symmetry

MDPI

Article

Convolution Properties of q-Janowski-Type Functions Associated with (x, y)-Symmetrical Functions

Fuad Alsarari [1,*] and Samirah Alzahrani [2]

1 Department of Mathematics and Statistics, Sciences College, Taibah University, P.O. Box 46421, Yanbu 41911, Saudi Arabia
2 Department of Mathematics and Statistics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia; samar.alz@tu.edu.sa
* Correspondence: alsrary@gmail.com or fsarari@taibahu.edu.sa

Abstract: The purpose of this paper is to define new classes of analytic functions by amalgamating the concepts of q-calculus, Janowski type functions and (x, y)-symmetrical functions. We use the technique of convolution and quantum calculus to investigate the convolution conditions which will be used as a supporting result for further investigation in our work, we deduce the sufficient conditions, Pólya-Schoenberg theorem and the application. Finally motivated by definition of the neighborhood, we give analogous definition of neighborhood for the classes $\widetilde{\mathcal{S}}_q^{x,y}(\alpha, \beta)$ and $\widetilde{\mathcal{K}}_q^{x,y}(\alpha, \beta)$, and then investigate the related neighborhood results, which are also pointed out.

Keywords: analytic functions; Hadamard product; (x, y)-symmetrical functions; q-calculus; (ρ, q)-neighborhood

Citation: Alsarari, F.; Alzahrani, S. Convolution Properties of q-Janowski-Type Functions Associated with (x, y)-Symmetrical Functions. *Symmetry* **2022**, *14*, 1406. https://doi.org/10.3390/sym14071406

Academic Editor: Erhan Deniz

Received: 16 June 2022
Accepted: 4 July 2022
Published: 8 July 2022

1. Introduction

Let $\mathcal{F}(k)$ denote the family of all functions that are analytic in the open unit disc $k = \{w \in \mathbb{C} : |w| < 1\}$ and let $\mathcal{F}$ represents a subfamily of class $h \in \mathcal{F}(k)$ which has the form

$$h(w) = w + \sum_{v=2}^{\infty} a_v w^v, \tag{1}$$

and suppose $\widetilde{\mathcal{S}}$ containing all the functions in $\mathcal{F}$ that are univalent k. The convolution or Hadamard product of two analytic functions $h, g \in \mathcal{F}$ where h is defined by (1) and $g(w) = w + \sum_{v=2}^{\infty} b_v w^v$, is

$$(h * g)(w) = w + \sum_{v=2}^{\infty} a_v b_v w^v.$$

In order to define new classes of q-Janowski symmetrical functions defined in k, we first recall the necessary notions and notations concerning, Janowski type functions, the theory of (x, y)-symmetrical functions and quantum calculus (or q-calculus).

Janowski in [1] introduced the class $\mathcal{P}[\alpha, \beta]$, a given $h \in \mathcal{F}$ and $h(0) = 1$ is said to be in $\mathcal{P}[\alpha, \beta]$ if and only if $p(w) = \dfrac{1 + \alpha s(w)}{1 + \beta s(w)}$, for $-1 \le \beta < \alpha \le 1$ and $s(w) \in \Delta$ where Δ denote for the family of *Schwarz functions*, that is

$$\Delta := \{s \in \mathcal{F},\ s(0) = 0, |s(w)| < 1,\ w \in k\}. \tag{2}$$

Let y be an arbitrarily fixed integer and for $\varepsilon = e^{\frac{2\pi i}{y}}$, a domain $\mathbf{G} \subset \mathbb{C}$ is said to be *y-fold symmetric domain* if $\varepsilon\mathbf{G} = \mathbf{G}$. A function h is called y-symmetrical function for each $w \in \mathbf{G}$ if $h(\varepsilon w) = \varepsilon h(w)$.

In 1995, Liczberski and Polubinski [2] constructed the concept of (x, y)-symmetrical functions for $(x = 0, 1, 2, \dots, y-1)$, and $(y = 2, 3, \dots)$. If $\mathbf{G}$ is y-fold symmetric domain and x any integer, then a function $h : \mathbf{G} \to \mathbb{C}$ is called (x, y)-symmetrical if for each $w \in \mathbf{G}$, $h(\varepsilon w) = \varepsilon^x h(w)$. The family of all (x, y)-symmetrical functions will be denoted by $\mathcal{F}_y^x$, we note that $\mathcal{F}_2^0$, $\mathcal{F}_2^1$ and $\mathcal{F}_y^1$ are families of even, odd and of y-symmetrical functions, respectively.

Theorem 1 ([2]). *For every mapping $h : k \mapsto \mathbb{C}$, and a y-fold symmetric set k, then*

$$h(w) = \sum_{x=0}^{y-1} h_{x,y}(w), \qquad h_{x,y}(w) = y^{-1} \sum_{r=0}^{y-1} \varepsilon^{-rx} h(\varepsilon^r w),\ w \in k. \tag{3}$$

Remark 1. *Equivalently,* (3) *may be written as*

$$h_{x,y}(w) = \sum_{v=1}^{\infty} \delta_{v,x} a_v w^v,\ \ a_1 = 1, \tag{4}$$

where

$$\delta_{v,x} = \frac{1}{y} \sum_{r=0}^{y-1} \varepsilon^{(v-x)r} = \begin{cases} 1, & v = ly + x; \\ 0, & v \neq ly + x; \end{cases}, \tag{5}$$

$$(l \in \mathbb{N},\ y = 1, 2, \ldots,\ x = 0, 1, 2, \ldots, y - 1).$$

Recently the authors of [3,4] obtained many interesting results for various classes using the concept of (x, y)-symmetrical functions and q-derivative.

In [5], Jackson introduced and studied the concept of the q-derivative operator $\partial_q h(w)$ as follows:

$$\partial_q h(w) = \begin{cases} \frac{h(w) - h(qw)}{w(1-q)}, & w \neq 0, \\ h'(0), & w = 0. \end{cases} \tag{6}$$

Equivalently (6), may be written as

$$\partial_q h(w) = 1 + \sum_{v=2}^{\infty} [v]_q a_v w^{v-1} \quad w \neq 0,$$

where

$$[v]_q = \frac{1 - q^v}{1 - q} = 1 + q + q^2 + \ldots + q^{v-1}. \tag{7}$$

Note that as $q \to 1^-$, $[v]_q \to v$. For a function $h(w) = w^v$, we can note that

$$\partial_q h(w) = \partial_q (w^v) = \frac{1 - q^v}{1 - q} w^{v-1} = [v]_q w^{v-1}.$$

Then

$$\lim_{q \to 1^-} \partial_q h(w) = \lim_{q \to 1^-} [v]_q w^{v-1} = v w^{v-1} = h'(w),$$

where $h'(w)$ is the ordinary derivative.

The q-integral of a function h presented by Jackson [6] As a right inverse as

$$\int_0^w h(z) d_q z = w(1 - q) \sum_{v=0}^{\infty} q^v h(wq^v),$$

provided that $\sum_{v=0}^{\infty} q^v h(wq^v)$ is converges.

Proposition 1. *If n and m any real (or complex) constants and $w \in k$, then we have*

1. $\partial_q (nh(w) \pm mg(w)) = n\partial_q h(w) \pm m\partial_q g(w)$,
2. $\partial_q (h(w)g(w)) = h(qw)\partial_q g(w) + \partial_q g h(w) g(w) = h(w)\partial_q g(w) + \partial_q h(w) g(qw)$,

3. $\partial_q\left(\frac{h(w)}{g(w)}\right) = \frac{g(w)\partial_q h(w) - h(w)\partial_q g(w)}{g(qw)g(w)}$.

In recent years, using quantum (or q-calculus) for studying diverse families of analytic functions. Srivastava et al. [7] found distortion and radius of univalent and starlikenss for several subclasses of q-starlike functions. Naeem et al. [8] investigated subfamilies of q-convex functions with respect to the Janowski functions connected with q-conic domain. Govindaraj and Sivasubramanian in [9] found subclasses connected with q-conic domain. In [10], we use the symmetric q-derivative operator to define a new subclass of analytic and bi-univalent function. Srivastava [11] published survey-cum-expository review paper which is useful for researchers and scholars.

Utilizing the ideas of q-derivative operator and the concept of (x, y)-symmetrical functions we introduce a new subclass $\widetilde{\mathcal{S}}_q^{x,y}(\alpha, \beta)$. This class is introduced by using the q-derivative operator with the concept to (x, y)-symmetric points.

Definition 1. *For arbitrary fixed numbers q, α, β and λ, $0 < q < 1$, $-1 \leq \beta < \alpha \leq 1$, let $\widetilde{\mathcal{S}}_q^{x,y}(\alpha, \beta)$ denote the family of functions $h \in \mathcal{F}$ which satisfies*

$$\Re\left\{\frac{w\partial_q h(w)}{h_{x,y}(w)}\right\} \in \mathcal{P}[\alpha, \beta], \quad w \in k, \tag{8}$$

where $h_{x,y}$ is defined in (3).

For special cases of the parameters q, α, β, x and y the class $\widetilde{\mathcal{S}}_q^{x,y}(\alpha, \beta)$ yield several known subclasses of $\mathcal{F}$, namely: $\widetilde{\mathcal{S}}_1^{x,y}(\alpha, \beta) := \widetilde{\mathcal{S}}^{x,y}(\alpha, \beta)$ introduced by the authors of [12]; $\widetilde{\mathcal{S}}_1^{1,y}(\alpha, \beta) := \widetilde{\mathcal{S}}_y(\alpha, \beta)$, introduced by the authors Latha and Darus [13]; $\widetilde{\mathcal{S}}_1^{1,y}(1, -1) := \widetilde{\mathcal{S}}_y$ as defined by Sakaguchi [14]; $\widetilde{\mathcal{S}}_1^{1,1}(\alpha, \beta) := \widetilde{\mathcal{S}}[\alpha, \beta]$ which reduce to a well-known class defined by Janowski [1]; $\widetilde{\mathcal{S}}_q^{1,1}(1 - 2\kappa, -1) = \widetilde{\mathcal{S}}_q(\kappa)$ which was introduced and studied by Agrawal and Sahoo in [15]; $\widetilde{\mathcal{S}}_q^{1,1}(1, -1) = \widetilde{\mathcal{S}}_q$ which was first introduced by Ismail et al. [16]; $\widetilde{\mathcal{S}}_1^{1,1}(1 - 2\kappa, -1) = \widetilde{\mathcal{S}}(\kappa)$ the well-known class of starlike function of order κ by Robertson [17]; and $\mathcal{S}_1^{1,1}(1, -1, 0) = \mathcal{S}^*$ the class introduced by Nevanlinna [18], etc.

We denote by $\widetilde{\mathcal{K}}_q^{x,y}(\alpha, \beta)$ the subclass of $\mathcal{F}$ consisting of all functions h such that

$$w\partial_q h(w) \in \widetilde{\mathcal{S}}_q^{x,y}(\alpha, \beta). \tag{9}$$

We need to recall the following neighborhood concept introduced by Goodman [19] and generalized by Ruscheweyh [20].

Definition 2. *For any $h \in \mathcal{F}$, ρ-neighborhood of function h can be defined as:*

$$\mathcal{N}_\rho(h) = \left\{ g \in \mathcal{F} : \; g(w) = w + \sum_{v=2}^{\infty} b_v w^v, \quad \sum_{v=2}^{\infty} v|a_v - b_v| \leq \rho \right\}, \; (\rho \geq 0). \tag{10}$$

For $e(w) = w$, we can see that

$$\mathcal{N}_\rho(e) = \left\{ g \; \in \mathcal{F} : g(w) = w + \sum_{v=2}^{\infty} b_v w^v, \quad \sum_{v=2}^{\infty} v|b_v| \leq \rho \right\}, \; (\rho \geq 0). \tag{11}$$

Ruscheweyh [20] proved, among other results, that for all $\eta \in \mathbb{C}$, with $|\eta| < \rho$,

$$\frac{h(w) + \eta w}{1 + \eta} \in \widetilde{\mathcal{S}}^* \Rightarrow \mathcal{N}_\rho(h) \subset \widetilde{\mathcal{S}}^*.$$

Lemma 1 ([21]). *Let ϕ be a convex and g a starlike, for F analytic in $\mathcal{U}$ with $F(0) = 1$, then*

$$\frac{\phi * Fg}{\phi * g}(\mathcal{U}) \subset \overline{CO}(F(\mathcal{U})),$$

where $\overline{CO}(F(\mathcal{U}))$ denotes the closed convex hull of $F(\mathcal{U})$.

The goal of this research to give a convolution conditions for a function h to be in the classes $\widetilde{\mathcal{S}}_q^{x,y}(\alpha, \beta)$ and $\widetilde{\mathcal{K}}_q^{x,y}(\alpha, \beta)$ which will be used to drive a sufficient conditions, Pólya–Schoenberg theorem and application. In the next section be the motivation of the Definition 2, we give analogous definition of neighborhood for the class $\widetilde{\mathcal{S}}_q^{x,y}(\alpha, \beta)$ and $\widetilde{\mathcal{K}}_q^{x,y}(\alpha, \beta)$, then investigate related neighborhood results.

2. Results

Theorem 2. *A function $h \in \widetilde{\mathcal{K}}_q^{x,y}(\alpha, \beta)$ if and only if*

$$\frac{1}{w}\left[h * \left(\frac{(w - qw^3)(1 + \beta e^{i\phi})}{(1 - w)(1 - qw)(1 - q^2 w)} - \frac{(1 + \alpha e^{i\phi})w}{(1 - u_x w)(1 - u_x q w)}\right)\right] \neq 0, \quad |w| < R \leq 1,$$

where $0 < q < 1$, $-1 \leq \beta < \alpha \leq 1, 0 \leq \phi < 2\pi$ and u_x is defined by (14).

Proof. We have, $h \in \widetilde{\mathcal{K}}_q^{x,y}(\alpha, \beta)$ if and only if

$$\frac{\partial_q(w\partial_q h(w))}{\partial_q h_{x,y}(w)} \neq \frac{1 + \alpha e^{i\phi}}{1 + \beta e^{i\phi}}, \quad |w| < R,$$

which implies

$$\partial_q(w\partial_q h(w))(1 + \beta e^{i\phi}) - \partial_q h_{x,y}(w)\{1 + \alpha e^{i\phi}\} \neq 0. \tag{12}$$

Setting $h(w) = w + \sum_{v=2}^{\infty} a_v w^v$, we have

$$\partial_q h = 1 + \sum_{v=2}^{\infty}[v]_q a_v w^{v-1}, \qquad \partial_q\left(w\partial_q h\right) = 1 + \sum_{v=2}^{\infty}[v]_q^2 a_v w^{v-1} = \partial_q h * \frac{1}{(1 - w)(1 - qw)}.$$

$$\partial_q h_{x,y}(w) = \partial_q h * \frac{1}{(1 - u_x w)} = \sum_{v=1}^{\infty}[v]_q u_x^v a_v w^{v-1}, \tag{13}$$

where

$$u_x^v = \delta_{v,x}, \quad \text{and } \delta_{v,x} \text{ is given by (5).} \tag{14}$$

The left hand side of (12) is equivalent to

$$\partial_q h * \left(\frac{1 + \beta e^{i\phi}}{(1 - w)(1 - qw)} - \frac{1 + \alpha e^{i\phi}}{1 - u_x w}\right), \tag{15}$$

simplifying (15) we obtain

$$\frac{1}{w}\left[w\partial_q h * \left(\frac{(1 + \beta e^{i\phi})w}{(1 - w)(1 - qw)} - \frac{(1 + \gamma e^{i\phi})w}{1 - u_x w}\right)\right] \neq 0, \tag{16}$$

since $w\partial_q h * g = h * w\partial_q g$, we can write the Equation (16) as

$$\frac{1}{w}\left[h * \left(\frac{(w - qw^3)(1 + \beta e^{i\phi})}{(1 - w)(1 - qw)(1 - q^2 w)} - \frac{(1 + \alpha e^{i\phi})w}{(1 - u_x w)(1 - u_x q w)}\right)\right] \neq 0.$$

□

Remark 2. *For $q \to 1^-$ and spacial values of x, y, α and β, we have following result proved by Ganesan and et al in [22] Silverman and et al in [23].*

Theorem 3. *A function $f \in \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta)$ if and only if*

$$\frac{1}{w}\left[h * \left(\frac{(1+\beta e^{i\phi})w}{(1-w)(1-qw)} - \frac{(1+\alpha e^{i\phi})w}{1-u_x w}\right)\right] \neq 0, \;\; |w|<1,$$

where $0<q<1$, $-1 \le \beta < \alpha \le 1, 0 \le \phi < 2\pi$ and u_x is defined by (14).

Proof. Since $h \in \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta)$ if and only if $g(w) = \int_0^w \frac{h(\zeta)}{\zeta} d_q\zeta \in \widetilde{\mathcal{K}}_q^{x,y}(\alpha,\beta)$, we have

$$\frac{1}{w}\left[g * \left(\frac{(w-qw^3)(1+\beta e^{i\phi})}{(1-w)(1-qw)(1-q^2w)} - \frac{(1+\alpha e^{i\phi})w}{(1-u_x w)(1-u_x qw)}\right)\right]$$
$$= \frac{1}{w}\left[h * \left(\frac{(1+\beta e^{i\phi})w}{(1-w)(1-qw)} - \frac{(1+\alpha e^{i\phi})w}{1-u_x w}\right)\right].$$

Thus the result follows from Theorem 3. □

Note that we can easily from Theorem 3 obtain that the equivalent condition for a function $h \in \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta)$ in the following Corollary.

Corollary 1. *For $q \in (0,1), -1 \le \beta < \alpha \le 1$ and $\phi \in [0,2\pi)$, then*

$$h \in \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta) \Leftrightarrow \frac{(h*g)(w)}{w} \neq 0, \quad ,w \in k, \tag{17}$$

where $g(w)$ has the form

$$g(w) = w + \sum_{v=2}^{\infty} t_v w^v,$$
$$t_v = \frac{[v]_q - \delta_{v,x} + ([v]_q\beta - \delta_{v,x}\alpha)e^{i\phi}}{(\beta-\alpha)e^{i\phi}}. \tag{18}$$

By using Corollary 1 we drive the sufficient condition theorem.

Theorem 4. *Let $h(w) = w + \sum_{v=2}^{\infty} a_v w^v$, be analytic in k, for $-1 \le \beta < \alpha \le 1$ and $0<q<1$, if*

$$\sum_{v=2}^{\infty}\left\{\frac{([v]_q - \delta_{v,x}) + |\alpha\delta_{v,x} - \beta[v]_q|}{|\alpha-\beta|}\right\}|a_v| \le 1, \tag{19}$$

then $h(w) \in \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta)$.

Proof. For the proof of Theorem 4, it suffices to show that $\frac{(h*g)(w)}{w} \neq 0$ where g is given by (18). Let $h(w) = w + \sum_{v=2}^{\infty} a_v w^v$ and $g(w) = w + \sum_{v=2}^{\infty} t_v w^v$. The convolution

$$\frac{(h*g)(w)}{w} = 1 + \sum_{v=2}^{\infty} t_v a_v w^{v-1}, w \in k.$$

From Corollary 1 that $h(w) \in \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta)$ if and only if $\frac{(h*g)(w)}{w} \neq 0$, for g given by (18). Using (18) and (19), we obtain

$$\left|\frac{(f*g)(w)}{w}\right| \ge 1 - \sum_{v=2}^{\infty} \frac{[v]_q - \delta_{v,x} + |[v]_q\beta - \delta_{v,x}\alpha|}{|\beta-\alpha|}|a_v||w|^{v-1} > 0, \; w \in k.$$

Thus, $h(w) \in \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta)$. □

Theorem 5. *Let f be a convex function and let $h(w) \in \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta)$ and satisfies inequality*

$$\sum_{v=2}^{\infty}\left\{\frac{([v]_q-\delta_{v,x})+|\alpha\delta_{v,x}-\beta[n]_q|}{|\alpha-\beta|}\right\}|a_v|<1, \tag{20}$$

then $(h*f) \in \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta)$.

Proof. Let $f(w)=w+\sum_{v=2}^{\infty}b_v w^v$ is a convex and $h(w)=w+\sum_{v=2}^{\infty}a_v w^v \in \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta)$ and satisfies inequality (20), therefore

$$1-\sum_{v=2}^{\infty}\frac{[v]_q-\delta_{v,x}+|[v]_q\beta-\delta_{v,x}\alpha|}{|\beta-\alpha|}|a_v|>0. \tag{21}$$

To prove that $(h*f) \in \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta)$ it is enough to show that $\frac{(h*f*g)(w)}{w} \neq 0$ where g is given by (18). Consider

$$\left|\frac{(h*f*g)(w)}{w}\right| \geq 1-\sum_{v=2}^{\infty}|a_v||b_v||t_v||w|^{v-1}.$$

Since $w \in k$ and g is convex, we obtain $|b_v| \leq 1$. Using (21), we obtain

$$\left|\frac{(h*f*g)(w)}{w}\right| \geq 1-\sum_{v=2}^{\infty}\frac{[v]_q-\delta_{v,x}+|[v]_q\beta-\delta_{v,x}\alpha|}{|\beta-\alpha|}|a_v|>0,\ w\in k.$$

Thus, $h*f \in \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta)$. □

3. Applications

Corollary 2. *Let $h \in \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta)$, and satisfies the inequality (20). Then*

$$F_i(w) \in \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta), (i=1,2,3,4),$$

where

$$F_1(w)=\int_0^w \frac{h(t)}{t}dt, \qquad F_2(w)=\int_0^w \frac{h(t)-h(zt)}{t-zt}dt, |z|\leq 1, z\neq 1,$$
$$F_3(w)=\frac{2}{w}\int_0^w h(t)dt, \qquad F_4(w)=\frac{m+1}{m}\int_0^w t^{m-1}h(t)dt, \Re m>0.$$

Proof. Since

$$F_1(w)=\phi_1(w)*h(w), \qquad \phi_1(w)=\sum_1^{\infty}\frac{1}{v}w^v=\log(1-w)^{-1},$$

$$F_2(w)=\phi_2(w)*h(w), \qquad \phi_2(w)=\sum_1^{\infty}\frac{1-z^v}{v(1-z)}w^v=\frac{1}{1-z}\log(\frac{1-zw}{1-w}), |z|\leq 1, z\neq 1,$$

$$F_3(w)=\phi_3(w)*h(w), \qquad \phi_3(w)=\sum_0^{\infty}\frac{2}{v+1}w^v=\frac{-2[w+\log(1-w)]}{w},$$

$$F_4(w)=\phi_4(w)*h(w), \qquad \phi_4(w)=\sum_0^{\infty}\frac{1+m}{v+m}w^v, \Re\{m\}>0.$$

We note that $\phi_i, i = 1,2,3,4.$ can easily be verified to be convex. Now, using Theorem 5 to obtain $F_i(w) \in \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta), (i = 1,2,3,4)$. $\square$

4. (ρ, q)-Neighborhoods for Functions in the Classes $\widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta)$ and $\widetilde{\mathcal{K}}_q^{x,y}(\alpha,\beta)$

By taking motivation from Definition 2 and to find some neighborhood results for our classes, we introduce the following concepts of neighborhood that analogous to those obtained by Ruscheweyh [20].

Definition 3. *For any $h \in \mathcal{F}$, ρ-neighborhood of function h can be defined as:*

$$\mathcal{N}_{\gamma,\rho}(h) = \left\{ f \in \mathcal{F} : f(w) = w + \sum_{v=2}^{\infty} b_v w^v, \quad \sum_{v=2}^{\infty} \gamma_v |a_v - b_v| \leq \rho \right\}, \ (\rho \geq 0). \tag{22}$$

For $e(w) = w$, we can see that

$$\mathcal{N}_{\gamma,\rho}(e) = \left\{ f \in \mathcal{F} : f(w) = w + \sum_{v=2}^{\infty} b_v w^v, \quad \sum_{v=2}^{\infty} \gamma_v |b_v| \leq \rho \right\}, \ (\rho \geq 0). \tag{23}$$

Remark 3.

1. *For $\gamma_v = v$ of Definition 3 we obtain Definition 2 of the neighborhood concept introduced by Goodman [19] and generalized by Ruscheweyh [20].*
2. *For $\gamma_v = [v]_q$ of Definition 3 we obtain the definition of neighborhood with q-derivative $\mathcal{N}^{\lambda}_{q,\rho}(h), \mathcal{N}^{\lambda}_{q,\rho}(e)$, where $[v]_q$ is given by Equation (7).*
3. *For $\gamma_v = \frac{([v]_q - \delta_{v,x}) + |\alpha\delta_{v,x} - \beta[v]_q|}{|\alpha-\beta|}$ of Definition 3 we obtain the definition of the neighborhood for the classes $\widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta)$ and $\widetilde{\mathcal{K}}_q^{x,y}(\alpha,\beta)$ which is $\mathcal{N}^{x,y}_{q,\rho}(\alpha,\beta;h)$.*

Theorem 6. *Let $h \in \mathcal{F}$, and for all complex number η, with $|\mu| < \rho$, if*

$$\frac{h(w) + \eta w}{1 + \eta} \in \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta). \tag{24}$$

Then

$$\mathcal{N}^{x,y}_{q,\rho}(\alpha,\beta;h) \subset \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta).$$

Proof. We assume that a function f defined by $f(w) = w + \sum_{v=2}^{\infty} b_v w^v$ is in the class $\mathcal{N}^{x,y}_{q,\rho}(\alpha,\beta;h)$. In order to prove the theorem, we only need to prove that $f \in \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta)$. We would prove this claim in next three steps.

From Theorem 3 we have

$$h \in \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta) \Leftrightarrow \frac{1}{w}[(h * g(w))] \neq 0, \quad w \in k, \tag{25}$$

where

$$g(w) = w + \sum_{v=2}^{\infty} \frac{[v]_q - \delta_{v,x} + ([v]_q\beta - \delta_{v,x}\alpha)e^{i\phi}}{(\beta - \alpha)e^{i\phi}} w^n,$$

where $0 \leq \phi < 2\pi, -1 \leq \beta < \alpha \leq 1$. We can write $g(w) = w + \sum_{v=2}^{\infty} t_v w^v$, where t_v is given by (18).

Secondly, we obtain that (24) is equivalent to

$$\left| \frac{h(w) * g(w)}{w} \right| \geq \rho, \tag{26}$$

because, if $h(w) = w + \sum_{v=2}^{\infty} a_v w^v \in \mathcal{F}$ and satisfy (24), then (25) is equivalent to

$$h \in \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta) \Leftrightarrow \frac{1}{w}\left[\frac{h(w)*g(w)}{1+\eta}\right] \neq 0, \qquad |\eta| < \rho.$$

Thirdly, letting $f(w) = w + \sum_{v=2}^{\infty} b_v w^v$ we notice that

$$\begin{aligned}\left|\frac{f(w)*g(w)}{w}\right| &= \left|\frac{h(w)*g(w)}{w} + \frac{(f(w)-h(w))*g(w)}{w}\right| \\ &\geq \rho - \left|\frac{(f(w)-h(w))*g(w)}{w}\right|,\end{aligned}$$

by using (26),

$$\begin{aligned}&= \rho - \left|\sum_{v=2}^{\infty}(b_v - a_v)t_v w^v\right| \\ &\geq \rho - |w|\sum_{v=2}^{\infty}\frac{[v]_q(1+|\beta|)}{|\beta-\alpha|}|b_v - a_v| \\ &\geq \rho - \rho|w| > 0.\end{aligned}$$

This proves that

$$\frac{(f*g)(w)}{w} \neq 0, \qquad w \in k.$$

In view of our observations (25), it follows that $f \in \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta)$. This completes the proof of the theorem. □

When $q \to 1^-, x = y = \alpha = 1$ and $\beta = -1$ in the above theorem we obtain the well-known result proved by Ruscheweyh in [20].

Theorem 7. *Let $h \in \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta)$, for $\rho_1 < c$. Then*

$$\mathcal{N}_{q,\rho_1}^{x,y}(\alpha,\beta;h) \subset \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta).$$

where c is a non-zero real number with $c \leq \left|\frac{(h*g)(w)}{w}\right|, w \in k$ and g is defined in Remark 1.

Proof. Let $f(w) = w + \sum_{v=2}^{\infty} b_v w^v \in \mathcal{N}_{q,\rho_1}^{x,y}(\alpha,\beta;h)$. For the proof of Theorem 7, it suffices to show that $\frac{(f*g)(w)}{w} \neq 0$ where g is given by (18). Consider

$$\left|\frac{f(w)*g(w)}{w}\right| \geq \left|\frac{h(w)*g(w)}{w}\right| - \left|\frac{(f(w)-h(w))*g(w)}{w}\right|. \tag{27}$$

Since $h \in \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta)$, therefore applying Theorem 4, we obtain

$$\left|\frac{(h*g)(w)}{w}\right| \geq c, \tag{28}$$

where c is a non-zero real number and $w \in k$. Now

$$\begin{aligned}\left|\frac{(f(w)-h(w))*g(w)}{w}\right| &= \left|\sum_{v=2}^{\infty}(b_v-a_v)t_v w^v\right| \\ &\leq \sum_{v=2}^{\infty}\frac{([v]_q-\delta_{v,x})+\left|\alpha\delta_{v,x}-\beta[v]_q\right|}{|\alpha-\beta|}|b_v-a_v| \\ &\leq \sum_{v=2}^{\infty}\frac{[v]_q(1+|\beta|)}{|\beta-\alpha|}|b_v-a_v| \\ &\leq \frac{\rho|\beta-\alpha|}{[v]_q(1+|\beta|)} = \rho_1, \end{aligned} \tag{29}$$

using (28) and (29) in (27), we obtain

$$\left|\frac{f(w)*g(w)}{w}\right| \geq c-\rho_1 > 0,$$

where $\rho_1 < c$. This completes the proof. □

Theorem 8. *Let $h \in \widetilde{\mathcal{K}}_q^{x,y}(\alpha,\beta)$, and for all complex number η, with $|\mu| < \frac{1}{4}$, we have*

$$H_\eta(w) = \frac{h(w)+\eta w}{1+\eta} \in \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta). \tag{30}$$

Proof. Let $h \in \widetilde{\mathcal{K}}_q^{x,y}(\alpha,\beta)$, for $\rho_1 < c$. Then

$$\begin{aligned} H_\eta(w) &= \frac{h(w)+\eta w}{1+\eta} \\ &= (h(w)*\psi(w)), w \in k. \end{aligned}$$

where

$$\psi(w) = \frac{w-\frac{\eta}{1+\eta}w^2}{1-w}.$$

Using the principle of convolution we obtain

$$h(w)*\psi(w) = w\partial_q h * \left(\psi(w)*\log\left(\frac{1}{1-w}\right)\right).$$

Since $h \in \widetilde{\mathcal{K}}_q^{x,y}(\alpha,\beta)$, $w\partial_q h \in \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta)$ and for $|\eta| < \frac{1}{4}$, ψ is in the class of starlike functions $\widetilde{\mathcal{S}}$, applying the convolution we obtain

$$\psi(w)*\log\left(\frac{1}{1-w}\right) = \int_0^w \frac{\psi(\zeta)}{\zeta}d_q\zeta. \tag{31}$$

Applying the Alexander relation in (31), we obtain $\psi(w)*\log\left(\frac{1}{1-w}\right)$ is in the class of convex functions $\widetilde{\mathcal{K}}$. Using Lemma 1 one can prove that $\widetilde{\mathcal{K}}*\widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta) \subset \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta)$. Hence

$$H_\eta(w) = w\partial_q h * \left(\psi(w)*\log\left(\frac{1}{1-w}\right)\right) \in \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta), |\eta| < \frac{1}{4}.$$

This completes the proof. □

Theorem 9. *Let $h \in \widetilde{\mathcal{K}}_q^{x,y}(\alpha,\beta)$. Then*

$$\mathcal{N}_{q,\rho}^{x,y}(\alpha,\beta;h) \subset \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta).$$

where $\rho = \frac{|\beta-\alpha|}{4(1+|\beta|)}$.

Proof. Let $h \in \widetilde{\mathcal{K}}_q^{x,y}(\alpha,\beta)$, then by Theorem 8 $H_\varsigma(w) \in \widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta), |\eta| < \frac{1}{4}$. Choosing $\rho = \frac{1}{4}$ and applying Theorem 6, we obtain our required result. □

5. Conclusions

Applications of the q-calculus have been the focal point in the recent times in various mentioned branches of mathematics and physics [11]. In this paper, we have applied the q-calculus for classes of analytic functions with respect to (x,y)-symmetric points. The new classes have been defined and studied. In particular, we have investigated some of its geometric properties such as a convolution conditions for the functions h to be in the classes $\widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta)$ and $\widetilde{\mathcal{K}}_q^{x,y}(\alpha,\beta)$ and a sufficient conditions, application of Pólya–Schoenberg by spatial examples and the neighborhood results related to the functions in the classes $\widetilde{\mathcal{S}}_q^{x,y}(\alpha,\beta)$ and $\widetilde{\mathcal{K}}_q^{x,y}(\alpha,\beta)$. The idea used in this article can easily be implemented to define several subclasses of analytic (odd-even-k-symmetrical) functions connected with different image domains. This will open up a lot of new opportunities for research in this and related fields. The generalized Janowski class and symmetric functions or using symmetric q-derivative operator, basic (or q-) series and basic (or q-) polynomials, especially the basic (or q-) hypergeometric functions and basic (or q-) hypergeometric polynomials are applicable particularly in several diverse areas.

Author Contributions: Methodology, F.A. and S.A.; software, F.A.; formal analysis, F.A. and S.A.; investigation, F.A.; writing original draft preparation, writing—review and editing, F.A.; project administration, F.A. and S.A.; funding acquisition, S.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received funding from Taif University, Researchers Supporting and Project number (TURSP-2020/207), Taif University, Taif, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Janowski, W. Some extremal problems for certain families of analytic functions. *Ann. Pol. Math.* **1973**, *28*, 297–326. [CrossRef]
2. Liczberski, P.; Połubiński, J. On (j,k)-symmetrical functions. *Math. Bohemca* **1995**, *120*, 13–28. [CrossRef]
3. Al-Sarari, F.; Latha, S.; Bulboacă, T. On Janowski functions associated with (n,m)-symmetrical functions. *J. Taibah Univ. Sci.* **2019**, *13*, 972–978. [CrossRef]
4. Al-Sarari, F.; Latha, S.; Frasin, B. A note on starlike functions associated with symmetric points. *Afr. Mat.* **2018**, *24*, 10–18. [CrossRef]
5. Jackson, F.H. On q-functions and a certain difference operator. *Trans. R. Soc. Edinb.* **1909**, *46*, 253–281. [CrossRef]
6. Jackson, F.H. On q-definite integrals. *Q. J. Pure Appl. Math.* **1910**, *41*, 193–203.
7. Srivastava, M.; Tahir, M.; Khan, B.; Ahmad, Z.; Khan, N. Some general classes of q-starlike functions associated with the Janowski functions. *Symmetry* **2019**, *11*, 292. [CrossRef]
8. Naeem, M.; Hussain, S.; Khan, S.; Mahmood, T.; Darus, M.; Shareef, Z. Janowski type q-convex and qclose-to-convex functions associated with q-conic domain. *Mathematics* **2020**, *8*, 440. [CrossRef]
9. Govindaraj, M.; Sivasubramanian, S. On a class of analytic functions related to conic domains involving q-calculus. *Anal. Math.* **2017**, *43*, 475–487. [CrossRef]
10. Khan, B.; Liu, Z.G.; Shaba, T.G.; Araci, S.; Khan, N.; Khan, M.G. Applications of q-Derivative Operator to the Subclass of Bi-Univalent Functions Involving-Chebyshev Polynomials. *J. Math.* **2022**, *2022*, 8162182. [CrossRef]
11. Srivastava, M. Operators of Basic (or q-) Calculus and Fractional q-Calculus and Their Applications in Geometric Function Theory of Complex Analysis. *Iran. Sci. Technol. Trans. Sci.* **2020**, *44*, 327–344. [CrossRef]
12. Al-Sarari, F.; Frasin, B.; AL-Hawary, T.; Latha, S. A few results on generalized Janowski type functions associated with (j,k)-symmetrical functions. *Acta Univ. Sapientiae Math.* **2016**, *8*, 195–205. [CrossRef]
13. Al-Sarari, F.; Latha, S.; Darus, M. A few results on Janowski functions associated with k-symmetric points. *Korean J. Math.* **2017**, *25*, 389–403. [CrossRef]
14. Sakaguchi, K. On a certain univalent mapping. *J. Math. Soc. Jpn.* **1959**, *11*, 72–75. [CrossRef]
15. Agrawal, S.; Sahoo, S.K. A generalization of starlike functions of order alpha. *Hokkaido Math. J.* **2017**, *46*, 15–27. [CrossRef]
16. Mourad, E.; Ismail, H.; Merkes, E.; Styer, D. A generalization of starlike functions. *Complex Var. Theory Appl.* **1990**, *14*, 77–84. [CrossRef]

17. Robertson, M.S. On the theory of univalent functions. *Ann. Math.* **1936**, *37*, 374–408. [CrossRef]
18. Nevanlinna, R. Uber Uber die konforme abbildung sterngebieten. *Over-Sikt Av Fin.-Vetensk. Soc. Forh.* **1920**, *63*, 1–21.
19. Goodman, A.W. Univalent functions and nonanalytic curves. *Proc. Am. Math. Soc.* **1975**, *8*, 598–601. [CrossRef]
20. Ruscheweyh, S. Neighborhoods of univalent functions. *Proc. Am. Math. Soc.* **1981**, *81*, 521–527. [CrossRef]
21. Ruscheweyh, S.; Sheil-Small, T. Hadamard products of Schlicht functions and the Polya-Schoenberg conjecture. *Comment. Math. Helv.* **1979**, *48*, 119–135. [CrossRef]
22. Ganesan, M.; Padmanabhan, K.S. Convolution conditions for certain classes of analytic functions. *Int. J. Pure Appl. Math.* **1984**, *15*, 777–780.
23. Silverman, H.; Silvia, E.M.; Telage, D. Convolution conditions for convexity, starlikeness and spiral-likeness. *Math. Z.* **1978**, *162*, 125–130. [CrossRef]