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Closed-Form Derivations of Infinite Sums and Products Involving Trigonometric Functions

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Abstract: We derive a closed-form expression for the infinite sum of the Hurwitz–Lerch zeta function using contour integration. This expression is used to evaluate infinite sum and infinite product formulae involving trigonometric functions expressed in terms of fundamental constants. These types of infinite sums and products have previously been and are currently studied by many mathematicians including Leonhard Euler. The results presented in this paper extend previous work by squaring parameters in the infinite sum of the Hurwitz–Lerch zeta function. This formula allows for new derivations featuring trigonometric functions with angles of powers of 2. The zero distribution of almost all Hurwitz–Lerch zeta functions is asymmetrical. A table of infinite products is produced highlighting the usefulness of this work and for easy reading by researchers interested in such formulae. Mathematica software was used in assisting with the numerical verification of the results in the tables produced.

Keywords: Hurwitz–Lerch zeta function; incomplete gamma function; Cauchy integral

MSC: Primary 30E20; 33-01; 33-03; 33-04

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1. Introduction

In 1744, Euler [1] discovered an infinite product formula which was concerned with an unexpected and deep connection between analysis and number theory. He was able to prove that the divergence of a harmonic series implies that the number of primes is infinite and vice versa. In the recent literature, Guillera et al. [2] produced more work on infinite products using the analytic continuation property of the Hurwitz–Lerch zeta function. In their work, Guillera et al. [2] were able to write fundamental constants in terms of infinite products. In the work by Nyblom [3], the application of double and triple angle identities for hyperbolic and trigonometric cosine functions were used to obtain closed-form evaluations for two families of infinite products involving nested radicals. The work by Olver [4] features the representation of special functions and the quotient of special functions in terms of infinite products. The book of Watson [5] includes the evaluation of a general class of infinite products by means of the evaluation of the Gamma function. The "connexion between the gamma function and the circular function" [5] has also been studied, including the derivation of the multiplication-theorem of Gauss and Legendre. The expansions for the logarithmic derivates of the gamma function and Euler's expression of $\Gamma(z)$ as an infinite integral have also been derived.

Based on the past and current literature involving infinite products expressed in terms of constants, we attempt to expand on this type of research by applying our contour integral method to a hyperbolic tangent function and express the infinite sum of the Hurwitz–Lerch zeta function in terms of the incomplete gamma and Hurwitz–Lerch zeta functions. This infinite sum is used to derive special cases of both infinite sums and infinite products in terms of fundamental constants such as Catalan's constant K and $\sqrt{2}$. Symmetry is an important property of a function. Even functions are symmetric with respect to the y-axis and odd functions are symmetric about the origin. In this article, we derive special cases involving the product of trigonometric functions, which themselves have symmetric properties.

We derive a new expression for the Hurwitz–Lerch zeta function in terms of the incomplete gamma and Hurwitz–Lerch zeta functions given by

$$\sum_{j=1}^{\infty} 2^{-j}\left(\log^k(a) - 2\left(2^{-j}\right)^k e^{2^{-j}m}\Phi\left(-e^{2^{-j}m}, -k, 2^j \log(a) + 1\right)\right)$$

$$= \frac{2a^{-m}(-m)^{-k}\Gamma(k+1, -m\log(a))}{m} + 2^{k+1}e^m\left(e^m\Phi\left(e^{2m}, -k, \frac{\log(a)}{2} + 1\right) + \Phi\left(e^{2m}, -k, \frac{1}{2}(\log(a)+1)\right)\right) + \log^k(a) \quad (1)$$

where the variables k, a and m are general complex numbers. The derivations follow the method used by us in [6]. This method involves using a form of the generalized Cauchy's integral formula given by

$$\frac{y^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}}dw, \quad (2)$$

where $y, w \in \mathbb{C}$ and C is in general an open contour in the complex plane where the bilinear concomitant [6] has the same value at the end points of the contour. This method involves using a form of Equation (2), then multiplying both sides by a function and taking the infinite sum of both sides. This yields an infinite sum in terms of a contour integral. Then, we multiply both sides of Equation (2) by another function and take the infinite sum of both sides such that the contour integral of both equations are the same.

2. The Hurwitz–Lerch Zeta and Incomplete Gamma Functions

We use Equation (1.11.3) in [7] where $\Phi(z, s, v)$ is the Lerch function, which is a generalization of the Hurwitz zeta $\zeta(s, v)$ and polylogarithm functions $Li_n(z)$. The Lerch function has a series representation given by

$$\Phi(z, s, v) = \sum_{n=0}^{\infty}(v+n)^{-s}z^n \quad (3)$$

where $|z| < 1, v \neq 0, -1, -2, -3, ...,$ and is continued analytically by its integral representation given by

$$\Phi(z, s, v) = \frac{1}{\Gamma(s)}\int_0^{\infty} \frac{t^{s-1}e^{-(v-1)t}}{e^t - z}dt \quad (4)$$

where $Re(v) > 0$ and either $|z| \leq 1, z \neq 1, Re(s) > 0$, or $z = 1, Re(s) > 1$.

The Incomplete Gamma Function

The multivalued incomplete gamma functions [8], $\gamma(s, z)$ and $\Gamma(s, z)$, are defined by

$$\gamma(s, z) = \int_0^z t^{s-1}e^{-t}dt,$$

and

$$\Gamma(s, z) = \int_z^{\infty} t^{s-1}e^{-t}dt,$$

where $Re(s) > 0$. The incomplete gamma function has a recurrence relation given by

$$\gamma(s, z) + \Gamma(s, z) = \Gamma(s),$$

where $s \neq 0, -1, -2, \ldots$ The incomplete gamma function is continued analytically by

$$\gamma(s, ze^{2m\pi i}) = e^{2\pi mia}\gamma(s, z),$$

and

$$\Gamma(s, ze^{2m\pi i}) = e^{2\pi mis}\Gamma(s, z) + (1 - e^{2\pi mis})\Gamma(s),$$

where $m \in \mathbb{Z}$. When $z \neq 0$, $\Gamma(s, z)$ is an entire function of s and $\gamma(s, z)$ is meromorphic with simple poles at $s = -n$ for $n = 0, 1, 2, \ldots$, with residue $\frac{(-1)^n}{n!}$. These definitions are listed in Section 8.2 (i) and (ii) in [8].

3. Double Infinite Sum of the Contour Integrals

3.1. Derivation of the First Contour Integral

We use the method in [6,9] where the cut starts from the origin and the contour goes round the origin with zero radius and is on opposite sides of the cut. The cut and contour are in the first quadrant of the complex w-plane with $0 < Re(w + m)$. Using a generalization of Cauchy's integral formula (2), we first replace $y \to \log(a) + 2^{-j}(y + 1)$, then multiply both sides by $2^{1-j}(-1)^y e^{2^{-j}m(y+1)}$ and take the infinite sums, respectively, over $y \in [0, \infty)$ and $j \in [1, \infty)$ and simplify in terms of the Hurwitz–Lerch zeta function to get

$$\sum_{j=1}^{\infty} \frac{2^{1-j}\left(2^{-j}\right)^k e^{2^{-j}m}\Phi\left(-e^{2^{-j}m}, -k, 2^j\log(a) + 1\right)}{\Gamma(k+1)}$$

$$= \frac{1}{2\pi i}\sum_{y=0}^{\infty}\sum_{j=1}^{\infty}\int_C 2^{1-j}(-1)^y a^w w^{-k-1} e^{2^{-j}(y+1)(m+w)}dw$$

$$= \frac{1}{2\pi i}\int_C \sum_{j=1}^{\infty}\sum_{y=0}^{\infty} 2^{1-j}(-1)^y a^w w^{-k-1} e^{2^{-j}(y+1)(m+w)}dw$$

$$= \frac{1}{2\pi i}\int_C \sum_{j=1}^{\infty}\left(2^{-j}a^w w^{-k-1}\tanh\left(2^{-j-1}(m+w)\right) + 2^{-j}a^w w^{-k-1}\right)dw$$

$$= \frac{1}{2\pi i}\int_C \left(-\frac{2a^w w^{-k-1}}{m+w} + a^w w^{-k-1}\coth(m+w) + a^w w^{-k-1}\text{csch}(m+w)\right)dw \quad (5)$$

from Equations (5.3.8.4) and (5.3.8.6) in [10], where $Re(m + w) > 0$ and $Im(m + w) > 0$ in order for the sum to converge. We apply Tonelli's theorem for multiple sums, see page 189 in [11], as the summand is of bounded measure over the space $\mathbb{C} \times [1, \infty) \times [0, \infty)$.

Derivation of the Additional Contour Integral

Using a generalization of Cauchy's integral formula (2), we first replace $y \to \log(a)$ and multiply both sides by 2^{-j} to simplify and get

$$\frac{2^{-j}\log^k(a)}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_C 2^{-j}a^w w^{-k-1}dw \quad (6)$$

4. Infinite Sum of the Contour Integral

4.1. Derivation of the First Contour Integral

We use the method in [6]. Using a generalization of Cauchy's integral formula (2), we first replace $y \to \log(a) + x$, then multiply both sides by e^{mx} and take the definite integral over $x \in [0, \infty)$ and finally, we simplify in terms of the incomplete gamma function to get

$$\frac{2a^{-m}(-m)^{-k-1}\Gamma(k+1, -m\log(a))}{\Gamma(k+1)} = -\frac{1}{2\pi i}\int_C \frac{2a^w w^{-k-1}}{m+w}dw \tag{7}$$

from Equation (3.382.4) in [12], where $|\arg\log(a)| < \pi$ and $Re(m+w) < 0$.

4.2. Derivation of the Second Contour Integral

We use the method in [6]. Using a generalization of Cauchy's integral formula (2), we first replace $y \to \log(a) + 2(y+1)$, then multiply both sides by $-2e^{2m(y+1)}$ and take the infinite sum over $y \in [0, \infty)$ and finally, we simplify in terms of the Hurwitz–Lerch zeta function to get

$$-\frac{2^{k+1}e^{2m}\Phi\left(e^{2m}, -k, \frac{\log(a)}{2}+1\right)}{\Gamma(k+1)}$$

$$= -\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C 2a^w w^{-k-1}e^{2(y+1)(m+w)}dw$$

$$= -\frac{1}{2\pi i}\int_C \sum_{y=0}^{\infty} 2a^w w^{-k-1}e^{2(y+1)(m+w)}dw$$

$$= \frac{1}{2\pi i}\int_C a^w w^{-k-1}\coth(m+w) + a^w w^{-k-1}dw \tag{8}$$

from Equation (1.232.1) in [12], where $Im(m+w) > 0$ in order for the sum to converge.

Derivation of the Additional Contour Integral

Using a generalization of Cauchy's integral formula (2), we first replace $y \to \log(a)$ and simplify to get

$$\frac{\log^k(a)}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_C a^w w^{-k-1}dw \tag{9}$$

4.3. Derivation of the Third Contour Integral

We use the method in [6]. Using a generalization of Cauchy's integral formula (2), we first replace $y \to \log(a) + 2y + 1$, then multiply both sides by $-2e^{m(2y+1)}$ and take the infinite sum over $y \in [0, \infty)$ and finally, we simplify in terms of the Hurwitz–Lerch zeta function to get

$$
-\frac{2^{k+1}e^{m}\Phi\left(e^{2m},-k,\frac{1}{2}(\log(a)+1)\right)}{\Gamma(k+1)}
$$

$$
=-\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_{C}2a^{w}w^{-k-1}e^{(2y+1)(m+w)}dw
$$

$$
=-\frac{1}{2\pi i}\int_{C}\sum_{y=0}^{\infty}2a^{w}w^{-k-1}e^{(2y+1)(m+w)}dw
$$

$$
=\frac{1}{2\pi i}\int_{C}a^{w}w^{-k-1}\operatorname{csch}(m+w)dw \quad (10)
$$

from Equation (1.232.3) in [12], where $Im(m+w)>0$ in order for the sum to converge.

5. Infinite Sum of the Hurwitz–Lerch Zeta Function in Terms of the Incomplete Gamma and Hurwitz–Lerch Zeta Functions

Theorem 1. *For all $k,a,m\in\mathbb{C}$ then,*

$$
\sum_{j=1}^{\infty}2^{-j}\left(\log^{k}(a)-2\left(2^{-j}\right)^{k}e^{2^{-j}m}\Phi\left(-e^{2^{-j}m},-k,2^{j}\log(a)+1\right)\right)
$$

$$
=\frac{2a^{-m}(-m)^{-k}\Gamma(k+1,-m\log(a))}{m}+2^{k+1}e^{m}\left(e^{m}\Phi\left(e^{2m},-k,\frac{\log(a)}{2}+1\right)\right.
$$

$$
\left.+\Phi\left(e^{2m},-k,\frac{1}{2}(\log(a)+1)\right)\right)+\log^{k}(a) \quad (11)
$$

Proof. Observe that the addition of the right-hand sides of Equations (5) and (6) is equal to the addition of the right-hand sides of Equations (7)–(10), so we may equate the left-hand sides and simplify the gamma function to yield the stated result. □

6. Special Cases

In this section, we evaluate Equation (11) for various values of the parameters involved to derive infinite product and infinite summation formulae in terms of fundamental constants.

Example 1. *The degenerate case.*

$$
\sum_{j=1}^{\infty}2^{-j}\tanh\left(2^{-j-1}m\right)=\coth\left(\frac{m}{2}\right)-\frac{2}{m} \quad (12)
$$

Proof. Use Equation (11), set $k=0$ and simplify using entry (4) of Table in Section (64:12) in [13]. □

Example 2. *An infinite product involving the exponential of trigonometric functions.*

$$\prod_{j=1}^{\infty} \exp\left(2^{-j}\left(\tanh\left(2^{-j-1}x\right) - \tanh\left(\frac{2^{-j-1}x}{\beta}\right)\right)\right) a^{2\log\left(e^{\frac{(\beta-1)2^{-j-1}x}{\beta}}\cosh\left(2^{-j-1}x\right)sech\left(\frac{2^{-j-1}x}{\beta}\right)\right)-\frac{(\beta-1)2^{-j}x}{\beta}}$$

$$= \exp\left(\frac{2\left(x\log(a)\log\left(\frac{\sinh\left(\frac{x}{2}\right)csch\left(\frac{x}{2\beta}\right)}{\beta}\right)+\beta-1\right)}{x} - \coth\left(\frac{x}{2\beta}\right) + \coth\left(\frac{x}{2}\right)\right) \tag{13}$$

Proof. Use Equation (11), set $k = 1, m = x$ and apply the method in Section (8) in [9]. □

Using Equation (13), we derive a few special case examples. These formulae are derived by forming two equations and multiplying and simplifying. The results are yielded when $x = \frac{\pi i}{3}, a = -1, \beta = \frac{1}{2}$ and $x = -\frac{\pi i}{3}, a = -1, \beta = \frac{1}{2}$ for the first equation and $x = \frac{\pi i}{2}, a = i, \beta = \frac{1}{3}$ and $x = -\frac{\pi i}{2}, a = i, \beta = \frac{1}{3}$ for the second equation, respectively.

$$\prod_{j=1}^{\infty} \cos\left(\frac{1}{3}\pi 2^{-j-1}\right)\sec\left(\frac{\pi 2^{-j}}{3}\right) = \frac{2}{\sqrt{3}} \tag{14}$$

and

$$\prod_{j=1}^{\infty} \frac{1}{\left(1-2\cos\left(\pi 2^{-j-1}\right)\right)^2} = 9 \tag{15}$$

Example 3. *Catalan's constant K.*

$$\sum_{j=1}^{\infty} 2^j e^{i\pi 2^{-j-1}}\Phi\left(-e^{i2^{-j-1}\pi}, 2, 1+2^j\right) = \frac{1}{2}\left(-2K - \pi Ei\left(\frac{i\pi}{2}\right) + 4 + \frac{23i\pi^2}{24}\right) \tag{16}$$

Proof. Use Equation (11) and set $k = -2, a = e, m = \frac{\pi i}{2}$ and simplify using Equation (25.14.3) in [8] and Equation (2.2.1.2.7) in [14]. □

Example 4. *An infinite sum involving the reciprocal of the hyperbolic cosine function.*

$$\sum_{j=1}^{\infty} \frac{4^{-j}}{\cosh\left(\theta 2^{-j}\right)+1} = \frac{2}{\theta^2} + \frac{1}{1-\cosh(\theta)} \tag{17}$$

Proof. Use Equation (11), set $k = 1, a = 1, m = \theta$ and simplify using entry (3) of table in Section (64:12) in [13]. □

Example 5. *An infinite product involving the exponential of trigonometric functions.*

$$\prod_{j=1}^{\infty}\left(e^{2^{-j}x}+1\right)^{-\frac{a^2}{2}}\left(e^{\frac{2^{-j}x}{\beta}}+1\right)^{\frac{a^2}{2}}$$
$$\exp\left(2^{-2j-3}\left(a2^{j+1}\left(\frac{a(\beta-1)x}{\beta}-\frac{4}{e^{\frac{2^{-j}x}{\beta}}+1}+\frac{4}{e^{2^{-j}x}+1}\right)+sech^2\left(\frac{2^{-j-1}x}{\beta}\right)-sech^2\left(2^{-j-1}x\right)\right)\right)$$
$$=x^{\frac{a^2}{2}}\left(\frac{x}{\beta}\right)^{-\frac{a^2}{2}}\sinh^{-\frac{a^2}{2}}\left(\frac{x}{2}\right)\sinh^{\frac{a^2}{2}}\left(\frac{x}{2\beta}\right)$$
$$\exp\left(\frac{1}{4}\left(-\frac{2(\beta-1)(2ax-\beta-1)}{x^2}+2a\coth\left(\frac{x}{2\beta}\right)-2a\coth\left(\frac{x}{2}\right)+\frac{1}{1-\cosh\left(\frac{x}{\beta}\right)}+\frac{1}{\cosh(x)-1}\right)\right) \quad (18)$$

Proof. Use Equation (11), set $k=2, m=x$ and apply the method in Section (8) in [9]. □

A few examples of special cases of Equation (18) are given below by forming two equations after substituting the values for $x=\frac{\pi i}{2}, a=-1, \beta=2$ and $x=-\frac{\pi i}{2}, a=-1$, $\beta = 2$ and $x=\frac{\pi i}{3}, a=i, \beta=2$ and $x=-\frac{\pi i}{3}, a=i, \beta=2$, respectively, and simplifying to get:

$$\prod_{j=1}^{\infty}\cos\left(\pi 2^{-j-3}\right)\sec\left(\pi 2^{-j-2}\right)$$
$$\left(\sinh\left(2^{-2(j+1)}\left(\sec^2\left(\pi 2^{-j-3}\right)-\sec^2\left(\pi 2^{-j-2}\right)\right)\right)\right.$$
$$\left.+\cosh\left(2^{-2(j+1)}\left(\sec^2\left(\pi 2^{-j-3}\right)-\sec^2\left(\pi 2^{-j-2}\right)\right)\right)\right)$$
$$=2\sqrt{2}e^{\frac{1}{2}+\frac{1}{\sqrt{2}}-\frac{12}{\pi^2}}\sin\left(\frac{\pi}{8}\right) \quad (19)$$

and

$$\prod_{j=1}^{\infty}\frac{\sqrt{\cos\left(\frac{\pi 2^{-j}}{3}\right)+1}\exp\left(2^{-2(j+1)}\left(\sec^2\left(\frac{1}{3}\pi 2^{-j-2}\right)-\sec^2\left(\frac{1}{3}\pi 2^{-j-1}\right)\right)\right)}{\sqrt{\cos\left(\frac{1}{3}\pi 2^{-j-1}\right)+1}}$$
$$=\frac{1}{2}\sqrt{2+\sqrt{3}}e^{1+\sqrt{3}-\frac{27}{\pi^2}} \quad (20)$$

7. Infinite Products Involving the Quotient of Exponentials

Example 6.

$$\prod_{j=1}^{\infty}\frac{e^{2^{-j}m}+1}{e^{2^{-j}r}+1}=\frac{r(e^m-1)}{m(e^r-1)} \quad (21)$$

Proof. Use Equation (11), form a second equation by replacing m by r, take their difference, set $k=-1, a=1$ and simplify using entry (3) of table in Section (64:12) in [13]. □

Table of Infinite Products

In this section, we evaluate Equation (21) for various imaginary values of the parameters m and r and multiply each step by their complex conjugate such that the right-hand side yields a real number listed in the table below. These evaluations involve the products of trigonometric and hyperbolic functions which may have symmetric properties. The results were mathematically validated using Wolfram Mathematica for both real, imaginary and complex values of the parameters in the products. For example, the first equation in Table 1 is obtained by multiplying two equations, when $m = \frac{\pi i}{2}, r = \pi i$ and $m = -\frac{\pi i}{2}, r = -\pi i$.

Table 1. Table of infinite products in terms of constants.

$\prod_{j=1}^{\infty} \cos\left(\pi 2^{-j-2}\right) \sec\left(\pi 2^{-j-1}\right)$	$\sqrt{2}$
$\prod_{j=1}^{\infty} \frac{1}{\left(1-2\cos\left(\frac{\pi 2^{-j}}{3}\right)\right)^2}$	$\frac{9}{4}$
$\prod_{j=1}^{\infty} \cos\left(\frac{1}{3}\pi 2^{-j-1}\right) \sec\left(\pi 2^{-j-2}\right)$	$\frac{3}{2\sqrt{2}}$
$\prod_{j=1}^{\infty} \cos\left(\frac{1}{3}\pi 2^{-j-1}\right) \sec\left(\pi 2^{-j-3}\right)$	$\frac{3}{4}\sqrt{\frac{1}{2}\left(2+\sqrt{2}\right)}$
$\prod_{j=1}^{\infty} \cos\left(\pi 2^{-j-2}\right) \sec\left(\pi 2^{-j-3}\right)$	$\frac{\sqrt{2+\sqrt{2}}}{2}$
$\prod_{j=1}^{\infty} \cos\left(\frac{1}{5}\pi 2^{-j-1}\right) \sec\left(\pi 2^{-j-3}\right)$	$\frac{5}{4}\sqrt{\frac{3-\sqrt{5}}{2\left(2-\sqrt{2}\right)}}$
$\prod_{j=1}^{\infty} \cos\left(\frac{\pi 2^{-j}}{3}\right) \sec\left(\pi 2^{-j-3}\right)$	$\frac{3}{8}\sqrt{\frac{3}{2}\left(2+\sqrt{2}\right)}$
$\prod_{j=1}^{\infty} \cos\left(\frac{\pi 2^{-j}}{3}\right) \sec\left(3\pi 2^{-j-3}\right)$	$\frac{9}{8}\sqrt{\frac{3}{2}\left(2-\sqrt{2}\right)}$
$\prod_{j=1}^{\infty} \frac{1}{\left(1-2\cos\left(\pi 2^{-j-2}\right)\right)^2}$	$27-18\sqrt{2}$
$\prod_{j=1}^{\infty} \cos\left(\frac{1}{5}\pi 2^{-j-1}\right) \sec\left(\frac{1}{5}\pi 2^{1-j}\right)$	$\sqrt{16-\frac{32}{\sqrt{5}}}$
$\prod_{j=1}^{\infty} \cos\left(\pi 2^{-j-2}\right) \operatorname{sech}\left(\frac{1}{5}\pi 2^{1-j}\right)$	$\frac{4}{5}\sqrt{2}\operatorname{csch}\left(\frac{2\pi}{5}\right)$
$\prod_{j=1}^{\infty} \frac{\left(2\cosh\left(\frac{1}{15}\pi 2^{1-j}\right)-2\cosh\left(\frac{\pi 2^{-j}}{15}\right)+1\right)^2}{\left(1-2\cosh\left(\frac{\pi 2^{-j}}{15}\right)\right)^2\left(1-2\cosh\left(\frac{\pi 2^{-j}}{5}\right)\right)^2}$	$\frac{81\left(1+2\cosh\left(\frac{\pi}{15}\right)+2\cosh\left(\frac{2\pi}{15}\right)\right)^2}{25\left(1+2\cosh\left(\frac{\pi}{15}\right)\right)^2\left(1+2\cosh\left(\frac{\pi}{5}\right)\right)^2}$
$\prod_{j=1}^{\infty} \cosh\left(\pi 2^{-j-2}\right) \operatorname{sech}\left(\pi 2^{-j-1}\right)$	$\operatorname{sech}\left(\frac{\pi}{4}\right)$
$\prod_{j=1}^{\infty} \cos\left(\frac{5}{3}\pi 2^{-j-2}\right) \sec\left(7\pi 2^{-j-4}\right)$	$\frac{21}{20}\sqrt{\frac{2+\sqrt{3}}{2+\sqrt{2+\sqrt{2}}}}$
$\prod_{j=1}^{\infty} \frac{\cos\left(\frac{7\pi 2^{-j}}{11}\right)+1}{\cos\left(\frac{3\pi 2^{-j}}{5}\right)+1}$	$\frac{4356\left(1+\sin\left(\frac{3\pi}{22}\right)\right)}{1225\left(3+\sqrt{5}\right)}$
$\prod_{j=1}^{\infty} \cos\left(\frac{1}{9}\pi 2^{2-j}\right) \sec\left(3\pi 2^{-j-3}\right)$	$\frac{27\cos\left(\frac{\pi}{18}\right)}{16\sqrt{2+\sqrt{2}}}$

8. Conclusions

The authors derived an infinite sum of the Hurwitz–Lerch zeta function in terms of the incomplete gamma function and the Hurwitz–Lerch zeta functions, where the parameter constraints were wide. Infinite products of involving trigonometric functions were also derived. The method applied in the derivation of the main theorem may be used to derive other sums and products in future work. Similar studies on infinite products involving trigonometric functions have been published in the work by Dieckmann [15].

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