved that M^3_t and S^4_t are λ-perfect. A result given by Bene Watts, Norin and Yepremyan [13] suggested that M^r_2 is not λ-perfect for $r \geq 4$. It is interesting to study the λ-perfect r-graphs. More results yielding λ-perfect r-graphs are in the papers [10,14–22]. It is worth mentioning that Yan and Peng [23] recently proved that the Lagrangian density of a 3-graph is an irrational number, and independently, Wu [24] showed that the Lagrangian density of M^4_3 is an irrational number. These two results give a positive answer to the question posed by Baber and Talbot [25]: whether there is an irrational Turán density of a single hypergraph. For more relevant Hypergraph Lagrangian results, one may refer to [10,17,22,26–38].

We call a graph *linear* if any two edges of it share at most one vertex in common. Our original motivation is to seek some new tools to study the Lagrangian density of linear 3-graphs, and to give brief proofs. Denote the disjoint union of two r-graphs G and H as $G \cup H$. In 2019, Yan and Peng [22] posed an interesting conjecture.

Conjecture 1 (Yan and Peng [22])**.**

(i) *For an integer $r \geq 3$, there exists $n_0(r)$ such that a linear r-graph on at least $n_0(r)$ vertices is λ-perfect.*

(ii) *For an integer $r \geq 3$, there exists $n_0(r)$ such that if G and H are two λ-perfect r-graphs with $v(G) \geq n_0(r)$ and $v(H) \geq n_0(r)$, then $G \cup H$ is λ-perfect.*

The other motivation of this paper is to find a class of λ-perfect 3-graphs to support Conjecture 1. We simplify S^3_t to S_t, i.e., $S_t = \{uv_iw_i : 1 \leq i \leq t$ and the v_i's and w_i's are all distinct$\}$. Let $S_{2,t} = S_2 \cup S_t$. In this paper, we show that $S_{2,t}$ is λ-perfect for all $t \geq 62$, proving that (ii) of Conjecture 1 holds for $G = S_2$ and $H = S_t$, also supporting (i) of Conjecture 1 in some sense.

Theorem 1. *Let $t \geq 62$ be an integer and G be a 3-graph. If G is $S_{2,t}$-free, then*

$$\lambda(G) \leq \lambda(K^3_{2t+5}) = \frac{(t+2)(2t+3)}{3(2t+5)^2}.$$

In particular, $S_{2,t}$ is λ-perfect.

The Lagrangian density problem is strongly related to the well-known Turán problem. For a given positive integer n and an r-graph F, define the *Turán number* of F as the maximum number of edges attained by an n-vertex F-free r-graph, and denote it as $ex(n, F)$. The *Turán density* of F is defined as

$$\pi(F) = \lim_{n \to \infty} \frac{ex(n, F)}{\binom{n}{r}},$$

such a limit is known to exist. Denote the *extension* of a graph F as H^F, which is an r-graph obtained from F by adding $(r-2)$ new vertices to each pair $\{v_i, v_j\}$ that is not contained in any edge of F. For example, $H^{\{123,456\}} = \{ijv_{ij} : 1 \leq i \leq 3, 4 \leq j \leq 6\}$, where all v_{ij} are different. A result of Sidorenko [9,10] yields that the Lagrangian density of F is equal to the Turán density of H^F. Hence, we can directly obtain the following corollary by Theorem 1.

Corollary 1. *Let $t \geq 62$ be an integer. Then $\pi(H^{S_{2,t}}) = \frac{2(t+2)(2t+3)}{(2t+5)^2}$.*

We remark that the lower bound for t can be improved slightly with some more tedious discussion, we omit it here.

2. Preliminaries

Let us list some useful results of the Lagrangian function. First, we obtain the following fact directly from the definition of Lagrangian.

Fact 1. *Let G be an r-graph. If F is a subgraph of G, then $\lambda(F) \leq \lambda(G)$.*

Call an r-graph G *dense* if $\lambda(F) < \lambda(G)$ for every proper subgraph F of G. Therefore, we may assume that G is dense when we estimate the Lagrangian upper bound of an F-free r-graph G. We say that F *covers pairs* if every pair of vertices is contained in some edge of F.

Fact 2 (Frankl and Rödl [7])**.** *If an r-graph G is dense, then G covers pairs.*

Let G be an r-graph, and $i, j \in V(G)$. Define

$$L_G(j \backslash i) = \left\{ g \in \binom{V(G) \backslash \{i, j\}}{r-1} : g \cup \{j\} \in E(G) \text{ and } g \cup \{i\} \notin E(G) \right\}.$$

There is a useful fact concerning the symmetry property of Lagrangian function. We omit its proof here; see [12].

Fact 3. *Let G be a dense r-graph on vertex set $[n]$ and $\mathbf{x}$ be an optimal weighting on G. If for some $i, j \in [n]$, $L_G(i \backslash j) = L_G(j \backslash i) = \emptyset$. Then $x_i = x_j$.*

Fact 4. $\lambda(K^r_n) = \binom{n}{r}\frac{1}{n^r}$.

Proof of Fact 4. By Fact 3, $(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$ is an optimum weighting of K_n^r, thus we have

$$\lambda(K_n^r) = \lambda(K_n^r, (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})) = \binom{n}{r}\frac{1}{n^r}.$$

□

Motzkin and Straus [6] proved that the Lagrangian of a simple graph G equals to the Lagrangian of its complete subgraph with maximum order, which implies a simple proof of Turán's classical theorem. Let $\omega(G)$ denote the *clique number* of G, i.e., $\omega(G) = \max\{s : K_s^2 \subseteq G\}$.

Theorem 1 (Motzkin and Straus, [6]). *Let G be a simple graph with $\omega(G) = t$,*

$$\lambda(G) = \lambda(K_t^2) = \frac{1}{2}\left(1 - \frac{1}{t}\right).$$

However, it is not easy to determine the Lagrangian of an r-graph when $r \geq 3$. The following result is useful to calculate the Lagrangian of hypergraphs.

Lemma 1 (Frankl and Rödl [7]). *Let G be an r-graph on vertex set $[n]$ and $\boldsymbol{x}$ be an optimum weighting on G. Then ($\frac{\partial w(G,x)}{\partial x_i}$ is the partial derivative of function $w(G, \boldsymbol{x})$ with respect to variable x)*

$$\frac{\partial w(G, \boldsymbol{x})}{\partial x_i} = r\lambda(G)$$

for every $i \in [n]$ with $x_i > 0$.

3. Proof of Theorem 1

For a given r-graph G and $U \subseteq V(G)$, define the *link graph* of U in G as the hypergraph with edge set $\{e \in \binom{V(G)\setminus U}{r-|U|} : e \cup U \in E(G)\}$, and denote as G_U. When $U = \{i\}$ or $U = \{i, j\}$, we simply write as G_i or G_{ij}.

Lemma 2. *Let G be a dense 3-graph on vertex set $[n]$. If $\lambda(G) > \lambda(K_{2t+5}^3)$, then for each vertex $i \in V(G)$, there is a clique on s vertices contained in G_i with $s > \frac{(2t+5)^2}{6t+13}$.*

Proof of Lemma 2. Let $\boldsymbol{x}$ be an optimum weighting on G. Then, by Lemma 1, for every vertex $i \in G$, we have

$$3\lambda(K_{2t+5}^3) < 3\lambda(G) = \frac{\partial w(G, \boldsymbol{x})}{\partial x_i} = w(G_i, \boldsymbol{x}).$$

Note that $x_i > 0$, which follows from G being dense. Since G_i is a simple graph, by Motzkin–Straus Theorem, $w(G_i, \boldsymbol{x}) = \lambda(K_s)(1 - x_i)^3 < \lambda(K_s)$, where s is the clique number of G_i. It follows that $3\lambda(K_{2t+5}^3) < \lambda(K_s)$. Thus, by Fact 4 and Motzkin–Straus Theorem, we have

$$\frac{1}{2} \cdot \frac{(2t+4)(2t+3)}{(2t+5)^2} < \frac{1}{2} \cdot \frac{s-1}{s}.$$

It yields that $\frac{1}{s} < 1 - \frac{(2t+4)(2t+3)}{(2t+5)^2}$, i.e., $\frac{1}{s} < \frac{6t+13}{(2t+5)^2}$. Therefore, $s > \frac{(2t+5)^2}{6t+13}$ and we are done. □

Let s be a positive integer. The s-fold enlargement of the graph F is obtained from F by adding the same s new vertices to every edge of F. For example, the 3-graph $\{123, 134, 145, \dots, 1t(t+1)\}$ is the 1-fold enlargement of $\{23, 34, 45, \dots, t(t+1)\}$, a path on t vertices. The following proposition is a consequence of a result of Sidorenko [10].

Proposition 1 (Sidorenko [10])**.** *Let G be a 3-graph. If $\lambda(G) > \lambda(K^3_{2t+5})$, then G contains a copy of the 1-fold enlargement of a path on $2t+5$ vertices.*

Let $A, B \in [n]^{(r)}$ be two distinct r-set. The *colex ordering* on $[n]^{(r)}$ is the ordering satisfying that $A < B$ if $\max((A\backslash B) \cup (B\backslash A)) \in B$. For instance, $236 < 146$ in $N^{(3)}$. Let $\mathcal{C}(m;r)$ be the r-graph consisting of the first m sets in the colex ordering of $N^{(r)}$. There is a famous conjecture proposed by Frankl and Füredi [39] in 1989. They conjectured that the Lagrangian of any r-graph with m edges is no more than $\lambda(\mathcal{C}(m;r))$.

Note that if $m = \binom{\ell}{3} + \binom{\ell-1}{2} = \binom{\ell+1}{3} - (\ell-1)$, then $\mathcal{C}(m;3) = [\ell+1]^3 \backslash \{\ell(\ell+1)i : i \in [\ell-1]\}$. Clearly, $\{\ell, \ell+1\}$ is not covering pairs in $\mathcal{C}(m;3)$. Thus $\mathcal{C}(m;3)$ is not dense by Fact 2. Hence $\lambda(\mathcal{C}(m;3)) \le \lambda(K^3_\ell)$. Moreover, we have $\lambda(\mathcal{C}(m;3))) \ge \lambda(K^3_\ell)$ since $K^3_\ell \subseteq \mathcal{C}(m;3))$. Therefore, $\lambda(\mathcal{C}(m;3))) = \lambda(K^3_\ell)$. For $r = 3$, Talbot [35] first showed the conjecture holds whenever $\binom{\ell}{3} \le m \le \binom{\ell}{3} + \binom{\ell-1}{2} - \ell$. After then, big progress has been made in [28–30,36,37,40].

Theorem 2 (Talbot [35])**.** *Let m and ℓ be integers satisfying*

$$\binom{\ell}{3} \le m \le \binom{\ell}{3} + \binom{\ell-1}{2} - \ell,$$

then for any 3-graph G with m edges, $\lambda(G) \le \lambda(\mathcal{C}(m;3))$. Moreover, $\lambda(\mathcal{C}(m;3)) = \lambda(K^3_l)$.

Corollary 2. *Let G be a 3-graph with m edges. If $\lambda(G) > \lambda(K^3_\ell)$, then $m \ge \binom{\ell}{3} + \binom{\ell-1}{2} - \ell + 1$.*

Proof of Corollary 2. For the contrary, suppose that $m \le \binom{l}{3} + \binom{l-1}{2} - l$. Let G' be the 3-graph obtained from G by adding arbitrarily s edges to G such that $\binom{l}{3} \le e(G') \le \binom{l}{3} + \binom{l-1}{2} - l$, where $s \ge 0$ is an integer. By Fact 1, we have $\lambda(G) \le \lambda(G')$. Moreover, by Theorem 2, we have $\lambda(G') \le \lambda(K^3_l)$. Thus, $\lambda(G) \le \lambda(K^3_l)$, which contradicts that $\lambda(G) > \lambda(K^3_l)$. $\square$

We now give two crucial lemmas.

Lemma 3. *Let $t \ge 62$ and $2t+6 \le n \le 2t+9$ be two positive integers. Let G be a dense n-vertex 3-graph. If G is $S_{2,t}$-free, then $\lambda(G) \le \lambda(K^3_{2t+5})$.*

Proof of Lemma 3. For the contrary, suppose that $\lambda(G) > \lambda(K^3_{2t+5})$. Since $\lambda(G) > \lambda(K^3_{2t+5})$, by Proposition 1, there exists a copy of the 1-fold enlargement of a path on $2t+5$ vertices in G. Denote this 3-graph as S with $V(S) = \{a_1, a_2, \dots, a_{2t+5}\} \cup \{o\}$ and $E(S) = \{oa_1a_2, oa_2a_3, \dots, oa_{2t+4}a_{2t+5}\}$. Clearly, $\{oa_1a_2, \dots, oa_{2i-1}a_{2i}, \dots, oa_{2t+3}a_{2t+4}\}$ forms a copy of S_{t+2} in S (see Figure 1).

For $1 \le k_1 < k_2 < k_3 < k_4 < k_5 \le t+3$, denote

$$U = U(k_1, \dots, k_5) = \{a_{2k_1-1}, a_{2k_2}, a_{2k_3-1}, a_{2k_4}, a_{2k_5-1}\} \text{ and } H = G[U].$$

Note that $V(S)\backslash U$ contains a copy of S_t with edge set $\cup_{i\in[6]} E_i$, where

$$\begin{aligned}
E_1 &= \{oa_1a_2, oa_3a_4, \dots, oa_{2k_1-3}a_{2k_1-2}\},\\
E_2 &= \{oa_{2k_1}a_{2k_1+1}, oa_{2k_1+2}a_{2k_1+3}, \dots, oa_{2k_2-2}a_{2k_2-1}\},\\
E_3 &= \{oa_{2k_2+1}a_{2k_2+2}, oa_{2k_2+3}a_{2k_2+4}, \dots, oa_{2k_3-3}a_{2k_3-2}\},\\
E_4 &= \{oa_{2k_3}a_{2k_3+1}, oa_{2k_3+2}a_{2k_3+3}, \dots, oa_{2k_4-2}a_{2k_4-1}\},\\
E_5 &= \{oa_{2k_4+1}a_{2k_4+2}, oa_{2k_4+3}a_{2k_4+4}, \dots, oa_{2k_5-3}a_{2k_5-2}\},\\
E_6 &= \{oa_{2k_5}a_{2k_5+1}, oa_{2k_5+2}a_{2k_5+3}, \dots, oa_{2t+4}a_{2t+5}\}.
\end{aligned}$$

Since G is $S_{2,t}$-free, H is S_2-free. We claim that $e(H) \leq 4$. Suppose that $e(H) \geq 5$, and relabel the vertex set of H as $[5]$. Without loss of generality, assume that $123 \in H$. Since H is S_2-free, thus there is no edge of H containing $\{4,5\}$. If there is no edge of H containing $\{4\}$ or $\{5\}$, then $H \subseteq K_4^3$, which contradicts that $e(H) \geq 5$. So there are at least two edges of H such that one contains 4 and the other contains 5. Without loss of generality, assume that $124 \in H$. Similarly, there is no edge of H containing $\{3,5\}$. So, $125 \in H$, and thus, there is no edge of H containing $\{3,4\}$. Therefore, $E(H) \subseteq \{123, 124, 125\}$, a contradiction. Hence, $e(H) \leq 4$.

There are $\binom{t+3}{5}$ such $\{k_1,k_2,k_3,k_4,k_5\}$, and for each $e \in \overline{H}$, there are at most $\binom{t}{2}$ such $\{k_1,k_2,k_3,k_4,k_5\}$ so that $e \subseteq \{a_{2k_1-1}, a_{2k_2}, a_{2k_3-1}, a_{2k_4}, a_{2k_5-1}\}$. Note that $\overline{H} \subseteq \overline{G}$ and $e(\overline{H}) = \binom{5}{3} - e(H) \geq 6$. Then

$$e(\overline{G}) \geq \frac{6 \cdot \binom{t+3}{5}}{\binom{t}{2}} = \frac{(t+3)(t+2)(t+1)}{10}. \tag{1}$$

By Corollary 2,

$$\begin{aligned} e(\overline{G}) &\leq \binom{n}{3} - \left(\binom{2t+5}{3} + \binom{2t+4}{2} - (2t+5) + 1\right) \\ &\leq \binom{2t+9}{3} - \left(\binom{2t+5}{3} + \binom{2t+4}{2} - (2t+5) + 1\right). \end{aligned}$$

Let

$$f(t) = \binom{2t+9}{3} - \left(\binom{2t+5}{3} + \binom{2t+4}{2} - (2t+5) + 1\right) - \frac{(t+3)(t+2)(t+1)}{10}.$$

Then

$$f(t) = \frac{-t^3 + 54t^2 + 419t + 714}{10} \text{ and } f'(t) = \frac{-3t^2 + 108t + 419}{10}.$$

The roots of $f'(t) = 0$ are $\frac{54-\sqrt{4173}}{3}$ and $\frac{54+\sqrt{4173}}{3}$. Since the quadratic function $f'(t)$ concave, $f(t)$ is increasing in $[0, \frac{54+\sqrt{4173}}{3})$, and decreasing in $(\frac{54+\sqrt{4173}}{3}, +\infty)$. By direct calculation, we have $\frac{54+\sqrt{4173}}{3} < 62$ and $f(62) < 0$. Hence, $f(t) < 0$ since $t \geq 62$, that is, $e(\overline{G}) < \frac{(t+3)(t+2)(t+1)}{10}$. This is a contradiction with (1). We complete the proof. $\square$

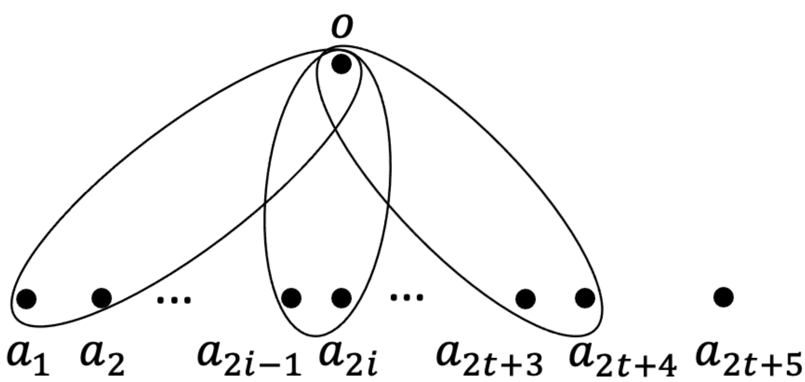

Figure 1. A copy of S_{t+2} in S.

Lemma 4. *Let $t \geq 62$ and $n \geq 2t+10$ be two positive integers. Let G be a dense n-vertex 3-graph. If G is $S_{2,t}$-free, then $\lambda(G) \leq \lambda(K^3_{2t+5})$.*

Proof of Lemma 4. For the contrary, suppose that $\lambda(G) > \lambda(K^3_{2t+5})$. For each $u \in V(G)$, denote a maximum clique in G_u as K^u. Since $\lambda(G) > \lambda(K^3_{2t+5})$ with $t \geq 62$, by Lemma 2,

$$v(K^u) \geq \frac{(2t+5)^2}{6t+13} \geq 43.$$

Furthermore, by Proposition 1, there exists a copy of the 1-fold enlargement of a path on $2t+5$ vertices in G. Denote this 3-graph as S with $V(S) = \{a_1, a_2, \dots, a_{2t+5}\} \cup \{o\}$ and $E(S) = \{oa_1a_2, oa_2a_3, \dots, oa_{2t+4}a_{2t+5}\}$. Clearly, $\{oa_1a_2, \dots, oa_{2i-1}a_{2i}, \dots, oa_{2t+3}a_{2t+4}\}$ forms a copy of S_{t+2} in S (see Figure 1). If we delete one vertex of $\{a_1, a_2, \dots, a_{2t+5}\}$ with an odd number of subscript, then we can still find a copy of S_{t+2} in S. However, If we delete one vertex of $\{a_1, a_2, \dots, a_{2t+5}\}$ with an even number of subscript, then we can only guarantee that there is a copy of S_{t+1} in S. The situation is always 'worse' when the subscript of the deleted vertex is even than when it is odd.

Since $n \geq 2t+10$, there are at least four vertices in $V(G)\backslash V(S)$, we denote four vertices among them as u_1, u_2, u_3, u_4 and denote $U = \{u_1, u_2, u_3, u_4\}$. Since G is dense, then G covers pairs by Fact 2. We consider $G_{u_iu_j}$ for all $1 \leq i < j \leq 4$, recall that $G_{u_iu_j} = \{v \in V(G) : vu_iu_j \in G\}$. □

Claim 1. *$G_{u_iu_j} \subseteq \{o, a_2, a_4, \dots, a_{2t+4}\}$ for all $1 \leq i < j \leq 4$.*

Proof of Claim 1. Suppose that there is $w \in V(G)\backslash\{o, a_2, a_4, \dots, a_{2t+4}\}$ such that $u_1u_2w \in G$. Recall that $v(K^w) \geq 43$, so we can pick two vertices in $V(K^w) - \{o, u_1, u_2\}$, say v_1, v_2. Thus, $\{u_1u_2w, wv_1v_2\}$ forms a copy of S_2 in $G - \{o\}$. We will show that there exists a copy of S_t in $S - \{w, u_1, u_2, v_1, v_2\}$. Recall that $w \notin \{a_2, a_4, \dots, a_{2t+4}\}$, the worst case (For the sake of brevity of the proof, we consider only the worst case and omit other cases where $\{v_1, v_2\} = \{a_{2i}, a_{2j+1}\}$ or $\{v_1, v_2\} = \{a_{2i+1}, a_{2j+1}\}$) is $w = a_{2k+1}$ and $\{v_1, v_2\} = \{a_{2i}, a_{2j}\}$, where $0 \leq k \leq t+2$ and $1 \leq i < j \leq t+2$. Assume that $i < j \leq k$, delete $a_{2i}, a_{2j}, a_{2k+1} \in \{a_1, a_2, \dots, a_{2t+5}\}$ and abandon a_{2i-1}, a_{2j-1}. We can find a copy of S_t with edge set $\{oa_1a_2, \dots, oa_{2i-3}a_{2i-2}\} \cup \{oa_{2i+1}a_{2i+2}, \dots, oa_{2j-3}a_{2j-2}\} \cup \{oa_{2j+1}a_{2j+2}, \dots, oa_{2k-1}a_{2k}\} \cup \{oa_{2k+2}a_{2k+3}, \dots, oa_{2t+4}a_{2t+5}\}$ in S. So, G contains a copy of $S_{2,t}$, a contradiction. The proofs for $k < i < j$ or $i \leq k < j$ are similar. □

Claim 2. *If $u_1u_2a_2 \in G$, then for each $k \in \{1, 3, \dots, 2t+5\}$, $G_{u_1a_k} = G_{u_2a_k} = \{o\}$.*

Proof of Claim 2. Fix $k \in \{1, 3, \dots, 2t+5\}$. First we prove that $G_{u_1a_k} \subseteq \{o, a_2\}$. Suppose that there is $w \notin \{o, a_2\}$ such that $u_1a_kw \in G$, then $\{a_2u_2u_1, u_1a_kw\}$ forms a copy of S_2. It is not hard to see that there exists a copy of S_t in $S - V(S_2)$. Thus, G contains a copy of $S_{2,t}$, a contradiction. Similarly, $G_{u_2a_k} \subseteq \{o, a_2\}$. Now we suppose that $u_1a_ka_2 \in G$. Then for $k' \in \{1, 3, 5, ..., 2t+5\}$ and $k' \neq k$, $G_{u_2a_{k'}} = \{o\}$. Otherwise $\{u_2a_{k'}a_2, a_2a_ku_1\}$ forms a copy of S_2, and clearly there exists a copy of S_t in $S - V(S_2)$. Thus, G contains a copy of $S_{2,t}$, a contradiction. Let $v, w \in V(K^{a_k}) - \{o, u_1, u_2, a_2\}$. $\{u_1a_2a_k, a_kvw\}$ forms a copy of S_2. Our goal is to find a copy of S_t in $S \cup \{u_2\} - \{a_k, a_2, v, w\}$, thus, we obtain a copy of $S_{2,t}$ in G and we are done. The worst case is $\{v, w\} = \{a_{2l}, a_{2l'}\}$, where $1 < l < l' \leq t+2$. Assume that $2l < k < 2l'$. In this case $k \neq 1$, we abandon $a_1, a_{2l-1}, a_{2l'+1}$ and find a copy of S_{t-1} with edge set $\{oa_3a_4, \dots, oa_{2l-3}a_{2l-2}\} \cup \{oa_{2l+1}a_{2l+2} \dots, oa_{k-2}a_{k-1}\} \cup \{oa_{k+1}a_{k+2}, \dots, oa_{2l'-2}a_{2l'-1}\} \cup \{oa_{2l'+2}a_{2l'+3}, \dots, oa_{2t+4}a_{2t+5}\}$ in S. This copy of S_{t-1} together with $\{ou_2a_1\}$ forms a copy of S_t, and we are done. The proofs for $k < 2l < 2l'$ or $l < l' < k$ are similar. □

Claim 3. *If $u_1u_2a_2 \in G$, then $u_iu_ja_2 \notin G$ for every pair $i \in \{1, 2\}$, $j \in \{3, 4\}$.*

Proof of Claim 3. Without loss of generality, assume that $u_1u_3a_2 \in G$. By Claim 2, $u_ia_ko \in G$ for every pair $i \in [3]$ and $k \in \{1, 3, \dots, 2t+5\}$. Let $v, w \in V(K^{a_2}) - \{a_1, u_1, u_2, u_3, u_4, o\}$. Then $\{u_1u_2a_2, a_2vw\}$ forms a copy of S_2 in G. Our goal is to find a copy of S_t in $S \cup \{u_3\} -$

$\{a_2, v, w\}$, thus we obtain a copy of $S_{2,t}$ in G and we are done. Similar to the proof of Claim 2, the worst case is $\{v, w\} = \{a_{2l}, a_{2l'}\}$, where $1 < l < l' \le t+2$, and there is a copy of S_{t-1} not containing a_1. This copy of S_{t-1} together with $\{ou_3a_1\}$ forms a copy of S_t, and we are done.

Now we continue to prove the lemma. We first claim that $\{u_1u_2o, u_3u_4o\}$ or $\{u_1u_3o, u_2u_4o\}$ or $\{u_1u_4o, u_2u_3o\}$ is contained in G. Otherwise, without loss of generality suppose that $u_1u_2o \notin G$. By Claim 1, we have $u_1u_2a_2 \in G$. If for some triple $i \in \{1,2\}$, $j \in \{3,4\}$ and $k \in \{4, 6, \dots, 2t+4\}$, $u_iu_ja_k \in G$, then $\{u_1u_2a_2, u_iu_ja_k\}$ forms a copy of S_2 in G. It is not hard to see that there is a copy of $S_{2,t}$ in $S - \{a_2, a_k\}$, a contradiction. So we may assume that $u_iu_ja_k \notin G$ for every triple $i \in \{1,2\}$, $j \in \{3,4\}$ and $k \in \{4, 6, \dots, 2t+4\}$. Moreover, by Claim 3, $u_iu_ja_2 \notin G$ for every pair $i \in \{1,2\}$, $j \in \{3,4\}$. Therefore, we have $u_1u_3o, u_2u_4o \in G$. Hence, we have shown that the claim holds. Without loss of generality, we assume that $\{u_1u_2o, u_3u_4o\}$ is contained in G. Let $c_1, c_2, c_3, c_4 \in V(K^{a_1}) - \{o, u_1, u_2, u_3, u_4\}$. Then $\{c_1c_2a_1, a_1c_3c_4\}$ forms a copy of S_2. Our goal is to find a copy of S_t in $S \cup \{u_1, u_2, u_3, u_4\} - \{a_1, c_1, c_2, c_3, c_4\}$, thus we obtain a copy of $S_{2,t}$ in G and we are done. Similar to the proof of Claim 2, the worst case is $\{c_1, c_2, c_3, c_4\} = \{a_{2i}, a_{2j}, a_{2k}, a_{2l}\}$, where $1 < i < j < k < l \le t+2$. It is not hard to see that there exists a copy of S_{t-2} in $S - \{a_1, c_1, c_2, c_3, c_4\}$. This copy of S_{t-1} together with $\{u_1u_2o, u_3u_4o\}$ forms a copy of S_t. Thus we complete the proof. □

Proof of Theorem 1. Let G be an $S_{2,t}$-free 3-graph with vertex set $[n]$. We can assume that G is dense, otherwise we replace G by a dense subgraph G' of G with $\lambda(G') = \lambda(G)$. So G covers pairs by Fact 2. If $n \le 2t+5$, then $\lambda(G) \le \lambda(K^3_{2t+5})$ by Fact 1. If $2t+6 \le n \le 2t+9$, then we are done by Lemma 3. Now assume that $n \ge 2t+10$. For the contrary, we suppose that $\lambda(G) > \lambda(K^3_{2t+5})$. By Lemma 4, there exists a copy of $S_{2,t}$ in G, a contradiction. So $\pi_\lambda(S_{2,t}) \le 3!\lambda(K^3_{2t+5})$. Since K^3_{2t+5} contains no $S_{2,t}$, we have $\pi_\lambda(S_{2,t}) \ge 3!\lambda(K^3_{2t+5})$. Hence, $\pi_\lambda(S_{2,t}) = 3!\lambda(K^3_{2t+5})$. Thus, we complete the proof. □

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