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Montgomery Identity and Ostrowski-Type Inequalities for Generalized Quantum Calculus through Convexity and Their Applications

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Abstract: The celebrated Montgomery identity has been studied extensively since it was established. We found a novel version of the Montgomery identity when we were working inside the framework of p- and q-calculus. We acquire a Montgomery identity through a definite (p, q)-integral from these results. Consequently, we establish specific Ostrowski-type (p, q)-integral inequalities by using Montgomery identity. In addition to the well-known repercussions, this novel study provides an opportunity to set up new boundaries in the field of comparative literature. The research that is being proposed on the (p, q)-integral includes some fascinating results that demonstrate the superiority and applicability of the findings that have been achieved. This highly successful and valuable strategy is anticipated to create a new venue in the contemporary realm of special relativity and quantum theory. These mathematical inequalities and the approaches that are related to them have applications in the areas that deal with symmetry. Additionally, an application to special means is provided in the conclusion.

Keywords: quantum montgomery identity; quantum calculus theory; post quantum calculus theory; Hölder's inequality; power mean inequality

Citation: Kalsoom, H.; Vivas-Cortez, M.; Abidin, M.Z.; Marwan, M.; Khan, Z.A. Montgomery Identity and Ostrowski-Type Inequalities for Generalized Quantum Calculus through Convexity and Their Applications. *Symmetry* **2022**, *14*, 1449. https://doi.org/10.3390/sym14071449

Academic Editor: Alexander Zaslavski

Received: 19 May 2022
Accepted: 20 June 2022
Published: 15 July 2022

1. Introduction

The field of quantum calculus has been established for a long time and has numerous applications in the fields of mathematics, physics, and engineering research. Researchers in number theory, hypergeometric functions, special functions, and other areas of mathematics have made extensive use of it, indicating that it played an important role in the field of mathematics. We would like to direct the attention of the readers to two essential publications on the fundamentals of q-calculus that were written by Ernst [1] and Kac and Cheung [2]. In the area of approximation theory, the first work was published in 1987, when the Romanian mathematician Lupaş [3] introduced the q analogof Bernstein polynomials. This marked the beginning of the discipline of approximation theory. He investigated several characteristics of the Bernstein polynomials using the q variation. However, sadly, the researchers of the period showed only a moderate amount of interest in the applications of quantum calculus in approximation theory. Nearly ten years later, Phillips [4] proposed another q variation of the Bernstein polynomials. Subsequently, academics began focusing their attention on this particular path of inquiry. The authors Ntouyas and Tariboon [5,6] examined how quantum derivatives and quantum integrals are solved across the intervals of the type $[\varphi_1, \varphi_2] \subset \mathbb{R}$ in their prior papers, which establish various quantum analogs for several well-known effects, including as the Hölder inequality, the Hermite–Hadamard inequality, and the Ostrowski inequality, as well as Cauchy–Bunyakovsky–Schwarz, Gruss, Gruss–Chebyshev, and other integral inequalities that make use of classical convexity. Research on q-calculus analysis has been conducted by a large number of mathematicians; for more information, the reader may refer to [7–16].

In addition, q-calculus can be extended to post-quantum calculus, more specifically (p,q)-calculus. In the most recent investigation of the quantum calculation, we dealt with q numbers using a single q basis. On the other hand, in post-quantum calculus, a p and a q number were used in conjunction with two independent variables, p and q. Since the publication of this study, researchers have discovered several additional (p,q)-analogs of classical inequality. For instance, Chakarabarti and Jagannathan [17] have suggested using this interpretation of the term. New post-quantum analogs of Hermite–Hadamard's inequality have been discovered by Kunt and colleagues [18]. To prove some recent parameterized inequalities, Luo et al. [19] employed a generalized integral identity that involved functions that could be differentiated by p and q. In 2015, Mursaleen et al. [20] proved the approximation Results by (p,q)-analogue of Bernstein-Stancu operators. Following is a description of the definitions of (p,q)-derivatives and (p,q)-integrals at finite intervals. In 2018, Duran [21] wrote a note on the (p,q) Hermite polynomials which is new direction in the field of post quantum calculus. These concepts were inspired by the most recent research on Tunç and Góv [22].

Definition 1 ([22]). *If $f:[\varphi_1,\varphi_2]\to\mathbb{R}$ is a continuous function, then the (p,q)-derivative of f at $\kappa\in[\varphi_1,\varphi_2]$ with $0<q<p\leq 1$ is defined as*

$$_{\varphi_1}D_{p,q}f(\kappa)=\frac{f(p\kappa+(1-p)\varphi_1)-f(q\kappa+(1-q)\varphi_1)}{(p-q)(\kappa-\varphi_1)},\quad \kappa\neq\varphi_1. \tag{1}$$

Since f is a arbitrary function, then we obtain $_{\varphi_1}D_{p,q}f(\varphi_1)=\lim\limits_{\kappa\to\varphi_1}D_{p,q}f(\kappa)$.

Note that for $p=1$

$$\lim_{q\to 1^-}D_qf(\kappa)=\frac{df(\kappa)}{d\kappa}, \tag{2}$$

if $f(\kappa)$ is differentiable.

Taking $\varphi_1=0$ in (1), then $d_{p,q}f=D_{p,q}f$, where $D_{p,q}f$ is well-known (p,q)-derivative of $f(\kappa)$, denoted as follows

$$D_{p,q}f(\kappa)=\frac{f(p\kappa)-f(q\kappa)}{(p-q)\kappa},\ \kappa\neq 0. \tag{3}$$

If we change $p=1$ in (3), then we will obtain an interpretation of the q- derivative, which can be found in [2], and it will be written as follows:

$$D_qf(\kappa)=\frac{f(\kappa)-f(q\kappa)}{(1-q)\kappa},\ \kappa\neq 0.$$

Example 1. *Let a mapping $f:[\varphi_1,\varphi_2]\to\mathbb{R}$ by $f(\kappa)=\kappa^2+\kappa+8$ with $0<q<p\leq 1$. Then, for $\kappa\neq\varphi_1$, we have*

$$\begin{aligned}_{\varphi_1}D_{p,q}(\kappa^2+\kappa+8) &= \frac{\left[(p\kappa+(1-p)\varphi_1)^2+(p\kappa+(1-p)\varphi_1)+8\right]}{(p-q)(\kappa-\varphi_1)}\\ &-\frac{\left[(q\kappa+(1-q)\varphi_1)^2+(q\kappa+(1-q)\varphi_1)+8\right]}{(p-q)(\kappa-\varphi_1)}\end{aligned}$$

$$= \frac{\kappa^2[2]_{p,q} + 2\varphi_1\kappa[1-[2]_{p,q}] + \varphi_1^2[[2]_{p,q}-2] + (\kappa-\varphi_1)}{(\kappa-\varphi_1)}$$
$$= \frac{\kappa[2]_{p,q}(\kappa-\varphi_1) + \varphi_1[2]_{p,q}(\kappa-\varphi_1) + 2\varphi_1(\kappa-\varphi_1) + (\kappa-\varphi_1)}{(\kappa-\varphi_1)}$$
$$= [2]_{p,q}(\kappa-\varphi_1) + 2\varphi_1 + 1.$$

Definition 2 ([22]). *Let $f : [\varphi_1, \varphi_2] \to \mathbb{R}$ be a arbitrary function, the (p,q)-integral on $[\varphi_1, \varphi_2]$, with $0 < q < p \leq 1$ is defined as follows:*

$$\int_{\varphi_1}^{\kappa} f(x)_{\varphi_1} d_{p,q}x = (p-q)(\kappa-\varphi_1) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}}\kappa + \left(1 - \frac{q^n}{p^{n+1}}\right)\varphi_1\right), \tag{4}$$

for all $\kappa \in [\varphi_1, \varphi_2]$.

Notice that if we consider $\varphi_1 = 0$ in (4), we obtain

$$\int_{0}^{\kappa} f(x)_0 d_{p,q}x = (p-q)\kappa \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}}\kappa\right).$$

If $c \in (\varphi_1, \lambda)$, then the definite (p,q)- integral on $[c, \kappa]$ is expressed as

$$\int_{c}^{\kappa} f(x)_{\varphi_1} d_{p,q}x = \int_{\varphi_1}^{\kappa} f(x)_{\varphi_1} d_{p,q}x - \int_{\varphi_1}^{c} f(x)_{\varphi_1} d_{p,q}x.$$

Example 2. *Define function $f : [\varphi_1, \varphi_2] \to \mathbb{R}$ with $f(\kappa) = 2\kappa + 5$ and $0 < q < p \leq 1$. Then,*

$$\begin{aligned}\int_{\varphi_1}^{\varphi_2} (2\kappa+5)\varphi_1 d_{p,q}x \qquad &= 2(\varphi_2-\varphi_1)(p-q) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left(\frac{q^n}{p^{n+1}}\varphi_2 + \left(1 - \frac{q^n}{p^{n+1}}\right)\varphi_1\right) \\ &+5(\varphi_2-\varphi_1)(p-q) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \\ &= \frac{2(\varphi_2-\varphi_1)(\varphi_2-\varphi_1(1-p-q))}{[2]_{p,q}} + 5(\varphi_2-\varphi_1).\end{aligned}$$

Generalized inequality, also known as (*p*,*q*)-Hermite–Hadamard-type inequality, was demonstrated by Kunt and colleagues [18].

Theorem 1 ([18]). *Suppose that a mapping $f : [\varphi_1, \varphi_2] \to \mathbb{R}$ ia convex differentiable on $[\varphi_1, \varphi_2]$ and $0 < q < p \leq 1$. Then, we have*

$$f\left(\frac{q\varphi_1 + p\varphi_2}{[2]_{p,q}}\right) \leq \frac{1}{p(\varphi_2-\varphi_1)} \int_{\varphi_1}^{p\varphi_2+(1-p)\varphi_1} f(x)_{\varphi_1} d_{p,q}x \leq \frac{qf(\varphi_1) + pf(\varphi_2)}{[2]_{p,q}}.$$

The Ostrowski-type inequality is the one of the most notable inequalities in the literature and was introduced by Dragomir et al. in [23].

Theorem 2. *Let $f : [\varphi_1, \varphi_2] \to \mathbb{R}$ be a differentiable mapping on (φ_1, φ_2) and $f' \in L[\varphi_1, \varphi_2]$. If $|f'(x)| \leq M$ where $x \in [\varphi_1, \varphi_2]$, then the following inequality holds*

$$\left| f(x) - \frac{1}{\varphi_2-\varphi_1} \int_{\varphi_1}^{\varphi_2} f(\kappa) d\kappa \right| \leq \frac{M}{\varphi_2-\varphi_1} \left[\frac{(x-\varphi_1)^2}{2} + \frac{(\varphi_2-x)^2}{2} \right] \tag{5}$$

for all $x \in [\varphi_1, \varphi_2]$.

The following well-known Montgomery identity is highly important for proving the Ostrowski-type inequality described above; for more information, see [24].

$$f(x)-\frac{1}{\varphi_2-\varphi_1}\int_{\varphi_1}^{\varphi_2} f(\kappa)d\kappa=\int_{\varphi_1}^{x}\frac{\kappa-\varphi_1}{\varphi_2-\varphi_1}f'(\kappa)d\kappa+\int_{x}^{\varphi_2}\frac{\kappa-\varphi_2}{\varphi_2-\varphi_1}f'(\kappa)d\kappa, \tag{6}$$

where $f(x)$ is a continuous function on $[\varphi_1,\varphi_2]$ with a continuous first derivative on (φ_1,φ_2).

By changing the variable, the Montgomery identity (6) can be expressed in the following way:

$$f(x)-\frac{1}{\varphi_2-\varphi_1}\int_{\varphi_1}^{\varphi_2} f(\kappa)d\kappa=(\varphi_2-\varphi_1)\int_0^1 K(\kappa)f'(\kappa\varphi_2+(1-\kappa)\varphi_1)d\kappa, \tag{7}$$

where

$$K(\kappa)=\begin{cases}\kappa, & \kappa\in\left[0,\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right];\\ \kappa-1, & \kappa\in\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1},1\right],\end{cases}$$

for all $x\in[\varphi_1,\varphi_2]$.

The aim of this paper is to investigate a Montgomery identity for post-quantum theory that is a generalization of the identity proved in [24]. We further use this identity to obtain generalizations of the Ostrowski type, midpoint type, etc., for (p,q)-calculus. We shall deal with functions whose derivatives in absolute values are the (p,q)-differentiable convex. Relevant connections of the results obtained in this work with those deduced in earlier published papers are considered. Additionally, an application to special means is provided in the conclusion. We hope our results will motivate further work in various fields of pure and applied sciences.

2. Main Results

Lemma 1. *Let $f:[\varphi_1,\varphi_2]\to\mathbb{R}$ be an arbitrary function, where ${}_{\varphi_1}D_{p,q}f$ is quantum-integrable on $[\varphi_1,\varphi_2]$, then the following quantum identity holds:*

$$\begin{aligned}&f(x)-\frac{1}{p^2(\varphi_2-\varphi_1)}\int_{\varphi_1}^{p\varphi_2+(1-p)\varphi_1} f(\kappa)\ {}_{\varphi_1}d_{p,q}\kappa\\ &=(\varphi_2-\varphi_1)\int_0^1 K_{p,q}(\kappa)\ {}_{\varphi_1}D_{p,q}f\ (\kappa\varphi_2+(1-\kappa)\varphi_1)\ d_{p,q}\kappa,\end{aligned} \tag{8}$$

with

$$K_{p,q}(\kappa)=\begin{cases}pq\kappa, & \kappa\in\left[0,\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right];\\ pq\kappa-p, & \kappa\in\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1},1\right],\end{cases}$$

for all $x\in[\varphi_1,\varphi_2]$.

Proof. Calculating the integral for the right side of (8), with the help of Definitions 1 and 2, gives us

$$(\varphi_2-\varphi_1)\int_0^1 K_{p,q}(\kappa)\ {}_{\varphi_1}D_{p,q}f\ (\kappa\varphi_2+(1-\kappa)\varphi_1)\ d_{p,q}\kappa$$

$$= (\varphi_2 - \varphi_1) \left[\int_0^{\frac{x-\varphi_1}{\varphi_2-\varphi_1}} pq\kappa \; {}_{\varphi_1}D_{p,q}f\;(\kappa\varphi_2 + (1-\kappa)\varphi_1)\, d_{p,q}\kappa \right.$$

$$\left. + \int_{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}^{1} (pq\kappa - p)\; {}_{\varphi_1}D_{p,q}f\;(\kappa\varphi_2 + (1-\kappa)\varphi_1)\, d_{p,q}\kappa \right]$$

$$= (\varphi_2 - \varphi_1) \left[\int_0^{\frac{x-\varphi_1}{\varphi_2-\varphi_1}} pq\kappa \; {}_{\varphi_1}D_{p,q}f\;(\kappa\varphi_2 + (1-\kappa)\varphi_1)\, d_{p,q}\kappa \right.$$

$$+ \int_0^1 (pq\kappa - p)\; {}_{\varphi_1}D_{p,q}f\;(\kappa\varphi_2 + (1-\kappa)\varphi_1)\, d_{p,q}\kappa$$

$$\left. - \int_0^{\frac{x-\varphi_1}{\varphi_2-\varphi_1}} (pq\kappa - p)\; {}_{\varphi_1}D_{p,q}f\;(\kappa\varphi_2 + (1-\kappa)\varphi_1)\, d_{p,q}\kappa \right]$$

$$= (\varphi_2 - \varphi_1) \int_0^1 (pq\kappa - p)\, {}_{\varphi_1}D_{p,q}f\;(\kappa\varphi_2 + (1-\kappa)\varphi_1)\, d_{p,q}\kappa$$

$$+(\varphi_2 - \varphi_1)p \int_0^{\frac{x-\varphi_1}{\varphi_2-\varphi_1}} {}_{\varphi_1}D_{p,q}f\;(\kappa\varphi_2 + (1-\kappa)\varphi_1)\, d_{p,q}\kappa$$

$$= (\varphi_2 - \varphi_1) \int_0^1 pq\kappa \frac{f\;(p\kappa\varphi_2 + (1-p\kappa)\varphi_1) - f\;(q\kappa\varphi_2 + (1-q\kappa)\varphi_1)}{(p-q)(\varphi_2-\varphi_1)\kappa}\; d_{p,q}\kappa$$

$$-p(\varphi_2 - \varphi_1) \int_0^1 \frac{f\;(p\kappa\varphi_2 + (1-p\kappa)\varphi_1) - f\;(q\kappa\varphi_2 + (1-q\kappa)\varphi_1)}{(p-q)\kappa(\varphi_2-\varphi_1)}\; d_{p,q}\kappa$$

$$+p(\varphi_2 - \varphi_1) \int_0^{\frac{x-\varphi_1}{\varphi_2-\varphi_1}} \frac{f\;(p\kappa\varphi_2 + (1-p\kappa)\varphi_1) - f\;(q\kappa\varphi_2 + (1-q\kappa)\varphi_1)}{(p-q)\kappa(\varphi_2-\varphi_1)}\; d_{p,q}\kappa$$

$$= \frac{pq}{p-q}\left[\int_0^1 f\;(p\kappa\varphi_2 + (1-p\kappa)\varphi_1)\, d_{p,q}\kappa - \int_0^1 f\;(q\kappa\varphi_2 + (1-q\kappa)\varphi_1)\, d_{p,q}\kappa\right]$$

$$-\frac{p}{p-q}\left[\int_0^1 \frac{f\;(p\kappa\varphi_2 + (1-p\kappa)\varphi_1)}{\kappa}\, d_{p,q}\kappa - \int_0^1 \frac{f\;(q\kappa\varphi_2 + (1-q\kappa)\varphi_1)}{\kappa}\; d_{p,q}\kappa\right]$$

$$+\frac{p}{p-q}\left[\int_0^{\frac{x-\varphi_1}{\varphi_2-\varphi_1}} \frac{f\;(p\kappa\varphi_2 + (1-p\kappa)\varphi_1)}{\kappa}\, d_{p,q}\kappa - \int_0^{\frac{x-\varphi_1}{\varphi_2-\varphi_1}} \frac{f\;(q\kappa\varphi_2 + (1-q\kappa)\varphi_1)}{\kappa}\; d_{p,q}\kappa\right]$$

$$= pq\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^n}\varphi_2 + \left(1-\frac{q^n}{p^n}\right)\varphi_1\right) - pq\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} f\left(\frac{q^{n+1}}{p^{n+1}}\varphi_2 + \left(1-\frac{q^{n+1}}{p^{n+1}}\right)\varphi_1\right)$$

$$-p\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} \frac{f\left(\frac{q^n}{p^n}\varphi_2 + \left(1-\frac{q^n}{p^n}\right)\varphi_1\right)}{\frac{q^n}{p^{n+1}}} + p\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} \frac{f\left(\frac{q^{n+1}}{p^{n+1}}\varphi_2 + \left(1-\frac{q^{n+1}}{p^{n+1}}\right)\varphi_1\right)}{\frac{q^n}{p^{n+1}}}$$

$$+p\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} \frac{f\left(\frac{q^n}{p^n}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)\varphi_2 + \left(1-\frac{q^n}{p^n}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)\right)\varphi_1\right)}{\frac{q^n}{p^{n+1}}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)}$$

$$-p\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} \frac{f\left(\frac{q^{n+1}}{p^{n+1}}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)\varphi_2 + \left(1-\frac{q^{n+1}}{p^{n+1}}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)\right)\varphi_1\right)}{\frac{q^n}{p^{n+1}}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)}$$

$$= pq\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^n}\varphi_2 + \left(1-\frac{q^n}{p^n}\right)\varphi_1\right) - pq\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} f\left(\frac{q^{n+1}}{p^{n+1}}\varphi_2 + \left(1-\frac{q^{n+1}}{p^{n+1}}\right)\varphi_1\right)$$

$$-p\sum_{n=0}^{\infty} f\left(\frac{q^n}{p^n}\varphi_2 + \left(1-\frac{q^n}{p^n}\right)\varphi_1\right) + p\sum_{n=0}^{\infty} f\left(\frac{q^{n+1}}{p^{n+1}}\varphi_2 + \left(1-\frac{q^{n+1}}{p^{n+1}}\right)\varphi_1\right)$$

$$+p\sum_{n=0}^{\infty} f\left(\frac{q^n}{p^n}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)\varphi_2 + \left(1-\frac{q^n}{p^n}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)\right)\varphi_1\right)$$

$$-p\sum_{n=0}^{\infty} f\left(\frac{q^{n+1}}{p^{n+1}}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)\varphi_2+\left(1-\frac{q^{n+1}}{p^{n+1}}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)\right)\varphi_1\right)$$
$$=q\sum_{n=0}^{\infty}\frac{q^n}{p^n} f\left(\frac{q^n}{p^n}\varphi_2+\left(1-\frac{q^n}{p^n}\right)\varphi_1\right)-p\sum_{n=1}^{\infty}\frac{q^n}{p^n} f\left(\frac{q^n}{p^n}\varphi_2+\left(1-\frac{q^n}{p^n}\right)\varphi_1\right)$$
$$-p\sum_{n=0}^{\infty} f\left(\frac{q^n}{p^n}\varphi_2+\left(1-\frac{q^n}{p^n}\right)\varphi_1\right)+p\sum_{n=1}^{\infty} f\left(\frac{q^n}{p^n}\varphi_2+\left(1-\frac{q^n}{p^n}\right)\varphi_1\right)$$
$$+p\sum_{n=0}^{\infty} f\left(\frac{q^n}{p^n}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)\varphi_2+\left(1-\frac{q^n}{p^n}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)\right)\varphi_1\right)$$
$$-p\sum_{n=1}^{\infty} f\left(\frac{q^n}{p^n}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)\varphi_2+\left(1-\frac{q^n}{p^n}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)\right)\varphi_1\right).$$

Simplifying the above calculations, we obtain the following result

$$=-(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^n} f\left(\frac{q^n}{p^n}\varphi_2+\left(1-\frac{q^n}{p^n}\right)\varphi_1\right)$$
$$+pf\left(\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)\varphi_2+\left(1-\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)\right)\varphi_1\right)$$
$$=pf(x)-(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^n} f\left(\frac{q^n}{p^n}\varphi_2+\left(1-\frac{q^n}{p^n}\right)\varphi_1\right)$$
$$=f(x)-\frac{1}{p^2(\varphi_2-\varphi_1)}\int_{\varphi_1}^{p\varphi_2+(1-p)\varphi_1} f(\kappa)\,{}_{\varphi_1}d_{p,q}\kappa,$$

which concludes the proof. □

Remark 1. *Choosing $q\to 1^-$ and $p=1$ in Lemma 1, we recapture the Montgomery identity (7).*

Remark 2. *Taking $x=\frac{q\varphi_1+p\varphi_2}{[2]_{p,q}}$ in Lemma 1, we have the Lemma proved in [18]*

$$f\left(\frac{q\varphi_1+p\varphi_2}{[2]_{p,q}}\right)-\frac{1}{p^2(\varphi_2-\varphi_1)}\int_{\varphi_1}^{p\varphi_2+(1-p)\varphi_1} f(\kappa)\,{}_{\varphi_1}d_{p,q}\kappa$$
$$=(\varphi_2-\varphi_1)\int_0^{\frac{p}{[2]_{p,q}}} pq\kappa\,{}_{\varphi_1}D_{p,q}f\,(\kappa\varphi_2+(1-\kappa)\varphi_1)\,d_{p,q}\kappa$$
$$+(\varphi_2-\varphi_1)\int_{\frac{p}{[2]_{p,q}}}^{1}(pq\kappa-p)\,{}_{\varphi_1}D_{p,q}f\,(\kappa\varphi_2+(1-\kappa)\varphi_1)\,d_{p,q}\kappa.$$

Remark 3. *Choosing $p=1$ in Remark 2, we obtain the Lemma proved in [25]*

$$f\left(\frac{q\varphi_1+\varphi_2}{[2]_q}\right)-\frac{1}{\varphi_2-\varphi_1}\int_{\varphi_1}^{\varphi_2} f(\kappa)\,{}_{\varphi_1}d_q\kappa$$
$$=(\varphi_2-\varphi_1)\int_0^{\frac{1}{[2]_q}} q\kappa\,{}_{\varphi_1}D_qf\,(\kappa\varphi_2+(1-\kappa)\varphi_1)\,d_q\kappa$$
$$+(\varphi_2-\varphi_1)\int_{\frac{1}{[2]_q}}^{1}(q\kappa-1)\,{}_{\varphi_1}D_qf\,(\kappa\varphi_2+(1-\kappa)\varphi_1)\,d_q\kappa.$$

Remark 4. *Choosing $q \to 1^-$ in Remark 3, we have the Lemma proved in [26]*

$$f\left(\frac{\varphi_1+\varphi_2}{2}\right)-\frac{1}{\varphi_2-\varphi_1}\int_{\varphi_1}^{\varphi_2} f(\kappa)d\kappa = (\varphi_2-\varphi_1)\int_0^{\frac{1}{2}} \kappa f'(\kappa\varphi_2+(1-\kappa)\varphi_1)d\kappa + (\varphi_2-\varphi_1)\int_{\frac{1}{2}}^1 (\kappa-1)f'(\kappa\varphi_2+(1-\kappa)\varphi_1)d\kappa.$$

Let us introduce some new (p,q)-integral inequalities by the help of Lemma 1 and the power means inequality:

Theorem 3. *Let $f : [\varphi_1, \varphi_2] \subset \mathbb{R} \to \mathbb{R}$ be a (p,q)-differentiable function on (φ_1, φ_2) and ${}_{\varphi_1}D_{p,q}f$ is quantum-integrable on $[\varphi_1, \varphi_2]$. If $\left|{}_{\varphi_1}D_{p,q}f\right|^r$ is a convex mapping on $[\varphi_1, \varphi_2]$ for $r \geq 1$ with $0 < q < p \leq 1$, then the following quantum-integral inequality holds:*

$$\begin{aligned}&\left|f(x)-\frac{1}{p^2(\varphi_2-\varphi_1)}\int_{\varphi_1}^{p\varphi_2+(1-p)\varphi_1} f(\kappa)\,{}_{\varphi_1}d_{p,q}\kappa\right| \leq (\varphi_2-\varphi_1)\\ &\times\left\{[H_1(x,p,q)]^{\frac{r-1}{r}}\left[\left|{}_{\varphi_1}D_{p,q}f\ (\varphi_2)\right|^r H_2(x,p,q)+\left|{}_{\varphi_1}D_{p,q}f\ (\varphi_1)\right|^r H_3(x,p,q)\right]^{\frac{1}{r}}\right.\\ &\left.+[H_4(x,p,q)]^{\frac{r-1}{r}}\left[\left|{}_{\varphi_1}D_{p,q}f\ (\varphi_2)\right|^r H_5(x,p,q)+\left|{}_{\varphi_1}D_{p,q}f\ (\varphi_1)\right|^r H_6(x,p,q)\right]^{\frac{1}{r}}\right\},\end{aligned} \tag{9}$$

for all $x \in [\varphi_1, \varphi_2]$, where

$$\begin{aligned}H_1(x,p,q)&=\frac{pq}{[2]_{p,q}}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)^2,\\ H_2(x,p,q)&=\frac{pq}{[3]_{p,q}}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)^3,\\ H_3(x,p,q)&=H_1(x,p,q)-H_2(x,p,q),\\ H_4(x,p,q)&=\frac{p^2-pq}{[2]_{p,q}}\left(\frac{\varphi_2-x}{\varphi_2-\varphi_1}\right)+\frac{pq}{[2]_{p,q}}\left(\frac{\varphi_2-x}{\varphi_2-\varphi_1}\right)^2,\\ H_5(x,p,q)&=\frac{p^3}{[2]_{p,q}[3]_{p,q}}-\frac{p}{[2]_{p,q}}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)^2+\frac{pq}{[3]_{p,q}}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)^3,\\ H_6(x,p,q)&=H_4(x,p,q)-H_5(x,p,q).\end{aligned}$$

Proof. By utilizing Lemma 1, the power means inequality, and the characteristics of the modulus, we obtain the following

$$\begin{aligned}&\left|f(x)-\frac{1}{p^2(\varphi_2-\varphi_1)}\int_{\varphi_1}^{p\varphi_2+(1-p)\varphi_1} f(\kappa)\,{}_{\varphi_1}d_{p,q}\kappa\right|\\ &\leq(\varphi_2-\varphi_1)\int_0^1\left|K_{p,q}(\kappa)\right|\left|{}_{\varphi_1}D_qf\ (\kappa\varphi_2+(1-\kappa)\varphi_1)\right|d_{p,q}\kappa\\ &\leq(\varphi_2-\varphi_1)\left[\int_0^{\frac{x-\varphi_1}{\varphi_2-\varphi_1}} pq\kappa+\int_{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}^1(p-pq\kappa)\right]\left|{}_{\varphi_1}D_{p,q}f\ (\kappa\varphi_2+(1-\kappa)\varphi_1)\right|d_{p,q}\kappa\\ &\leq(\varphi_2-\varphi_1)\left\{\left(\int_0^{\frac{x-\varphi_1}{\varphi_2-\varphi_1}} pq\kappa\ \ d_{p,q}\kappa\right)^{\frac{r-1}{r}}\right.\end{aligned}$$

$$\times\left(\int_0^{\frac{x-\varphi_1}{\varphi_2-\varphi_1}} pq\kappa\left|{}_{\varphi_1}D_{p,q}f\left(\kappa\varphi_2+(1-\kappa)\varphi_1\right)\right|^r d_{p,q}\kappa\right)^{\frac{1}{r}}$$

$$+\left(\int_{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}^1 (p-pq\kappa)\,d_{p,q}\kappa\right)^{\frac{r-1}{r}}$$

$$\times\left(\int_{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}^1 (p-pq\kappa)\left|{}_{\varphi_1}D_{p,q}f\left(\kappa\varphi_2+(1-\kappa)\varphi_1\right)\right|^r d_{p,q}\kappa\right)^{\frac{1}{r}}\Bigg\}.$$

Utilizing the convexity of $\left|{}_{\varphi_1}D_{p,q}f\right|^r$

$$\left|{}_{\varphi_1}D_{p,q}f(\kappa\varphi_2+(1-\kappa)\varphi_1)\right|^r \le \kappa\left|{}_{\varphi_1}D_{p,q}f(\varphi_2)\right|^r+(1-\kappa)\left|{}_{\varphi_1}D_{p,q}f(\varphi_1)\right|^r,$$

we obtain the resulting inequality:

$$\le(\varphi_2-\varphi_1)\Bigg\{\left(\left(\int_0^{\frac{x-\varphi_1}{\varphi_2-\varphi_1}} pq\kappa\,d_{p,q}\kappa\right)^{\frac{r-1}{r}}\left(\left|{}_{\varphi_1}D_{p,q}f\left(\varphi_2\right)\right|^r\int_0^{\frac{x-\varphi_1}{\varphi_2-\varphi_1}} pq\kappa^2\;\;d_{p,q}\kappa\right.\right.$$

$$\left.+\left|{}_{\varphi_1}D_{p,q}f\left(\varphi_1\right)\right|^r\int_0^{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}\left(pq\kappa-p\kappa^2\right)\;d_{p,q}\kappa\right)^{\frac{1}{r}}+\left(\int_{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}^1 (p-pq\kappa)\;\;d_{p,q}\kappa\right)^{\frac{r-1}{r}}$$

$$\times\left(\left|{}_{\varphi_1}D_{p,q}f\left(\varphi_2\right)\right|^r\int_{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}^1 \left(p\kappa-pq\kappa^2\right)\;d_{p,q}\kappa\right.$$

$$\left.+\left|{}_{\varphi_1}D_{p,q}f\left(\varphi_1\right)\right|^r\int_{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}^1 \left((p-pq\kappa)-\left(p\kappa-pq\kappa^2\right)\right)\;d_{p,q}\kappa\;\right)^{\frac{1}{r}}\Bigg\}.$$

We evaluate the required definite (p,q)–integrals as follows:

$$H_1(x,p,q)=\int_0^{\frac{x-\varphi_1}{\varphi_2-\varphi_1}} pq\kappa\;\;d_{p,q}\kappa=pq(p-q)\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\frac{q^n}{p^{n+1}}\right)$$
$$=pq(p-q)\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)^2\frac{1}{p^2-q^2}=\frac{pq}{[2]_{p,q}}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)^2,$$

$$H_2(x,p,q)=\int_0^{\frac{x-\varphi_1}{\varphi_2-\varphi_1}} pq\kappa^2\;\;d_{p,q}\kappa=pq(p-q)\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\frac{q^n}{p^{n+1}}\right)^2$$
$$=pq(p-q)\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)^3\frac{1}{p^3-q^3}=\frac{pq}{[3]_{p,q}}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)^3,$$

$$H_3(x,p,q)=\int_0^{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}\left(pq\kappa-pq\kappa^2\right)\;\;d_{p,q}\kappa=H_1(x,p,q)-H_2(x,p,q),$$

$$H_4(x,p,q)=\int_{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}^1 (p-pq\kappa)\,d_{p,q}\kappa=\int_0^1(p-pq\kappa)\,d_{p,q}\kappa-\int_0^{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}(p-pq\kappa)\,d_{p,q}\kappa$$

$$= (p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(p-\frac{q^{n+1}}{p^n}\right)-(p-q)\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(p-\frac{q^{n+1}}{p^n}\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)$$
$$= p(p-q)\left[\frac{1}{p-q}-\frac{q}{p^2-q^2}\right]-p(p-q)\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)\left[\frac{1}{p-q}-\frac{q}{p^2-q^2}\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right]$$
$$= p\left[1-\frac{q}{[2]_{p,q}}\right]-p\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)\left[1-\frac{q}{[2]_{p,q}}\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right]$$
$$= \frac{p^2}{[2]_{p,q}}-p\left(1-\frac{\varphi_2-x}{\varphi_2-\varphi_1}\right)\left[\frac{p}{[2]_{p,q}}+\frac{q}{[2]_{p,q}}+\frac{q}{[2]_{p,q}}\left(1-\frac{\varphi_2-x}{\varphi_2-\varphi_1}\right)\right]$$
$$= \frac{p^2}{[2]_{p,q}}-p\left(1-\frac{\varphi_2-x}{\varphi_2-\varphi_1}\right)\left[\frac{p}{[2]_{p,q}}-\frac{q}{[2]_{p,q}}\left(\frac{\varphi_2-x}{\varphi_2-\varphi_1}\right)\right]$$
$$= \frac{p^2-pq}{[2]_{p,q}}\left(\frac{\varphi_2-x}{\varphi_2-\varphi_1}\right)+\frac{pq}{[2]_{p,q}}\left(\frac{\varphi_2-x}{\varphi_2-\varphi_1}\right)^2,$$
$$H_5(x,p,q)=\int_{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}^{1}\left(p\kappa-pq\kappa^2\right)d_{p,q}\kappa=\int_0^1\left(p\kappa-pq\kappa^2\right)d_{p,q}\kappa-\int_0^{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}\left(p\kappa-pq\kappa^2\right)d_{p,q}\kappa$$
$$= (p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^n}-\frac{q^{2n+1}}{p^n}\right)$$
$$-(p-q)\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^n}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)-\frac{q^{2n+1}}{p^n}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)^2\right)$$
$$= p(p-q)\left[\frac{1}{p^2-q^2}-\frac{q}{p^3-q^3}\right]$$
$$-p(p-q)\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)\left[\frac{1}{p^2-q^2}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)-\frac{q}{p^3-q^3}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)^2\right]$$
$$= p\left[\frac{1}{[2]_{p,q}}-\frac{q}{[3]_{p,q}}\right]$$
$$-p\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)\left[\frac{1}{[2]_{p,q}}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)-\frac{q}{[3]_{p,q}}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)^2\right]$$
$$= \frac{p^3}{[2]_{p,q}[3]_{p,q}}-\frac{p}{[2]_{p,q}}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)^2+\frac{pq}{[3]_{p,q}}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)^3,$$

$$H_6(x,p,q)=\int_{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}^{1}\left((p-pq\kappa)-\left(p\kappa-pq\kappa^2\right)\right)d_{p,q}\kappa=H_4(x,p,q)-H_5(x,p,q).$$

The proof is thus accomplished. □

Here, we derive some special cases from Theorem 3.

Corollary 1. *I. By Using $\left|{}_{\varphi_1}D_{p,q}f\,(\varphi)\right|\le M$ and putting $r=1$ in Theorem 3, we obtain the following special case*

$$\left|f(x)-\frac{1}{p^2(\varphi_2-\varphi_1)}\int_{\varphi_1}^{p\varphi_2+(1-p)\varphi_1}f(\kappa)\ {}_{\varphi_1}d_{p,q}\kappa\right|$$
$$\le\frac{M}{[2]_{p,q}(\varphi_2-\varphi_1)}\left[pq(x-\varphi_1)^2+\left(p^2-pq\right)(\varphi_2-x)(\varphi_2-\varphi_1)+pq(\varphi_2-x)^2\right]. \tag{10}$$

Moreover, if the limit is taken as $q \to 1^-$ and $p = 1$ in (10), the inequality (10) reduces to (5).
II. Choosing $x = \frac{q\varphi_1 + p\varphi_2}{[2]_{p,q}}$ in Theorem 3, we obtain

$$\left| f\left(\frac{q\varphi_1 + p\varphi_2}{[2]_{p,q}}\right) - \frac{1}{p^2(\varphi_2 - \varphi_1)} \int_{\varphi_1}^{p\varphi_2 + (1-p)\varphi_1} f(\kappa)\, {}_{\varphi_1}d_{p,q}\kappa \right|$$
$$\le (\varphi_2 - \varphi_1)\left(\frac{p^3 q}{([2]_{p,q})^3}\right)^{\frac{r-1}{r}} \left\{ \left[\left| {}_{\varphi_1}D_{p,q} f(\varphi_2) \right|^r \frac{p^4 q}{([2]_{p,q})^3 [3]_{p,q}} \right.\right.$$
$$\left. + \left| {}_{\varphi_1}D_{p,q} f(\varphi_1) \right|^r \frac{qp^5 + q^2p^4 + q^3p^3 - qp^4}{([2]_{p,q})^3 [3]_{p,q}} \right]^{\frac{1}{r}}$$
$$\left. + \left[\left| {}_{\varphi_1}D_{p,q} f(\varphi_2) \right|^r \frac{2p^4 q}{([2]_{p,q})^3 [3]_{p,q}} + \left| {}_{\varphi_1}D_{p,q} f(\varphi_1) \right|^r \frac{p^2 q(-2p + [3]_{p,q})}{([2]_{p,q})^3 [3]_{p,q}} \right]^{\frac{1}{r}} \right\}.$$

III. Choosing $p = 1$ and $x = \frac{q\varphi_1 + \varphi_2}{[2]_q}$ in Theorem 3, we obtain following inequality proved in [25]

$$\left| f\left(\frac{q\varphi_1 + \varphi_2}{[2]_q}\right) - \frac{1}{\varphi_2 - \varphi_1} \int_{\varphi_1}^{\varphi_2} f(\kappa)\, {}_{\varphi_1}d_q\kappa \right| \le (\varphi_2 - \varphi_1)\left(\frac{q}{([2]_q)^3}\right)^{\frac{r-1}{r}}$$
$$\times \left\{ \left[\left| {}_{\varphi_1}D_q f(\varphi_2) \right|^r \frac{q}{([2]_q)^3([3]_q)} + \left| {}_{\varphi_1}D_q f(\varphi_1) \right|^r \frac{q^2 + q^3}{([2]_q)^3([3]_q)} \right]^{\frac{1}{r}} \right.$$
$$\left. + \left[\left| {}_{\varphi_1}D_q f(\varphi_2) \right|^r \frac{2q}{([2]_q)^3([3]_q)} + \left| {}_{\varphi_1}D_q f(\varphi_1) \right|^r \frac{-q + q^2 + q^3}{([2]_q)^3([3]_q)} \right]^{\frac{1}{r}} \right\}.$$

IV. Choosing $q \to 1^-$ in Corollary 1 part III, we obtain following inequality proved in [26]

$$\left| f\left(\frac{\varphi_1 + \varphi_2}{2}\right) - \frac{1}{(\varphi_2 - \varphi_1)} \int_{\varphi_1}^{\varphi_2} f(\kappa)\ d\kappa \right| \le \frac{(\varphi_2 - \varphi_1)}{2^{3 - \frac{3}{r}}}$$
$$\times \left[\left(\frac{1}{24}\left|f'(\varphi_2)\right|^r + \frac{1}{12}\left|f'(\varphi_1)\right|^r \right)^{\frac{1}{r}} + \left(\frac{1}{12}\left|f'(\varphi_2)\right|^r + \frac{1}{24}\left|f'(\varphi_1)\right|^r \right)^{\frac{1}{r}} \right].$$

V. Choosing $r = 1$ and $x = \frac{q\varphi_1 + p\varphi_2}{[2]_{p,q}}$ in Theorem 3, we obtain

$$\left| f\left(\frac{q\varphi_1 + p\varphi_2}{[2]_{p,q}}\right) - \frac{1}{p^2(\varphi_2 - \varphi_1)} \int_{\varphi_1}^{p\varphi_2 + (1-p)\varphi_1} f(\kappa)\, {}_{\varphi_1}d_{p,q}\kappa \right|$$
$$\le (\varphi_2 - \varphi_1)\left\{ \left| {}_{\varphi_1}D_{p,q} f(\varphi_2) \right| H_2\left(\frac{q\varphi_1 + p\varphi_2}{[2]_{p,q}}, p, q\right) + \left| {}_{\varphi_1}D_{p,q} f(\varphi_1) \right| H_3\left(\frac{q\varphi_1 + p\varphi_2}{[2]_{p,q}}, p, q\right) \right.$$
$$\left. \times \left| {}_{\varphi_1}D_{p,q} f(\varphi_2) \right| H_5\left(\frac{q\varphi_1 + p\varphi_2}{[2]_{p,q}}, p, q\right) + \left| {}_{\varphi_1}D_{p,q} f(\varphi_1) \right| H_6\left(\frac{q\varphi_1 + p\varphi_2}{[2]_{p,q}}, p, q\right) \right\}$$
$$\le (\varphi_2 - \varphi_1)\left\{ \left| {}_{\varphi_1}D_{p,q} f(\varphi_2) \right| \frac{qp^4}{\left([2]_{p,q}\right)^3 [3]_{p,q}} + \left| {}_{\varphi_1}D_{p,q} f(\varphi_1) \right| \frac{qp^5 + q^2p^4 + q^3p^3 - qp^4}{\left([2]_{p,q}\right)^3 [3]_{p,q}} \right.$$
$$\left. \times \left| {}_{\varphi_1}D_{p,q} f(\varphi_2) \right| \frac{2qp^4}{\left([2]_{p,q}\right)^3 [3]_{p,q}} + \left| {}_{\varphi_1}D_{p,q} f(\varphi_1) \right| \frac{p^2 q(-2p + [3]_{p,q})}{\left([2]_{p,q}\right)^3 [3]_{p,q}} \right\}$$
$$\left| f\left(\frac{q\varphi_1 + p\varphi_2}{[2]_{p,q}}\right) - \frac{1}{p(\varphi_2 - \varphi_1)} \int_{\varphi_1}^{p\varphi_2 + (1-p)\varphi_1} f(\kappa)\ {}_{\varphi_1}d_{p,q}\kappa \right|$$

$$\leq (\varphi_2 - \varphi_1)\Big\{\big|{}_{\varphi_1}D_{p,q}f\ (\varphi_2)\big|\frac{3qp^4}{\big([2]_{p,q}\big)^3[3]_{p,q}}$$

$$+\big|{}_{\varphi_1}D_{p,q}f\ (\varphi_1)\big|\frac{-3qp^4+2qp^5+2q^2p^4+2q^3p^5}{\big([2]_{p,q}\big)^3[3]_{p,q}}\Big\}.$$

VI. Choosing $r = 1 = p$ and $x = \frac{q\varphi_1+\varphi_2}{[2]_q}$ in Theorem 3, we obtain

$$\left|f\left(\frac{q\varphi_1+\varphi_2}{[2]_q}\right) - \frac{1}{\varphi_2-\varphi_1}\int_{\varphi_1}^{\varphi_2} f(\kappa)\ {}_{\varphi_1}d_q\kappa\right|$$

$$\leq (\varphi_2 - \varphi_1)\Big\{\big|{}_{\varphi_1}D_{q}f\ (\varphi_2)\big|\frac{3q}{\big([2]_{q}\big)^3\big([2]_q+q^2\big)}$$

$$+\big|{}_{\varphi_1}D_{q}f\ (\varphi_1)\big|\frac{2q^2+2q^3-q}{\big([2]_{p,q}\big)^3\big([3]_q\big)}\Big\}.$$

VII. Choosing $q \to 1^-$ in Corollary 1 part VI, we obtain the following inequality proved in [26]

$$\left|f\left(\frac{\varphi_1+\varphi_2}{2}\right) - \frac{1}{(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_2} f(\kappa)\ d\kappa\right| \leq \frac{(\varphi_2-\varphi_1)[f'(\varphi_1)+f'(\varphi_2)]}{8}.$$

Here, we introduce some new (p,q)-integral inequalities by the help of Lemma 1 and the Hölder's inequality:

Theorem 4. *Let $f : [\varphi_1,\varphi_2] \subset \mathbb{R} \to \mathbb{R}$ be a (p,q)-differentiable function on (φ_1,φ_2) and ${}_{\varphi_1}D_{p,q}f$ be quantum integrable on $[\varphi_1,\varphi_2]$. If $\big|{}_{\varphi_1}D_{p,q}f\big|^r$ is a convex mapping on $[\varphi_1,\varphi_2]$ for $r > 1$ and $\frac{1}{r}+\frac{1}{s} = 1$, with $0 < q < p \leq 1$, then the following quantum integral inequality holds:*

$$\left|f(x) - \frac{1}{p^2(\varphi_2-\varphi_1)}\int_{\varphi_1}^{p\varphi_2+(1-p)\varphi_1} f(\kappa)\ {}_{\varphi_1}d_{p,q}\kappa\right| \leq (\varphi_2-\varphi_1)$$

$$\times\Big\{(N_1(x,p,q))^{\frac{1}{s}}\Big(\big|{}_{\varphi_1}D_{p,q}f\ (\varphi_2)\big|^r N_2(x,p,q) + \big|{}_{\varphi_1}D_{p,q}f\ (\varphi_1)\big|^r N_3(x,p,q)\ \Big)^{\frac{1}{r}} \quad (11)$$

$$+(N_4(x,p,q))^{\frac{1}{s}}\Big(\big|{}_{\varphi_1}D_{p,q}f\ (\varphi_2)\big|^r N_5(x,p,q) + \big|{}_{\varphi_1}D_{p,q}f\ (\varphi_1)\big|^r N_6(x,p,q)\Big)^{\frac{1}{r}}\Big\},$$

where

$$N_1(x,p,q) = p^s q^s\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)^{s+1}\left(\frac{p-q}{p^{s+1}-q^{s+1}}\right),$$

$$N_2(x,p,q) = \frac{1}{[2]_{p,q}}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)^2,$$

$$N_3(x,p,q) = \left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right) - \frac{1}{[2]_{p,q}}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)^2,$$

$$N_4(x,p,q) = p^s q^s (p-q)\left[\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}-\frac{1}{q}\right)^s\right.$$

$$-\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)-\frac{1}{q}\right)^s\Bigg],$$

$$N_5(x,p,q)=\frac{1}{[2]_{p,q}}\left(1-\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)^2\right),$$

$$N_6(x,p,q)=\frac{[2]_{p,q}-1}{[2]_{p,q}}-\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)-\frac{1}{[2]_{p,q}}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)^2,$$

for all $x\in[\varphi_1,\varphi_2]$.

Proof. By using Lemma 1, Hölder's inequality and properties of modulus, we obtain

$$\left|f(x)-\frac{1}{p^2(\varphi_2-\varphi_1)}\int_{\varphi_1}^{p\varphi_2+(1-p)\varphi_1}f(\kappa)\ {}_{\varphi_1}d_{p,q}\kappa\right|$$

$$\leq(\varphi_2-\varphi_1)\int_0^1\left|K_{p,q}(\kappa)\right|\left|{}_{\varphi_1}D_qf\ (\kappa\varphi_2+(1-\kappa)\varphi_1)\right|d_{p,q}\kappa$$

$$\leq(\varphi_2-\varphi_1)\left(\int_0^{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}pq\kappa+\int_{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}^1(p-pq\kappa)\right)\left|{}_{\varphi_1}D_{p,q}f\ (\kappa\varphi_2+(1-\kappa)\varphi_1)\right|d_{p,q}\kappa$$

$$\leq(\varphi_2-\varphi_1)\left\{\left(\int_0^{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}(pq\kappa)^s\,d_{p,q}\kappa\right)^{\frac{1}{s}}\right.$$

$$\times\left(\int_0^{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}\left|{}_{\varphi_1}D_{p,q}f\ (\kappa\varphi_2+(1-\kappa)\varphi_1)\right|^r d_{p,q}\kappa\right)^{\frac{1}{r}}$$

$$+\left(\int_{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}^1(p-pq\kappa)^s\,d_{p,q}\kappa\right)^{\frac{1}{s}}$$

$$\left.\times\left(\int_{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}^1\left|{}_{\varphi_1}D_{p,q}f\ (\kappa\varphi_2+(1-\kappa)\varphi_1)\right|^r d_{p,q}\kappa\right)^{\frac{1}{r}}\right\}.$$

Utilizing the convexity of $\left|{}_{\varphi_1}D_{p,q}f\right|^r$, we obtain

$$\leq(\varphi_2-\varphi_1)\left\{\left(\int_0^{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}(pq\kappa)^s\,d_{p,q}\kappa\right)^{\frac{1}{s}}\left(\left|{}_{\varphi_1}D_{p,q}f\ (\varphi_2)\right|^r\int_0^{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}\kappa\,d_{p,q}\kappa\right.\right.$$

$$\left.+\left|{}_{\varphi_1}D_{p,q}f\ (\varphi_1)\right|^r\int_0^{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}(1-\kappa)\,d_{p,q}\kappa\right)^{\frac{1}{r}}+\left(\int_{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}^1(p-pq\kappa)^s\,d_{p,q}\kappa\right)^{\frac{1}{s}}$$

$$\left.\times\left(\left|{}_{\varphi_1}D_{p,q}f\ (\varphi_2)\right|^r\int_{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}^1\kappa\,d_{p,q}\kappa+\left|{}_{\varphi_1}D_{p,q}f\ (\varphi_1)\right|^r\int_{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}^1(1-\kappa)\,d_{p,q}\kappa\right)^{\frac{1}{r}}\right\}.$$

In order to determine the essential definite (p,q)-integrals,

$$N_1(x,p,q)=p^sq^s\int_0^{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}\kappa^s\,d_{p,q}\kappa=p^sq^s\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)^{s+1}\left(\frac{p-q}{p^{s+1}-q^{s+1}}\right),$$

$$N_2(x,p,q)=\int_0^{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}\kappa\,d_{p,q}\kappa=\frac{1}{[2]_{p,q}}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)^2,$$

$$N_3(x,p,q) = \int_0^{\frac{x-\varphi_1}{\varphi_2-\varphi_1}} (1-\kappa)\, d_{p,q}\kappa = \left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right) - \frac{1}{[2]_{p,q}}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)^2,$$

$$N_4(x,p,q) = p^s q^s \int_{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}^{1} \left(\kappa - \frac{1}{q}\right)^s d_{p,q}\kappa = p^s q^s (p-q)$$

$$\times\left[\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^{n+1}} - \frac{1}{q}\right)^s - \left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right) - \frac{1}{q}\right)^s\right],$$

$$N_5(x,p,q) = \int_{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}^{1} \kappa\, d_{p,q}\kappa = \frac{1}{[2]_{p,q}}\left(1 - \left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)^2\right),$$

$$N_6(x,p,q) = \int_{\frac{x-\varphi_1}{\varphi_2-\varphi_1}}^{1} (1-\kappa)\, d_{p,q}\kappa$$

$$= \frac{[2]_{p,q}-1}{[2]_{p,q}} - \left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right) + \frac{1}{[2]_{p,q}}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right)^2.$$

The proof is completed. □

Here, we derive some special cases from Theorem 4.

Corollary 2. *I. By using $\left|{}_{\varphi_1}D_{p,q}f\ (\varphi)\right| \le M$ in Theorem 4, we obtain the following special cases*

$$\left|f(x) - \frac{1}{p^2(\varphi_2-\varphi_1)}\int_{\varphi_1}^{p\varphi_2+(1-p)\varphi_1} f(\kappa)\ {}_{\varphi_1}d_{p,q}\kappa\right| \le M(\varphi_2-\varphi_1)$$
$$\times\left\{(N_1(x,p,q))^{\frac{1}{s}}(N_2(x,p,q)+N_3(x,p,q)\ )^{\frac{1}{r}} + (N_4(x,p,q))^{\frac{1}{s}}(N_5(x,p,q)+N_6(x,p,q))^{\frac{1}{r}}\right\}. \tag{12}$$

II. Choosing $x = \frac{q\varphi_1+p\varphi_2}{[2]_{p,q}}$ in Theorem 4, we obtain

$$\left|f\left(\frac{q\varphi_1+p\varphi_2}{[2]_{p,q}}\right) - \frac{1}{p^2(\varphi_2-\varphi_1)}\int_{\varphi_1}^{p\varphi_2+(1-p)\varphi_1} f(\kappa)\ {}_{\varphi_1}d_{p,q}\kappa\right|$$

$$\le pq(\varphi_2-\varphi_1)\left\{\left[\left(\frac{p}{[2]_{p,q}}\right)^{s+1}\left(\frac{p-q}{p^{s+1}-q^{s+1}}\right)\right]^{\frac{1}{s}}\right.$$

$$\times\left[\left|{}_{\varphi_1}D_{p,q}f\ (\varphi_2)\right|^r \frac{p^2}{([2]_{p,q})^3} + \left|{}_{\varphi_1}D_{p,q}f\ (\varphi_1)\right|^r \frac{p^3+2p^2q+pq^2-p^2}{([2]_{p,q})^3}\right]^{\frac{1}{r}}$$

$$+\left(N_7\left(\frac{q\varphi_1+p\varphi_2}{[2]_{p,q}}, p, q\right)\right)^{\frac{1}{s}} \times$$

$$\left.\left[\left|{}_{\varphi_1}D_{p,q}f\ (\varphi_2)\right|^r \frac{2pq+q^2}{([2]_{p,q})^3} + \left|{}_{\varphi_1}D_{p,q}f\ (\varphi_1)\right|^r \frac{p^2q+2pq^2-2pq-q^2+q^3}{([2]_{p,q})^3}\right]^{\frac{1}{r}}\right\},$$

where

$$N_7\left(\frac{q\varphi_1+p\varphi_2}{[2]_{p,q}}, p, q\right) = \int_{\frac{p}{[2]_{p,q}}}^{1} \left(\kappa - \frac{1}{q}\right)^s d_{p,q}\kappa = (p-q)$$

$$\times\left[\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^{n+1}} - \frac{1}{q}\right)^s - \left(\frac{p}{[2]_{p,q}}\right)\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}\left(\frac{p}{[2]_{p,q}}\right) - \frac{1}{q}\right)^s\right].$$

III. Choosing $p = 1$ and $x = \frac{q\varphi_1+\varphi_2}{[2]_q}$ in Theorem 4, we obtain following inequality proved in [25]

$$\left|f\left(\frac{q\varphi_1+\varphi_2}{[2]_q}\right) - \frac{1}{\varphi_2-\varphi_1}\int_{\varphi_1}^{\varphi_2} f(\kappa)\,{}_{\varphi_1}d_q\kappa\right|$$
$$\leq q(\varphi_2-\varphi_1)\left\{\left[\left(\frac{1}{[2]_q}\right)^{s+1}\left(\frac{1-q}{1-q^{s+1}}\right)\right]^{\frac{1}{s}}\right.$$
$$\times\left[\left|{}_{\varphi_1}D_qf\,(\varphi_2)\right|^r\frac{1}{([2]_q)^3} + \left|{}_{\varphi_1}D_qf\,(\varphi_1)\right|^r\frac{2q+q^2}{([2]_q)^3}\right]^{\frac{1}{r}}$$
$$+\left(N_8\left(\frac{q\varphi_1+\varphi_2}{[2]_q},1,q\right)\right)^{\frac{1}{s}}$$
$$\left.\times\left[\left|{}_{\varphi_1}D_qf\,(\varphi_2)\right|^r\frac{2q+q^2}{([2]_q)^3} + \left|{}_{\varphi_1}D_qf\,(\varphi_1)\right|^r\frac{q^3+q^2-q}{([2]_q)^3}\right]^{\frac{1}{r}}\right\},$$

where

$$N_8\left(\frac{q\varphi_1+\varphi_2}{[2]_q},1,q\right) = \int_{\frac{1}{[2]_q}}^{1}\left(\kappa-\frac{1}{q}\right)^s d_q\kappa = (1-q)$$
$$\times\left[\sum_{n=0}^{\infty} q^n\left(q^n-\frac{1}{q}\right)^s - \frac{1}{[2]_q}\sum_{n=0}^{\infty} q^n\left(q^n\left(\frac{1}{[2]_q}\right)-\frac{1}{q}\right)^s\right].$$

VI. Choosing $q \to 1^-$ in Corollary 2 part III, we obtain following inequality proved in [26]

$$\left|f\left(\frac{\varphi_1+\varphi_2}{2}\right) - \frac{1}{(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_2} f(\kappa)d\kappa\right| \leq \frac{(\varphi_2-\varphi_1)}{16}\left(\frac{4}{s+1}\right)^{\frac{1}{s}}$$
$$\times\left[\left(\left|f'(\varphi_2)\right|^r + 3\left|f'(\varphi_1)\right|^r\right)^{\frac{1}{r}} + \left(3\left|f'(\varphi_2)\right|^r + \left|f'(\varphi_1)\right|^r\right)^{\frac{1}{r}}\right].$$

3. Application to Special Means

The following is an example of how the special means for positive real numbers would be used:

1. Arithmetic mean

$$\mathcal{A}(\varphi_1,\varphi_2) = \frac{\varphi_1+\varphi_2}{2}.$$

2. Generalized logarithmic mean

$$\mathcal{L}_k(\varphi_1,\varphi_2) = \left(\frac{\varphi_2^{k+1}-\varphi_1^{k+1}}{(k+1)(\varphi_2-\varphi_1)}\right)^{\frac{1}{k}},\quad k\in\mathbb{R}\backslash\{-1,0\}.$$

Proposition 1. *If $k > 1$ and φ_1,φ_2 are two positive real numbers such that $\varphi_1 < \varphi_2$, then*

$$\left|\mathcal{A}(\varphi_1^k,\varphi_2^k) - \frac{1}{\varphi_2-\varphi_1}\mathcal{L}_k^k(\varphi_1,\varphi_2)\right| \leq \frac{k(\varphi_2-\varphi_1)}{4}\mathcal{A}\left(\varphi_1^{k-1},\varphi_2^{k-1}\right). \tag{13}$$

Proof. Let $f(x) = x^k$ for $k > 1$. Then, we have

$$\int_{\varphi_1}^{(1-p)\varphi_1+p\varphi_2} x^k\,{}_{\varphi_1}d_{p,q}x = \left[\frac{p-q}{p^{k+1}-q^{k+1}}\right]\left(\frac{((1-p)\varphi_1+p\varphi_2)^{k+1}-\varphi_1^{k+1}}{((1-p)\varphi_1+p\varphi_2)-\varphi_1}\right),$$

$$ {}_{\varphi_1}D_{p,q}f(\varphi_1) = {}_{\varphi_1}D_{p,q}\varphi_1^k = \left[\frac{p^k - q^k}{p - q}\right]\varphi_1^{k-1} $$

and

$$ {}_{\varphi_1}D_{p,q}f(\varphi_2) = {}_{\varphi_1}D_{p,q}\varphi_2^k = \left[\frac{p^k - q^k}{p - q}\right]\varphi_2^{k-1}. $$

So, using the Corollary 1 part VI, we have

$$ \begin{aligned} &\left| \frac{q\varphi_1^k + p\varphi_2^k}{[2]_{p,q}} - \frac{1}{p^2(\varphi_2 - \varphi_1)}\left[\frac{p-q}{p^{k+1} - q^{k+1}}\right]\left(\frac{((1-p)\varphi_1 + p\varphi_2)^{k+1} - \varphi_1^{k+1}}{((1-p)\varphi_1 + p\varphi_2) - \varphi_1}\right)\right| \\ &\leq (\varphi_2 - \varphi_1)\left[\frac{3qp^4}{([2]_{p,q})^3[3]_{p,q}}\left[\frac{p^k - q^k}{p-q}\right]\varphi_2^{k-1}\right. \\ &\qquad \left. + \frac{-3qp^4 + 2qp^5 + 2q^2p^4 + 2q^3p^5}{([2]_{p,q})^3[3]_{p,q}}\left[\frac{p^k - q^k}{p-q}\right]\varphi_1^{k-1}\right]. \end{aligned} $$

By taking the limit when $q \to 1^-$ and $p = 1$, we are able to obtain the inequality that we are looking for (13). □

Remark 5. *Utilizing the same concept as in Proposition 1 while employing Theorems 3 and 4, and associated respective corollaries, and taking suitable functions, such as* $f(x) = x^k$, $k > 1$ *and* $x > 0$; $f(x) = \frac{1}{x}$, $x > 0$; $f(x) = e^x$, $x \in \mathbb{R}$, *etc., we are able to obtain many fresh new interesting inequalities by making use of a variety of distinct methods. We do not include their proofs; therefore, it is the reader's responsibility to seek out the material they require in order to satisfy their curiosity.*

4. Conclusions

In order to use functions whose first derivatives' absolute values are (p,q)–differentiable convex, the authors formulate some Ostrowski-type integral inequalities in terms of the identified Montgomery identity. In addition, we investigated the essential relevant links between the results achieved in this work with all those developed in the previous peer-reviewed papers. Some of the sub-results can be generated from our main results by taking a unique and valuable variable for $x \in [\varphi_1, \varphi_2]$, some fixed value for r, $p = 1$, and $q \to 1^-$. To the best of our knowledge, these results are new in the literature. Since quantum calculus has large applications in many mathematical areas, we hope that our results can be applied in umbral calculus, oscillations in q-calculus, interpolation theory, quantum groups, quantum algebras, hypergeometric series, complex analysis, and particle physics. The results obtained by the future plan are all exhilarating compared to the results available in the literature.

Author Contributions: Writing-original, writing-review, and editing, software, H.K.; software, validation, funding acquisition, M.V.-C.; writing-review and editing, M.Z.A.; formal analysis, methodology, M.M.; supervision, formal analysis, Z.A.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We would like to express our gratitude to the "Dirección de investigación" from Pontificia Universidad Católica del Ecuador for their technical support to our research project entitled "Algunas desigualdades integrales para funciones convexas generalizadas y aplicaciones". Princess

Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2022R8). Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

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