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Symmetric Toeplitz Matrices for a New Family of Prestarlike Functions

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Abstract: By making use of prestarlike functions, we introduce in this paper a certain family of normalized holomorphic functions defined in the open unit disk, and we establish coefficient estimates for the first four determinants of the symmetric Toeplitz matrices $T_2(2)$, $T_2(3)$, $T_3(2)$ and $T_3(1)$ for the functions belonging to this family. We also mention some known and new results that follow as special cases of our results.

Keywords: prestarlike functions; univalent functions; Toeplitz matrices; coefficient estimates

MSC: 30C45; 30C20

check for updates

Citation: Cotîrlă, L.-I.; Wanas, A.K. Symmetric Toeplitz Matrices for a New Family of Prestarlike Functions. *Symmetry* **2022**, *14*, 1413. https://doi.org/10.3390/sym14071413

Academic Editor: Alexander Zaslavski

Received: 16 June 2022
Accepted: 8 July 2022
Published: 9 July 2022

1. Introduction

Let $\mathcal{A}$ stand for the family of functions f of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are holomorphic in the open unit disk $U = \{z \in C : |z| < 1\}$. Let S indicate the family of all functions in $\mathcal{A}$ that are univalent in U.

Ruscheweyh [1] studied and investigated the family of prestarlike functions of order γ, which is the set of functions f such that $f * I_\gamma$ is a starlike function of order γ, where

$$I_\gamma(z) = \frac{z}{(1-z)^{2(1-\gamma)}} \quad (0 \leqq \gamma < 1;\ z \in U),$$

and $*$ stands the "Hadamard product". The function I_γ can be written in the form:

$$I_\gamma(z) = z + \sum_{n=2}^{\infty} \varphi_n(\gamma) z^n,$$

where

$$\varphi_n(\gamma) = \frac{\prod_{i=2}^{n}(i-2\gamma)}{(n-1)!}, \quad n \geqq 2.$$

We note that $\varphi_n(\gamma)$ is a decreasing function in γ and satisfies

$$\lim_{n\to\infty} \varphi_n(\gamma) = \begin{cases} \infty, & \text{if } \gamma < \frac{1}{2} \\ 1, & \text{if } \gamma = \frac{1}{2} \\ 0, & \text{if } \gamma > \frac{1}{2} \end{cases}.$$

The so-called class of prestarlike functions was further extended and studied by various authors (see [2–5]).

In univalent function theory, extensive focus has been given to estimate the bounds of Hankel matrices. Hankel matrices and determinants play an important role in several branches of mathematics and have many applications [6]. Toeplitz determinants are closely related to Hankel determinants. Hankel matrices have constant entries along the reverse diagonal, whereas Toeplitz matrices have constant entries along the diagonal.

Recently, Thomas and Halim [7] introduced the symmetric Toeplitz determinant $T_q(n)$ for $f \in \mathcal{A}$, defined by

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix},$$

where $n \geqq 1$, $q \geqq 1$ and $a_1 = 1$. In particular,

$$T_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}, \quad T_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_3 \end{vmatrix},$$

and

$$T_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix}, \quad T_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix}.$$

Very recently, several authors established estimates of the Toeplitz determinant $|T_q(n)|$ for functions belonging to various families of univalent functions (see, for example, [7–13]).

In recent years, studies estimating the coefficient bounds for the Toeplitz determinants for the class of univalent functions and its subclasses have been done by several researchers, such as Srivastava et al. (2019) [12], Ramachand and Kavita [11], Al-Khafaji et al. (2020) [14], Radnika et al. (2016, 2018) [9,10], Sivasupramanian et al. (2016) [15], Zhang et al. (2019) [16] and Ali et al. (2018) [17].

Recently, Aleman and Constantin [18] provided a nice connection between univalent function theory and fluid dynamics. They sought explicit solutions to the incompressible two-dimensional Euler equations by means of a univalent harmonic map. More precisely, the problem of finding all solutions describing the particle paths of the flow in Lagrangian variables was reduced to finding harmonic functions satisfying an explicit nonlinear differential system in C^n with $n = 3$ or $n = 4$ (see also [19]).

We need the following results.

Lemma 1 ([20])**.** *If the function $p \in \mathcal{P}$ is given by the series $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$, then the sharp estimate $|p_k| \leqq 2$ $(k = 1, 2, 3, \cdots)$ holds.*

Lemma 2 ([21])**.** *If the function $p \in \mathcal{P}$, then*

$$2p_2 = p_1^2 + \left(4 - p_1^2\right)x$$

$$4p_3 = p_1^3 + 2p_1\left(4 - p_1^2\right)x - p_1\left(4 - p_1^2\right)x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right)z,$$

for some x, z with $|x| \leqq 1$ and $|z| \leqq 1$.

In the next section, we define a new family of holomorphic and prestarlike functions. We denote this family by $\mathcal{W}(\lambda, \gamma)$. For this family, we generate Taylor–Maclaurin coefficient estimates for the coefficients a_2, a_3, a_4 and for the first four determinants of the Toeplitz matrices $T_2(2), T_2(3), T_3(2)$ and $T_3(1)$ for the functions belonging to this newly introduced family.

2. Main Results

We define the family $\mathcal{W}(\lambda,\gamma)$ as follows:

Definition 1. *We say that the family $\mathcal{W}(\lambda,\gamma)$ $(0 \leqq \lambda \leqq 1,\ 0 \leqq \gamma < 1)$ contains all the functions $f \in \mathcal{A}$ if the condition is satisfied:*

$$Re\left\{(1-\lambda)\frac{z(f*I_\gamma)'(z)}{(f*I_\gamma)(z)} + \lambda\left(1+\frac{z(f*I_\gamma)''(z)}{(f*I_\gamma)'(z)}\right)\right\} > 0, \quad (z \in U).$$

Theorem 1. *Let function $f \in \mathcal{W}(\lambda,\gamma)$ be given by relation (1). Then*

$$|a_2| \leqq \frac{1}{(1-\gamma)(\lambda+1)},$$

$$|a_3| \leqq \frac{1}{(1-\gamma)(3-2\gamma)(1+2\lambda)} + \frac{2(1-\gamma)^2(1+3\lambda)}{(-\gamma+1)^3(3-2\gamma)(1+2\lambda)(1+\lambda)^2}$$

and

$$|a_4| \leqq \frac{1}{(-\gamma+2)(1-\gamma)(3-2\gamma)(3\lambda+1)} + \frac{3(1-\gamma)^2(3-2\gamma)(5\lambda+1)}{(-\gamma+1)^3(-\gamma+2)(-2\gamma+3)^2(\lambda+1)(2\lambda+1)(3\lambda+1)} + \frac{12(1-\gamma)^4(3-2\gamma)(5\lambda+1)(3\lambda+1)-(1-\gamma)(3-2\gamma)(2\lambda+1)}{6(1-\gamma)^5(2-\gamma)(3-2\gamma)^2(\lambda+1)^3(2\lambda+1)(3\lambda+1)}.$$

Proof. Let function $f \in \mathcal{W}(\lambda,\gamma)$. Then there exists $p \in \mathcal{P}$ such that

$$(-\lambda+1)\frac{z(f*I_\gamma)'(z)}{(f*I_\gamma)(z)} + \lambda\left(1+\frac{z(f*I_\gamma)''(z)}{(f*I_\gamma)'(z)}\right) = f(z)p(z) \tag{2}$$

where

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$$

By equating the coefficients in (2), we have the relations

$$2(1-\gamma)(\lambda+1)a_2 = p_1, \tag{3}$$

$$2(1-\gamma)(3-2\gamma)(2\lambda+1)a_3 - 4(1-\gamma)^2(3\lambda+1)a_2^2 = p_2 \tag{4}$$

and

$$2(1-\gamma)(2-\gamma)(3-2\gamma)(3\lambda+1)a_4 - 6(1-\gamma)^2(3-2\gamma)(5\lambda+1)a_2a_3 + 8(1-\gamma)^3(7\lambda+1)a_2^3 = p_3. \tag{5}$$

From the relations (3), (4) and (5), we obtain

$$a_2 = \frac{1}{2(1-\gamma)(\lambda+1)}p_1, \tag{6}$$

$$a_3 = \frac{1}{2(1-\gamma)(3-2\gamma)(2\lambda+1)}p_2 + \frac{(1-\gamma)^2(3\lambda+1)}{2(1-\gamma)^3(3-2\gamma)(2\lambda+1)(\lambda+1)^2}p_1^2 \tag{7}$$

and

$$a_4 = \frac{1}{2(1-\gamma)(2-\gamma)(3-2\gamma)(3\lambda+1)} p_3 + \frac{3(1-\gamma)^2(3-2\gamma)(5\lambda+1)}{4(1-\gamma)^3(-\gamma+2)(-2\gamma+3)^2(1+\lambda)(1+2\lambda)(1+3\lambda)} p_1 p_2 + \frac{12(1-\gamma)^4(3-2\gamma)(5\lambda+1)(3\lambda+1)-(1-\gamma)(3-2\gamma)(2\lambda+1)}{48(1-\gamma)^5(2-\gamma)(3-2\gamma)^2(\lambda+1)^3(2\lambda+1)(3\lambda+1)} p_1^3, \quad (8)$$

and by applying Lemma 1, we get

$$|a_2| \leqq \frac{1}{(1-\gamma)(\lambda+1)},$$

$$|a_3| \leqq \frac{1}{(1-\gamma)(3-2\gamma)(1+2\lambda)} + \frac{2(1-\gamma)^2(1+3\lambda)}{(1-\gamma)^3(3-2\gamma)(1+2\lambda)(1+\lambda)^2}$$

and

$$|a_4| \leqq \frac{1}{(-\gamma+1)(-\gamma+2)(-2\gamma+3)(3\lambda+1)} + \frac{3(-\gamma+1)^2(-2\gamma+3)(5\lambda+1)}{(1-\gamma)^3(-\gamma+2)(-2\gamma+3)^2(3\lambda+1)(2\lambda+1)(\lambda+1)} + \frac{12(1-\gamma)^4(3-2\gamma)(5\lambda+1)(3\lambda+1)-(1-\gamma)(3-2\gamma)(2\lambda+1)}{6(1-\gamma)^5(2-\gamma)(3-2\gamma)^2(\lambda+1)^3(2\lambda+1)(3\lambda+1)}.$$

□

Theorem 2. *Let $f \in \mathcal{W}(\lambda,\gamma)$ be given by (1). Then*

$$|T_2(2)| \leqq \frac{2\left[(\lambda+1)^4+4(\lambda+1)^2(3\lambda+1)+4(3\lambda+1)^2\right]}{(1-\gamma)^2(3-2\gamma)^2(1+\lambda)^4(2\lambda+1)^2} - \frac{1}{(\lambda+1)^2(1-\gamma)^2}. \quad (9)$$

Proof. In view of (6) and (7), it easy to see that

$$|T_2(2)| = \left|a_3^2-a_2^2\right| = \left|\frac{p_2^2}{4(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2} + \frac{8(3\lambda+1)p_1^2p_2}{16(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2(\lambda+1)^2} - \frac{(3\lambda+1)^2p_1^4}{4(1-\gamma)^4(3-2\gamma)^2(2\lambda+1)^2(\lambda+1)^4} - \frac{p_1^2}{4(1-\gamma)^2(\lambda+1)^2}\right|.$$

By applying Lemma 2 to express p_2 in terms p_1, it follows that

$$\left|a_3^2-a_2^2\right| = \left|\frac{\left[(\lambda+1)^4+4(\lambda+1)^2(3\lambda+1)+4(3\lambda+1)^2\right]p_1^4}{8(1-\gamma)^2(3-2\gamma)^2(\lambda+1)^4(2\lambda+1)^2} - \frac{p_1^2}{4(1-\gamma)^2(\lambda+1)^2} + \frac{((\lambda+1)^2+2(3\lambda+1))p_1^2x(4-p_1^2)}{8(1-\gamma)^2(3-2\gamma)^2(\lambda+1)^2(2\lambda+1)^2} + \frac{x^2(4-p_1^2)^2}{16(-\gamma+1)^2(-2\gamma+3)^2(2\lambda+1)^2}\right|.$$

For convenience of notation, we choose $p_1 = p$, and since p is in the family $\mathcal{P}$ simultaneously, we can suppose without loss of generality that $p \in [0,2]$. Thus, by applying the triangle inequality with $P = 4-p^2$, we deduce that

$$\left|a_3^2-a_2^2\right| \leqq \left|\frac{\left[(\lambda+1)^4+4(\lambda+1)^2(3\lambda+1)+4(3\lambda+1)^2\right]p^4}{8(3-2\gamma)^2(1-\gamma)^2(1+\lambda)^4(2\lambda+1)^2}-\frac{p^2}{4(1-\gamma)^2(1+\lambda)^2}\right|$$
$$+\frac{((\lambda+1)^2+2(3\lambda+1))c^2|x|P}{8(-\gamma+1)^2(3-2\gamma)^2(\lambda+1)^2(2\lambda+1)^2}+\frac{|x|^2P^2}{16(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2}=:F(p,|x|).$$

It is obvious that $F'(p,|x|)>0$ on $[0,1]$, and thus $F(p,|x|)\leqq F(p,|1|)$. Trivially, when $p=2$, we note that the expression $F(|x|)$ has a maximum value on $[0,2]$. Consequently

$$|T_2(2)|=\left|a_3^2-a_2^2\right| \leqq \frac{2\left[(\lambda+1)^4+4(3\lambda+1)(\lambda+1)^2+4(1+3\lambda)^2\right]}{(1-\gamma)^2(-2\gamma+3)^2(1+\lambda)^4(2\lambda+1)^2}-\frac{1}{(1-\gamma)^2(1+\lambda)^2}.$$

This concludes the proof. □

Remark 1. *Choosing $\gamma=\frac{1}{2}$ and $\lambda=0$ in Theorem 2 gives the result in Theorem 1, which was investigated by Thomas and Halim [7].*

Theorem 3. *Let $f\in\mathcal{W}(\lambda,\gamma)$ be given by (1). Then*

$$|T_2(3)|=|a_4^2-a_3^2| \leqq \frac{\Upsilon_1}{(1-\gamma)^2(2-\gamma)^2(3-2\gamma)^2(3\lambda+1)^2}-\frac{2\left[(\lambda+1)^4+4(\lambda+1)^2(3\lambda+1)+4(3\lambda+1)^2\right]}{(1-\gamma)^2(-2\gamma+3)^2(1+\lambda)^4(1+2\lambda)^2},$$

where

$$\Upsilon_1=1+\frac{6(5\lambda+1)}{(\lambda+1)(2\lambda+1)}+\frac{12(1-\gamma)^3(5\lambda+1)(3\lambda+1)-2\lambda-1}{3(1-\gamma)^3(\lambda+1)^3(2\lambda+1)}$$
$$+\frac{9(5\lambda+1)^2}{(2\lambda+1)^2(\lambda+1)^2}+\frac{\left[12(1-\gamma)^3(5\lambda+1)(3\lambda+1)-2\lambda-1\right]^2}{36(1-\gamma)^6(2\lambda+1)^2(\lambda+1)^6}$$
$$+\frac{(5\lambda+1)\left[12(1-\gamma)^3(5\lambda+1)(1+3\lambda)-1-2\lambda\right]}{(1-\gamma)^3(2\lambda+1)^2(\lambda+1)^4}.$$

Proof. Applying (7), (8) and using Lemma 2, we have

$$|a_4^2-a_3^2|=\left|\frac{\left[(\lambda+1)^4+4(\lambda+1)^2(3\lambda+1)+4(3\lambda+1)^2\right]p_1^4}{8(1-\gamma)^2(3-2\gamma)^2(\lambda+1)^4(2\lambda+1)^2}+\frac{\Upsilon_1 p_1^6}{64(1-\gamma)^2(2-\gamma)^2(3-2\gamma)^2(3\lambda+1)^2}\right.$$
$$-\frac{\left[(\lambda+1)^2+2(3\lambda+1)\right]p_1^2x(4-p_1^2)}{8(1-\gamma)^2(3-2\gamma)^2(\lambda+1)^2(2\lambda+1)^2}+\frac{\Upsilon_2xp_1^4(4-p_1^2)}{16(-\gamma+1)^2(2-\gamma)^2(3-2\gamma)^2(3\lambda+1)^2}$$
$$-\frac{\left[6(1-\gamma)^3(2\lambda+1)(\lambda+1)^3+6(1-\gamma)^3(5\lambda+1)\left(3(\lambda+1)^2+2(3\lambda+1)\right)-2\lambda-1\right]p_1^4(4-p_1^2)x^2}{192(1-\gamma)^5(2-\gamma)^2(3-2\gamma)^2(3\lambda+1)^2(2\lambda+1)(\lambda+1)^3}$$
$$-\frac{x^2(4-p_1^2)^2}{16(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2}+\frac{\left[4(2\lambda+1)(\lambda+1)((2\lambda+1)(\lambda+1)+3(5\lambda+1))+9(5\lambda+1)^2\right]p_1^2(4-p_1^2)^2x^2}{64(1-\gamma)^2(-\gamma+2)^2(3-2\gamma)^2(3\lambda+1)^2(2\lambda+1)^2(\lambda+1)^2}$$

$$
- \frac{(2(2\lambda+1)(\lambda+1)+3(5\lambda+1))p_1^2(4-p_1^2)^2x^3}{32(1-\gamma)^2(-2\gamma+3)^2(2-\gamma)^2(3\lambda+1)^2(2\lambda+1)(\lambda+1)} + \frac{p_1^2(4-p_1^2)^2x^4}{64(1-\gamma)^2(-\gamma+2)^2(3-2\gamma)^2(3\lambda+1)^2}
$$
$$
+ \frac{\left[6(1-\gamma)^3(2\lambda+1)(\lambda+1)^3+6(1-\gamma)^3(5\lambda+1)\Big(3(1+\lambda)^2+2(1+3\lambda)\Big)-1-2\lambda\right]p_1^3(4-p_1^2)(1-|x|^2)z}{96(1-\gamma)^5(2-\gamma)^2(-2\gamma+3)^2(1+3\lambda)^2(1+2\lambda)(1+\lambda)^3}
$$
$$
+ \frac{(2(1+2\lambda)(\lambda+1)+3(5\lambda+1))p_1(4-p_1^2)^2(1-|x|^2)xz}{16(1-\gamma)^2(2-\gamma)^2(3-2\gamma)^2(3\lambda+1)^2(2\lambda+1)(\lambda+1)}
$$
$$
- \frac{p_1(4-p_1^2)^2(1-|x|^2)x^2z}{16(1-\gamma)^2(-\gamma+2)^2(3-2\gamma)^2(3\lambda+1)^2} + \frac{(4-p_1^2)^2(1-|x|^2)^2z^2}{16(1-\gamma)^2(-\gamma+2)^2(3-2\gamma)^2(3\lambda+1)^2}\Bigg|,
$$

where

$$
Y_2 = 1 + \frac{9(5\lambda+1)}{2(\lambda+1)(2\lambda+1)} + \frac{12(1-\gamma)^3(5\lambda+1)(3\lambda+1)-2\lambda-1}{6(1-\gamma)^3(\lambda+1)^3(2\lambda+1)}
+ \frac{9(5\lambda+1)^2}{2(2\lambda+1)^2(\lambda+1)^2} + \frac{(5\lambda+1)\left[12(1-\gamma)^3(1+5\lambda)(3\lambda+1)-2\lambda-1\right]}{8(1-\gamma)^3(1+2\lambda)^2(1+\lambda)^4}.
$$

We select $p_1 = p$ for ease of notation, and because the function p is in the family $\mathcal{P}$ at the same time, we may assume that $p \in [0,2]$ without losing generality. As a result, using the triangle inequality with $P = 4 - p^2$ and $Z = (1-|x|^2)z$, we may conclude

$$
|a_4^2 - a_3^2| = \left|\frac{Y_1 p^6}{64(1-\gamma)^2(2-\gamma)^2(3-2\gamma)^2(3\lambda+1)^2} - \frac{\left[(\lambda+1)^4+4(\lambda+1)^2(3\lambda+1)+4(3\lambda+1)^2\right]p^4}{8(1-\gamma)^2(3-2\gamma)^2(\lambda+1)^4(2\lambda+1)^2}\right|
$$
$$
+ \frac{\left[(\lambda+1)^2+2(3\lambda+1)\right]p^2|x|P}{8(1-\gamma)^2(3-2\gamma)^2(\lambda+1)^2(2\lambda+1)^2} + \frac{Y_2p^4P|x|}{16(1-\gamma)^2(2-\gamma)^2(3-2\gamma)^2(3\lambda+1)^2}
$$
$$
+ \frac{\left[6(1-\gamma)^3(2\lambda+1)(\lambda+1)^3+6(1-\gamma)^3(5\lambda+1)\Big(3(\lambda+1)^2+2(3\lambda+1)\Big)-2\lambda-1\right]p^4P|x|^2}{192(1-\gamma)^5(2-\gamma)^2(3-2\gamma)^2(3\lambda+1)^2(2\lambda+1)(\lambda+1)^3}
$$
$$
+ \frac{|x|^2P^2}{16(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2} + \frac{\left[4(\lambda+1)(1+2\lambda)((2\lambda+1)(\lambda+1)+3(1+5\lambda))+9(1+5\lambda)^2\right]p^2P^2|x|^2}{64(1-\gamma)^2(2-\gamma)^2(-2\gamma+3)^2(1+3\lambda)^2(1+2\lambda)^2(1+\lambda)^2}
$$
$$
+ \frac{(2(1+\lambda)(1+2\lambda)+3(5\lambda+1))p^2P^2|x|^3}{32(1-\gamma)^2(2-\gamma)^2(3-2\gamma)^2(3\lambda+1)^2(2\lambda+1)(\lambda+1)} + \frac{p^2P^2|x|^4}{64(1-\gamma)^2(2-\gamma)^2(3-2\gamma)^2(3\lambda+1)^2}
$$
$$
+ \frac{\left[6(1-\gamma)^3(2\lambda+1)(\lambda+1)^3+6(1-\gamma)^3(5\lambda+1)\Big(3(\lambda+1)^2+2(3\lambda+1)\Big)-2\lambda-1\right]p^3PZ}{96(1-\gamma)^5(2-\gamma)^2(3-2\gamma)^2(3\lambda+1)^2(2\lambda+1)(\lambda+1)^3}
$$
$$
+ \frac{(2(2\lambda+1)(\lambda+1)+3(5\lambda+1))p|x|P^2Z}{16(1-\gamma)^2(2-\gamma)^2(3-2\gamma)^2(3\lambda+1)^2(2\lambda+1)(\lambda+1)}
$$
$$
+ \frac{p|x|^2P^2Z}{16(1-\gamma)^2(2-\gamma)^2(3-2\gamma)^2(3\lambda+1)^2} + \frac{P^2Z^2}{16(1-\gamma)^2(2-\gamma)^2(3-2\gamma)^2(3\lambda+1)^2} =: F_1(p,|x|).
$$

Using elementary calculus to differentiate $F_1(p,|x|)$ with respect to $|x|$, we have

$$
\begin{aligned}
\frac{\partial F_1(p,|x|)}{\partial |x|} &= \frac{\left[(\lambda+1)^2+2(3\lambda+1)\right]p^2(4-p^2)}{8(1-\gamma)^2(3-2\gamma)^2(\lambda+1)^2(2\lambda+1)^2} + \frac{\Upsilon_2(4-p^2)p^4}{16(1-\gamma)^2(-\gamma+2)^2(3-2\gamma)^2(3\lambda+1)^2} \\
&- \frac{\left[6(1-\gamma)^3(2\lambda+1)(\lambda+1)^3+6(1-\gamma)^3(5\lambda+1)\left(3(\lambda+1)^2+2(3\lambda+1)\right)-2\lambda-1\right]p^3(4-p^2)|x|}{48(1-\gamma)^5(2-\gamma)^2(3-2\gamma)^2(3\lambda+1)^2(2\lambda+1)(\lambda+1)^3} \\
&+ \frac{\left[6(1-\gamma)^3(2\lambda+1)(\lambda+1)^3+6(1-\gamma)^3(5\lambda+1)\left(3(\lambda+1)^2+2(1+3\lambda)\right)-2\lambda-1\right]p^4(4-p^2)|x|}{96(1-\gamma)^5(2-\gamma)^2(3-2\gamma)^2(3\lambda+1)^2(1+2\lambda)(1+\lambda)^3} \\
&+ \frac{\left[4(1+2\lambda)(1+\lambda)((1+2\lambda)(1+\lambda)+3(1+5\lambda))+9(1+5\lambda)^2\right]p^2(4-p^2)^2|x|}{32(1-\gamma)^2(3-2\gamma)^2(-\gamma+2)^2(3\lambda+1)^2(2\lambda+1)^2(\lambda+1)^2} \\
&- \frac{p(2(2\lambda+1)(\lambda+1)+3(5\lambda+1))(4-p^2)^2|x|^2}{8(1-\gamma)^2(2-\gamma)^2(3-2\gamma)^2(3\lambda+1)^2(2\lambda+1)(\lambda+1)} \\
&+ \frac{3(2(2\lambda+1)(\lambda+1)+3(5\lambda+1))p^2(4-p^2)^2|x|^2}{32(1-\gamma)^2(2-\gamma)^2(3-2\gamma)^2(1+3\lambda)^2(1+2\lambda)(1+\lambda)} \\
&- \frac{p(4-p^2)^2|x|^3}{4(1-\gamma)^2(2-\gamma)^2(3-2\gamma)^2(3\lambda+1)^2} + \frac{p^2(4-p^2)^2|x|^3}{16(1-\gamma)^2(2-\gamma)^2(3-2\gamma)^2(3\lambda+1)^2} \\
&+ \frac{(2(2\lambda+1)(\lambda+1)+3(5\lambda+1))p(4-p^2)^2(1-|x|^2)}{16(1-\gamma)^2(2-\gamma)^2(3-2\gamma)^2(3\lambda+1)^2(2\lambda+1)(\lambda+1)} \\
&- \frac{|x|(4-p^2)^2(1-|x|^2)}{4(-\gamma+2)^2(1-\gamma)^2(3-2\gamma)^2(1+3\lambda)^2} \\
&+ \frac{p|x|(4-p^2)^2(1-|x|^2)}{8(1-\gamma)^2(2-\gamma)^2(3-2\gamma)^2(3\lambda+1)^2}.
\end{aligned}
$$

It is shown that $(\partial F_1(p,|x|)/\partial |x|) \geqq 0$ for $|x| \in [0,1]$ and fixed $p \in [0,2]$. As a result, $F_1(p,|x|)$ is an increasing function of $|x|$. So, $F_1(p,|x|) \leq F_1(p,|1|)$. Therefore,

$$
\begin{aligned}
|a_4^2 - a_3^2| \leqq& \left| \frac{\Upsilon_1 p^6}{64(-\gamma+2)^2(1-\gamma)^2(3-2\gamma)^2(3\lambda+1)^2} - \frac{\left[(\lambda+1)^4+4(3\lambda+1)(\lambda+1)^2+4(1+3\lambda)^2\right]p^4}{8(1-\gamma)^2(3-2\gamma)^2(\lambda+1)^4(2\lambda+1)^2} \right| \\
&+ \frac{\left[(\lambda+1)^2+2(3\lambda+1)\right]p^2(4-p^2)}{8(1-\gamma)^2(3-2\gamma)^2(\lambda+1)^2(2\lambda+1)^2} + \frac{(4-p^2)^2}{16(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2} \\
&+ \frac{\left[12\Upsilon_2(-\gamma+1)^3(2\lambda+1)(\lambda+1)^3+6(1-\gamma)^3(2\lambda+1)(\lambda+1)^3+6(1-\gamma)^3(1+5\lambda)\left(3(1+\lambda)^2+2(3\lambda+1)\right)-2\lambda-1\right]}{192(1-\gamma)^5(2-\gamma)^2(3-2\gamma)^2(3\lambda+1)^2(1+2\lambda)(1+\lambda)^3} \cdot \\
&\cdot [p^4(4-p^2)] + \\
&+ \frac{[4(1+2\lambda)(1+\lambda)((1+2\lambda)(1+\lambda)+3(5\lambda+1))][p^2(4-p^2)^2]}{64(1-\gamma)^2(2-\gamma)^2(3-2\gamma)^2(3\lambda+1)^2(2\lambda+1)^2(\lambda+1)^2} + \\
&+ \frac{[9(5\lambda+1)^2+2[2(2\lambda+1)(\lambda+1)+3(5\lambda+1)](2\lambda+1)(\lambda+1)+(2\lambda+1)^2(\lambda+1)^2][p^2(4-p^2)^2]}{64(2-\gamma)^2(-\gamma+1)^2(3-2\gamma)^2(3\lambda+1)^2(2\lambda+1)^2(\lambda+1)^2}.
\end{aligned}
$$

Now, on $[0,2]$ at $P=2$, we have

$$
\begin{aligned}
|a_4^2 - a_3^2| \leqq& \frac{\Upsilon_1}{(1-\gamma)^2(2-\gamma)^2(3-2\gamma)^2(3\lambda+1)^2} \\
&- \frac{2\left[(\lambda+1)^4+4(\lambda+1)^2(3\lambda+1)+4(3\lambda+1)^2\right]}{(1-\gamma)^2(-2\gamma+3)^2(1+\lambda)^4(1+2\lambda)^2}.
\end{aligned}
$$

□

Remark 2. *Choosing $\gamma = \frac{1}{2}$ and $\lambda = 0$ in Theorem 3 gives the result in Theorem 2, which was investigated by Thomas and Halim [7].*

Theorem 4. *Let $f \in \mathcal{W}(\lambda, \gamma)$ be given by (1). Then*

$$|T_3(2)| = |(a_2 - a_4)(a_2^2 - 2a_3^2 + a_2a_4)| \leqq \left[\frac{1}{(1-\gamma)(\lambda+1)} - \frac{\left(3(1-\gamma)^2(\lambda+1)^2(2\lambda+1) + 9(1-\gamma)^2(\lambda+1)(5\lambda+1) + \left[12(1-\gamma)^3(5\lambda+1)(3\lambda+1) - 2\lambda - 1\right]\right)p^3}{3(1-\gamma)^3(2-\gamma)(3-2\gamma)(3\lambda+1)(2\lambda+1)(\lambda+1)^2}\right] \left[\frac{1}{(1-\gamma)^2(\lambda+1)^2} - \frac{Y_3}{(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2}\right], \quad (10)$$

where

$$Y_3 = 2 + \frac{8(3\lambda+1)}{(\lambda+1)^2} + \frac{8(3\lambda+1)^2}{(\lambda+1)^4} - \frac{(3-2\gamma)(2\lambda+1)^2}{(2-\gamma)(3\lambda+1)(\lambda+1)} - \frac{6(1-\gamma)(3-2\gamma)^2(2\lambda+1)^2(5\lambda+1)}{(-\gamma+2)(1+3\lambda)(1+\lambda)^2} - \frac{(3-2\gamma)(2\lambda+1)\left[12(1-\gamma)^3(5\lambda+1)(1+3\lambda) - 1 - 2\lambda\right]}{6(1-\gamma)^3(2-\gamma)(3\lambda+1)(\lambda+1)^4}$$

and

$$Y_4 = 4(2-\gamma)(3-2\gamma)(3\lambda+1)[(\lambda+1)^2 + 2(3\lambda+1)] - 2(3-2\gamma)^2(\lambda+1)(2\lambda+1)^2 - 3(3-2\gamma)^2(5\lambda+1)(2\lambda+1).$$

Proof. From (6), (8) and applying Lemma 2, we have

$$|a_2 - a_4| = \left|\frac{p_1}{2(1-\gamma)(\lambda+1)} - \frac{p_1^3}{8(1-\gamma)(2-\gamma)(3-2\gamma)(3\lambda+1)} - \frac{p_1(4-p_1^2)x}{4(1-\gamma)(2-\gamma)(3-2\gamma)(3\lambda+1)} + \frac{p_1(4-p_1^2)x^2}{8(1-\gamma)(2-\gamma)(3-2\gamma)(3\lambda+1)} - \frac{(4-p_1^2)(1-|x|^2)z}{4(1-\gamma)(2-\gamma)(3-2\gamma)(3\lambda+1)} - \frac{3(5\lambda+1)p_1^3}{8(1-\gamma)(2-\gamma)(3-2\gamma)(3\lambda+1)(2\lambda+1)(\lambda+1)} - \frac{3(5\lambda+1)p_1(4-p_1^2)x}{8(1-\gamma)(2-\gamma)(3-2\gamma)(3\lambda+1)(2\lambda+1)(\lambda+1)} - \frac{\left[12(1-\gamma)^3(5\lambda+1)(1+3\lambda) - 2\lambda - 1\right]p_1^3}{24(1-\gamma)^3(2-\gamma)(-2\gamma+3)(3\lambda+1)(2\lambda+1)(1+\lambda)^2}\right|.$$

Applying triangle inequality and $p_1 = p$, we have

$$|a_2 - a_4| \leqq \left|\frac{p}{2(1-\gamma)(\lambda+1)} - \frac{\left(3(1-\gamma)^2(\lambda+1)^2(2\lambda+1) + 9(1-\gamma)^2(\lambda+1)(5\lambda+1) + \left[12(1-\gamma)^3(5\lambda+1)(3\lambda+1) - 2\lambda - 1\right]\right)p^3}{24(1-\gamma)^3(2-\gamma)(3-2\gamma)(3\lambda+1)(2\lambda+1)(\lambda+1)^2}\right| + \frac{p(2(2\lambda+1)(\lambda+1) + 3(5\lambda+1))|x|P}{2(2\lambda+1)(\lambda+1)} + \frac{p|x|^2P}{8(1-\gamma)(2-\gamma)(3-2\gamma)(3\lambda+1)} + \frac{PZ}{4(1+3\lambda)(2-\gamma)(1-\gamma)(-2\gamma+3)} + \frac{3(5\lambda+1)p|x|P}{8(-\gamma+2)(1-\gamma)(3-2\gamma)(3\lambda+1)(2\lambda+1)(\lambda+1)}.$$

Using the same methods as Theorems 2 and 3, we have

$$|a_2 - a_4| \leqq \frac{1}{(1-\gamma)(\lambda+1)} - \frac{\left(3(1-\gamma)^2(\lambda+1)^2(2\lambda+1) + 9(1-\gamma)^2(\lambda+1)(5\lambda+1) + \left[12(1-\gamma)^3(5\lambda+1)(3\lambda+1) - 2\lambda - 1\right]\right)p^3}{3(1-\gamma)^3(2-\gamma)(3-2\gamma)(3\lambda+1)(2\lambda+1)(\lambda+1)^2}. \quad (11)$$

Further, using (6), (7), (8) and applying Lemma 2 and taking $p_1 = p \in [0,2]$, we have

$$|a_2^2 - 2a_3^2 + a_2a_4| \leqq \left|\frac{p^2}{4(1-\gamma)^2(\lambda+1)^2} - \frac{\Upsilon_3 p^4}{16(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2}\right| + \frac{\Upsilon_4 p^2(4-p^2)|x|}{16(1-\gamma)(2-\gamma)(3-2\gamma)^3(1+3\lambda)(1+2\lambda)^2(1+\lambda)^2} + \frac{p^2(4-p^2)|x|^2}{16(-\gamma+1)^2(2-\gamma)(3-2\gamma)(1+3\lambda)(1+\lambda)} + \frac{(4-p^2)^2|x|^2}{8(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2} + \frac{p(4-p^2)(1-|x|^2)}{8(1-\gamma)^2(2-\gamma)(3-2\gamma)(3\lambda+1)(\lambda+1)} := F_2(p,|x|).$$

On the closed area $[0,2] \times [0,1]$, we need to find the maximum value of $F_2(p,|x|)$. Assume that a maximum of $[0,2] \times [0,1]$ exists at an interior point $(p_0,|x|)$. After that, by differentiating $F_2(p,|x|)$ with respect to $|x|$, we have

$$\frac{\partial F_2(p,|x|)}{\partial |x|} = \frac{\Upsilon_4 p^2(4-p^2)}{16(1-\gamma)(2-\gamma)(3-2\gamma)^3(3\lambda+1)(2\lambda+1)^2(\lambda+1)^2} + \frac{p^2(4-p^2)|x|}{8(1-\gamma)^2(2-\gamma)(3-2\gamma)(3\lambda+1)(\lambda+1)} + \frac{(4-p^2)^2|x|}{4(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2} - \frac{p(4-p^2)|x|)}{4(1-\gamma)^2(2-\gamma)(3-2\gamma)(3\lambda+1)(\lambda+1)}.$$

If $p = 0$,

$$F_2(0,|x|) = \frac{2}{(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2}|x|^2 \leqq \frac{2}{(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2}$$

If $p = 2$,

$$F_2(2,|x|) = \frac{1}{(1-\gamma)^2(\lambda+1)^2} - \frac{\Upsilon_3}{(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2}.$$

If $|x| = 0$,

$$F_2(p,0) = \left|\frac{p^2}{4(1-\gamma)^2(\lambda+1)^2} - \frac{\Upsilon_3 p^4}{16(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2}\right| + \frac{p(4-p^2)}{8(1-\gamma)^2(2-\gamma)(3-2\gamma)(3\lambda+1)(\lambda+1)},$$

which has the highest possible value

$$\frac{1}{(1-\gamma)^2(\lambda+1)^2} - \frac{\Upsilon_3}{(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2}$$

on $[0,2]$. Further, if $|x| = 1$, we have

$$F_2(p,1) = \left| \frac{p^2}{4(1-\gamma)^2(\lambda+1)^2} - \frac{\Upsilon_3 p^4}{16(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2} \right| + \frac{\Upsilon_4 p^2(4-p^2)}{16(1-\gamma)(2-\gamma)(3-2\gamma)^3(3\lambda+1)(2\lambda+1)^2(\lambda+1)^2} + \frac{p^2(4-p^2)}{16(1-\gamma)^2(2-\gamma)(3-2\gamma)(3\lambda+1)(\lambda+1)} + \frac{(4-p^2)^2}{8(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2},$$

which has the highest possible value

$$\frac{1}{(1-\gamma)^2(\lambda+1)^2} - \frac{\Upsilon_3}{(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2}$$

on $[0,2]$. So,

$$|T_3(2)| = |(a_2-a_4)(a_2^2-2a_3^2+a_2a_4)| \leqq \left[\frac{1}{(1-\gamma)(\lambda+1)} - \frac{\Big(3(1-\gamma)^2(\lambda+1)^2(2\lambda+1) + 9(1-\gamma)^2(\lambda+1)(5\lambda+1) + \Big[12(1-\gamma)^3(5\lambda+1)(3\lambda+1) - 2\lambda - 1\Big]\Big)p^3}{3(1-\gamma)^3(2-\gamma)(3-2\gamma)(3\lambda+1)(2\lambda+1)(\lambda+1)^2}\right] \left[\frac{1}{(1-\gamma)^2(\lambda+1)^2} - \frac{\Upsilon_3}{(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2}\right].$$

□

Remark 3. *Choosing $\gamma = \frac{1}{2}$ and $\lambda = 0$ in Theorem 4 gives the result in Theorem 3, which was investigated by Thomas and Halim [7].*

Theorem 5. *Let $f \in \mathcal{W}(\lambda,\gamma)$ be given by (1). Then*

$$\begin{aligned} |T_3(1)| &= |1 + 2a_2^2(a_3-1) - a_3^2| \\ &\leqq 1 + \frac{4}{(1-\gamma)^2(3-2\gamma)(2\lambda+1)(\lambda+1)} + \frac{4(3\lambda+1)}{(1-\gamma)^3(3-2\gamma)(2\lambda+1)(\lambda+1)^4} \\ &\quad - \frac{1}{(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2} - \frac{4(3\lambda+1)}{(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2(\lambda+1)^2} \\ &\quad - \frac{4(3\lambda+1)^2}{(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2(\lambda+1)^4} - \frac{2}{(1-\gamma)^2(\lambda+1)^2}. \end{aligned} \tag{12}$$

Proof. From (6), (7) and applying Lemma 2 and some calculations, we have

$$\begin{aligned} |T_3(1)| = \Bigg| 1 &+ \frac{p_1^4}{4(1-\gamma)^2(3-2\gamma)(2\lambda+1)(\lambda+1)} + \frac{p_1^2 x(4-p_1^2)}{4(1-\gamma)^2(3-2\gamma)(2\lambda+1)(\lambda+1)} \\ &+ \frac{(3\lambda+1)p_1^4}{16(1-\gamma)^3(3-2\gamma)(2\lambda+1)(\lambda+1)^4} - \frac{p_1^2}{2(1-\gamma)^2(\lambda+1)^2} \\ &- \frac{\Big[(\lambda+1)^4 + 4(3\lambda+1)(\lambda+1)^2 + 4(3\lambda+1)^2\Big]p_1^4}{16(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2(\lambda+1)^4} \\ &- \frac{\Big[(\lambda+1)^2 + 2(3\lambda+1)\Big]p_1^2 x(4-p_1^2)}{8(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2(\lambda+1)^2} - \frac{x^2(4-p_1^2)^2}{16(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2} \Bigg|. \end{aligned}$$

We select $p_1 = p$ for ease of notation, and because the function p is in the family $\mathcal{P}$ at the same time, we may assume that $p \in [0,2]$ without losing generality. As a result, using the triangle inequality with $P = 4 - r^2$, we have

$$|T_3(1)| \leqq \Bigg|1 + \Bigg[\frac{1}{4(1-\gamma)^2(3-2\gamma)(2\lambda+1)(\lambda+1)} + \frac{(3\lambda+1)}{4(1-\gamma)^3(3-2\gamma)(2\lambda+1)(\lambda+1)^4} - \frac{1}{16(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2} - \frac{(3\lambda+1)}{4(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2(\lambda+1)^2} - \frac{(3\lambda+1)^2}{4(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2(\lambda+1)^4}\Bigg]p^4 - \frac{2p^2}{4(1-\gamma)^2(\lambda+1)^2}\Bigg| + \frac{\left[(\lambda+1)^2 + 2(3\lambda+1)\right]p^2(4-p^2)}{8(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2(\lambda+1)^2} + \frac{(4-p^2)^2}{16(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2}.$$

Hence, at $p = 2$, we have

$$|T_3(1)| \leqq 1 + \frac{4}{(1-\gamma)^2(3-2\gamma)(2\lambda+1)(\lambda+1)} + \frac{4(3\lambda+1)}{(1-\gamma)^3(3-2\gamma)(2\lambda+1)(\lambda+1)^4} - \frac{1}{(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2} - \frac{4(3\lambda+1)}{(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2(\lambda+1)^2} - \frac{4(3\lambda+1)^2}{(1-\gamma)^2(3-2\gamma)^2(2\lambda+1)^2(\lambda+1)^4} - \frac{2}{(1-\gamma)^2(\lambda+1)^2}. \quad (13)$$

□

Remark 4. *Choosing $\gamma = \frac{1}{2}$ and $\lambda = 0$ in Theorem 5 gives the result in Theorem 4, which was investigated by Thomas and Halim [7].*

3. Conclusions

The objective of this paper was to create a new family $\mathcal{W}(\lambda, \gamma)$ of holomorphic and prestarlike functions. We generate Taylor–Maclaurin coefficient estimates for the first four determinants of the Toeplitz matrices $T_2(2)$, $T_2(3)$, $T_3(2)$ and $T_3(1)$ for the functions belonging to this newly introduced family.

Author Contributions: Conceptualization, A.K.W. and L.-I.C.; methodology, A.K.W. and L.-I.C.; software, A.K.W. and L.-I.C.; validation, A.K.W. and L.-I.C.; formal analysis, A.K.W. and L.-I.C.; investigation, A.K.W. and L.-I.C.; resources, A.K.W. and L.-I.C.; data curation, A.K.W. and L.-I.C.; writing—original draft preparation, A.K.W. and L.-I.C.; writing—review and editing, A.K.W. and L.-I.C.; visualization, A.K.W. and L.-I.C.; supervision, A.K.W. and L.-I.C.; project administration, A.K.W. and L.-I.C.; funding acquisition, A.K.W. and L.-I.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank the referees for their careful reading and helpful comments.

Conflicts of Interest: The authors declare no conflict of interest in this paper.

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