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On the Dragomir Extension of Furuta's Inequality and Numerical Radius Inequalities

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Abstract: In this work, some numerical radius inequalities based on the recent Dragomir extension of Furuta's inequality are obtained. Some particular cases are also provided. Among others, the celebrated Kittaneh inequality reads: $w(T) \leq \frac{1}{2}\||T| + |T^*|\|$. It is proved that $w(T) \leq \frac{1}{2}\||T| + |T^*|\| - \frac{1}{2}\inf_{\|x\|=1}\left(\langle |T|x,x\rangle^{\frac{1}{2}} - \langle |T^*|x,x\rangle^{\frac{1}{2}}\right)^2$, which improves on the Kittaneh inequality for symmetric and non-symmetric Hilbert space operators. Other related improvements to well-known inequalities in literature are also provided.

Keywords: mixed Schwarz inequality; Furuta inequality; numerical radius inequalities

MSC: 47A30; 47A12; 15A60; 47A63

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1. Introduction

Let $\mathscr{B}(\mathscr{H})$ be the Banach algebra of all bounded linear operators defined on a complex Hilbert space $(\mathscr{H};\langle\cdot,\cdot\rangle)$ with the identity operator $1_{\mathscr{H}}$ in $\mathscr{B}(\mathscr{H})$. When $\mathscr{H} = \mathbb{C}^n$, we identify $\mathscr{B}(\mathscr{H})$ with the algebra $\mathcal{M}_{n\times n}$ of n-by-n complex matrices. The cone of n-by-n positive semidefinite matrices is then $\mathcal{M}^+_{n\times n}$. This is adopted for all matrices, whether self-adjoint (symmetric) or not.

The numerical range $W(T)$ of a bounded linear operator T on a Hilbert space $\mathscr{H}$ is the image of the unit sphere of $\mathscr{H}$ associated with the operator under the quadratic form $x \to \langle Tx,x\rangle$. More precisely, we have

$$W(T) = \{\langle Tx,x\rangle : x \in \mathscr{H}, \|x\| = 1\}.$$

Furthermore, the numerical radius is

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\} = \sup_{\|x\|=1} |\langle Tx,x\rangle|.$$

The spectral radius of an operator T is indicated as

$$r(T) = \sup\{|\lambda| : \lambda \in \mathrm{sp}(T)\}.$$

We recall that the usual operator norm of an operator T is defined as

$$\|T\| = \sup\{\|Tx\| : x \in H, \|x\| = 1\},$$

and

$$\ell(T) := \inf\{\|Tx\| : x \in \mathscr{H}, \|x\| = 1\}$$
$$= \inf\{|\langle Tx, y\rangle| : x, y \in \mathscr{H}, \|x\| = \|y\| = 1\}.$$

It is well-known that the numerical radius is not submultiplicative, but it satisfies

$$w(TS) \leq 4w(T)w(S)$$

for all $T, S \in \mathscr{B}(\mathscr{H})$. In particular, if T and S commute, then

$$w(TS) \leq 2w(T)w(S).$$

Moreover, if T and S are normal, then $w(\cdot)$ is submultiplicative $w(TS) \leq w(T)w(S)$.

The absolute value of the operator T is denoted by $|T| = (T^*T)^{1/2}$. Then we have $w(|T|) = \|T\|$. It is convenient to mention that the numerical radius norm is weakly unitarily invariant, i.e., $w(U^*TU) = w(T)$ for all unitary U. Furthermore, let us not miss the chance to mention the important properties that $w(T) = w(T^*)$ and $w(T^*T) = w(TT^*)$ for every $T \in \mathscr{B}(\mathscr{H})$.

A popular problem is the following: does the numerical radius of the product of operators commute, i.e., $w(TS) = w(ST)$ for any operators $T, S \in \mathscr{B}(\mathscr{H})$?

This problem has been given serious attention by many authors and in several resources (see [1], for example). Fortunately, it has been shown recently that for any bounded linear operators $A, B \in \mathscr{B}(\mathscr{H})$, AZ and ZB always have the same numerical radius for all rank one $Z \in \mathscr{B}(\mathscr{H})$ if and only if $A = e^{it}B$ is a multiple of a unitary operator for some $t \in [0, 2\pi)$. This fact was proved by Chien et al. in [2]. For other related problems involving numerical ranges and radiuses, see [2,3] as well as the elegant work of Li [4] and the references therein. For more classical and recent properties of numerical range and radiuses, see [2–4] and the comprehensive books [5–7].

On the other hand, $w(\cdot)$ is well-known to define an operator norm on $\mathscr{B}(\mathscr{H})$, which is equivalent to the operator norm $\|\cdot\|$. Moreover, we have

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\| \tag{1}$$

for any $T \in \mathscr{B}(\mathscr{H})$. The inequality is sharp.

In 2003, Kittaneh [8] refined the right-hand side of (1), where he proved that

$$w(T) \leq \frac{1}{2}\||T| + |T^*|\| \tag{2}$$

for any $T \in \mathscr{B}(\mathscr{H})$.

After that, in 2005, the same author in [9] proved that

$$\frac{1}{4}\|A^*A + AA^*\| \leq w^2(A) \leq \frac{1}{2}\|A^*A + AA^*\|. \tag{3}$$

These inequalities were also reformulated and generalized in [10] but in terms of Cartesian decomposition. Both of them have been generalized recently in [11,12], respectively.

In 2007, Yamazaki [13] improved (1) by proving that

$$w(T) \leq \frac{1}{2}\left(\|T\| + w\left(\widetilde{T}\right)\right) \leq \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{1/2}\right), \tag{4}$$

where $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$ with unitary U.

In 2008, Dragomir [14] used the Buzano inequality to improve (1), where he proved that

$$w^2(T) \leq \frac{1}{2}\left(\|T\| + w\left(T^2\right)\right). \tag{5}$$

This result was also recently generalized by Sattari et al. in [15]. This result was also recently generalized by Sattari et al. in [15] and Alomari in [16–19]. For more recent results about the numerical radius, see the recent monograph study in [14,20–22].

According to the Schwarz inequality for positive operators, for any positive operator A in $\mathscr{B}(\mathscr{H})$, we have

$$|\langle Ax, y\rangle|^2 \leq \langle Ax, x\rangle\langle Ay, y\rangle \tag{6}$$

for any vectors $x, y \in \mathscr{H}$.

In 1951, Reid [23] proved an inequality, which in some senses considered a variant of the Schwarz inequality. In fact, he proved that for all operators $A \in \mathscr{B}(\mathscr{H})$ such that A is positive and AB is self-adjoint, then

$$|\langle ABx, y\rangle| \leq \|B\|\langle Ax, x\rangle, \tag{7}$$

for all $x \in \mathscr{H}$. In [24], Halmos presented his stronger version of the Reid inequality (7) by substituting $r(B)$ for $\|B\|$.

In 1952, Kato [25] introduced a companion inequality of (6), called the mixed Schwarz inequality, which asserts

$$|\langle Ax, y\rangle|^2 \leq \left\langle |A|^{2\alpha}x, x\right\rangle\left\langle |A^*|^{2(1-\alpha)}y, y\right\rangle, \qquad 0 \leq \alpha \leq 1, \tag{8}$$

for every operators $A \in \mathscr{B}(\mathscr{H})$ and any vectors $x, y \in \mathscr{H}$, where $|A| = (A^*A)^{1/2}$.

In 1988, Kittaneh [26] proved a very interesting extension combining both the Halmos–Reid Inequality (2) and the mixed Schwarz Inequality (3). His result says that

$$|\langle ABx, y\rangle| \leq r(B)\|f(|A|)x\|\|g(|A^*|)y\| \tag{9}$$

for any vectors $x, y \in \mathscr{H}$, where $A, B \in \mathscr{B}(\mathscr{H})$ such that $|A|B = B^*|A|$ and f, g are nonnegative continuous functions defined on $[0, \infty)$ satisfying that $f(t)g(t) = t$ $(t \geq 0)$. Clearly, if we choose $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$ with $B = 1_{\mathscr{H}}$, then we may refer to (8). Moreover, choosing $\alpha = \frac{1}{2}$, some manipulations refer to the Halmos version of the Reid inequality. The cartesian decomposition form of (9) was recently proved by Alomari in [16].

In 1994, Furuta [27] proved another attractive generalization of Kato's inequality (3), as follows:

$$\left|\left\langle T|T|^{\alpha+\beta-1}x, y\right\rangle\right|^2 \leq \left\langle |T|^{2\alpha}x, x\right\rangle\left\langle |T|^{2\beta}y, y\right\rangle \tag{10}$$

for any $x, y \in \mathscr{H}$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \geq 1$.

The inequality (5) was generalized for any $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1$ by Dragomir in [22]. Indeed, as noted by Dragomir, the condition $\alpha, \beta \in [0, 1]$ was assumed by Furuta to fit with the Heinz–Kato inequality, which reads:

$$|\langle Tx, y\rangle| \leq \|A^\alpha x\|\left\|B^{1-\alpha}y\right\|$$

for any $x, y \in \mathscr{H}$ and $\alpha \in [0, 1]$, where A and B are positive operators such that $\|Tx\| \leq \|Ax\|$ and $\|T^*y\| \leq \|By\|$ for any $x, y \in \mathscr{H}$.

In the same work [22], Dragomir provides a useful extension of Furuta's inequality, as follows:

$$|\langle DCBAx, y\rangle|^2 \leq \left\langle A^*|B|^2 Ax, x\right\rangle \left\langle D|C^*|^2 D^* y, y\right\rangle \tag{11}$$

for any $A, B, C, D \in \mathscr{B}(\mathscr{H})$ and any vectors $x, y \in \mathscr{H}$. The equality in (11) holds iff the vectors BAx and C^*D^*y are linearly dependent in $\mathscr{H}$.

Indeed, since $A^*|B|^2 A = A^*B^*BA = (A^*B^*)(BA) = (BA)^*(BA) = |BA|^2$ and $D|C^*|^2 D^* = DCC^*D^* = (DC)(C^*D^*) = (DC)(DC)^* = |(DC)^*|^2 = |C^*D^*|^2$, the Inequality (11) can be rewritten as

$$|\langle DCBAx, y\rangle|^2 \leq \left\langle |BA|^2 x, x\right\rangle \left\langle |C^*D^*|^2 y, y\right\rangle. \tag{12}$$

If one setting $D = U$ (U is unitary), $B = 1_{\mathscr{H}}$, $C = |T|^\beta$ and $A = |T|^\alpha$ such that $\alpha + \beta \geq 1$, then we recapture (10).

Based on the most recent Dragomir extension of Furuta's inequality, various numerical radius inequalities are derived in this paper. Additionally, several specific examples are given.

The rest of the paper is composed of the following sections: Section 2 presents some crucial lemmas. Numerical radius inequalites are determined and proved in Section 3. The conclusion is made in Section 4.

2. Lemmas

2.1. Preliminaries

In order to prove our main result, we need the following Lemmas:

Lemma 1. *Let $S \in \mathscr{B}(\mathscr{H})$, $S \geq 0$ and $x \in \mathscr{H}$ be a unit vector. Then, the operator Jensen's inequality states that*

$$\langle Sx, x\rangle^r \leq (\geq) \langle S^r x, x\rangle, \qquad r \geq 1 \qquad (0 \leq r \leq 1). \tag{13}$$

Kittaneh and Manasrah [28] obtained the following result, which is a refinement of the scalar Young inequality.

Lemma 2. *Let $a, b \geq 0$, and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then we have*

$$ab + \min\left\{\frac{1}{p}, \frac{1}{q}\right\}(a^{\frac{p}{2}} - b^{\frac{q}{2}})^2 \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{14}$$

Manasrah and Kittaneh have generalized (15) in [29], as follows:

Lemma 3. *If $a, b > 0$, and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then for $m = 1, 2, 3, \ldots,$*

$$(a^{\frac{1}{p}} b^{\frac{1}{q}})^m + r_0^m (a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 \leq \left(\frac{a^r}{p} + \frac{b^r}{q}\right)^{\frac{m}{r}}, \ r \geq 1, \tag{15}$$

where $r_0 = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$. In particular, if $p = q = 2$, then we have

$$(a^{\frac{1}{2}} b^{\frac{1}{2}})^m + \frac{1}{2^m}(a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 \leq 2^{-\frac{m}{r}}(a^r + b^r)^{\frac{m}{r}}.$$

For $m = 1$, we obtain

$$(a^{\frac{1}{2}} b^{\frac{1}{2}}) + \frac{1}{2}(a^{\frac{1}{2}} - b^{\frac{1}{2}})^2 \leq 2^{-\frac{1}{r}}(a^r + b^r)^{\frac{1}{r}}.$$

Lemma 4 ([30])**.** *Let f be a twice differentiable function on $[a,b]$. If f is convex such that $f'' \geq \lambda := \min_{x\in[a,b]} f(x) > 0$, then we have*

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2} - \frac{1}{8}\lambda(b-a)^2. \tag{16}$$

Lemma 5 ([31])**.** *Let f be a convex function defined on a real interval I. Then for every self-adjoint operator $A \in \mathscr{B}(\mathscr{H})$ whose $\mathrm{sp}(A) \subset I$, we have*

$$f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle$$

for all vectors $x \in \mathscr{H}$.

2.2. Extensions of the Dragomir—Furuta Inequality

In this section, we provide some key lemmas that play the main role in the proof of our main results.

Lemma 6. *Let $A, B, C, D \in \mathscr{B}(\mathscr{H})$. Let f be a positive, increasing and convex function on $\mathbb{R}$. If f is twice differentiable such that $f'' \geq \lambda > 0$, then*

$$\begin{aligned} f(|\langle DCBAx, y\rangle|) \leq \frac{1}{2}\Big[\Big\langle f\Big(A^*|B|^2A\Big)x, x\Big\rangle + \Big\langle f\Big(D|C^*|^2D^*\Big)y, y\Big\rangle\Big] \\ - \frac{1}{8}\lambda\Big(\Big\langle A^*|B|^2Ax, x\Big\rangle - \Big\langle D|C^*|^2D^*y, y\Big\rangle\Big)^2 \end{aligned} \tag{17}$$

for all vectors $x, y \in \mathscr{H}$.

Proof. Employing the monotonicity and convexity of f for the Inequality (6), we have

$$\begin{aligned} f(|\langle DCBAx, y\rangle|) &\leq f\Big(\Big\langle A^*|B|^2Ax, x\Big\rangle^{\frac{1}{2}}\Big\langle D|C^*|^2D^*y, y\Big\rangle^{\frac{1}{2}}\Big) && (f \text{ increasing}) \\ &\leq f\left(\frac{\Big\langle A^*|B|^2Ax, x\Big\rangle + \Big\langle D|C^*|^2D^*y, y\Big\rangle}{2}\right) && (\text{by AM-GM inequality}) \\ &\leq \frac{f\Big(\Big\langle A^*|B|^2Ax, x\Big\rangle\Big) + f\Big(\Big\langle D|C^*|^2D^*y, y\Big\rangle\Big)}{2} && (\text{by Lemma 4}) \\ &\quad - \frac{1}{8}\lambda\Big(\Big\langle A^*|B|^2Ax, x\Big\rangle - \Big\langle D|C^*|^2D^*y, y\Big\rangle\Big)^2 \\ &\leq \frac{1}{2}\Big[\Big\langle f\Big(A^*|B|^2A\Big)x, x\Big\rangle + \Big\langle f\Big(D|C^*|^2D^*\Big)y, y\Big\rangle\Big] && (\text{by Lemma 5}) \\ &\quad - \frac{1}{8}\lambda\Big(\Big\langle A^*|B|^2Ax, x\Big\rangle - \Big\langle D|C^*|^2D^*y, y\Big\rangle\Big)^2 \end{aligned}$$

for all vectors $x, y \in \mathscr{H}$, which proves the result. □

Corollary 1. *Let $T \in \mathscr{B}(\mathscr{H})$. Let f be a positive, increasing and convex function on $\mathbb{R}$. If f is twice differentiable such that $f'' \geq \lambda > 0$, then we have*

$$\begin{aligned} f\Big(\Big|\Big\langle T|T|^{\alpha+\beta-1}x, y\Big\rangle\Big|\Big) \leq \frac{1}{2}\Big[\Big\langle f\Big(|T|^{2\alpha}\Big)x, x\Big\rangle + \Big\langle f\Big(|T^*|^{2\beta}\Big)y, y\Big\rangle\Big] \\ - \frac{1}{8}\lambda\Big(\Big\langle |T|^{2\alpha}x, x\Big\rangle - \Big\langle |T^*|^{2\beta}y, y\Big\rangle\Big)^2 \end{aligned} \tag{18}$$

for all vectors $x, y \in \mathscr{H}$ and all $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \geq 1$.

Proof. Let $D = U$, $B = 1_{\mathscr{H}}$, $C = |T|^{\beta}$ and $A = |T|^{\alpha}$ such that $\alpha + \beta \geq 1$ in (17), then we have

$$DCBA = U|T|^{\beta}|T|^{\alpha} = U|T||T|^{\alpha+\beta-1} = T|T|^{\alpha+\beta-1},$$

also, we have $A^*|B|^2A = |T|^{2\alpha}$ and $D|C^*|^2D^* = U|T|^{2\beta}U^* = |T|^{2\beta}$, and this proves the required result. □

Lemma 7. *Let f be a positive, increasing, convex and supermultiplicative function on $\mathbb{R}$, i.e., $f(ts) \leq f(t)f(s)$ for all $t, s \in \mathbb{R}$. Then we have*

$$f\left(|\langle DCBAx, y\rangle|^2\right) \leq \frac{1}{p}\left\langle f^p\left(A^*|B|^2A\right)x, x\right\rangle + \frac{1}{q}\left\langle f^q\left(D|C^*|^2D^*\right)y, y\right\rangle - r_0\left(\left\langle f\left(A^*|B|^2A\right)x, x\right\rangle^{\frac{p}{2}} - \left\langle f\left(D|C^*|^2D^*\right)y, y\right\rangle^{\frac{q}{2}}\right)^2 \tag{19}$$

for all vectors $x, y \in \mathscr{H}$.

Proof. From (6), we obtain

$$\begin{aligned}
&f\left(|\langle DCBAx, y\rangle|^2\right) \\
&\leq f\left(\left\langle A^*|B|^2Ax, x\right\rangle\left\langle D|C^*|^2D^*y, y\right\rangle\right) && (f \text{ increasing}) \\
&\leq f\left(\left\langle A^*|B|^2Ax, x\right\rangle\right)f\left(\left\langle D|C^*|^2D^*y, y\right\rangle\right) && (f \text{ supermultiplicative}) \\
&\leq \left\langle f\left(A^*|B|^2A\right)x, x\right\rangle\left\langle f\left(D|C^*|^2D^*\right)y, y\right\rangle && (\text{by Lemma 5}) \\
&\leq \frac{1}{p}\left\langle f\left(A^*|B|^2A\right)x, x\right\rangle^p + \frac{1}{q}\left\langle f\left(D|C^*|^2D^*\right)y, y\right\rangle^q && (\text{by Lemma 2}) \\
&\quad - r_0\left(\left\langle f\left(A^*|B|^2A\right)x, x\right\rangle^{\frac{p}{2}} - \left\langle f\left(D|C^*|^2D^*\right)y, y\right\rangle^{\frac{q}{2}}\right)^2 \\
&\leq \frac{1}{p}\left\langle f^p\left(A^*|B|^2A\right)x, x\right\rangle + \frac{1}{q}\left\langle f^q\left(D|C^*|^2D^*\right)y, y\right\rangle && (\text{by Lemma 1}) \\
&\quad - r_0\left(\left\langle f\left(A^*|B|^2A\right)x, x\right\rangle^{\frac{p}{2}} - \left\langle f\left(D|C^*|^2D^*\right)y, y\right\rangle^{\frac{q}{2}}\right)^2
\end{aligned}$$

for all vectors $x, y \in \mathscr{H}$. □

Corollary 2. *Let f be a positive, increasing, convex and supermultiplicative function on $\mathbb{R}$, i.e., $f(ts) \leq f(t)f(s)$ for all $t, s \in \mathbb{R}$. Then, we have*

$$f\left(\left|\left\langle T|T|^{\alpha+\beta-1}x, y\right\rangle\right|^2\right) \leq \frac{1}{p}\left\langle f^p\left(|T|^{2\alpha}\right)x, x\right\rangle + \frac{1}{q}\left\langle f^q\left(|T^*|^{2\beta}\right)y, y\right\rangle - r_0\left(\left\langle f\left(|T|^{2\alpha}\right)x, x\right\rangle^{\frac{p}{2}} - \left\langle f\left(|T^*|^{2\alpha}\right)y, y\right\rangle^{\frac{q}{2}}\right)^2 \tag{20}$$

for all vectors $x, y \in \mathscr{H}$.

Proof. The proof proceeds similarly to the proof of Corollary 1, taking into account Lemma 7. □

Lemma 8. *Let f be a positive, increasing, convex and supermultiplicative function on $\mathbb{R}$, i.e., $f(ts) \le f(t)f(s)$ for all $t,s \in \mathbb{R}$. Then, we have*

$$f\left(|\langle DCBAx,y\rangle|^2\right) \le 2^{-\frac{2}{r}}\left(\left\langle f^r\left(A^*|B|^2A\right)x,x\right\rangle + \left\langle f^r\left(D|C^*|^2D^*\right)y,y\right\rangle\right)^{\frac{2}{r}} - \frac{1}{4}\left[\left\langle f\left(A^*|B|^2A\right)x,x\right\rangle - \left\langle f\left(D|C^*|^2D^*\right)y,y\right\rangle\right] \tag{21}$$

for all $r \ge 1$. In particular, we have

$$f\left(|\langle DCBAx,y\rangle|^2\right) \le \frac{1}{4}\left(\left\langle f\left(A^*|B|^2A\right)x,x\right\rangle + \left\langle f\left(D|C^*|^2D^*\right)y,y\right\rangle\right)^2 - \frac{1}{4}\left[\left\langle f\left(A^*|B|^2A\right)x,x\right\rangle - \left\langle f\left(D|C^*|^2D^*\right)y,y\right\rangle\right] \tag{22}$$

for all vectors $x,y \in \mathscr{H}$.

Proof. Since f is increasing and convex, then by applying Lemma 3, with $p = q = 2$ and $m = 2$, we obtain

$$\begin{aligned}
& f\left(|\langle DCBAx,y\rangle|^2\right) \\
& \le f\left(\left\langle A^*|B|^2Ax,x\right\rangle\left\langle D|C^*|^2D^*y,y\right\rangle\right) && (f \text{ increasing}) \\
& \le f\left(\left\langle A^*|B|^2Ax,x\right\rangle\right)f\left(\left\langle D|C^*|^2D^*y,y\right\rangle\right) && (f \text{ supermultiplicative}) \\
& \le \left\langle f\left(A^*|B|^2A\right)x,x\right\rangle\left\langle f\left(D|C^*|^2D^*\right)y,y\right\rangle && (\text{by Lemma } 5) \\
& \le 2^{-\frac{2}{r}}\left(\left\langle f\left(A^*|B|^2A\right)x,x\right\rangle^r + \left\langle f\left(D|C^*|^2D^*\right)y,y\right\rangle^r\right)^{\frac{2}{r}} && (\text{by Lemma } 3) \\
& \quad - \frac{1}{4}\left[\left\langle f\left(A^*|B|^2A\right)x,x\right\rangle - \left\langle f\left(D|C^*|^2D^*\right)y,y\right\rangle\right] \\
& \le 2^{-\frac{2}{r}}\left(\left\langle f^r\left(A^*|B|^2A\right)x,x\right\rangle + \left\langle f^r\left(D|C^*|^2D^*\right)y,y\right\rangle\right)^{\frac{2}{r}} && (\text{by Lemma } 1) \\
& \quad - \frac{1}{4}\left[\left\langle f\left(A^*|B|^2A\right)x,x\right\rangle - \left\langle f\left(D|C^*|^2D^*\right)y,y\right\rangle\right]
\end{aligned}$$

for all vectors $x,y \in \mathscr{H}$. □

Corollary 3. *Let f be a positive, increasing, convex and supermultiplicative function on $\mathbb{R}$, i.e., $f(ts) \le f(t)f(s)$ for all $t,s \in \mathbb{R}$. Then, we have*

$$f\left(\left|\left\langle T|T|^{\alpha+\beta-1}x,y\right\rangle\right|^2\right) \le 2^{-\frac{2}{r}}\left(\left\langle f^r\left(|T|^{2\alpha}\right)x,x\right\rangle + \left\langle f^r\left(|T^*|^{2\beta}\right)y,y\right\rangle\right)^{\frac{2}{r}} - \frac{1}{4}\left[\left\langle f\left(|T|^{2\alpha}\right)x,x\right\rangle - \left\langle f\left(|T^*|^{2\beta}\right)y,y\right\rangle\right]. \tag{23}$$

As a particular case, we have

$$f\left(\left|\left\langle T|T|^{\alpha+\beta-1}x,y\right\rangle\right|^2\right) \le \frac{1}{4}\left(\left\langle f\left(|T|^{2\alpha}\right)x,x\right\rangle + \left\langle\left(|T^*|^{2\beta}\right)y,y\right\rangle\right)^2 - \frac{1}{4}\left[\left\langle f\left(|T|^{2\alpha}\right)x,x\right\rangle - \left\langle f\left(|T^*|^{2\beta}\right)y,y\right\rangle\right] \tag{24}$$

for all vectors $x,y \in \mathscr{H}$.

Proof. The proof of (19) proceeds similarly to the proof of Corollary 1, taking into account Lemma 8. □

3. Numerical Radius Inequalities

In this section, we provide some numerical radius inequalities. Let us begin with the following key result.

Theorem 1. *Let $A, B, C, D \in \mathscr{B}(\mathscr{H})$. Let f be a positive, increasing and convex function on $\mathbb{R}$. If f is twice differentiable such that $f'' \geq \lambda > 0$, then*

$$f(w(DCBA)) \leq \frac{1}{2}\left\| f\left(A^*|B|^2A\right) + f\left(D|C^*|^2D^*\right)\right\| - \inf_{\|x\|=1} \eta(x), \tag{25}$$

where $\eta(x) := \frac{1}{8}\lambda\left\langle \left[A^*|B|^2A - D|C^*|^2D^*\right]x, x\right\rangle^2$.

Proof. Let $y = x$ in (17), then we obtain

$$\begin{aligned} f(|\langle DCBAx, x\rangle|) &\leq \frac{1}{2}\left[\left\langle f\left(A^*|B|^2A\right)x, x\right\rangle + \left\langle f\left(D|C^*|^2D^*\right)x, x\right\rangle\right] \\ &\quad - \frac{1}{8}\lambda\left(\left\langle A^*|B|^2Ax, x\right\rangle - \left\langle D|C^*|^2D^*x, x\right\rangle\right)^2 \\ &= \frac{1}{2}\left\langle \left[f\left(A^*|B|^2A\right) + f\left(D|C^*|^2D^*\right)\right]x, x\right\rangle \\ &\quad - \frac{1}{8}\lambda\left\langle \left[A^*|B|^2A - D|C^*|^2D^*\right]x, x\right\rangle^2. \end{aligned}$$

Taking the supremum over all unit vectors $x \in \mathscr{H}$, we obtain the required result. □

Corollary 4. *Let $A, B, C, D \in \mathscr{B}(\mathscr{H})$. Then we have*

$$w^2(DCBA) \leq \frac{1}{2}\left\|\left(A^*|B|^2A\right)^2 + \left(D|C^*|^2D^*\right)^2\right\| - \inf_{\|x\|=1} \frac{1}{4}\left\langle \left[A^*|B|^2A - D|C^*|^2D^*\right]x, x\right\rangle^2$$

Proof. Take $f(x) = x^2$ in Theorem 1, in such a way that the required λ would be '2'. □

Corollary 5. *Let $T \in \mathscr{B}(\mathscr{H})$. Let f be a positive, increasing and convex function on $\mathbb{R}$. If f is twice differentiable such that $f'' \geq \lambda > 0$, then we have*

$$f\left(w\left(T|T|^{\alpha+\beta-1}\right)\right) \leq \frac{1}{2}\left\| f\left(|T|^{2\alpha}\right) + f\left(|T^*|^{2\beta}\right)\right\| - \inf_{\|x\|=1} \xi(x), \tag{26}$$

where $\xi(x) := \frac{1}{8}\lambda\left\langle \left[|T|^{2\alpha} - |T^*|^{2\beta}\right]x, x\right\rangle^2$, for all $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \geq 1$.

Proof. Let $y = x$ in (18), we obtain

$$\begin{aligned} f\left(\left|\left\langle T|T|^{\alpha+\beta-1}x, x\right\rangle\right|\right) &\leq \frac{1}{2}\left[\left\langle f\left(|T|^{2\alpha}\right)x, x\right\rangle + \left\langle f\left(|T^*|^{2\beta}\right)x, x\right\rangle\right] \\ &\quad - \frac{1}{8}\lambda\left(\left\langle |T|^{2\alpha}x, x\right\rangle - \left\langle |T^*|^{2\beta}x, x\right\rangle\right)^2. \end{aligned}$$

Taking the supremum over all unit vectors $x \in \mathscr{H}$, we obtain the required result. □

Corollary 6. *Let $A, B \in \mathscr{B}(\mathscr{H})$. Let f be a positive, increasing and convex function on $\mathbb{R}$. If f is twice differentiable such that $f'' \geq \lambda > 0$, then we have*

$$f\left(w\left((BA)^2\right)\right) \leq \frac{1}{2}\left\|f\left(A^*|B|^2A\right) + f\left(B|A^*|^2B^*\right)\right\| - \inf_{\|x\|=1} \eta_1(x),$$

where $\eta_1(x) := \frac{1}{8}\lambda\left\langle\left[A^*|B|^2A - B|A^*|^2B^*\right]x, x\right\rangle^2$.

Proof. Setting $D = B$ and $C = A$ in (25), we establish the stated result. □

Corollary 7. *Let $A, B \in \mathscr{B}(\mathscr{H})$. Let f be a positive, increasing and convex function on $\mathbb{R}$. If f is twice differentiable such that $f'' \geq \lambda > 0$, then we have*

$$f\left(w\left(A^*B^2A\right)\right) \leq \frac{1}{2}\left\|f\left(A^*|B|^2A\right) + f\left(A^*|B^*|^2A\right)\right\| - \inf_{\|x\|=1} \eta_2(x),$$

where $\eta_2(x) := \frac{1}{8}\lambda\left\langle\left[A^*|B|^2A - A^*|B^*|^2A\right]x, x\right\rangle^2$.

Proof. Setting $D = A$ and $C = B$ in (25), we obtain the desired result. □

Corollary 8. *Let $A \in \mathscr{B}(\mathscr{H})$. Let f be a positive, increasing and convex function on $\mathbb{R}$. If f is twice differentiable such that $f'' \geq \lambda > 0$, then we have*

$$f\left(w\left(A^4\right)\right) \leq \frac{1}{2}\left\|f\left(A^*|A|^2A\right) + f\left(A|A^*|^2A^*\right)\right\| - \inf_{\|x\|=1} \eta(x),$$

where $\eta(x) := \frac{1}{8}\lambda\left\langle\left[A^*|A|^2A - A|A^*|^2A^*\right]x, x\right\rangle^2$.

Proof. Setting $D = C = B = A$ in (25), the desired result follows. □

Theorem 2. *Let $A, B, C, D \in \mathscr{B}(\mathscr{H})$. Let f be a positive, increasing, convex and supermultiplicative function on $\mathbb{R}$, i.e., $f(ts) \leq f(t)f(s)$ for all $t, s \in \mathbb{R}$. Then, we have*

$$f\left(w^2(DCBA)\right) \leq \left\|\frac{1}{p}f^p\left(A^*|B|^2A\right) + \frac{1}{q}f^q\left(D|C^*|^2D^*\right)\right\| - \inf_{\|x\|=1} \psi(x). \tag{27}$$

For all $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \geq 1$ and all $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, where

$$\psi(x) := r_0\left(\left\langle f\left(A^*|B|^2A\right)x, x\right\rangle^{\frac{p}{2}} - \left\langle f\left(D|C^*|^2D^*\right)x, x\right\rangle^{\frac{q}{2}}\right)^2.$$

Proof. Let $y = x$ in (19), we obtain

$$\begin{aligned} f\left(|\langle DCBAx, x\rangle|^2\right) \leq &\left\langle\left[\frac{1}{p}f^p\left(A^*|B|^2A\right) + \frac{1}{q}f^q\left(D|C^*|^2D^*\right)\right]x, x\right\rangle \\ &- r_0\left(\left\langle f\left(A^*|B|^2A\right)x, x\right\rangle^{\frac{p}{2}} - \left\langle f\left(D|C^*|^2D^*\right)x, x\right\rangle^{\frac{q}{2}}\right)^2. \end{aligned}$$

Taking the supremum over all unit vectors $x \in \mathscr{H}$, we obtain the required result. □

Corollary 9. *Let $T \in \mathscr{B}(\mathscr{H})$. Let f be a positive, increasing, convex and supermultiplicative function on $\mathbb{R}$, i.e., $f(ts) \leq f(t)f(s)$ for all $t, s \in \mathbb{R}$. Then we have*

$$f\left(w^2\left(T|T|^{\alpha+\beta-1}\right)\right) \leq \left\|\frac{1}{p}f^p\left(|T|^{2\alpha}\right) + \frac{1}{q}f^q\left(|T^*|^{2\beta}\right)\right\| - \inf_{\|x\|=1}\psi_1(x). \tag{28}$$

For all $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \geq 1$ and all $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, where

$$\psi_1(x) := r_0\left(\left\langle f\left(|T|^{2\alpha}\right)x, x\right\rangle^{\frac{p}{2}} - \left\langle f\left(|T^*|^{2\beta}\right)x, x\right\rangle^{\frac{q}{2}}\right)^2.$$

Proof. Let $y = x$ in (20), and then taking the supremum over all unit vectors $x \in \mathscr{H}$, we obtain the required result. □

Corollary 10. *Let $T \in \mathscr{B}(\mathscr{H})$. Then we have*

$$w^{2r}\left(T|T|^{\alpha+\beta-1}\right) \leq \left\|\frac{1}{p}|T|^{2rp\alpha} + \frac{1}{q}|T^*|^{2rq\beta}\right\| - \inf_{\|x\|=1}\psi_1(x) \tag{29}$$

for all $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \geq 1$, where

$$\psi_1(x) := r_0\left(\left\langle |T|^{2r\alpha}x, x\right\rangle^{\frac{p}{2}} - \left\langle |T^*|^{2r\beta}x, x\right\rangle^{\frac{q}{2}}\right)^2$$

for all $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Applying Corollary 9 for the convex increasing function $f(t) = t^r$, $(t > 0)$ $r \geq 1$, we obtain the stated result. □

Remark 1. *In (29), let $p = q = 2$, we obtain*

$$w^{2r}\left(T|T|^{\alpha+\beta-1}\right) \leq \frac{1}{2}\left\||T|^{4r\alpha} + |T^*|^{4r\beta}\right\| - \inf_{\|x\|=1}\psi_2(x) \tag{30}$$

for all $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \geq 1$, where

$$\psi_2(x) := \frac{1}{2}\left(\left\langle |T|^{2r\alpha}x, x\right\rangle - \left\langle |T^*|^{2r\beta}x, x\right\rangle\right)^2.$$

In particular, for $\alpha = \beta = \frac{1}{2}$, we have

$$w^{2r}(T) \leq \frac{1}{2}\left\||T|^{2r} + |T^*|^{2r}\right\| - \frac{1}{2}\inf_{\|x\|=1}\left(\langle |T|^r x, x\rangle - \langle |T^*|^r x, x\rangle\right)^2 \tag{31}$$

for all $r \geq 1$.

Example 1. *Let $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$. Applying (31) with $r = 1$, simple calculations yield that $\omega(A) = 7.049$, $\left\||T|^2 + |T^*|^2\right\| = 99.8911$ and $\inf_{\|x\|=1}(\langle |T|x, x\rangle - \langle |T^*|x, x\rangle)^2 = 0.3048$. Thus, we have*

$$
\begin{aligned}
w(T) &\le \sqrt{\frac{1}{2}\left\| |T|^2 + |T^*|^2 \right\| - \frac{1}{2}\inf_{\|x\|=1}\left(\langle |T|x,x\rangle - \langle |T^*|x,x\rangle\right)^2} \\
&= 7.056 \\
&\le \frac{1}{2}\left\| |T|^2 + |T^*|^2 \right\| = 7.067,
\end{aligned}
$$

which means that (31) *is a non-trivial improvement of the right-hand side of* (10).

Theorem 3. *Let $A,B,C,D \in \mathscr{B}(\mathscr{H})$. Let f be a positive, increasing, convex and supermultiplicative function on $\mathbb{R}$, i.e., $f(ts) \le f(t)f(s)$ for all $t,s \in \mathbb{R}$. Then we have*

$$
f\left(w^2(DCBA)\right) \le 2^{-\frac{2}{r}}\left\| f^r\left(A^*|B|^2A\right) + f^r\left(D|C^*|^2D^*\right)\right\|^{\frac{2}{r}} - \inf_{\|x\|=1}\phi(x), \tag{32}
$$

where

$$
\phi(x) := \frac{1}{4}\left[\left\langle \left[f\left(A^*|B|^2A\right) - f\left(D|C^*|^2D^*\right)\right]x,x\right\rangle\right].
$$

As a special case, we have

$$
f\left(w^2(DCBA)\right) \le \frac{1}{4}\left\| f\left(A^*|B|^2A\right) + f\left(D|C^*|^2D^*\right)\right\|^2 - \inf_{\|x\|=1}\phi(x). \tag{33}
$$

Proof. Let $y = x$ in (21), we obtain

$$
\begin{aligned}
f\left(|\langle DCBAx,x\rangle|^2\right) \le{}& 2^{-\frac{2}{r}}\left(\left\langle f^r\left(A^*|B|^2A\right)x,x\right\rangle + \left\langle f^r\left(D|C^*|^2D^*\right)x,x\right\rangle\right)^{\frac{2}{r}} \\
&- \frac{1}{4}\left[\left\langle f\left(A^*|B|^2A\right)x,x\right\rangle - \left\langle f\left(D|C^*|^2D^*\right)x,x\right\rangle\right]
\end{aligned}
$$

Taking the supremum over all unit vectors $x \in \mathscr{H}$, we obtain the required result. The particular case follows by setting $y = x$ in (22) and then taking the supremum over all unit vectors $x \in \mathscr{H}$. □

Corollary 11. *Let $A,B,C,D \in \mathscr{B}(\mathscr{H})$. Then, we have*

$$
w^{2\lambda}(DCBA) \le 2^{-\frac{2}{r}}\left\| \left(A^*|B|^2A\right)^{r\lambda} + \left(D|C^*|^2D^*\right)^{r\lambda}\right\|^{\frac{2}{r}} - \inf_{\|x\|=1}\phi(x), \tag{34}
$$

where

$$
\phi_1(x) := \frac{1}{4}\left[\left\langle \left[\left(A^*|B|^2A\right)^{\lambda} - \left(D|C^*|^2D^*\right)^{\lambda}\right]x,x\right\rangle\right].
$$

In this particular case, we have

$$
w^{2\lambda}(DCBA) \le \frac{1}{4}\left\| \left(A^*|B|^2A\right)^{\lambda} + \left(D|C^*|^2D^*\right)^{\lambda}\right\|^2 - \inf_{\|x\|=1}\phi_1(x). \tag{35}
$$

Proof. Applying Theorem 3 for $f(t) = t^{\lambda}$ $(\lambda \ge 1)$, we obtain the required result. □

Corollary 12. *Let $T \in \mathscr{B}(\mathscr{H})$. Let f be a positive, increasing, convex and supermultiplicative function on $\mathbb{R}$, i.e., $f(ts) \le f(t)f(s)$ for all $t, s \in \mathbb{R}$. Then we have*

$$f\left(w^2\left(T|T|^{\alpha+\beta-1}\right)\right) \le 2^{-\frac{2}{r}}\left\|f^r\left(|T|^{2\alpha}\right) + f^r\left(|T^*|^{2\beta}\right)\right\|^{\frac{2}{r}} - \inf_{\|x\|=1} \Psi(x), \tag{36}$$

where

$$\Psi(x) := \frac{1}{4}\left[\left\langle\left[f\left(|T|^{2\alpha}\right) - f\left(|T^*|^{2\beta}\right)\right]x, x\right\rangle\right].$$

Proof. The proof follows by considering $D = U$, $B = 1_{\mathscr{H}}$, $C = |T|^{\beta}$ and $A = |T|^{\alpha}$ such that $\alpha + \beta \ge 1$ in (32). $\square$

Corollary 13. *Let $T \in \mathscr{B}(\mathscr{H})$. Then we have*

$$w^{2\lambda}\left(T|T|^{\alpha+\beta-1}\right) \le 2^{-\frac{2}{r}}\left\||T|^{2r\alpha\lambda} + |T^*|^{2r\beta\lambda}\right\|^{\frac{2}{r}} - \inf_{\|x\|=1} \Psi_1(x) \tag{37}$$

for all $\alpha, \beta \ge 0$ such that $\alpha + \beta \ge 1$, where

$$\Psi_1(x) := \frac{1}{4}\left\langle\left[|T|^{2\alpha\lambda} - |T^*|^{2\beta\lambda}\right]x, x\right\rangle.$$

Proof. Setting $f(t) = t^{\lambda}$ $(\lambda \ge 1)$ in Corollary 12, we obtain the required result. $\square$

Remark 2. *By choosing $\alpha = \beta = \frac{1}{2}$ in (37), we obtain*

$$w^{2\lambda}(T) \le 2^{-\frac{2}{r}}\left\||T|^{r\lambda} + |T^*|^{r\lambda}\right\|^{\frac{2}{r}} - \frac{1}{4}\inf_{\|x\|=1}\left\langle\left[|T|^{\lambda} - |T^*|^{\lambda}\right]x, x\right\rangle \tag{38}$$

for all $r, \lambda \ge 1$.

Furthermore, for $r = 1$ in (38), we obtain

$$w^{2\lambda}(T) \le \frac{1}{4}\left\||T|^{\lambda} + |T^*|^{\lambda}\right\|^2 - \frac{1}{4}\inf_{\|x\|=1}\left\langle\left[|T|^{\lambda} - |T^*|^{\lambda}\right]x, x\right\rangle$$

for all $\lambda \ge 1$.

In general, for $\lambda = 1$ in (38), we have

$$w^2(T) \le 2^{-\frac{2}{r}}\||T|^r + |T^*|^r\|^{\frac{2}{r}} - \frac{1}{4}\inf_{\|x\|=1}\langle[|T| - |T^*|]x, x\rangle$$

for all $r \ge 1$. In particular, for $r = 1$, we have

$$\begin{aligned} w^2(T) &\le \frac{1}{4}\||T| + |T^*|\|^2 - \frac{1}{4}\inf_{\|x\|=1}\langle[|T| - |T^*|]x, x\rangle \\ &= \left\|\left(\frac{|T| + |T^*|}{2}\right)^2\right\| - \frac{1}{4}\inf_{\|x\|=1}\langle[|T| - |T^*|]x, x\rangle \\ &\le \left\|\frac{|T|^2 + |T^*|^2}{2}\right\| - \frac{1}{4}\inf_{\|x\|=1}\langle[|T| - |T^*|]x, x\rangle \\ &\le \frac{1}{2}\left\||T|^2 + |T^*|^2\right\|, \end{aligned} \tag{39}$$

which refines the right-hand side of (10)*, where we have used the fact that*

$$\left\| f\left(\frac{T+S}{2}\right) \right\| \le \left\| \frac{f(T)+f(S)}{2} \right\|$$

for every non-negative convex function f and all positive operators $T, S \in \mathscr{B}(\mathscr{H})$ (see [32]), in the second inequality above.

Example 2. *Let $A = \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}$. Applying* (39)*, simple calculations yield that $\omega(A) = 7.0276$, $\||T| + |T^*|\| = 14.0553$, $\left\||T|^2 + |T^*|^2\right\| = 99.2769$, and $\inf_{\|x\|=1} \langle [|T| - |T^*|]x, x \rangle = -0.993883$. Thus, we have*

$$\begin{aligned} w(T) &\le \sqrt{\frac{1}{4}\||T| + |T^*|\|^2 - \frac{1}{4} \inf_{\|x\|=1} \langle [|T| - |T^*|]x, x \rangle} \\ &= 7.045348 \\ &\le \sqrt{\frac{1}{2}\left\||T|^2 + |T^*|^2\right\|} = 7.045457, \end{aligned}$$

which means that (39) *is a non-trivial improvement of the right-hand side of the celebrated Kittaneh Inequality* (10).

The numerical radius inequality of special type of Hilbert space operators for commutators can be established as follows:

Lemma 9. *Let $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2 \in \mathscr{B}(\mathscr{H})$. Then, for all $r \ge 1$, the following inequality:*

$$\begin{aligned} &|\langle (D_1C_1B_1A_1 + D_2C_2B_2A_2)x, y \rangle| \\ &\le 2^{-\frac{1}{r}} \left(\left\langle \left(A_1^*|B_1|^2A_1 \right)^r x, x \right\rangle + \left\langle \left(D_1|C_1^*|^2D_1^* \right)^r y, y \right\rangle \right)^{\frac{1}{r}} \\ &\qquad - \frac{1}{2}\left(\left\langle A_1^*|B_1|^2A_1x, x \right\rangle^{\frac{1}{2}} - \left\langle D_1|C_1^*|^2D_1^*y, y \right\rangle^{\frac{1}{2}} \right)^2 \\ &\quad + 2^{-\frac{1}{r}} \left(\left\langle \left(A_2^*|B_2|^2A_2 \right)^r x, x \right\rangle + \left\langle \left(D_2|C_2^*|^2D_2^* \right)^r y, y \right\rangle \right)^{\frac{1}{r}} \\ &\qquad - \frac{1}{2}\left(\left\langle A_2^*|B_2|^2A_2x, x \right\rangle^{\frac{1}{2}} - \left\langle D_2|C_2^*|^2D_2^*y, y \right\rangle^{\frac{1}{2}} \right)^2 \end{aligned} \tag{40}$$

holds for all vectors $x, y \in \mathscr{H}$.

Proof. Employing the triangle inequality and the Inequality (6), we have

$$\begin{aligned}
&|\langle (D_1C_1B_1A_1 + D_2C_2B_2A_2)x, y\rangle| \\
&\leq |\langle (D_1C_1B_1A_1)x, y\rangle| + |\langle (D_2C_2B_2A_2)x, y\rangle| \\
&\leq \left\langle A_1^*|B_1|^2A_1x, x\right\rangle^{\frac{1}{2}} \left\langle D_1|C_1^*|^2D_1^*y, y\right\rangle^{\frac{1}{2}} \\
&\qquad + \left\langle A_2^*|B_2|^2A_2x, x\right\rangle^{\frac{1}{2}} \left\langle D_2|C_2^*|^2D_2^*y, y\right\rangle^{\frac{1}{2}} \\
&\leq 2^{-\frac{1}{r}} \left(\left\langle A_1^*|B_1|^2A_1x, x\right\rangle^{r} + \left\langle D_1|C_1^*|^2D_1^*y, y\right\rangle^{r} \right)^{\frac{1}{r}} \\
&\qquad\qquad - \frac{1}{2}\left(\left\langle A_1^*|B_1|^2A_1x, x\right\rangle^{\frac{1}{2}} - \left\langle D_1|C_1^*|^2D_1^*y, y\right\rangle^{\frac{1}{2}} \right)^2 \\
&\qquad + 2^{-\frac{1}{r}} \left(\left\langle A_2^*|B_2|^2A_2x, x\right\rangle^{r} + \left\langle D_2|C_2^*|^2D_2^*y, y\right\rangle^{r} \right)^{\frac{1}{r}} \\
&\qquad\qquad - \frac{1}{2}\left(\left\langle A_2^*|B_2|^2A_2x, x\right\rangle^{\frac{1}{2}} - \left\langle D_2|C_2^*|^2D_2^*y, y\right\rangle^{\frac{1}{2}} \right)^2 \\
&\leq 2^{-\frac{1}{r}} \left(\left\langle \left(A_1^*|B_1|^2A_1\right)^{r}x, x\right\rangle + \left\langle \left(D_1|C_1^*|^2D_1^*\right)^{r}y, y\right\rangle \right)^{\frac{1}{r}} \\
&\qquad\qquad - \frac{1}{2}\left(\left\langle A_1^*|B_1|^2A_1x, x\right\rangle^{\frac{1}{2}} - \left\langle D_1|C_1^*|^2D_1^*y, y\right\rangle^{\frac{1}{2}} \right)^2 \\
&\qquad + 2^{-\frac{1}{r}} \left(\left\langle \left(A_2^*|B_2|^2A_2\right)^{r}x, x\right\rangle + \left\langle \left(D_2|C_2^*|^2D_2^*\right)^{r}y, y\right\rangle \right)^{\frac{1}{r}} \\
&\qquad\qquad - \frac{1}{2}\left(\left\langle A_2^*|B_2|^2A_2x, x\right\rangle^{\frac{1}{2}} - \left\langle D_2|C_2^*|^2D_2^*y, y\right\rangle^{\frac{1}{2}} \right)^2
\end{aligned}$$

for all vectors $x, y \in \mathscr{H}$, which proves the result. □

Corollary 14. *Let $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2 \in \mathscr{B}(\mathscr{H})$. Then, the following inequality:*

$$\begin{aligned}
&w((D_1C_1B_1A_1 + D_2C_2B_2A_2)) \\
&\leq 2^{-\frac{1}{r}} \left\| \left(A_1^*|B_1|^2A_1\right)^{r} + \left(D_1|C_1^*|^2D_1^*\right)^{r} \right\|^{\frac{1}{r}} \\
&\quad + 2^{-\frac{1}{r}} \left\| \left(A_2^*|B_2|^2A_2\right)^{r} + \left(D_2|C_2^*|^2D_2^*\right)^{r} \right\|^{\frac{1}{r}} \\
&\quad - \frac{1}{2}\inf_{\|x\|=1}\left(\left\langle A_1^*|B_1|^2A_1x, x\right\rangle^{\frac{1}{2}} - \left\langle D_1|C_1^*|^2D_1^*x, x\right\rangle^{\frac{1}{2}} \right)^2 \\
&\quad - \frac{1}{2}\inf_{\|x\|=1}\left(\left\langle A_2^*|B_2|^2A_2x, x\right\rangle^{\frac{1}{2}} - \left\langle D_2|C_2^*|^2D_2^*x, x\right\rangle^{\frac{1}{2}} \right)^2
\end{aligned} \tag{41}$$

holds for all $r \geq 1$.

Proof. Let $y = x$ in (40) and then taking the supremum over all unit vectors $x \in \mathscr{H}$, we obtain the mentioned result. □

Corollary 15. *Let $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2 \in \mathscr{B}(\mathscr{H})$. Then we have*

$$\begin{aligned} &w((D_1C_1B_1A_1 + D_2C_2B_2A_2)) \\ &\leq \frac{1}{2}\left\| A_1^*|B_1|^2A_1 + D_1|C_1^*|^2D_1^* + A_2^*|B_2|^2A_2 + D_2|C_2^*|^2D_2^* \right\| \\ &\quad - \frac{1}{2}\inf_{\|x\|=1}\left(\left\langle A_1^*|B_1|^2A_1x, x\right\rangle^{\frac{1}{2}} - \left\langle D_1|C_1^*|^2D_1^*x, x\right\rangle^{\frac{1}{2}}\right)^2 \\ &\quad - \frac{1}{2}\inf_{\|x\|=1}\left(\left\langle A_2^*|B_2|^2A_2x, x\right\rangle^{\frac{1}{2}} - \left\langle D_2|C_2^*|^2D_2^*x, x\right\rangle^{\frac{1}{2}}\right)^2 \end{aligned} \tag{42}$$

for all vectors $x \in \mathscr{H}$.

Proof. Let $y = x$ in (40) and consider $r = 1$. In the proof of (42), combining the inner products, then taking the supremum over all unit vectors $x \in \mathscr{H}$, we obtain the required result. □

In special cases, a particular choice of A, B, C, D in the Corollaries 14 and 15 would give the following result:

Corollary 16. *Let $T, S \in \mathscr{B}(\mathscr{H})$, $\alpha, \beta, \gamma, \delta \geq 0$ such that $\alpha + \beta \geq 1$ and $\gamma + \delta \geq 1$. Then we have*

$$\begin{aligned} &w\left(T|T|^{\alpha+\beta-1} + S|S|^{\gamma+\delta-1}\right) \\ &\leq 2^{-\frac{1}{r}}\left\| |T|^{2r\alpha} + |T^*|^{2r\beta} \right\|^{\frac{1}{r}} + 2^{-\frac{1}{r}}\left\| |S|^{2r\gamma} + |S^*|^{2r\delta} \right\|^{\frac{1}{r}} \\ &\quad - \frac{1}{2}\inf_{\|x\|=1}\left(\left\langle |T|^{2\alpha}x, x\right\rangle^{\frac{1}{2}} - \left\langle |T^*|^{2\beta}x, x\right\rangle^{\frac{1}{2}}\right)^2 \\ &\quad - \frac{1}{2}\inf_{\|x\|=1}\left(\left\langle |S|^{2\gamma}x, x\right\rangle^{\frac{1}{2}} - \left\langle |S^*|^{2\delta}x, x\right\rangle^{\frac{1}{2}}\right)^2 \end{aligned} \tag{43}$$

for all $r \geq 1$.

Proof. Let $D = U$, $B = 1_{\mathscr{H}}$, $C = |T|^{\beta}$ and $A = |T|^{\alpha}$ such that $\alpha + \beta \geq 1$ in (42), then we have

$$DCBA = U|T|^{\beta}|T|^{\alpha} = U|T||T|^{\alpha+\beta-1} = T|T|^{\alpha+\beta-1},$$

also, we have $A^*|B|^2A = |T|^{2\alpha}$ and $D|C^*|^2D^* = U|T|^{2\beta}U^* = |T|^{2\beta}$. □

Corollary 17. *Let $T, S \in \mathscr{B}(\mathscr{H})$, $\alpha, \beta, \gamma, \delta \geq 0$ such that $\alpha + \beta \geq 1$ and $\gamma + \delta \geq 1$. Then we have*

$$\begin{aligned} w\left(T|T|^{\alpha+\beta-1} + S|S|^{\gamma+\delta-1}\right) \leq &\frac{1}{2}\left\| |T|^{2\alpha} + |T^*|^{2\beta} + |S|^{2\gamma} + |S^*|^{2\delta} \right\| \\ &- \frac{1}{2}\inf_{\|x\|=1}\left(\left\langle |T|^{2\alpha}x, x\right\rangle^{\frac{1}{2}} - \left\langle |T^*|^{2\beta}x, x\right\rangle^{\frac{1}{2}}\right)^2 \\ &- \frac{1}{2}\inf_{\|x\|=1}\left(\left\langle |S|^{2\gamma}x, x\right\rangle^{\frac{1}{2}} - \left\langle |S^*|^{2\delta}x, x\right\rangle^{\frac{1}{2}}\right)^2. \end{aligned} \tag{44}$$

Proof. It is enough to consider $D = U$, $B = 1_{\mathscr{H}}$, $C = |T|^{\beta}$ and $A = |T|^{\alpha}$ such that $\alpha + \beta \geq 1$ in (42). □

Remark 3. *Setting $\alpha = \beta = \gamma = \delta = \frac{1}{2}$ in (44), we obtain*

$$w(T+S) \le \frac{1}{2}\||T| + |T^*| + |S| + |S^*|\| - \frac{1}{2}\inf_{\|x\|=1}\left(\langle |T|x,x\rangle^{\frac{1}{2}} - \langle |T^*|x,x\rangle^{\frac{1}{2}}\right)^2 - \frac{1}{2}\inf_{\|x\|=1}\left(\langle |S|x,x\rangle^{\frac{1}{2}} - \langle |S^*|x,x\rangle^{\frac{1}{2}}\right)^2.$$

In particular, take $S = T$, we obtain

$$w(T) \le \frac{1}{2}\||T| + |T^*|\| - \frac{1}{2}\inf_{\|x\|=1}\left(\langle |T|x,x\rangle^{\frac{1}{2}} - \langle |T^*|x,x\rangle^{\frac{1}{2}}\right)^2 \tag{45}$$

Example 3. *Let $A = \begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix}$. Applying (45), simple calculations yield that $\omega(A) = 7.162$, $\||T| + |T^*|\| = 14.3819$, and $\inf_{\|x\|=1}\left(\langle |T|x,x\rangle^{\frac{1}{2}} - \langle |T^*|x,x\rangle^{\frac{1}{2}}\right)^2 = 0.0083657$. Thus, we have*

$$\begin{aligned} w(T) &\le \frac{1}{2}\||T| + |T^*|\| - \frac{1}{2}\inf_{\|x\|=1}\left(\langle |T|x,x\rangle^{\frac{1}{2}} - \langle |T^*|x,x\rangle^{\frac{1}{2}}\right)^2 \\ &= 7.1867 \\ &\le \frac{1}{2}\||T| + |T^*|\| = 7.1909, \end{aligned}$$

which means that (45) is a non-trivial improvement of the celebrated Kittaneh Inequality (2).

Remark 4. *Setting $\alpha = \beta = \gamma = \delta = 1$ in (45), we obtain*

$$\begin{aligned} w(T|T| + S|S|) \le \frac{1}{2}\left\||T|^2 + |T^*|^2 + |S|^2 + |S^*|^2\right\| &- \frac{1}{2}\inf_{\|x\|=1}\left(\left\langle |T|^2x,x\right\rangle^{\frac{1}{2}} - \left\langle |T^*|^2x,x\right\rangle^{\frac{1}{2}}\right)^2 \\ &- \frac{1}{2}\inf_{\|x\|=1}\left(\left\langle |S|^2x,x\right\rangle^{\frac{1}{2}} - \left\langle |S^*|^2x,x\right\rangle^{\frac{1}{2}}\right)^2 \end{aligned}$$

In particular, take $S = T$, we obtain

$$\begin{aligned} w(T|T|) &\le \frac{1}{2}\left\||T|^2 + |T^*|^2\right\| - \frac{1}{2}\inf_{\|x\|=1}\left(\left\langle |T|^2x,x\right\rangle^{\frac{1}{2}} - \left\langle |T^*|^2x,x\right\rangle^{\frac{1}{2}}\right)^2 \\ &= \frac{1}{2}\|T^*T + TT^*\| - \frac{1}{2}\inf_{\|x\|=1}\left(\left\langle |T|^2x,x\right\rangle^{\frac{1}{2}} - \left\langle |T^*|^2x,x\right\rangle^{\frac{1}{2}}\right)^2. \end{aligned}$$

4. Conclusions

In this work, some numerical radius inequalities based on the recent Dragomir extension of Furuta's inequality are obtained. Some particular cases are also provided. Among others, the celebrated Kittaneh inequality reads:

$$w(T) \le \frac{1}{2}\||T| + |T^*|\|.$$

It is proven that

$$w(T) \le \frac{1}{2}\||T| + |T^*|\| - \frac{1}{2}\inf_{\|x\|=1}\left(\langle |T|x,x\rangle^{\frac{1}{2}} - \langle |T^*|x,x\rangle^{\frac{1}{2}}\right)^2,$$

which improves the Kittaneh inequality for symmetric and non-symmetric Hilbert space operators. Other related improvements to well-known inequalities in literature are also provided. Namely, inequalities for the numerical radius of the product of several Hilbert space operators are refined and improved.

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