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A Semi-Analytical Method to Investigate Fractional-Order Gas Dynamics Equations by Shehu Transform

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Abstract: This work aims at a new semi-analytical method called the variational iteration transformation method for solving nonlinear homogeneous and nonhomogeneous fractional-order gas dynamics equations. The Shehu transformation and the iterative technique are applied to solve the suggested problems. The proposed method has an advantage over existing approaches because it does not require additional materials or computations. Four problems are used to test the authenticity of the proposed method. Using the suggested method, the solution proves to be more accurate. The proposed method can be implemented to solve many nonlinear fractional order problems because it has a straightforward implementation.

Keywords: variational iteration method; Shehu transform; gas dynamics equation; Mittag–Leffler function; Caputo derivatives

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1. Introduction

A branch of applied mathematics known as fractional calculus deals with derivatives and integrals of any order. Fractional calculus has been used in a variety of seemingly unrelated scientific and technical domains over the past ten years. In order to describe issues in electromagnetism, diffusion, signal processing, biology, fluid mechanics, and many other physical phenomena, fractional differential equations are increasingly being used [1–4]. Gas dynamics (GD) equations are mathematical representations of mass conservation, energy conservation, momentum conservation, and other physical laws. In shock fronts, unique factions, and connection discontinuities, nonlinear fractional-order GD equations are used. The basic concepts of continuum flow and hydrodynamics, gas dynamics is a branch of fluid mechanics that examines gas flow and its effects on fundamental structures. Several examples of this research include, but are not limited to, blocks tending to flow in shock waves around aircraft, aerodynamics heating on atmospheric re-entry vehicles, and gas fuel streams in a rocket engine [5,6].

Consider the nonlinear fractional GD equation

$$\frac{\partial^{\delta}\mu}{\partial\varsigma^{\delta}} + \mu\frac{\partial\mu}{\partial\vartheta} - \mu(1-\mu) = 0, \quad \vartheta \in \mathrm{R}, \quad 0 < \varrho \leq 1, \tag{1}$$

The initial condition is $\Phi(\xi, 0) = f(\xi)$, where ϱ is a parameter that expresses the fractional derivative and ς and ϑ are time and space coordinates. When $\varrho = 1$, Equation (1) enhances the standard GD equation. In the last century, numerous numerical analytical techniques have been used to determine the gas dynamic equations [5,6]. The differential transformation approach [7] was utilized to solve the homogeneous and nonhomogeneous nonlinear gas dynamics equations. Some methods have been investigated by gas dynamics equations, such as the homotopy perturbation transformation method [8], fractional reduced differential transform method [9], Adomian decomposition method [10], q-homotopy analysis method [11], fractional homotopy analysis transformation method [12], variational iteration method [13,14], homotopy perturbation method applying Laplace transformation [15], and natural decomposition method [16].

Because some physical procedures, as well as procedures in fields of science, can be briefly explained by utilizing the fractional calculus principle to create models, the answer to fractional equations eventually converges to classical equations, a current research topic [17–21]. Fractional differentiation is particularly efficient due to a wide scope of implementations for mathematical issues in applied sciences models, such as flow traffic designs, earthquake designing, diffusion models, and relaxation procedures [22–26]. The investigation of partial differential equations, particularly those derived from financial math, is where the elegance of symmetry evaluation is most apparent [27–29]. The key of nature is symmetry, but the majority of natural occurrences lack symmetry. The trend of unexpected symmetry breaking is an efficient method for hiding symmetry [30].

The variational iteration method (VIM) [31] is a numerical technique for solving fractional- and integer-order linear and nonlinear partial differential equations without small perturbations or linearization. The theory of the VIM came from the Lagrange multiplier technique [32], and the basic implementation was also suggested to solve nonlinear examples in quantum mechanics. In the VIM, a correction function is constructed by a general Lagrange multiplier, which can be defined using a variational concept. An approximated result can be achieved from its trial function with possible unknown constants that can be recognized by successively imposing the boundary conditions. The current technique has been shown to solve nonlinear models efficiently, straightforwardly, and appropriately, with approximates quickly convergent to accurate results [33–40]. Recently, a technique mixture with the Laplace transform and VIM was suggested [41,42] and Wu introduced a method modified via fractional calculus and Laplace transforms [43]: the Laplace variational iteration method for investigating nonlinear partial differential equations [44] and the schemes of fractional partial differential equations [45].

In this article, the variational iteration transform method is coupled with the Shehu transform and the variational iteration method to solve time-fractional gas dynamics equations. Applying the initial condition, approximative analytical expressions for various fractional Brownian motions and standard motion are derived. The approximate solution is computed numerically and illustrated graphically. This article's elegance can be attributed to its straightforward approach to finding an approximate analytical solution to the problem. In Section 2, some basic definitions of the Shehu transform are given. In Section 3, we generally discuss the variational iteration transform method. In Section 4, we give current solutions to suggested equations, explaining how to implement the suggested technique. Finally, we present the conclusion.

2. Basic Definitions

Definition 1. *The fractional Riemann–Liouville integral is defined as [46,47]*

$$I_0^{\varrho} h(\tau) = \frac{1}{\Gamma(\wp)} \int_0^{\varsigma} (\varsigma - s)^{\wp-1} h(s) ds. \quad (2)$$

Definition 2. *The Caputo fractional derivative of $h(\varsigma)$ is given as [46,47]*

$$\begin{aligned} D_{\varsigma}^{\wp} h(\varsigma) = & I^{\imath-\wp} f^{\imath}, \ \imath - 1 < \wp < \imath, \quad \imath \in \mathbb{N} \\ & \frac{d^{\imath}}{d\varsigma^{\imath}} h(\varsigma), \quad \wp = \imath, \quad \imath \in \mathbb{N}. \end{aligned} \quad (3)$$

Definition 3. *The nth derivative of the Shehu transform is defined as [48–50]*

$$S\left\{\nu^{(\imath)}(\varsigma)\right\} = \frac{s^{\imath}}{u^{\imath}}V(s,u) - \sum_{k=0}^{\imath-1}\left(\frac{s}{u}\right)^{\imath-k-1}\nu^{(k)}(0). \tag{4}$$

Definition 4. *The fractional-order Shehu transform derivative is defined as [48–50]*

$$S\left\{\nu^{(\wp)}(\varsigma)\right\} = \frac{s^{\wp}}{u^{\wp}}V(s,u) - \sum_{k=0}^{\imath-1}\left(\frac{s}{u}\right)^{\wp-k-1}\nu^{(k)}(0), \ \ 0 < \wp \leq \imath. \tag{5}$$

Definition 5. *The Mittag–Leffler function of $E_{\wp}(z)$ for $\wp > 0$ is defined as*

$$E_{\wp}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\wp k+1)}, \qquad \wp > 0, \quad z \in \mathbb{C}.$$

3. The Methodology of Variational Iteration Transform Technique

Consider the fractional-order partial differential equation

$$D_{\varsigma}^{\wp}\mu(\vartheta,\varsigma) + \bar{\mathcal{M}}(\vartheta,\varsigma) + \mathcal{N}(\vartheta,\varsigma) - \mathcal{P}(\vartheta,\varsigma) = 0, \ \ \imath - 1 < \wp \leq \imath, \tag{6}$$

with the initial sources

$$\mu(\vartheta,0) = g_1(\vartheta). \tag{7}$$

where $\wp$ the Caputo fractional derivative, $\bar{\mathcal{M}}$ is a linear and $\mathcal{N}$ is a nonlinear term, respectively, and the source term is $\mathcal{P}$.

We first apply the Shehu transformation to Equation (6). We obtain

$$S[D_{\varsigma}^{\wp}\mu(\vartheta,\varsigma)] + S[\bar{\mathcal{M}}(\vartheta,\varsigma) + \mathcal{N}(\vartheta,\varsigma) - \mathcal{P}(\vartheta,\varsigma)] = 0, \tag{8}$$

$$S[\mu(\vartheta,\varsigma)] - \sum_{k=0}^{\imath-1}\frac{s^{\wp-k-1}}{u^{\wp-k}}\frac{\partial^k\mu(\vartheta,\varsigma)}{\partial^k\varsigma}|_{\varsigma=0} = -S\big[\bar{\mathcal{M}}(\vartheta,\varsigma) + \mathcal{N}(\vartheta,\varsigma) - \mathcal{P}(\vartheta,\varsigma)\big], \tag{9}$$

We use the Lagrange multiplier in the present method,

$$\begin{aligned} S[\mu_{\imath+1}(\vartheta,\varsigma)] = S[\mu_{\imath}(\vartheta,\varsigma)] + \lambda(s)\Bigg[\frac{s^{\wp}}{u^{\wp}}\mu_{\imath}(\vartheta,\varsigma) - \sum_{k=0}^{\imath-1}\frac{s^{\wp-k-1}}{u^{\wp-k}}\frac{\partial^k\mu(\vartheta,\varsigma)}{\partial^k\varsigma}|_{\varsigma=0} \\ -S[\mathcal{P}(\vartheta,\varsigma)] - S\{\bar{\mathcal{M}}(\vartheta,\varsigma) + \mathcal{N}(\vartheta,\varsigma)\}\Bigg]. \end{aligned} \tag{10}$$

The Lagrange multiplier is defined

$$\lambda(s) = -\frac{u^{\wp}}{s^{\wp}}, \tag{11}$$

Applying the inverse Shehu transform S^{-1} from Equation (10),

$$\begin{aligned} \mu_{\imath+1}(\vartheta,\varsigma) = \mu_{\imath}(\vartheta,\varsigma) - S^{-1}\Bigg[\frac{u^{\wp}}{s^{\wp}}\Bigg[\sum_{k=0}^{\imath-1}\frac{s^{\wp-k-1}}{u^{\wp-k}}\frac{\partial^k\mu(\vartheta,\varsigma)}{\partial^k\varsigma}|_{\varsigma=0} \\ -S[\mathcal{P}(\vartheta,\varsigma)] - S\{\bar{\mathcal{M}}(\vartheta,\varsigma) + \mathcal{N}(\vartheta,\varsigma)\}\Bigg]\Bigg], \end{aligned} \tag{12}$$

the initial value can be defined as

$$\mu_0(\vartheta,\varsigma) = S^{-1}\left[\frac{u^{\wp}}{s^{\wp}}\left\{\sum_{k=0}^{\imath-1}\frac{s^{\wp-k-1}}{u^{\wp-k}}\frac{\partial^k\mu(\vartheta,\varsigma)}{\partial^k\varsigma}|_{\varsigma=0}\right\}\right]. \tag{13}$$

4. Numerical Results

Problem 1. *Consider the homogeneous fractional gas dynamics equation [51]*

$$\frac{\partial^{\wp}\mu}{\partial\varsigma^{\wp}} + \mu\frac{\partial\mu}{\partial\varsigma} - \mu(1-\mu) = 0, \quad 0 < \wp \leq 1, \tag{14}$$

with the initial condition

$$\mu(\vartheta, 0) = e^{-\vartheta}. \tag{15}$$

Using VITM on Equation (14), we obtain

$$\mu_{m+1}(\vartheta,\varsigma) = S^{-1}\left[\frac{\mu_m(\vartheta,\varsigma)}{s}\right] + S^{-1}\left[\lambda(s)S\left\{\mu_m\frac{\partial\mu_m}{\partial\vartheta} - \mu_m(1-\mu_m)\right\}\right], \tag{16}$$

where the Lagrange multiplier is

$$\lambda(s) = -\frac{u^{\wp}}{s^{\wp}},$$

$$\mu_{m+1}(\vartheta,\varsigma) = S^{-1}\left[\frac{\mu_m(\vartheta,\varsigma)}{s}\right] - S^{-1}\left[\frac{u^{\wp}}{s^{\wp}}S\left\{\mu_m\frac{\partial\mu_m}{\partial\vartheta} - \mu_m(1-\mu_m)\right\}\right]. \tag{17}$$

Now take

$$u_0(\vartheta,\varsigma) = e^{-\vartheta},$$

and consequently, we obtain

$$m = 0, 1, 2, 3\cdots$$

$$\mu_1(\vartheta,\varsigma) = S^{-1}\left[\frac{\mu_0(\vartheta,\varsigma)}{s}\right] - S^{-1}\left[\frac{u^{\wp}}{s^{\wp}}S\left\{\mu_0\frac{\partial\mu_0}{\partial\vartheta} - \mu_0(1-\mu_0)\right\}\right],$$

$$u_1(\vartheta,\varsigma) = e^{-\vartheta} + \frac{e^{-\vartheta}\varsigma^{\wp}}{\Gamma(\wp+1)},$$

$$\mu_2(\vartheta,\varsigma) = S^{-1}\left[\frac{\mu_1(\vartheta,\varsigma)}{s}\right] - S^{-1}\left[\frac{u^{\wp}}{s^{\wp}}S\left\{\mu_1\frac{\partial\mu_1}{\partial\vartheta} - \mu_1(1-\mu_1)\right\}\right],$$

$$u_2(\vartheta,\varsigma) = e^{-\vartheta} + \frac{e^{-\vartheta}\varsigma^{\wp}}{\Gamma(\wp+1)} + \frac{e^{-\vartheta}\varsigma^{2\wp}}{\Gamma(2\wp+1)},$$

and

$$\mu_3(\vartheta,\varsigma) = S^{-1}\left[\frac{\mu_2(\vartheta,\varsigma)}{s}\right] - S^{-1}\left[\frac{u^{\wp}}{s^{\wp}}S\left\{\mu_2\frac{\partial\mu_2}{\partial\vartheta} - \mu_2(1-\mu_2)\right\}\right],$$

$$u_3(\vartheta,\varsigma) = e^{-\vartheta} + \frac{e^{-\vartheta}\varsigma^{\wp}}{\Gamma(\wp+1)} + \frac{e^{-\vartheta}\varsigma^{2\wp}}{\Gamma(2\wp+1)} + \frac{e^{-\vartheta}\varsigma^{3\wp}}{\Gamma(3\wp+1)}.$$

The analytical result of Equation (14) can be achieved as

$$u(\vartheta,\varsigma) = e^{-\vartheta} + \frac{e^{-\vartheta}\varsigma^{\wp}}{\Gamma(\wp+1)} + \frac{e^{-\vartheta}\varsigma^{2\wp}}{\Gamma(2\wp+1)} + \frac{e^{-\vartheta}\varsigma^{3\wp}}{\Gamma(3\wp+1)} + \cdots + \frac{e^{-\vartheta}\varsigma^{m\wp}}{\Gamma(m\wp+1)},$$

$$u(\vartheta,\varsigma) = e^{-\vartheta}\sum_{m=0}^{\infty}\frac{(\varsigma^{\wp})^m}{\Gamma(m\wp+1)} = e^{-\vartheta}E_{\wp}(\varsigma^{\wp}). \tag{18}$$

The exact solution of Equation (14) is

$$u(\vartheta,\varsigma) = e^{-\vartheta+\varsigma}. \tag{19}$$

In Figure 1, we show the analytical and exact solution graphs of Problem 1 at $\wp = 1$*. From the provided graphs, it can be determined that the actual and approximate solutions are closely related. In addition, in Figures 2 and 3, the analytical solutions of Problem 1 are calculated at fractional-order* $\wp = 0.8, 0.6,$ *and* 0.4*.*

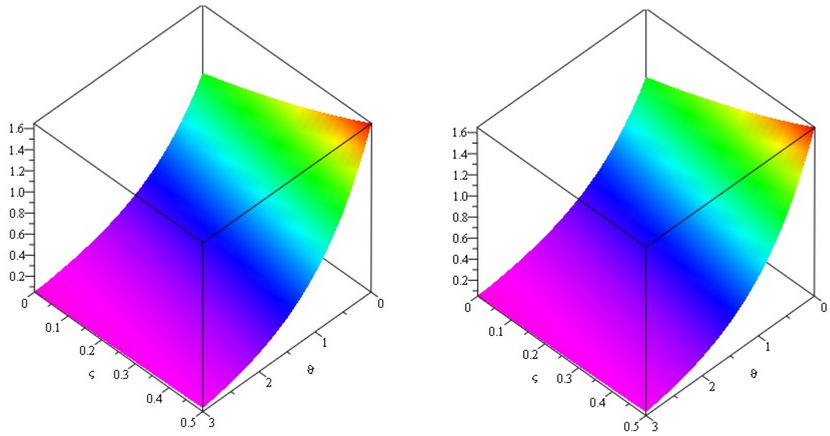

Figure 1. The analytical and exact solutions figures of Example 3.1.

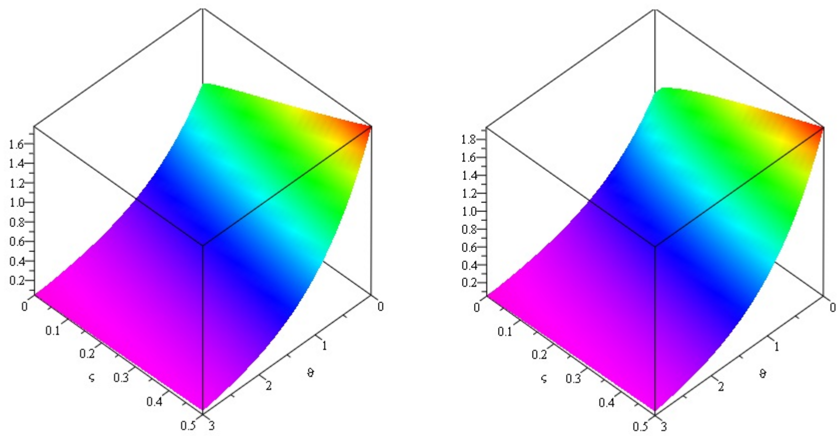

Figure 2. The different fractional order of $\wp = 0.8$ and 0.6 of Example 3.1.

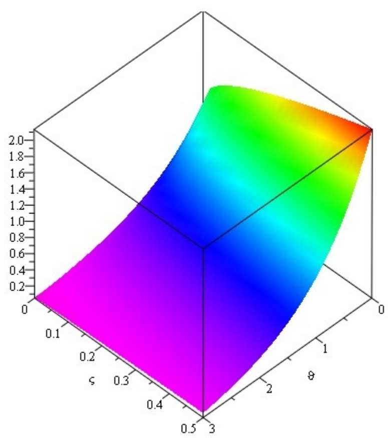

Figure 3. The noninteger order of $\wp = 0.4$ of Example 3.1.

Problem 2. *Consider the fractional gas dynamics equation [51]*

$$\frac{\partial^{\wp}\mu}{\partial\varsigma^{\wp}} + \mu\frac{\partial\mu}{\partial\varsigma} - \mu(1-\mu)\log b = 0, \quad b > 0 \quad 0 < \wp \leq 1, \tag{20}$$

with the initial condition

$$\mu(\vartheta,0)=b^{-\vartheta}. \tag{21}$$

Using VITM on Equation (20), we have

$$\mu_{m+1}(\vartheta,\varsigma)=S^{-1}\left[\frac{\mu_m(\vartheta,\varsigma)}{s}\right]+S^{-1}\left[\lambda(s)S\left\{\mu\frac{\partial\mu}{\partial\vartheta}-\mu(1-\mu)\log b\right\}\right], \tag{22}$$

where the Lagrange multiplier $\lambda(s)$ is

$$\lambda(s)=-\frac{u^{\wp}}{s^{\wp}},$$

$$\mu_{m+1}(\vartheta,\varsigma)=S^{-1}\left[\frac{\mu_m(\vartheta,\varsigma)}{s}\right]-S^{-1}\left[\frac{u^{\wp}}{s^{\wp}}S\left\{\mu_m\frac{\partial\mu_m}{\partial\vartheta}-\mu_m(1-\mu_m)\log b\right\}\right]. \tag{23}$$

Now take

$$u_0(\vartheta,\varsigma)=b^{-\vartheta},$$

and consequently, we obtain

$$m=0,1,2,3\cdots$$

$$\mu_1(\vartheta,\varsigma)=S^{-1}\left[\frac{\mu_0(\vartheta,\varsigma)}{s}\right]-S^{-1}\left[\frac{u^{\wp}}{s^{\wp}}S\left\{\mu_0\frac{\partial\mu_0}{\partial\vartheta}-\mu_0(1-\mu_0)\log b\right\}\right],$$

$$\mu_1(\vartheta,\varsigma)=b^{-\vartheta}+b^{-\vartheta}\frac{\log b\varsigma^{\wp}}{\Gamma(\wp+1)},$$

$$\mu_2(\vartheta,\varsigma)=S^{-1}\left[\frac{\mu_1(\vartheta,\varsigma)}{s}\right]-S^{-1}\left[\frac{u^{\wp}}{s^{\wp}}S\left\{\mu_1\frac{\partial\mu_1}{\partial\vartheta}-\mu_1(1-\mu_1)\log b\right\}\right],$$

$$\mu_2(\vartheta,\varsigma)=b^{-\vartheta}+b^{-\vartheta}\frac{\log b\varsigma^{\wp}}{\Gamma(\wp+1)}+b^{-\vartheta}\frac{(\log b)^2\varsigma^{2\wp}}{\Gamma(2\wp+1)},$$

and

$$\mu_3(\vartheta,\varsigma)=S^{-1}\left[\frac{\mu_2(\vartheta,\varsigma)}{s}\right]-S^{-1}\left[\frac{u^{\wp}}{s^{\wp}}S\left\{\mu_2\frac{\partial\mu_2}{\partial\vartheta}-\mu_2(1-\mu_2)\log b\right\}\right],$$

$$\mu_3(\vartheta,\varsigma)=b^{-\vartheta}+b^{-\vartheta}\frac{\log b\varsigma^{\wp}}{\Gamma(\wp+1)}+b^{-\vartheta}\frac{(\log b)^2\varsigma^{2\wp}}{\Gamma(2\wp+1)}+b^{-\vartheta}\frac{(\log b)^3\varsigma^{3\wp}}{\Gamma(3\wp+1)},$$

$$\vdots$$

The analytical solution of Equation (20) can be obtained as

$$\begin{aligned}u(\vartheta,\varsigma)=b^{-\vartheta}+b^{-\vartheta}\tfrac{\log b\varsigma^{\wp}}{\Gamma(\wp+1)}+b^{-\vartheta}\tfrac{(\log b)^2\varsigma^{2\wp}}{\Gamma(2\wp+1)}+b^{-\vartheta}\tfrac{(\log b)^3\varsigma^{3\wp}}{\Gamma(3\wp+1)}+\cdots+b^{-\vartheta}\tfrac{(\log b\varsigma^{\wp})^m}{\Gamma(m\wp+1)},\\ =b^{-\vartheta}\textstyle\sum_{m=0}^{\infty}\tfrac{(\log b\varsigma^{\wp})^m}{\Gamma(m\wp+1)}=b^{-\vartheta}E_{\wp}(\log b\varsigma^{\wp}).\end{aligned} \tag{24}$$

The exact result of Equation (20) is

$$u(\vartheta,\varsigma)=b^{-\vartheta+\varsigma}. \tag{25}$$

In Figure 4, the actual and analytical solution graphs of Problem 2 at $\wp=1$ are given. From the provided graphs, it can be determined that the actual and approximate solutions are closely related. In addition, in Figures 5 and 6, the analytical solutions of Problem 2 are calculated at fractional orders $\wp=0.8, 0.6,$ and 0.4.

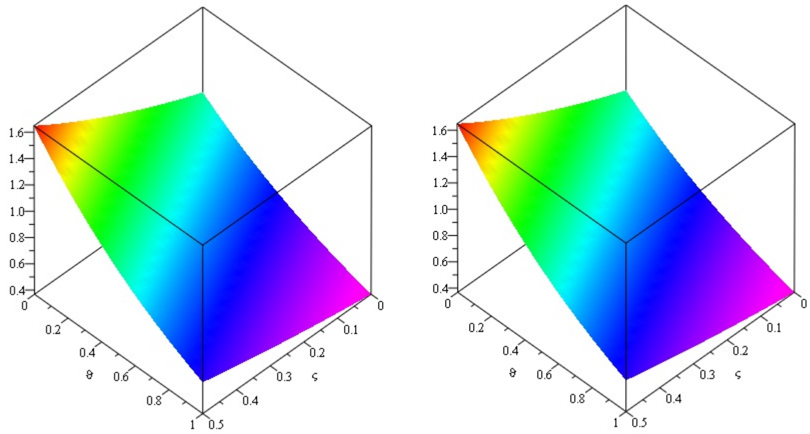

Figure 4. The first graph of the actual and the second graph of the approximate result of Example 3.2.

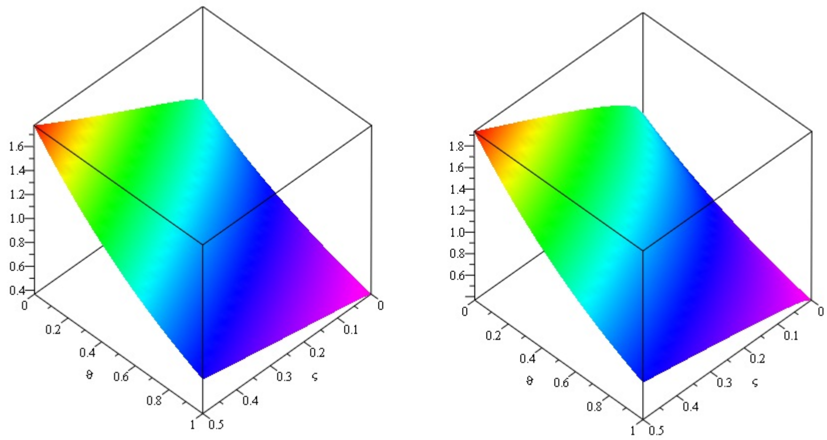

Figure 5. The first graph of the fractional order of $\wp = 0.8$ and the second graph of $\wp = 0.6$ of Example 3.2.

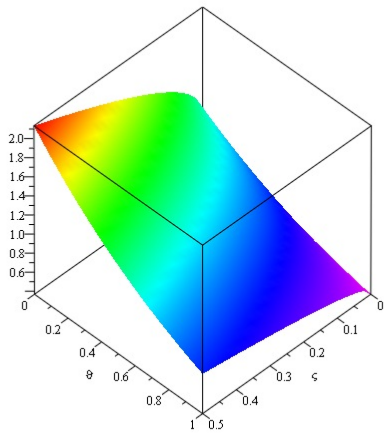

Figure 6. The fractional order of $\wp = 0.4$ of Example 3.2.

Problem 3. *Consider the fractional-order nonhomogenous gas dynamics equation [51]*

$$\frac{\partial^{\wp}\mu}{\partial\varsigma^{\wp}} + \mu\frac{\partial\mu}{\partial\varsigma} + \mu^2(1+\varsigma)^2 - \vartheta^2 = 0, \quad 0 < \wp \leq 1, \tag{26}$$

with the initial condition

$$\mu(\vartheta, 0) = \vartheta. \tag{27}$$

Using VITM on Equation (26), we obtain

$$\mu_{m+1}(\vartheta,\varsigma) = S^{-1}\left[\frac{\mu_m(\vartheta,\varsigma)}{s}\right] + S^{-1}\left[\lambda(s)S\left\{\mu_m\frac{\partial\mu_m}{\partial\vartheta} + \mu_m^2(1+\varsigma)^2 - \vartheta^2\right\}\right], \tag{28}$$

where the Lagrange multiplier is $\lambda(s)$

$$\lambda(s) = -\frac{u^{\wp}}{s^{\wp}},$$

$$\mu_{m+1}(\vartheta,\varsigma) = S^{-1}\left[\frac{\mu_m(\vartheta,\varsigma)}{s}\right] - S^{-1}\left[\frac{u^{\wp}}{s^{\wp}}S\left\{\mu_m\frac{\partial\mu_m}{\partial\vartheta} + \mu_m^2(1+\varsigma)^2 - \vartheta^2\right\}\right], \tag{29}$$

Now take

$$\mu_0(\vartheta,\varsigma) = \vartheta,$$

and consequently, we obtain,

$$m = 0,1,2,3\cdots$$

$$\mu_1(\vartheta,\varsigma) = S^{-1}\left[\frac{\mu_0(\vartheta,\varsigma)}{s}\right] - S^{-1}\left[\frac{u^{\wp}}{s^{\wp}}S\left\{\mu_0\frac{\partial\mu_0}{\partial\vartheta} + \mu_0^2(1+\varsigma)^2 - \vartheta^2\right\}\right],$$

$$\mu_1(\vartheta,\varsigma) = \vartheta - \vartheta\frac{\varsigma^{\wp}}{\Gamma(\wp+1)} - 2\vartheta\frac{\varsigma^{\wp+1}}{\Gamma(\wp+2)} - 2\vartheta\frac{\varsigma^{\wp+2}}{\Gamma(\wp+3)},$$

and

$$\mu_2(\vartheta,\varsigma) = \vartheta - \vartheta\frac{\varsigma^{\wp}}{\Gamma(\wp+1)} - 2\vartheta\frac{\varsigma^{\wp+1}}{\Gamma(\wp+2)} - 2\vartheta\frac{\varsigma^{\wp+2}}{\Gamma(\wp+3)} - (-2\vartheta-2\vartheta^2)\frac{\varsigma^{\wp+1}}{\Gamma(\wp+2)} - \Gamma(\wp+2) -$$
$$(-6\vartheta^2 - 2\vartheta^3 + \vartheta)\frac{\varsigma^{\wp+2}}{\Gamma(\wp+3)}\Gamma(\wp+3)(-\frac{8\vartheta^3}{3} + 2\vartheta^2)\frac{\varsigma^{\wp+3}}{\Gamma(\wp+4)} - \Gamma(\wp+4)$$
$$(\frac{10\vartheta^3}{3} + \vartheta^4 + 2\vartheta^2)\frac{\varsigma^{\wp+4}}{\Gamma(\wp+5)} - \Gamma(\wp+5)(4\vartheta^3 + \frac{8}{3}\vartheta^4)\frac{\varsigma^{\wp+5}}{\Gamma(\wp+6)} - \Gamma(\wp+6)(\frac{8\vartheta^3}{9}$$
$$+ \frac{22\vartheta^4}{9})\frac{\varsigma^{\wp+6}}{\Gamma(\wp+7)} - 4480\vartheta^4\frac{\varsigma^{\wp+7}}{\Gamma(\wp+8)} - 4480\vartheta^4\frac{\varsigma^{\wp+8}}{\Gamma(\wp+9)}.$$

$$\vdots$$

The analytical result of Equation (26) can be achieved as

$$u(\vartheta,\varsigma) = \vartheta - \vartheta\frac{\varsigma^{\wp}}{\Gamma(\wp+1)} - 2\vartheta\frac{\varsigma^{\wp+1}}{\Gamma(\wp+2)} - 2\vartheta\frac{\varsigma^{\wp+2}}{\Gamma(\wp+3)} - (-2\vartheta-2\vartheta^2)\frac{\varsigma^{\wp+1}}{\Gamma(\wp+2)} - \Gamma(\wp+2) -$$
$$(-6\vartheta^2 - 2\vartheta^3 + \vartheta)\frac{\varsigma^{\wp+2}}{\Gamma(\wp+3)}\Gamma(\wp+3)(-\frac{8\vartheta^3}{3} + 2\vartheta^2)\frac{\varsigma^{\wp+3}}{\Gamma(\wp+4)} - \Gamma(\wp+4)$$
$$(\frac{10\vartheta^3}{3} + \vartheta^4 + 2\vartheta^2)\frac{\varsigma^{\wp+4}}{\Gamma(\wp+5)} - \Gamma(\wp+5)(4\vartheta^3 + \frac{8}{3}\vartheta^4)\frac{\varsigma^{\wp+5}}{\Gamma(\wp+6)} - \Gamma(\wp+6)(\frac{8\vartheta^3}{9}$$
$$+ \frac{22\vartheta^4}{9})\frac{\varsigma^{\wp+6}}{\Gamma(\wp+7)} - 4480\vartheta^4\frac{\varsigma^{\wp+7}}{\Gamma(\wp+8)} - 4480\vartheta^4\frac{\varsigma^{\wp+8}}{\Gamma(\wp+9)} + \cdots,$$

The exact solution of Equation (26) is

$$u(\vartheta,\varsigma) = \frac{\vartheta}{1+\varsigma}, \tag{30}$$

In Figure 7, the exact and approximate solution graphs of Problem 3 at $\wp = 1$ *are shown. From the provided graphs, it can be determined that the actual and approximate solutions are closely related. In addition, in Figure 8, the analytical solutions of Problem 3 are calculated at fractional orders* $\wp = 0.8, 0.6,$ *and* 0.4.

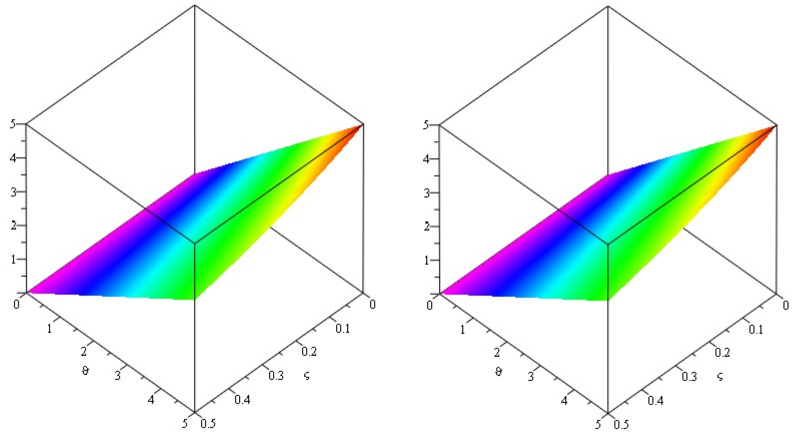

Figure 7. The actual and analytical results of Example 3.3.

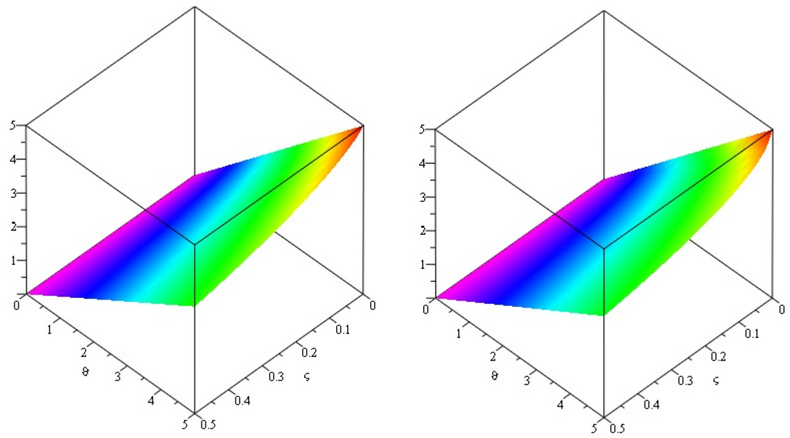

Figure 8. The first graph shows the fractional order of $\wp = 0.8$ and the second graph shows $\wp = 0.6$ of Example 3.3.

Problem 4. *Consider the fractional-order nonlinear homogenous gas dynamics equation [51]*

$$\frac{\partial^{\wp}\mu}{\partial\varsigma^{\wp}} + \mu\frac{\partial\mu}{\partial\varsigma} - \mu(1-\mu) + e^{-\vartheta+\varsigma} = 0, \;\; b > 0 \quad 0 < \wp \leq 1, \tag{31}$$

with initial condition

$$\mu(\vartheta, 0) = 1 - e^{-\vartheta}. \tag{32}$$

Applying VITM on Equation (31), we obtain

$$\mu_{m+1}(\vartheta,\varsigma) = S^{-1}\left[\frac{\mu_m(\vartheta,\varsigma)}{s}\right] + S^{-1}\left[\lambda(s)S\left\{\mu_m\frac{\partial\mu_m}{\partial\vartheta} - \mu_m(1-\mu_m) + e^{-\vartheta+\varsigma}\right\}\right], \tag{33}$$

where the Lagrange multiplier $\lambda(s)$ is

$$\lambda(s) = -\frac{u^{\wp}}{s^{\wp}},$$

$$\mu_{m+1}(\vartheta,\varsigma) = S^{-1}\left[\frac{\mu_m(\vartheta,\varsigma)}{s}\right] - S^{-1}\left[\frac{u^{\wp}}{s^{\wp}}S\left\{\mu_m\frac{\partial\mu_m}{\partial\vartheta} - \mu_m(1-\mu_m) + e^{-\vartheta+\varsigma}\right\}\right], \tag{34}$$

Now take

$$\mu_0(\vartheta,\varsigma) = 1 - e^{-\vartheta},$$

and consequently, we obtain

$$m = 0, 1, 2, 3 \cdots$$

$$\mu_1(\vartheta, \varsigma) = S^{-1}\left[\frac{\mu_0(\vartheta, \varsigma)}{s}\right] - S^{-1}\left[\frac{u^{\wp}}{s^{\wp}} S\left\{\mu_0 \frac{\partial \mu_0}{\partial \vartheta} - \mu_0(1 - \mu_0) + e^{-\vartheta + \varsigma}\right\}\right],$$

$$\mu_1(\vartheta, \varsigma) = 1 - e^{-\vartheta} - e^{-\vartheta}\frac{\varsigma^{\wp}}{\Gamma(\wp + 1)} - e^{-\vartheta}\frac{\varsigma^{2\wp}}{\Gamma(2\wp + 1)},$$

and

$$\mu_2(\vartheta, \varsigma) = S^{-1}\left[\frac{\mu_1(\vartheta, \varsigma)}{s}\right] - S^{-1}\left[\frac{u^{\wp}}{s^{\wp}} S\left\{\mu_1 \frac{\partial \mu_1}{\partial \vartheta} - \mu_1(1 - \mu_1) + e^{-\vartheta + \varsigma}\right\}\right],$$

$$\mu_2(\vartheta, \varsigma) = 1 - e^{-\vartheta} - e^{-\vartheta}\frac{\varsigma^{\wp}}{\Gamma(\wp + 1)} - e^{-\vartheta}\frac{\varsigma^{2\wp}}{\Gamma(2\wp + 1)} - e^{-\vartheta}\frac{\varsigma^{3\wp}}{\Gamma(3\wp + 1)} - \cdots$$

$$\vdots$$

The analytical solution of Equation (31) can be obtained as

$$u(\vartheta, \varsigma) = 1 - e^{-\vartheta} - e^{-\vartheta}\frac{\varsigma^{\wp}}{\Gamma(\wp + 1)} - e^{-\vartheta}\frac{\varsigma^{2\wp}}{\Gamma(2\wp + 1)} - e^{-\vartheta}\frac{\varsigma^{3\wp}}{\Gamma(3\wp + 1)} - \cdots - e^{-\vartheta}\frac{\varsigma^{m\wp}}{\Gamma(m\wp + 1)},$$

$$u(\vartheta, \varsigma) = e^{-\vartheta}\sum_{m=0}^{\infty}\frac{(\varsigma^{\wp})^m}{\Gamma(m\wp + 1)} = e^{-\vartheta}E_{\wp}(\varsigma^{\wp}). \tag{35}$$

The actual solution of Equation (31) is

$$u(\vartheta, \varsigma) = 1 - e^{-\vartheta + \varsigma}. \tag{36}$$

In Figure 9, we show the actual and approximate result graphs of Problem 4 at $\wp = 1$. *From the provided graphs, it can be determined that the actual and approximate solutions are closely related. In addition, in Figures 10 and 11, the analytical solutions of Problem 4 are calculated at fractional orders* $\wp = 0.8, 0.6,$ *and* 0.4.

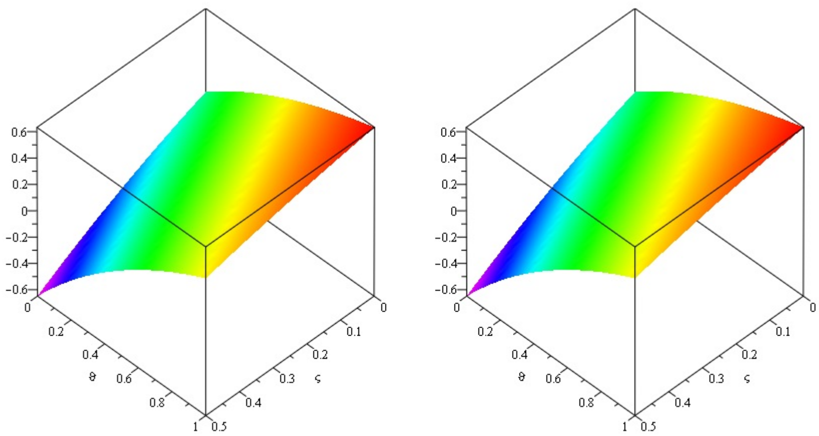

Figure 9. The actual and analytical results of Example 3.4.

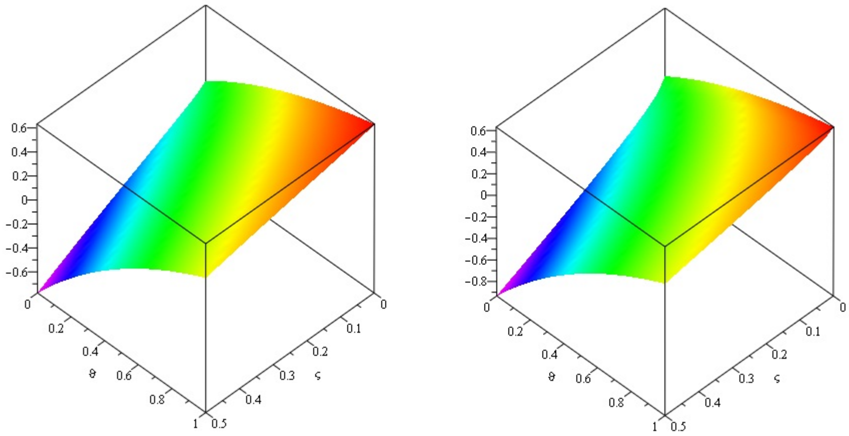

Figure 10. The first graph shows the fractional order of $\wp = 0.8$ and the second graph shows $\wp = 0.6$ of Example 3.4.

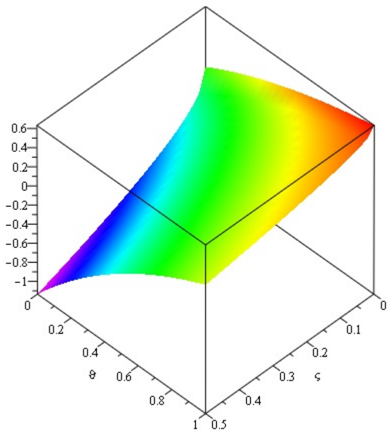

Figure 11. This graph shows the fractional order of $\wp = 0.4$ of Example 3.4.

5. Conclusions

In this article, the variational iteration transformation method is applied to achieve an analytical solution of fractional-order gas dynamics equations. The actual brilliance of the work is exemplified by its emphatic application of the Caputo fractional-order time derivative to classical equations, resulting in the production of very accurate solutions by known series solutions, even for more complex nonlinear fractional partial differential equations. Furthermore, the method is capable of decreasing computational effort and making fractional nonlinear equations much easier to solve. Therefore, the method is a very efficient and powerful technique for solving various types of linear and nonlinear fractional differential equations arising in different areas of engineering and science.

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