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# Fractional View Analysis of Kuramoto–Sivashinsky Equations with Non-Singular Kernel Operators

**Azzh Saad Alshehry [1], Muhammad Imran [2], Adnan Khan [3], Rasool Shah [3] and Wajaree Weera [4,*]**

1 Department of Mathematical Sciences, Faculty of Sciences, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia; asalshihry@pnu.edu.sa
2 School of Huminities and Social Sciences (H&SS), Bahria University, Karachi 75300, Pakistan; muhammadimran.bukc@bahria.edu.pk
3 Department of Mathematics, Abdul Wali Khan University, Mardan 23200, Pakistan; adnanmummand@gmail.com (A.K.); rasoolshahawkum@gmail.com (R.S.)
4 Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand
* Correspondence: wajawe@kku.ac.th

**Abstract:** In this article, we investigate the nonlinear model describing the various physical and chemical phenomena named the Kuramoto–Sivashinsky equation. We implemented the natural decomposition method, a novel technique, mixed with the Caputo–Fabrizio (CF) and Atangana–Baleanu deriavatives in Caputo manner (ABC) fractional derivatives for obtaining the approximate analytical solution of the fractional Kuramoto–Sivashinsky equation (FKS). The proposed method gives a series form solution which converges quickly towards the exact solution. To show the accuracy of the proposed method, we examine three different cases. We presented proposed method results by means of graphs and tables to ensure proposed method validity. Further, the behavior of the achieved results for the fractional order is also presented. The results we obtain by implementing the proposed method shows that our technique is extremely efficient and simple to investigate the behaviour of nonlinear models found in science and technology.

**Keywords:** Caputo–Fabrizio and Atangana–Baleanu operators; Adomian decomposition method; natural transform; fractional Kuramoto–Sivashinsky equations



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## 1. Introduction

Although fractional calculus (FC) has been in existence since classical calculus, it has recently attracted a lot of attention as a result of how it relates to the fundamental principles. Fractional calculus was first introduced by Leibniz and L'Hospital, but the idea has since been adopted by many scholars with numerous applications in several fields. Subsequently, it has been frequently employed to look at a range of phenomena. However, many researchers have emphasised the drawbacks of using this operator, specifically the derivative of a non-zero constant and the physical significance of the initial condition. Caputo then presented a new and unique fractional operator that covered all of the previous restrictions. The Caputo operator is used to study the majority of the models studied and examined under the FC framework. Some basic works of fractional calculus on different aspects are given by Podlubny [1], Momani and Shawagfeh [2], Kiryakova [3], Jafari and Seifi [4,5], Miller and Ross [6], Oldham and Spanier [7], Diethelm et al. [8], Kilbas and Trujillo [9], and Kemple and Beyer [10].

Physical and engineering processes have been modeled using fractional calculus, and fractional differential equations have been determined to give the best description of these models. Fractional differential equations (FDEs) have grown in popularity and significance as a result of their proven applications in a wide range of largely disparate fields of science and engineering. Fractional differential equations are applied to model a wide range of physical problems, including signal processing [11], electrodynamics [12], fluid and continuum mechanics [13], chaos theory [14], biological population models [15], finance [16], optics [17] and financial models [18]. Here, in particular, [19] presents a homotopy perturbation technique for nonlinear transport equations, papers [20–26] give the application of ADM to different transport models, also including fractional and non-linear cases, works [27–32] provide reviews or/and developments of various numerical approaches to transport/advection-diffusion problems, while [33] proposes perturbational approach to construct analytical approximations based on the double-parameter transformation perturbation expansion method. Finally, the review paper [34] contains an exhaustive review of various modern fractional calculus applications. Symmetry is the cornerstone of nature, yet the vast majority of natural phenomena lack symmetry. Breaking unexpected symmetry is an effective approach for concealing symmetry. Two types of symmetry exist: finite and infinitesimal. Finite symmetries can have discrete or continuous symmetries. Space is a constant transformation, whereas natural symmetries such as symmetry and time reversal are discrete. Patterns have interested mathematicians for millennia. In the nineteenth century, systematic classifications of planar and spatial patterns emerged. Solving fractional nonlinear differential equations with precision has proven to be rather challenging [29,35,36].

In nature, most of the complicated phenomenons are nonlinear. The world's most important processes are represented by nonlinear equations. Nonlinear FDEs solutions are of tremendous interest in both mathematics and real-world applications. In applied mathematics and physics, it is still a major problem to get the the nonlinear differential equations exact solution importance of finding the nonlinear differential equations exact solution that needs new approaches to obtain new exact or approximate solutions [37–39]. In the literature, several analytical techniques have been introduced for solving these equations, such as the fractional Adomian decomposition method (FADM) [40], reduced differential transform method (RDTM) [41], iterative Laplace transform method (ILTM) [42], the fractional variational iteration method (FVIM) [43], the fractional natural decomposition method (FNDM) [44], Elzaki transform decomposition method (ETDM) [45,46], Yang transform decomposition method (YTDM) [47], homotopy transform method and the Laplace transform (HLTM) [48], and the homotopy perturbation transform method (HPTM) [49], among many others.

Plasma instabilities, chemical reaction-diffusion, flame front propagation viscous flow problems, and magnetised plasmas are all modeled by the Kuramoto–Sivashinsky equation [50,51]. Our focus in this article is to investigate the FKS equation [52,53]

$$D_{\tau}^{\rho}\xi(\chi,\tau) + \xi\xi_{\chi} + \mathbf{x}\xi_{\chi\chi} + \mathbf{y}\xi_{\chi\chi\chi} + \mathbf{z}\xi_{\chi\chi\chi\chi} = 0, \quad 0 < \rho \leq 1, \chi \in [u,v], \tau > 0 \tag{1}$$

having the initial source

$$\xi(\chi,0) = g(\chi),$$

and boundary conditions $\xi(u,\tau) = \Psi_1(\chi), \xi(v,\tau) = \Psi_2(\chi), \xi_{\chi}(u,\tau) = \xi_{\chi}(v,\tau) = \xi_{\chi\chi}(u,\tau) = \xi_{\chi\chi}(v,\tau)$ where $g(\chi)$, $\Psi_1(\chi)$ and $\Psi_2(\chi)$ are known functions, and **x**,**y** and **z** are constants. The nonlinear advection term $\xi\xi_{\chi}$, as well as the dissipation terms, show the energy transfer mechanism.

To investigate the classical order KS equation, many authors used techniques such as finite-difference discretization [54], Chebyshev spectral collocation methods [55], homotopy analysis method [56], He's variational iteration method [57], the Lattice–Boltzmann method [58], cubic B-spline finite difference-collocation method [59] and other schemes [60,61]. To address the difficulties in this issue, these approaches require a complex procedure and a large amount of computing; however, it is relatively easy to use the proposed method to obtain and examine the solution for nonlinear problems. As a result, we used the natural transform decomposition method (NTDM) to determine the approximate solution of the Kuramoto–Sivashinsky problem with arbitrary order using two distinct fractional derivatives. The natural transform decomposition method (NTDM) combines the well-known natural transform and the Adomian decomposition method. This new method is regarded to be the finest tool for solving specific classes of coupled nonlinear partial differential equations in a straightforward and fast manner. This method produces a solution in the form of a quick convergence series, which can be exact or approximate. Many physical phenomena which are modeled by fractional PDEs are solved by using NTDM [62–64].

The present work is structured as follows: In Section 2, we provide some useful definitions for fractional derivatives, which we will employ in our current work. The basic concept of the natural decomposition approach in connection with two separate fractional derivatives is described in Section 3 for the FKS equation. Section 5 gives the execution of the proposed method for solving various FKS equation problems. A short summary of the entire work is provided at the end.

## 2. Basic Preliminaries

The natural transform, the natural transform of fractional derivatives, and the fractional calculus that will be used throughout the study are introduced here along with certain definitions and basic properties.

**Definition 1.** *A real function $j(\kappa), \kappa > 0$, will be in the space $C_v, v \in \mathbb{R}$ if there exists a real number $q > v$ with $j(\kappa) = \kappa^q g(\kappa)$, where $g \in C[0,\infty)$, and will be in the space $C_v^m$ if $j^{(m)} \in C_v$, $m \in \mathbb{N}$.*

**Definition 2.** *The fractional Riemann–Liouville integral for a function $j \in C_v, v \geq -1$ is stated as [65]*

$$I^{\rho} j(\kappa) = \frac{1}{\Gamma(\rho)} \int_0^{\kappa} (\kappa - \psi)^{\rho-1} j(\psi) d\psi, \;\; \rho > 0, \;\; \kappa > 0.$$
$$\text{and} \;\; I^0 j(\kappa) = j(\kappa) \tag{2}$$

**Definition 3.** *The Caputo derivative of $j(\kappa)$ having fractional order is stated as [65]*

$${}^{C}D_{\kappa}^{\rho} j(\kappa) = I^{m-\rho} D^m j(\kappa) = \frac{1}{\Gamma(m-\rho)} \int_0^{\kappa} (\kappa - \psi)^{m-\rho-1} D(j(\psi)) d\psi \tag{3}$$

*for $m - 1 < \rho \leq m, \;\; m \in N, \;\; \kappa > 0, j \in C_v^m, v \geq -1$.*

**Definition 4.** *The Caputo–Fabrizio derivative of $j(\kappa)$ having fractional order is stated as [65]*

$${}^{CF}D_{\kappa}^{\rho} j(\kappa) = \frac{F(\rho)}{1-\rho} \int_0^{\kappa} \exp\left( \frac{-\rho(\kappa - \psi)}{1-\rho} \right) D(j(\psi)) d\psi \tag{4}$$

*where $0 < \rho < 1$ and $F(\rho)$ is the normalization function having $F(0) = F(1) = 1$.*

**Definition 5.** *The Atangana–Baleanu–Caputo derivative of $j(\kappa)$ having fractional order is stated as [65]*

$${}^{ABC}D_{\kappa}^{\rho} j(\kappa) = \frac{B(\rho)}{1-\rho} \int_0^{\kappa} E_{\rho} \left( \frac{-\rho(\kappa - \psi)^{\rho}}{1-\rho} \right) D(j(\psi)) d\psi \tag{5}$$

*where $0 < \rho < 1$, where $B(\rho)$ represents the normalization function and $E_{\rho}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\rho^m + 1)}$ represents the Mittag–Leffler function.*

**Definition 6.** *On employing the natural transform to $\xi(\tau)$, we have*

$$\mathcal{N}(\xi(\tau)) = \mathcal{V}(\wp, \gamma) = \int_{-\infty}^{\infty} e^{-\wp \tau} \xi(\gamma \tau) d\tau, \;\; \wp, \gamma \in (-\infty, \infty). \tag{6}$$

*The natural transform of $\xi(\tau)$ for $\tau \in (0,\infty)$ is given as*

$$\mathcal{N}(\xi(\tau)H(\tau)) = \mathcal{N}^{+} = \mathcal{V}^{+}(\wp,\gamma) = \int_{-\infty}^{\infty} e^{-\wp\tau}\xi(\gamma\tau)d\tau, \ \ \wp,\gamma \in (0,\infty). \tag{7}$$

*where $H(\tau)$ is the Heaviside function.*

**Definition 7.** *On employing the inversenNatural transform the function $\mathcal{V}(\wp,\gamma)$ is defined as*

$$\mathcal{N}^{-1}[\mathcal{V}(\wp,\gamma)] = \xi(\tau), \ \ \forall\tau \geq 0. \tag{8}$$

**Lemma 1.** *The linearity property of the natural transform for $\xi_1(\tau)$ is $\xi_1(\wp,\gamma)$ and $\xi_2(\tau)$ is $\xi_2(\wp,\gamma)$. Then*

$$\mathcal{N}[c_1\xi_1(\tau) + c_2\xi_2(\tau)] = c_1\mathcal{N}[\xi_1(\tau)] + c_2\mathcal{N}[\xi_2(\tau)] = c_1\xi_1(\wp,\gamma) + c_2\xi_2(\wp,\gamma), \tag{9}$$

*where $c_1$ and $c_2$ are constants.*

**Lemma 2.** *If the inverse natural transforms of $\mathcal{V}_1(\wp,\gamma)$ and $\mathcal{V}_2(\wp,\gamma)$ are $\xi_1(\tau)$ and $\xi_2(\tau)$, respectively, then*

$$\mathcal{N}^{-1}[c_1\mathcal{V}_1(\wp,\gamma) + c_2\mathcal{V}_2(\wp,\gamma)] = c_1\mathcal{N}^{-1}[\mathcal{V}_1(\wp,\gamma)] + c_2\mathcal{N}^{-1}[\mathcal{V}_2(\wp,\gamma)] = c_1\xi_1(\tau) + c_2\xi_2(\tau), \tag{10}$$

*where $c_1$ and $c_2$ are constants.*

**Definition 8.** *The natural transform of $D_{\tau}^{\rho}\xi(\tau)$ in the Caputo sense is defined as [65]*

$$\mathcal{N}[{}^{C}D_{\tau}^{\rho}\xi(\tau)] = \left(\frac{\wp}{\gamma}\right)^{\rho}\left(\mathcal{N}[\xi(\tau)] - \left(\frac{1}{\wp}\right)\xi(0)\right), \quad \rho \leq 1. \tag{11}$$

**Definition 9.** *The natural transform of $D_{\tau}^{\rho}\xi(\tau)$ in the sense of Caputo–Fabrizio is defined as [65]*

$$\mathcal{N}[{}^{CF}D_{\tau}^{\rho}\xi(\tau)] = \frac{1}{1-\rho+\rho(\frac{\gamma}{\wp})}\left(\mathcal{N}[\xi(\tau)] - \left(\frac{1}{\wp}\right)\xi(0)\right), \quad \rho \leq 1. \tag{12}$$

**Definition 10.** *The natural transform of $D_{\tau}^{\rho}\xi(\tau)$ in the sense of Atangana–Baleanu–Caputo is defined as [65]*

$$\mathcal{N}[{}^{ABC}D_{\tau}^{\rho}\xi(\tau)] = \frac{B(\rho)}{1-\rho+\rho(\frac{\gamma}{\wp})^{\rho}}\left(\mathcal{N}[\xi(\tau)] - \left(\frac{1}{\wp}\right)\xi(0)\right), \quad \rho \leq 1. \tag{13}$$

**Definition 11.** *The inverse natural transform $\mathcal{N}^{-1}$ is stated as*

$$\mathcal{N}^{-1}[\mathcal{V}(\wp,\gamma)] = \xi(\tau) = \lim_{T\to\infty}\frac{1}{2\pi\imath}\int_{\sigma-\imath T}^{\sigma+\imath T} e^{\frac{\wp\tau}{\gamma}}\mathcal{V}(\wp,\gamma)d\wp. \tag{14}$$

## 3. General Procedure

Here, we discuss the general procedure of the proposed method to solve the below equation.

$$D_{\tau}^{\rho}\xi(\chi,\tau) = \mathcal{L}(\xi(\chi,\tau)) + \mathbb{N}(\xi(\chi,\tau)) + h(\chi,\tau), \tag{15}$$

subjected to the initial condition

$$\xi(\chi,0) = \phi(\psi), \tag{16}$$

with $\mathcal{L}, \mathbb{N}$ representing the linear and nonlinear terms and $h(\chi,\tau)$ as the source function.

### 3.1. Case I ($NTDM_{CF}$)

On taking the natural transform and using the fractional Caputo–Fabrizio derivative, Equation (15) is determined as

$$\frac{1}{p(\rho,\gamma,\wp)}\left(\mathcal{N}[\xi(\chi,\tau)] - \frac{\phi(\chi)}{\wp}\right) = \mathcal{N}\left[\mathcal{L}(\xi(\chi,\tau)) + \mathbb{N}(\xi(\chi,\tau)) + h(\chi,\tau)\right], \tag{17}$$

with

$$p(\rho,\gamma,\wp) = 1 - \rho + \rho(\frac{\gamma}{\wp}). \tag{18}$$

On taking the inverse natural transform, Equation (17) is stated as

$$\xi(\chi,\tau) = \mathcal{N}^{-1}\left(\frac{\phi(\chi)}{\wp} + p(\rho,\gamma,\wp)\mathcal{N}[h(\chi,\tau)]\right) + \mathcal{N}^{-1}\left[p(\rho,\gamma,\wp)\mathcal{N}\left(\mathcal{L}(\xi(\chi,\tau)) + \mathbb{N}(\xi(\chi,\tau))\right)\right], \tag{19}$$

$\mathbb{N}(\xi(\chi,\tau))$ can be decomposed as

$$\mathbb{N}(\xi(\chi,\tau)) = \sum_{i=0}^{\infty} A_i. \tag{20}$$

The approximate solution for $\xi^{CF}(\chi,\tau)$ in series form is defined as

$$\xi^{CF}(\chi,\tau) = \sum_{i=0}^{\infty} \xi_i^{CF}(\chi,\tau). \tag{21}$$

Substituting Equations (20) and (21) into (19), we get

$$\begin{aligned}\sum_{i=0}^{\infty} \xi_i(\chi,\tau) =& \mathcal{N}^{-1}\left(\frac{\phi(\chi)}{\wp} + p(\rho,\gamma,\wp)\mathcal{N}[h(\chi,\tau)]\right) \\ &+ \mathcal{N}^{-1}\left(p(\rho,\gamma,\wp)\mathcal{N}\left[\sum_{i=0}^{\infty}(\mathcal{L}(\xi_i(\chi,\tau)) + A_i)\right]\right).\end{aligned} \tag{22}$$

From (22), we have

$$\begin{aligned}\xi_0^{CF}(\chi,\tau) =& \mathcal{N}^{-1}\left(\frac{\phi(\chi)}{\wp} + p(\rho,\gamma,\wp)\mathcal{N}[h(\chi,\tau)]\right), \\ \xi_1^{CF}(\chi,\tau) =& \mathcal{N}^{-1}(p(\rho,\gamma,\wp)\mathcal{N}[\mathcal{L}(\xi_0(\chi,\tau)) + A_0]), \\ &\vdots \\ \xi_{l+1}^{CF}(\chi,\tau) =& \mathcal{N}^{-1}(p(\rho,\gamma,\wp)\mathcal{N}[\mathcal{L}(\xi_l(\chi,\tau)) + A_l]), \;\; l = 1,2,3,\cdots\end{aligned} \tag{23}$$

Hence, we get the solution to (15) in the $NTDM_{CF}$ manner by substituting (23) into (21):

$$\xi^{CF}(\chi,\tau) = \xi_0^{CF}(\chi,\tau) + \xi_1^{CF}(\chi,\tau) + \xi_2^{CF}(\chi,\tau) + \cdots \tag{24}$$

### 3.2. Case II ($NTDM_{ABC}$)

On taking the natural transform and using the fractional Atangana–Baleanu–Caputo derivative, Equation (15) is determined as

$$\frac{1}{q(\rho,\gamma,\wp)}\left(\mathcal{N}[\xi(\chi,\tau)]-\frac{\phi(\chi)}{\wp}\right)=\mathcal{N}\left[\mathcal{L}(\xi(\chi,\tau))+\mathbb{N}(\xi(\chi,\tau))+h(\chi,\tau)\right], \tag{25}$$

with

$$q(\rho,\gamma,\wp)=\frac{1-\rho+\rho(\frac{\gamma}{\wp})^{\rho}}{B(\rho)}. \tag{26}$$

On taking the inverse natural transform, Equation (25) is stated as

$$\xi(\chi,\tau)=\mathcal{N}^{-1}\left(\frac{\phi(\chi)}{\wp}+q(\rho,\gamma,\wp)\mathcal{N}[h(\chi,\tau)]\right)+\mathcal{N}^{-1}\left[q(\rho,\gamma,\wp)\mathcal{N}\left(\mathcal{L}(\xi(\chi,\tau))+\mathbb{N}(\xi(\chi,\tau))\right)\right]. \tag{27}$$

$\mathbb{N}(\xi(\chi,\tau))$ can be decomposed as

$$\mathbb{N}(\xi(\chi,\tau))=\sum_{i=0}^{\infty}A_i. \tag{28}$$

The approximate solution for $\xi^{ABC}(\chi,\tau)$ in series form is defined as

$$\xi^{ABC}(\chi,\tau)=\sum_{i=0}^{\infty}\xi_i^{ABC}(\chi,\tau). \tag{29}$$

Substituting Equations (28) and (29) into (27), we get

$$\begin{aligned}\sum_{i=0}^{\infty}\xi_i(\chi,\tau)=&\mathcal{N}^{-1}\left(\frac{\phi(\chi)}{\wp}+q(\rho,\gamma,\wp)\mathcal{N}[h(\chi,\tau)]\right)\\&+\mathcal{N}^{-1}\left(q(\rho,\gamma,\wp)\mathcal{N}\left[\sum_{i=0}^{\infty}(\mathcal{L}(\xi_i(\chi,\tau))+A_i)\right]\right).\end{aligned} \tag{30}$$

From (22), we have

$$\begin{aligned}\xi_0^{ABC}(\chi,\tau)=&\mathcal{N}^{-1}\left(\frac{\phi(\chi)}{\wp}+q(\rho,\gamma,\wp)\mathcal{N}[h(\chi,\tau)]\right),\\\xi_1^{ABC}(\chi,\tau)=&\mathcal{N}^{-1}(q(\rho,\gamma,\wp)\mathcal{N}[\mathcal{L}(\xi_0(\chi,\tau))+A_0]),\\&\vdots\\\xi_{l+1}^{ABC}(\chi,\tau)=&\mathcal{N}^{-1}(q(\rho,\gamma,\wp)\mathcal{N}[\mathcal{L}(\xi_l(\chi,\tau))+A_l]),\ \ l=1,2,3,\cdots\end{aligned} \tag{31}$$

Hence, we get the solution to (15) in an $NTDM_{ABC}$ manner by substituting (31) into (29):

$$\xi^{ABC}(\chi,\tau)=\xi_0^{ABC}(\chi,\tau)+\xi_1^{ABC}(\chi,\tau)+\xi_2^{ABC}(\chi,\tau)+\cdots \tag{32}$$

## 4. Applications

In this part, we obtain the approximate solution of fractional Kuramoto–Sivashinsky equations.

**Example 1.** *Consider the FKS Equation (1) for $x=-1$, $y=0$, $z=1$,*

$$D_\tau^{\rho}\xi(\chi,\tau)+\xi\xi_\chi-\xi_{\chi\chi}+\xi_{\chi\chi\chi\chi}=0,\quad 0<\rho\le 1, \tag{33}$$

*subjected to the initial condition*

$$\xi(\chi,0)=\delta+\frac{15\tanh^3[\ell(\chi-\gamma)]-45\tanh[\ell(\chi-\gamma)]}{19\sqrt{19}}. \tag{34}$$

*where $\delta,\ell$ and $\gamma$ are real constants. On taking the inverse natural transform, Equation (33) is stated as*

$$\mathcal{N}[D^{\rho}_{\tau}\xi(\chi,\tau)]=-\mathcal{N}\Bigg\{\xi\xi_{\chi}\Bigg\}+\mathcal{N}\Bigg\{\xi_{\chi\chi}\Bigg\}-\mathcal{N}\Bigg\{\xi_{\chi\chi\chi\chi}\Bigg\}. \tag{35}$$

*By the transformation property, we have*

$$\frac{1}{\wp^{\rho}}\mathcal{N}[\xi(\chi,\tau)]-\wp^{2-\rho}\xi(\chi,0)=\mathcal{N}\Bigg[-\xi\xi_{\chi}+\xi_{\chi\chi}-\xi_{\chi\chi\chi\chi}\Bigg]. \tag{36}$$

*We obtain after simplification*

$$\mathcal{N}[\xi(\chi,\tau)]=\wp^2\Bigg[\delta+\frac{15\tanh^3[\ell(\chi-\gamma)]-45\tanh[\ell(\chi-\gamma)]}{19\sqrt{19}}\Bigg]+\frac{\rho(\wp-\rho(\wp-\rho))}{\wp^2}\Bigg[-\xi\xi_{\chi}+\xi_{\chi\chi}-\xi_{\chi\chi\chi\chi}\Bigg]. \tag{37}$$

*On taking the inverse natural transform, Equation (37) is stated as*

$$\begin{aligned}\xi(\chi,\tau)=&\Bigg[\delta+\frac{15\tanh^3[\ell(\chi-\gamma)]-45\tanh[\ell(\chi-\gamma)]}{19\sqrt{19}}\Bigg]\\&+\mathcal{N}^{-1}\Bigg[\frac{\rho(\wp-\rho(\wp-\rho))}{\wp^2}\mathcal{N}\Bigg\{-\xi\xi_{\chi}+\xi_{\chi\chi}-\xi_{\chi\chi\chi\chi}\Bigg\}\Bigg].\end{aligned} \tag{38}$$

**Now we implement *NDM<sub></sub>***$_{CF}$.

*The approximate solution for $\xi(\chi,\tau)$ in series form is defined as*

$$\xi(\chi,\tau)=\sum_{l=0}^{\infty}\xi_l(\chi,\tau). \tag{39}$$

*Additionally, by Adomian polynomials, the nonlinear terms $\xi\xi_{\chi}=\sum_{l=0}^{\infty}\mathcal{A}_l$, are calculated as*

$$A_0=\xi_0(\xi_0)_{\chi},A_1=\xi_1(\xi_0)_{\chi}+\xi_0(\xi_1)_{\chi}.$$

*Thus, Equation (38) is stated as*

$$\begin{aligned}\sum_{l=0}^{\infty}\xi_{l+1}(\chi,\tau)=&\,\delta+\frac{15\tanh^3[\ell(\chi-\gamma)]-45\tanh[\ell(\chi-\gamma)]}{19\sqrt{19}}\\&+\mathcal{N}^{-1}\Bigg[\frac{\rho(\wp-\rho(\wp-\rho))}{\wp^2}\mathcal{N}\Bigg\{-\sum_{l=0}^{\infty}\mathcal{A}_l+\sum_{l=0}^{\infty}\xi_{l\chi\chi}-\sum_{l=0}^{\infty}\xi_{l\chi\chi\chi\chi}\Bigg\}\Bigg].\end{aligned} \tag{40}$$

*On comparison of Equation (40) on both sides, we get*

$$\xi_0(\chi,\tau)=\delta+\frac{15\tanh^3[\ell(\chi-\gamma)]-45\tanh[\ell(\chi-\gamma)]}{19\sqrt{19}},$$

$$\begin{aligned}\xi_1(\chi,\tau)=&\frac{45\ell\,\mathrm{sech}^7[\ell(\chi-\gamma)]}{27436\sqrt{19}}\Bigg(-1083\delta\cosh[\ell(\chi-\gamma)]-361\delta\cosh[3\ell(\chi-\gamma)]+4(30\sqrt{19}-722\ell\\&-31768\ell^3)+\Big(15\sqrt{19}-722\ell+11552\ell^3\Big)\cosh[2\ell(\chi-\gamma)]\sinh[\ell(\chi-\gamma)]\Bigg)(\rho(\tau-1)+1),\end{aligned} \tag{41}$$

$$\xi_2(\chi,\tau) = \frac{45\ell\,\mathrm{sech}^{11}[\ell(\chi-\gamma)]}{39617584}\Bigg(-21660\Big(-27+38\sqrt{19}\ell(1+32\ell^2)\Big)\delta\cosh[\ell(\chi-\gamma)]+1444\Big(45+152\sqrt{19}\ell(-1+31\ell^2)\Big)\delta\cosh[3\ell(\chi-\gamma)]+1444(-75+76\sqrt{19}\ell(1+74\ell^2))\delta\cosh[5\ell(\chi-\gamma)]-1444(15+38\sqrt{19}\ell(-1+16\ell^2))\delta\cosh[7\ell(\chi-\gamma)]-5\Bigg(5776\ell\Bigg(-69+2\ell\Bigg(19\sqrt{19}+4\ell\Bigg(-2031+19\sqrt{19}\ell(49+11014\ell^2)\Bigg)\Bigg)\Bigg)\Bigg)+\sqrt{19}(10620-6859\delta^2)\sinh[\ell(\chi-\gamma)]+3\Bigg(2888\ell\Bigg(220+19\ell\Bigg(-19\sqrt{19}+8\ell\Bigg(-65+\sqrt{19}\ell\Bigg(-109+12442\ell^2\Bigg)\Bigg)\Bigg)\Bigg)\Bigg)+3\sqrt{19}(-1500+6859\delta^2)\sinh[\ell(\chi-\gamma)]-5\Bigg(2888\ell\Bigg(12+\ell\Bigg(19\sqrt{19}+8\ell\Bigg(111+19\sqrt{19}\ell(-31+734\ell^2)\Bigg)\Bigg)\Bigg)\Bigg)-\sqrt{19}(900+6859\delta^2)\sinh[5\ell(\chi-\gamma)]+(5776\ell(-1+16\ell^2))\Bigg(15+19\sqrt{19}(-1+16\ell^2)\Bigg)+\sqrt{19}(900+6859\delta^2)\sinh[7\ell(\chi-\gamma)]\Bigg(\Big((1-\rho)^2+2\rho(1-\rho)\tau+\frac{\rho^2\tau^2}{2}\Big). \tag{42}$$

*In this way, the other terms $\xi_l$ for $(l \geq 3)$ are easy to obtain.*

$$\xi(\chi,\tau) = \sum_{l=0}^{\infty}\xi_l(\chi,\tau) = \xi_0(\chi,\tau)+\xi_1(\chi,\tau)+\xi_2(\chi,\tau)+\cdots,$$

$$\xi(\chi,\tau) = \delta+\frac{15\tanh^3[\ell(\chi-\gamma)]-45\tanh[\ell(\chi-\gamma)]}{19\sqrt{19}}-\frac{45\ell\,\mathrm{sech}^7[\ell(\chi-\gamma)]}{27436\sqrt{19}}\Bigg(-1083\delta\cosh[\ell(\chi-\gamma)]-361\delta\cosh[3\ell(\chi-\gamma)]+4(30\sqrt{19}-722\ell-31768\ell^3)+\Big(15\sqrt{19}-722\ell+11552\ell^3\Big)\cosh[2\ell(\chi-\gamma)]\sinh[\ell(\chi-\gamma)]\Bigg)(\rho(\tau-1)+1)+\frac{45\ell\,\mathrm{sech}^{11}[\ell(\chi-\gamma)]}{39617584}\Bigg(-21660\Big(-27+38\sqrt{19}\ell(1+32\ell^2)\Big)\delta\cosh[\ell(\chi-\gamma)]+1444\Big(45+152\sqrt{19}\ell(-1+31\ell^2)\Big)\delta\cosh[3\ell(\chi-\gamma)]+1444(-75+76\sqrt{19}\ell(1+74\ell^2))\delta\cosh[5\ell(\chi-\gamma)]-1444(15+38\sqrt{19}\ell(-1+16\ell^2))\delta\cosh[7\ell(\chi-\gamma)]-5\Bigg(5776\ell\Bigg(-69+2\ell\Bigg(19\sqrt{19}+4\ell\Bigg(-2031+19\sqrt{19}\ell(49+11014\ell^2)\Bigg)\Bigg)\Bigg)\Bigg)+\sqrt{19}(10620-6859\delta^2)\sinh[\ell(\chi-\gamma)]+3\Bigg(2888\ell\Bigg(220+19\ell\Bigg(-19\sqrt{19}+8\ell\Bigg(-65+\sqrt{19}\ell(-109+12442\ell^2)\Bigg)\Bigg)\Bigg)\Bigg)+3\sqrt{19}(-1500+6859\delta^2)\sinh[\ell(\chi-\gamma)]-5\Bigg(2888\ell\Bigg(12+\ell\Bigg(19\sqrt{19}+8\ell\Bigg(111+19\sqrt{19}\ell(-31+734\ell^2)\Bigg)\Bigg)\Bigg)\Bigg)-\sqrt{19}(900+6859\delta^2)\sinh[5\ell(\chi-\gamma)]+(5776\ell(-1+16\ell^2))\Bigg(15+19\sqrt{19}(-1+16\ell^2)\Bigg)+\sqrt{19}(900+6859\delta^2)\sinh[7\ell(\chi-\gamma)]\Big((1-\rho)^2+2\rho(1-\rho)\tau+\frac{\rho^2\tau^2}{2}\Big)+\cdots. \tag{43}$$

***Now we implement $NDM_{ABC}$.***

*The approximate solution for $\xi(\chi,\tau)$ in series form is defined as*

$$\xi(\chi,\tau) = \sum_{l=0}^{\infty} \xi_l(\chi,\tau). \tag{44}$$

*Additionally, by Adomian polynomials, the nonlinear terms $\xi\xi_\chi = \sum_{l=0}^{\infty} \mathcal{A}_l$ are calculated as*

$$A_0 = \xi_0(\xi_0)_\chi,\, A_1 = \xi_1(\xi_0)_\chi + \xi_0(\xi_1)_\chi.$$

*Thus, Equation (38) is stated as*

$$\begin{aligned}\sum_{l=0}^{\infty} \xi_{l+1}(\chi,\tau) = \delta + \frac{15\tanh^3[\ell(\chi-\gamma)] - 45\tanh[\ell(\chi-\gamma)]}{19\sqrt{19}} \\ + \mathcal{N}^{-1}\left[\frac{\gamma^\rho(\wp^\rho + \rho(\gamma^\rho - \wp^\rho))}{\wp^{2\rho}}\mathcal{N}\left\{-\sum_{l=0}^{\infty}\mathcal{A}_l + \sum_{l=0}^{\infty}\xi_{l\chi\chi} - \sum_{l=0}^{\infty}\xi_{l\chi\chi\chi\chi}\right\}\right].\end{aligned} \tag{45}$$

*On comparison of Equation (45) on both sides, we obtain*

$$\xi_0(\chi,\tau) = \delta + \frac{15\tanh^3[\ell(\chi-\gamma)] - 45\tanh[\ell(\chi-\gamma)]}{19\sqrt{19}},$$

$$\begin{aligned}\xi_1(\chi,\tau) = \frac{45\ell\,\text{sech}^7[\ell(\chi-\gamma)]}{27436\sqrt{19}}\Bigg(-1083\delta\cosh[\ell(\chi-\gamma)] - 361\delta\cosh[3\ell(\chi-\gamma)] + 4(30\sqrt{19} - 722\ell - \\ 31768\ell^3) + \left(15\sqrt{19} - 722\ell + 11552\ell^3\right)\cosh[2\ell(\chi-\gamma)]\sinh[\ell(\chi-\gamma)]\Bigg)\left(1 - \rho + \frac{\rho\tau^\rho}{\rho(\rho+1)}\right),\end{aligned} \tag{46}$$

$$\begin{aligned}\xi_2(\chi,\tau) = \frac{45\ell\,\text{sech}^{11}[\ell(\chi-\gamma)]}{39617584}\Bigg(-21660\left(-27 + 38\sqrt{19}\ell(1+32\ell^2)\right)\delta\cosh[\ell(\chi-\gamma)] + 1444\Bigg(45 + \\ 152\sqrt{19}\ell(-1+31\ell^2)\Bigg)\delta\cosh[3\ell(\chi-\gamma)] + 1444(-75 + 76\sqrt{19}\ell(1+74\ell^2))\delta\cosh[5\ell(\chi-\gamma)] - 1444(15 \\ + 38\sqrt{19}\ell(-1+16\ell^2))\delta\cosh[7\ell(\chi-\gamma)] - 5\Bigg(5776\ell\Bigg(-69 + 2\ell\Bigg(19\sqrt{19} + 4\ell\Bigg(-2031 + 19\sqrt{19}\ell(49 + \\ 11014\ell^2)\Bigg)\Bigg)\Bigg)\Bigg) + \sqrt{19}(10620 - 6859\delta^2)\sinh[\ell(\chi-\gamma)] + 3(2888\ell(220 + 19\ell(-19\sqrt{19} + 8\ell(-65 + \\ \sqrt{19}\ell(-109 + 12442\ell^2))))) + 3\sqrt{19}(-1500 + 6859\delta^2)\sinh[\ell(\chi-\gamma)] - 5\Bigg(2888\ell\Bigg(12 + \ell\Bigg(19\sqrt{19} + 8\ell \\ \Bigg(111 + 19\sqrt{19}\ell(-31 + 734\ell^2)\Bigg)\Bigg)\Bigg)\Bigg) - \sqrt{19}(900 + 6859\delta^2)\sinh[5\ell(\chi-\gamma)] + (5776\ell(-1+16\ell^2))\Bigg(15 + \\ 19\sqrt{19}(-1+16\ell^2)\Bigg) + \sqrt{19}(900 + 6859\delta^2)\sinh[7\ell(\chi-\gamma)]\left[\frac{\rho^2\tau^{2\rho}}{\rho(2\rho+1)} + 2\rho(1-\rho)\frac{\tau^\rho}{\rho(\rho+1)} + (1-\rho)^2\right].\end{aligned} \tag{47}$$

*In this way, the other terms $\xi_l$ for $(l \geq 3)$ are easy to obtain.*

$$
\begin{aligned}
&\xi(\chi,\tau)=\sum_{l=0}^{\infty}\xi_l(\chi,\tau)=\xi_0(\chi,\tau)+\xi_1(\chi,\tau)+\xi_2(\chi,\tau)+\cdots,\\
&\xi(\chi,\tau)=\delta+\frac{15\tanh^3[\ell(\chi-\gamma)]-45\tanh[\ell(\chi-\gamma)]}{19\sqrt{19}}-\frac{45\ell\operatorname{sech}^7[\ell(\chi-\gamma)]}{27436\sqrt{19}}\Bigg(-1083\delta\cosh[\ell(\chi-\gamma)]-\\
&361\delta\cosh[3\ell(\chi-\gamma)]+4(30\sqrt{19}-722\ell-31768\ell^3)+\Big(15\sqrt{19}-722\ell+11552\ell^3\Big)\cosh[2\ell(\chi-\gamma)]\sinh\\
&[\ell(\chi-\gamma)]\Bigg)\bigg(1-\rho+\frac{\rho\tau^{\rho}}{\rho(\rho+1)}\bigg)+\frac{45\ell\operatorname{sech}^{11}[\ell(\chi-\gamma)]}{39617584}\Bigg(-21660\Bigg(-27+38\sqrt{19}\ell(1+32\ell^2)\Bigg)\delta\\
&\cosh[\ell(\chi-\gamma)]+1444\Bigg(45+152\sqrt{19}\ell(-1+31\ell^2)\Bigg)\delta\cosh[3\ell(\chi-\gamma)]+1444(-75+76\sqrt{19}\ell(1+74\ell^2))\\
&\delta\cosh[5\ell(\chi-\gamma)]-1444(15+38\sqrt{19}\ell(-1+16\ell^2))\delta\cosh[7\ell(\chi-\gamma)]-5\Bigg(5776\ell\Bigg(-69+2\ell\Bigg(19\sqrt{19}+\\
&4\ell\Bigg(-2031+19\sqrt{19}\ell(49+11014\ell^2)\Bigg)\Bigg)\Bigg)\Bigg)+\sqrt{19}(10620-6859\delta^2)\sinh[\ell(\chi-\gamma)]+3\Bigg(2888\ell\Bigg(220+19\ell\\
&\Bigg(-19\sqrt{19}+8\ell\Bigg(-65+\sqrt{19}\ell(-109+12442\ell^2)\Bigg)\Bigg)\Bigg)\Bigg)+3\sqrt{19}(-1500+6859\delta^2)\sinh[\ell(\chi-\gamma)]\\
&-5\Bigg(2888\ell\Bigg(12+\ell\Bigg(19\sqrt{19}+8\ell\Bigg(111+19\sqrt{19}\ell(-31+734\ell^2)\Bigg)\Bigg)\Bigg)\Bigg)-\sqrt{19}(900+6859\delta^2)\sinh[5\ell\\
&(\chi-\gamma)]+(5776\ell(-1+16\ell^2))\Bigg(15+19\sqrt{19}(-1+16\ell^2)\Bigg)+\sqrt{19}(900+6859\delta^2)\sinh[7\ell(\chi-\gamma)]\\
&\left[\frac{\rho^2\tau^{2\rho}}{\rho(2\rho+1)}+2\rho(1-\rho)\frac{\tau^{\rho}}{\rho(\rho+1)}+(1-\rho)^2\right]+\cdots.
\end{aligned}
\tag{48}
$$

*By taking $\rho=1$, we obtain the exact solution as*

$$\xi(\chi,\tau)=\delta+\frac{15\tanh^3[\ell(\chi-\delta\tau-\gamma)]-45\tanh[\ell(\chi-\delta\tau-\gamma)]}{19\sqrt{19}}, \tag{49}$$

**Example 2.** *Consider the FKS Equation (1) for **x**=1, **y**=0, **z**=1,*

$$D_{\tau}^{\rho}\xi(\chi,\tau)+\xi\xi_{\chi}+\xi_{\chi\chi}+\xi_{\chi\chi\chi\chi}=0,\quad 0<\rho\leq 1, \tag{50}$$

*subjected to the initial condition*

$$\xi(\chi,0)=\delta+\frac{15}{9}\sqrt{\frac{11}{9}}(-9\tanh[\ell(\chi-\gamma)]+11\tanh^3[\ell(\chi-\gamma)]). \tag{51}$$

*On taking the inverse natural transform, Equation (50) is stated as*

$$\mathcal{N}[D_{\tau}^{\rho}\xi(\chi,\tau)]=-\mathcal{N}\Bigg\{\xi\xi_{\chi}\Bigg\}-\mathcal{N}\Bigg\{\xi_{\chi\chi}\Bigg\}-\mathcal{N}\Bigg\{\xi_{\chi\chi\chi\chi}\Bigg\}. \tag{52}$$

*By the transformation property, we have*

$$\frac{1}{\wp^{\rho}}\mathcal{N}[\xi(\chi,\tau)]-\wp^{2-\rho}\xi(\chi,0)=\mathcal{N}\Bigg[-\xi\xi_{\chi}-\xi_{\chi\chi}-\xi_{\chi\chi\chi\chi}\Bigg]. \tag{53}$$

*We obtain after simplification*

$$\mathcal{N}[\xi(\chi,\tau)] = \wp^2\left[\delta + \frac{15}{9}\sqrt{\frac{11}{9}}(-9\tanh[\ell(\chi-\gamma)] + 11\tanh^3[\ell(\chi-\gamma)])\right] + \frac{\rho(\wp-\rho(\wp-\rho))}{\wp^2}\left[-\xi\xi_\chi - \xi_{\chi\chi} - \xi_{\chi\chi\chi\chi}\right]. \tag{54}$$

*On taking the inverse natural transform, Equation (54) is stated as*

$$\xi(\chi,\tau) = \left[\delta + \frac{15}{9}\sqrt{\frac{11}{9}}(-9\tanh[\ell(\chi-\gamma)] + 11\tanh^3[\ell(\chi-\gamma)])\right] + \mathcal{N}^{-1}\left[\frac{\rho(\wp-\rho(\wp-\rho))}{\wp^2}\mathcal{N}\left\{-\xi\xi_\chi - \xi_{\chi\chi} - \xi_{\chi\chi\chi\chi}\right\}\right]. \tag{55}$$

***Now, we apply NDM$_{CF}$.***
*The approximate solution for $\xi(\chi,\tau)$ in series form is defined as*

$$\xi(\chi,\tau) = \sum_{l=0}^{\infty}\xi_l(\chi,\tau). \tag{56}$$

*Additionally, by Adomian polynomials, the nonlinear terms $\xi\xi_\chi = \sum_{l=0}^{\infty}\mathcal{A}_l$ are calculated as*

$$A_0 = \xi_0(\xi_0)_\chi, A_1 = \xi_1(\xi_0)_\chi + \xi_0(\xi_1)_\chi.$$

*Thus, Equation (55) is stated as*

$$\sum_{l=0}^{\infty}\xi_{l+1}(\chi,\tau) = \delta + \frac{15}{9}\sqrt{\frac{11}{9}}(-9\tanh[\ell(\chi-\gamma)] + 11\tanh^3[\ell(\chi-\gamma)]) + \mathcal{N}^{-1}\left[\frac{\rho(\wp-\rho(\wp-\rho))}{\wp^2}\mathcal{N}\left\{-\sum_{l=0}^{\infty}\mathcal{A}_l - \sum_{l=0}^{\infty}\xi_{l\chi\chi} - \sum_{l=0}^{\infty}\xi_{l\chi\chi\chi\chi}\right\}\right]. \tag{57}$$

*On comparison of Equation (57) on both sides, we obtain*

$$\xi_0(\chi,\tau) = \delta + \frac{15}{9}\sqrt{\frac{11}{19}}(-9\tanh[\ell(\chi-\gamma)] + 11\tanh^3[\ell(\chi-\gamma)]),$$

$$\xi_1(\chi,\tau) = \frac{45\sqrt{11}\ell\,\mathrm{sech}^7 h[\ell(\chi-\gamma)]}{27436\sqrt{19}}(1444\delta\cosh^3[\ell(\chi-\gamma)](-7+\cosh[2\ell(\chi-\gamma)]) + 6(955\sqrt{209} - 722\ell(-3+368\ell^2))\sinh[\ell(\chi-\gamma)] + 2\Big(-735\sqrt{209} + 2888\ell(2+53\ell^2)\Big)\sinh[3\ell(\chi-\gamma)] + 4\Big(15\sqrt{209} - 361(\ell+4\ell^3)\Big)\sinh[5\ell(\chi-\gamma)])(\rho(\tau-1)+1)$$

$$\xi_2(\chi,\tau) = -\frac{45\ell^2\,\mathrm{sech}^{11}[\ell(\chi-\gamma)]}{39617584}(4332\Big(-39105 + 38\sqrt{209}\ell(-41+1576\ell^2)\Big)\delta \cosh[\ell(\chi-\gamma)] - 33212\Big(-2145 + 38\sqrt{209}\ell(1+64\ell^2)\Big)\delta\cosh[3\ell(\chi-\gamma)] - 137180(-429+304\ell^2)) \times \delta\cosh[5\ell(\chi-\gamma)] + 1444\Big(-8580 + 19\sqrt{209}\ell(19+436\ell^2)\Big)\delta\cosh[7\ell(\chi-\gamma)] - 1444\Big(-165+$$

$$19\sqrt{209}\ell(1+4\ell^2)\Big)\delta\cosh[9\ell(\chi-\gamma)] + (2888\ell\Big(456060+\ell\Big(3743\sqrt{209}+8\ell\Big(-22074195+19\sqrt{209}\ell$$
$$(-5173+1200214\ell^2)\Big)\Big)\Big) + \sqrt{209}(356390100-281219\delta^2)) \times \sinh[\ell(\chi-\gamma)] + (-2888\ell\Big(-267960+$$
$$\ell\Big(-4883\sqrt{209}+8\ell\Big(-8086155+19\sqrt{209}\ell \times (3313+417866\ell^2)\Big)\Big)\Big) - \sqrt{209}(147341700+486989\delta^2))$$
$$\sinh[3\ell(\chi-\gamma)] + 5(2888\ell\Big(-34584+\ell\Big(133\sqrt{209}+8\ell\Big(-191631+19\sqrt{209}\ell(349+8834\ell^2)\Big)\Big)\Big)+$$
$$\sqrt{209}(4468860-48013\delta^2))\sinh[5\ell(\chi-\gamma)] - 4(361\ell\Big(-30030+\ell\Big(931\sqrt{209}+8\ell\Big(-53625+19\sqrt{209}\ell$$
$$(229+1898\ell^2)\Big)\Big)\Big) + \sqrt{209}(247500+6859\delta^2))\sinh[7\ell(\chi-\gamma)] + \Big(1444\ell(1+4\ell^2)\Big(-330+19\sqrt{209}\ell$$
$$(1+4\ell^2)\Big) + \sqrt{209}(9900+6859\delta^2)\Big)\sinh[9\ell(\chi-\gamma)])\Big((1-\rho)^2+2\rho(1-\rho)\tau+\frac{\rho^2\tau^2}{2}\Big).$$

*In this way, the other terms $\xi_l$ for $(l \geq 3)$ are easy to obtain.*

$$\xi(\chi,\tau) = \sum_{l=0}^{\infty}\xi_l(\chi,\tau) = \xi_0(\chi,\tau)+\xi_1(\chi,\tau)+\xi_2(\chi,\tau)+\cdots,$$
$$\xi(\chi,\tau) = \delta + \frac{15}{9}\sqrt{\frac{11}{19}}(-9\tanh[\ell(\chi-\gamma)]+11\tanh^3[\ell(\chi-\gamma)]) - \frac{45\sqrt{11}\ell\,\mathrm{sech}^7 h[\ell(\chi-\gamma)]}{27436\sqrt{19}}(1444\delta$$
$$\cosh^3[\ell(\chi-\gamma)](-7+\cosh[2\ell(\chi-\gamma)]) + 6(955\sqrt{209}-722\ell(-3+368\ell^2))\sinh[\ell(\chi-\gamma)] + 2$$
$$\Big(-735\sqrt{209}+2888\ell(2+53\ell^2)\Big)\sinh[3\ell(\chi-\gamma)] + 4\Big(15\sqrt{209}-361(\ell+4\ell^3)\Big)\sinh[5\ell(\chi-\gamma)])\Big(\rho(\tau-$$
$$1)+1\Big) - \frac{45\ell^2\,\mathrm{sech}^{11}[\ell(\chi-\gamma)]}{39617584}(4332\Big(-39105+38\sqrt{209}\ell(-41+1576\ell^2)\Big)\delta\cosh[\ell(\chi-\gamma)] - 33212$$
$$\Big(-2145+38\sqrt{209}\ell(1+64\ell^2)\Big)\delta\cosh[3\ell(\chi-\gamma)] - 137180(-429+304\ell^2)) \times \delta\cosh[5\ell(\chi-\gamma)] + 1444 \quad (58)$$
$$\Big(-8580+19\sqrt{209}\ell(19+436\ell^2)\Big)\delta\cosh[7\ell(\chi-\gamma)] - 1444\Big(-165+19\sqrt{209}\ell(1+4\ell^2)\Big)\delta\cosh[9\ell(\chi-$$
$$\gamma)] + (2888\ell\Big(456060+\ell\Big(3743\sqrt{209}+8\ell\Big(-22074195+19\sqrt{209}\ell(-5173+1200214\ell^2)\Big)\Big)\Big) + \sqrt{209}$$
$$(356390100-281219\delta^2)) \times \sinh[\ell(\chi-\gamma)] + (-2888\ell\Big(-267960+\ell\Big(-4883\sqrt{209}+8\ell\Big(-8086155+$$
$$19\sqrt{209}\ell \times (3313+417866\ell^2)\Big)\Big)\Big) - \sqrt{209}(147341700+486989\delta^2))\sinh[3\ell(\chi-\gamma)] + 5(2888\ell\Big(-34584$$
$$+\ell\Big(133\sqrt{209}+8\ell\Big(-191631+19\sqrt{209}\ell(349+8834\ell^2)\Big)\Big)\Big) + \sqrt{209}(4468860-48013\delta^2))\sinh[5\ell(\chi-$$

$$\gamma)] - 4(361\ell\Big( - 30030 + \ell\Big(931\sqrt{209} + 8\ell\Big( - 53625 + 19\sqrt{209}\ell(229 + 1898\ell^2)\Big)\Big)\Big) + \sqrt{209}(247500+ 6859\delta^2))\sinh[7\ell(\chi-\gamma)] + \Big(1444\ell(1+4\ell^2)\Big( - 330 + 19\sqrt{209}\ell(1+4\ell^2)\Big) + \sqrt{209}(9900 + 6859\delta^2)\Big) \sinh[9\ell(\chi-\gamma)])\Big((1-\rho)^2 + 2\rho(1-\rho)\tau + \frac{\rho^2\tau^2}{2}\Big) + \cdots. \tag{59}$$

***Now, we apply*** $NDM_{ABC}$

*The approximate solution for* $\xi(\chi,\tau)$ *in series form is defined as*

$$\xi(\chi,\tau) = \sum_{l=0}^{\infty} \xi_l(\chi,\tau). \tag{60}$$

*Additionally, by Adomian polynomials, the nonlinear terms* $\xi\xi_\chi = \sum_{l=0}^{\infty} \mathcal{A}_l$ *are calculated as*

$$A_0 = \xi_0(\xi_0)_\chi, A_1 = \xi_1(\xi_0)_\chi + \xi_0(\xi_1)_\chi.$$

*Thus, Equation (55) is stated as*

$$\sum_{l=0}^{\infty} \xi_l(\chi,\tau) = \delta + \frac{15}{9}\sqrt{\frac{11}{9}}(-9\tanh[\ell(\chi-\gamma)] + 11\tanh^3[\ell(\chi-\gamma)]) + \mathcal{N}^{-1}\left[\frac{\gamma^\rho(\wp^\rho + \rho(\gamma^\rho - \wp^\rho))}{\wp^{2\rho}}\mathcal{N}\left\{ - \sum_{l=0}^{\infty}\mathcal{A}_l - \sum_{l=0}^{\infty}\xi_{l\chi\chi} - \sum_{l=0}^{\infty}\xi_{l\chi\chi\chi\chi}\right\}\right]. \tag{61}$$

*On comparison of Equation (61) on both sides, we obtain*

$$\xi_0(\chi,\tau) = \delta + \frac{15}{9}\sqrt{\frac{11}{19}}(-9\tanh[\ell(\chi-\gamma)] + 11\tanh^3[\ell(\chi-\gamma)]),$$

$$\xi_1(\chi,\tau) = \frac{45\sqrt{11}\ell\,\mathrm{sech}^7\,h[\ell(\chi-\gamma)]}{27436\sqrt{19}}(1444\delta\cosh^3[\ell(\chi-\gamma)](-7+\cosh[2\ell(\chi-\gamma)]) + 6(955\sqrt{209} - 722\ell(-3+368\ell^2))\sinh[\ell(\chi-\gamma)] + 2\Big( - 735\sqrt{209} + 2888\ell(2+53\ell^2)\Big)\sinh[3\ell(\chi-\gamma)] + 4\Big(15\sqrt{209} - 361(\ell+4\ell^3)\Big)\sinh[5\ell(\chi-\gamma)])\Big(1-\rho+\frac{\rho\tau^\rho}{\rho(\rho+1)}\Big),$$

$$\xi_2(\chi,\tau) = -\frac{45\ell^2\,\mathrm{sech}^{11}[\ell(\chi-\gamma)]}{39617584}(4332\Big( - 39105 + 38\sqrt{209}\ell(-41+1576\ell^2)\Big)\delta\cosh[\ell(\chi-\gamma)] - 33212\Big( - 2145 + 38\sqrt{209}\ell(1+64\ell^2)\Big)\delta\cosh[3\ell(\chi-\gamma)] - 137180(-429+304\ell^2)) \times \delta\cosh[5\ell(\chi-\gamma)] + 1444\Big( - 8580 + 19\sqrt{209}\ell(19+436\ell^2)\Big)\delta\cosh[7\ell(\chi-\gamma)] - 1444\Big( - 165 + 19\sqrt{209}\ell(1+4\ell^2)\Big) \delta\cosh[9\ell(\chi-\gamma)] + (2888\ell\Big(456060 + \ell\Big(3743\sqrt{209} + 8\ell\Big( - 22074195 + 19\sqrt{209}\ell(-5173 + 1200214\ell^2)$$

$$\Bigg)\Bigg)\Bigg) + \sqrt{209}(356390100 - 281219\delta^2)) \times \sinh[\ell(\chi-\gamma)] + (-2888\ell\Bigg( - 267960 + \ell\Bigg( - 4883\sqrt{209} + 8\ell \Bigg( - 8086155 + 19\sqrt{209}\ell \times (3313 + 417866\ell^2)\Bigg)\Bigg)\Bigg) - \sqrt{209}(147341700 + 486989\delta^2)) \sinh[3\ell(\chi-\gamma)] + 5(2888\ell\Bigg( - 34584 + \ell\Bigg(133\sqrt{209} + 8\ell\Bigg( - 191631 + 19\sqrt{209}\ell(349 + 8834\ell^2)\Bigg)\Bigg)\Bigg) + \sqrt{209}(4468860 - 48013\delta^2)) \sinh[5\ell(\chi-\gamma)] - 4(361\ell\Bigg( - 30030 + \ell\Bigg(931\sqrt{209} + 8\ell\Bigg( - 53625 + 19\sqrt{209}\ell(229 + 1898\ell^2)\Bigg)\Bigg)\Bigg) + \sqrt{209}(247500 + 6859\delta^2)) \sinh[7\ell(\chi-\gamma)] + \Bigg(1444\ell(1+4\ell^2)\Bigg( - 330 + 19\sqrt{209}\ell(1+4\ell^2)\Bigg) + \sqrt{209}(9900 + 6859\delta^2)\Bigg) \sinh[9\ell(\chi-\gamma)]\left[\frac{\rho^2\tau^{2\rho}}{\rho(2\rho+1)} + 2\rho(1-\rho)\frac{\tau^\rho}{\rho(\rho+1)} + (1-\rho)^2\right].$$

*In this way, the other terms $\xi_l$ for $(l \geq 3)$ are easy to obtain.*

$$\xi(\chi,\tau) = \sum_{l=0}^{\infty} \xi_l(\chi,\tau) = \xi_0(\chi,\tau) + \xi_1(\chi,\tau) + \xi_2(\chi,\tau) + \cdots,$$

$$\xi(\chi,\tau) = \delta + \frac{15}{9}\sqrt{\frac{11}{19}}(-9\tanh[\ell(\chi-\gamma)] + 11\tanh^3[\ell(\chi-\gamma)]) - \frac{45\sqrt{11}\ell\,\mathrm{sech}^7 h[\ell(\chi-\gamma)]}{27436\sqrt{19}}(1444\delta\cosh^3[\ell(\chi-\gamma)](-7 + \cosh[2\ell(\chi-\gamma)]) + 6(955\sqrt{209} - 722\ell(-3 + 368\ell^2))\sinh[\ell(\chi-\gamma)] + 2\Big( - 735\sqrt{209} + 2888\ell(2 + 53\ell^2)\Big)\sinh[3\ell(\chi-\gamma)] + 4\Big(15\sqrt{209} - 361(\ell + 4\ell^3)\Big)\sinh[5\ell(\chi-\gamma)])\left(1 - \rho + \frac{\rho\tau^\rho}{\rho(\rho+1)}\right) - \frac{45\ell^2\,\mathrm{sech}^{11}[\ell(\chi-\gamma)]}{39617584}(4332\Bigg( - 39105 + 38\sqrt{209}\ell(-41 + 1576\ell^2)\Bigg)\delta\cosh[\ell(\chi-\gamma)] - 33212\Bigg( - 2145 + 38\sqrt{209}\ell(1 + 64\ell^2)\Bigg)\delta\cosh[3\ell(\chi-\gamma)] - 137180(-429 + 304\ell^2)) \times \delta\cosh[5\ell(\chi-\gamma)] + 1444\Bigg( - 8580 + 19\sqrt{209}\ell(19 + 436\ell^2)\Bigg)\delta\cosh[7\ell(\chi-\gamma)] - 1444\Bigg( - 165 + 19\sqrt{209}\ell(1 + 4\ell^2)\Bigg)\delta\cosh[9\ell(\chi-\gamma)] + (2888\ell \Bigg(456060 + \ell\Bigg(3743\sqrt{209} + 8\ell\Bigg( - 22074195 + 19\sqrt{209}\ell(-5173 + 1200214\ell^2)\Bigg)\Bigg)\Bigg) + \sqrt{209}(356390100 - 281219\delta^2)) \times \sinh[\ell(\chi-\gamma)] + (-2888\ell(-267960 + \ell(-4883\sqrt{209} + 8\ell(-8086155 + 19\sqrt{209}\ell \times (3313 + 417866\ell^2)))) - \sqrt{209}(147341700 + 486989\delta^2))\sinh[3\ell(\chi-\gamma)] + 5(2888\ell\Bigg( - 34584 + \ell\Bigg(133\sqrt{209} + 8\ell\Bigg( - 191631 + 19\sqrt{209}\ell(349 + 8834\ell^2)\Bigg)\Bigg)\Bigg) + \sqrt{209}(4468860 - 48013\delta^2))\sinh[5\ell(\chi-\gamma)] - 4(361\ell\Bigg( - 30030 + \ell\Bigg(931\sqrt{209} + 8\ell\Bigg( - 53625 + 19\sqrt{209}\ell(229 + 1898\ell^2)\Bigg)\Bigg)\Bigg) + \sqrt{209}(247500 + 6859\delta^2))\sinh[7\ell(\chi-\gamma)] + \Bigg(1444\ell(1+4\ell^2)\Bigg( - 330 + 19\sqrt{209}\ell(1+4\ell^2)\Bigg) + \sqrt{209}(9900 + 6859\delta^2)\Bigg) \tag{62}$$

$$\sinh[9\ell(\chi-\gamma)]\left[\frac{\rho^2\tau^{2\rho}}{\rho(2\rho+1)}+2\rho(1-\rho)\frac{\tau^\rho}{\rho(\rho+1)}+(1-\rho)^2\right]+\cdots. \tag{63}$$

*By taking $\rho = 1$, we obtain the exact solution as*

$$\xi(\chi,\tau)=\delta+\frac{15}{9}\sqrt{\frac{11}{19}}(-9\tanh[\ell(\chi-\delta\tau-\gamma)]+11\tanh^3[\ell(\chi-\delta\tau-\gamma)]). \tag{64}$$

**Example 3.** *Consider the FKS Equation (1) at $x = 1$, $y = 4$, $z = 1$,*

$$D_\tau^\rho\xi(\chi,\tau)+\xi\xi_\chi+\xi_{\chi\chi}+4\xi_{\chi\chi\chi}+\xi_{\chi\chi\chi\chi}=0, \quad 0<\rho\le 1, \tag{65}$$

*subjected to the initial condition*

$$\xi(\chi,0)=\delta+9-15(\tanh[\ell(\chi-\gamma)]+\tanh^2[\ell(\chi-\gamma)]-\tanh^3[\ell(\chi-\gamma)]). \tag{66}$$

*On taking the inverse natural transform, Equation (65) is stated as*

$$\mathcal{N}[D_\tau^\rho\xi(\chi,\tau)]=-\mathcal{N}\left\{\xi\xi_\chi\right\}-\mathcal{N}\left\{\xi_{\chi\chi}\right\}-\mathcal{N}\left\{4\xi_{\chi\chi\chi}\right\}-\mathcal{N}\left\{\xi_{\chi\chi\chi\chi}\right\}. \tag{67}$$

*By the transformation property, we have*

$$\frac{1}{\wp^\rho}\mathcal{N}[\xi(\chi,\tau)]-\wp^{2-\rho}\xi(\chi,0)=\mathcal{N}\left[-\xi\xi_\chi-\xi_{\chi\chi}-4\xi_{\chi\chi\chi}-\xi_{\chi\chi\chi\chi}\right]. \tag{68}$$

*We obtain after simplification*

$$\begin{aligned}\mathcal{N}[\xi(\chi,\tau)]=\wp^2\left[\delta+9-15(\tanh[\ell(\chi-\gamma)]+\tanh^2[\ell(\chi-\gamma)]-\tanh^3[\ell(\chi-\gamma)])\right]+\frac{\rho(\wp-\rho(\wp-\rho))}{\wp^2}\\ \left[-\xi\xi_\chi-\xi_{\chi\chi}-4\xi_{\chi\chi\chi}-\xi_{\chi\chi\chi\chi}\right].\end{aligned} \tag{69}$$

*On taking the inverse natural transform, Equation (69) is stated as*

$$\begin{aligned}\xi(\chi,\tau)=&\left[\delta+9-15(\tanh[\ell(\chi-\gamma)]+\tanh^2[\ell(\chi-\gamma)]-\tanh^3[\ell(\chi-\gamma)])\right]\\&+\mathcal{N}^{-1}\left[\frac{\rho(\wp-\rho(\wp-\rho))}{\wp^2}\mathcal{N}\left\{-\xi\xi_\chi-\xi_{\chi\chi}-4\xi_{\chi\chi\chi}-\xi_{\chi\chi\chi\chi}\right\}\right].\end{aligned} \tag{70}$$

***Now, we apply NDM$_{CF}$.***
*The approximate solution for $\xi(\chi,\tau)$ in series form is defined as*

$$\xi(\chi,\tau)=\sum_{l=0}^{\infty}\xi_l(\chi,\tau). \tag{71}$$

*Additionally, by Adomian polynomials, the nonlinear terms $\xi\xi_\chi=\sum_{l=0}^{\infty}\mathcal{A}_l$, are calculated as*

$$A_0=\xi_0(\xi_0)_\chi,\, A_1=\xi_1(\xi_0)_\chi+\xi_0(\xi_1)_\chi.$$

*Thus, Equation (70) is stated as*

$$\sum_{l=0}^{\infty}\xi_{l+1}(\chi,\tau)=\delta+9-15(\tanh[\ell(\chi-\gamma)]+\tanh^2[\ell(\chi-\gamma)]-\tanh^3[\ell(\chi-\gamma)])$$
$$+\mathcal{N}^{-1}\left[\frac{\rho(\wp-\rho(\wp-\rho))}{\wp^2}\mathcal{N}\left\{-\sum_{l=0}^{\infty}\mathcal{A}_l-\sum_{l=0}^{\infty}\xi_{l\chi\chi}-4\sum_{l=0}^{\infty}\xi_{l\chi\chi\chi}-\sum_{l=0}^{\infty}\xi_{l\chi\chi\chi\chi}\right\}\right]. \quad (72)$$

*On comparison of Equation (72) on both sides, we obtain*

$$\xi_0(\chi,\tau)=\delta+9-15(\tanh[\ell(\chi-\gamma)]+\tanh^2[\ell(\chi-\gamma)]-\tanh^3[\ell(\chi-\gamma)]),$$
$$\xi_1(\chi,\tau)=-\frac{15\ell\,\mathrm{sech}^6[\ell(\chi-\gamma)]}{8}(4(-36+2\ell-32\ell^2+344\ell^3+\delta)\cosh[2\ell(\chi-\gamma)]+(-6-10\ell+112\ell^2$$
$$-136\ell^3+\delta)\cosh[4\ell(\chi-\gamma)]+3(74+6\ell-80\ell^2-456\ell^3+\delta+2(14-2\ell-112\ell^2+120\ell^3+\delta)$$
$$\sinh[2\ell(\chi-\gamma)]+(-6-2\ell+48\ell^2-40\ell^3+\delta)\sinh[4\ell(\chi-\gamma)]))\times(-1+\tanh[\ell(\chi-\gamma)])$$
$$(\rho(\tau-1)+1)$$

$$\xi_2(\chi,\tau)=\frac{15\ell^2\,\mathrm{sech}^{10}[\ell(\chi-\gamma)]}{64}(2(5(4\ell(966+\ell(-77693+8\ell(-25495+\ell(-2205+31238\ell(4+\ell)))))+$$
$$4\ell(19+4(38-245\ell)\ell)\delta)-5\delta^2+60(693+5\delta))-2(127332+4\ell(-282+\ell(-242299+8\ell(-70727+\ell(693+$$
$$88234\ell(4+\ell)))))+276\delta+4\ell(-43+4\ell(-86+77\ell))\delta+17\delta^2)\cosh[2\ell(\chi-\gamma)]+8(6624+4\ell(-471+$$
$$4\ell(-3313+\ell(-5503+2\ell(1071+3652\ell(4+\ell)))))-198\delta+16\ell(-1+\ell(-8+119\ell))\delta-\delta^2)\cosh[4\ell(\chi$$
$$-\gamma)]-2(1404+4\ell(-246+\ell(-2917+8\ell(-329+\ell(531+502\ell(4+\ell)))))-228\delta+4\ell(11+4\ell(22+59\ell))$$
$$\delta-\delta^2)\cosh[6\ell(\chi-\gamma)]+(-6+2\ell(1+4\ell(2+\ell))+\delta)^2\cosh[8\ell(\chi-\gamma)]+2(4\ell(-3558+\ell(-157321+$$
$$8\ell(-87713+\ell(50967+45304\ell+504046\ell^2))))-4\ell(-23+868\ell(2+\ell))\delta+23\delta^2+36(3223+59\delta))\sinh$$
$$[2\ell(\chi-\gamma)]-2(4(-1+2\ell)(-6507+\ell(-11136+\ell(34637+2\ell(77025+81222\ell+97708\ell^2))))+4(-1+$$
$$2\ell)(159+67\ell(5+2\ell))\delta-17\delta^2)\times\sinh[4\ell(\chi-\gamma)]+6(516+4\ell(-66+\ell(-1223+8\ell(-99+\ell(369+$$
$$328\ell+338\ell^2))))-92\delta+4\ell(1+36\ell(2+\ell))\delta+\delta^2)\sinh[6\ell(\chi-\gamma)]-(-6+2\ell(1+4\ell(2+\ell))+\delta)^2$$
$$\times\sinh[8\ell(\chi-\gamma)]-43200(5-208\ell^3+1344\ell^6)\tanh[\ell(\chi-\gamma)])\left((1-\rho)^2+2\rho(1-\rho)\tau+\frac{\rho^2\tau^2}{2}\right).$$

*In this way, the other terms $\xi_l$ for $(l\geq 3)$ are easy to obtain.*

$$\xi(\chi,\tau)=\sum_{l=0}^{\infty}\xi_l(\chi,\tau)=\xi_0(\chi,\tau)+\xi_1(\chi,\tau)+\xi_2(\chi,\tau)+\cdots,$$
$$\xi(\chi,\tau)=\delta+9-15(\tanh[\ell(\chi-\gamma)]+\tanh^2[\ell(\chi-\gamma)]-\tanh^3[\ell(\chi-\gamma)])-\frac{15\ell\,\mathrm{sech}^6[\ell(\chi-\gamma)]}{8}$$
$$(4(-36+2\ell-32\ell^2+344\ell^3+\delta)\cosh[2\ell(\chi-\gamma)]+(-6-10\ell+112\ell^2-136\ell^3+\delta)\cosh[4\ell(\chi-\gamma)]+$$
$$3(74+6\ell-80\ell^2-456\ell^3+\delta+2(14-2\ell-112\ell^2+120\ell^3+\delta)\sinh[2\ell(\chi-\gamma)]+(-6-2\ell+48\ell^2-$$
$$40\ell^3+\delta)\sinh[4\ell(\chi-\gamma)]))\times(-1+\tanh[\ell(\chi-\gamma)])(\rho(\tau-1)+1)+\frac{15\ell^2\,\mathrm{sech}^{10}[\ell(\chi-\gamma)]}{64}(2(5(4\ell \quad (73)$$
$$(966+\ell(-77693+8\ell(-25495+\ell(-2205+31238\ell(4+\ell)))))+4\ell(19+4(38-245\ell)\ell)\delta)-5\delta^2+60$$
$$(693+5\delta))-2(127332+4\ell(-282+\ell(-242299+8\ell(-70727+\ell(693+88234\ell(4+\ell)))))+276\delta+4\ell$$
$$(-43+4\ell(-86+77\ell))\delta+17\delta^2)\cosh[2\ell(\chi-\gamma)]+8(6624+4\ell(-471+4\ell(-3313+\ell(-5503+2\ell(1071$$
$$+3652\ell(4+\ell)))))-198\delta+16\ell(-1+\ell(-8+119\ell))\delta-\delta^2)\cosh[4\ell(\chi-\gamma)]-2(1404+4\ell(-246+\ell$$

$$(-2917+8\ell(-329+\ell(531+502\ell(4+\ell)))))-228\delta+4\ell(11+4\ell(22+59\ell))\delta-\delta^2)\cosh[6\ell(\chi-\gamma)]+ (-6+2\ell(1+4\ell(2+\ell))+\delta)^2\cosh[8\ell(\chi-\gamma)]+2(4\ell(-3558+\ell(-157321+8\ell(-87713+\ell(50967 +45304\ell+504046\ell^2))))-4\ell(-23+868\ell(2+\ell))\delta+23\delta^2+36(3223+59\delta))\sinh[2\ell(\chi-\gamma)]-2 (4(-1+2\ell)(-6507+\ell(-11136+\ell(34637+2\ell(77025+81222\ell+97708\ell^2))))+4(-1+2\ell)(159+67\ell (5+2\ell))\delta-17\delta^2)\times\sinh[4\ell(\chi-\gamma)]+6(516+4\ell(-66+\ell(-1223+8\ell(-99+\ell(369+328\ell+338\ell^2))))- 92\delta+4\ell(1+36\ell(2+\ell))\delta+\delta^2)\sinh[6\ell(\chi-\gamma)]-(-6+2\ell(1+4\ell(2+\ell))+\delta)^2\times\sinh[8\ell(\chi-\gamma)]- 43200(5-208\ell^3+1344\ell^6)\tanh[\ell(\chi-\gamma)])\left((1-\rho)^2+2\rho(1-\rho)\tau+\frac{\rho^2\tau^2}{2}\right)+\cdots. \tag{74}$$

***Now, we apply NDM$_{ABC}$.***

*The approximate solution for $\xi(\chi,\tau)$ in series form is defined as*

$$\xi(\chi,\tau)=\sum_{l=0}^{\infty}\xi_l(\chi,\tau). \tag{75}$$

*Additionally, by Adomian polynomials, the nonlinear terms $\xi\xi_\chi=\sum_{l=0}^{\infty}\mathcal{A}_l$ are calculated as*

$$A_0=\xi_0(\xi_0)_\chi,\ A_1=\xi_1(\xi_0)_\chi+\xi_0(\xi_1)_\chi.$$

$$\begin{aligned}\sum_{l=0}^{\infty}\xi_l(\chi,\tau)&=\delta+9-15(\tanh[\ell(\chi-\gamma)]+\tanh^2[\ell(\chi-\gamma)]-\tanh^3[\ell(\chi-\gamma)])\\&+\mathcal{N}^{-1}\left[\frac{\gamma^\rho(\wp^\rho+\rho(\gamma^\rho-\wp^\rho))}{\wp^{2\rho}}\mathcal{N}\left\{-\sum_{l=0}^{\infty}\mathcal{A}_l-\sum_{l=0}^{\infty}\xi_{l\chi\chi}-4\sum_{l=0}^{\infty}\xi_{l\chi\chi\chi}-\sum_{l=0}^{\infty}\xi_{l\chi\chi\chi\chi}\right\}\right].\end{aligned} \tag{76}$$

*On comparison of Equation (76) on both sides, we obtain*

$$\begin{aligned}\xi_0(\chi,\tau)&=\delta+9-15(\tanh[\ell(\chi-\gamma)]+\tanh^2[\ell(\chi-\gamma)]-\tanh^3[\ell(\chi-\gamma)]),\\ \xi_1(\chi,\tau)&=-\frac{15\ell\,\mathrm{sech}^6[\ell(\chi-\gamma)]}{8}(4(-36+2\ell-32\ell^2+344\ell^3+\delta)\cosh[2\ell(\chi-\gamma)]+(-6-10\ell+112\ell^2\\&-136\ell^3+\delta)\cosh[4\ell(\chi-\gamma)]+3(74+6\ell-80\ell^2-456\ell^3+\delta+2(14-2\ell-112\ell^2+120\ell^3+\delta)\\&\sinh[2\ell(\chi-\gamma)]+(-6-2\ell+48\ell^2-40\ell^3+\delta)\sinh[4\ell(\chi-\gamma)]))\times(-1+\tanh[\ell(\chi-\gamma)])\\&\left(1-\rho+\frac{\rho\tau^\rho}{\rho(\rho+1)}\right),\end{aligned}$$

$$\xi_2(\chi,\tau)=\frac{15\ell^2\,\mathrm{sech}^{10}[\ell(\chi-\gamma)]}{64}(2(5(4\ell(966+\ell(-77693+8\ell(-25495+\ell(-2205+31238\ell(4+\ell)))))+ 4\ell(19+4(38-245\ell)\ell)\delta)-5\delta^2+60(693+5\delta))-2(127332+4\ell(-282+\ell(-242299+8\ell(-70727+\ell (693+88234\ell(4+\ell)))))+276\delta+4\ell(-43+4\ell(-86+77\ell))\delta+17\delta^2)\cosh[2\ell(\chi-\gamma)]+8(6624+4\ell(-471 +4\ell(-3313+\ell(-5503+2\ell(1071+3652\ell(4+\ell)))))-198\delta+16\ell(-1+\ell(-8+119\ell))\delta-\delta^2)\cosh[4\ell(\chi -\gamma)]-2(1404+4\ell(-246+\ell(-2917+8\ell(-329+\ell(531+502\ell(4+\ell)))))-228\delta+4\ell(11+4\ell(22+59\ell)) \delta-\delta^2)\cosh[6\ell(\chi-\gamma)]+(-6+2\ell(1+4\ell(2+\ell))+\delta)^2\cosh[8\ell(\chi-\gamma)]+2(4\ell(-3558+\ell(-157321+ 8\ell(-87713+\ell(50967+45304\ell+504046\ell^2))))-4\ell(-23+868\ell(2+\ell))\delta+23\delta^2+36(3223+59\delta))\sinh[2\ell (\chi-\gamma)]-2(4(-1+2\ell)(-6507+\ell(-11136+\ell(34637+2\ell(77025+81222\ell+97708\ell^2))))+4(-1+2\ell)$$

$$(159+67\ell(5+2\ell))\delta-17\delta^2)\times\sinh[4\ell(\chi-\gamma)]+6(516+4\ell(-66+\ell(-1223+8\ell(-99+\ell(369+328\ell+338\ell^2))))-92\delta+4\ell(1+36\ell(2+\ell))\delta+\delta^2)\sinh[6\ell(\chi-\gamma)]-(-6+2\ell(1+4\ell(2+\ell))+\delta)^2\times\sinh[8\ell(\chi-\gamma)]-43200(5-208\ell^3+1344\ell^6)\tanh[\ell(\chi-\gamma)])\left[\frac{\rho^2\tau^{2\rho}}{\rho(2\rho+1)}+2\rho(1-\rho)\frac{\tau^\rho}{\rho(\rho+1)}+(1-\rho)^2\right].$$

*In this way, the other terms $\xi_l$ for $(l\geq 3)$ are easy to obtain.*

$$\xi(\chi,\tau)=\sum_{l=0}^{\infty}\xi_l(\chi,\tau)=\xi_0(\chi,\tau)+\xi_1(\chi,\tau)+\xi_2(\chi,\tau)+\cdots,$$

$$\xi(\chi,\tau)=\delta+9-15(\tanh[\ell(\chi-\gamma)]+\tanh^2[\ell(\chi-\gamma)]-\tanh^3[\ell(\chi-\gamma)])-\frac{15\ell\,\mathrm{sech}^6[\ell(\chi-\gamma)]}{8}(4(-36+2\ell-32\ell^2+344\ell^3+\delta)\cosh[2\ell(\chi-\gamma)]+(-6-10\ell+112\ell^2-136\ell^3+\delta)\cosh[4\ell(\chi-\gamma)]+3(74+6\ell-80\ell^2-456\ell^3+\delta+2(14-2\ell-112\ell^2+120\ell^3+\delta)\sinh[2\ell(\chi-\gamma)]+(-6-2\ell+48\ell^2-40\ell^3+\delta)\sinh[4\ell(\chi-\gamma)]))\times(-1+\tanh[\ell(\chi-\gamma)])\left(1-\rho+\frac{\rho\tau^\rho}{\rho(\rho+1)}\right)+\frac{15\ell^2\,\mathrm{sech}^{10}[\ell(\chi-\gamma)]}{64}(2(5(4\ell(966+\ell(-77693+8\ell(-25495+\ell(-2205+31238\ell(4+\ell)))))+4\ell(19+4(38-245\ell)\ell)\delta)-5\delta^2+60(693+5\delta))-2(127332+4\ell(-282+\ell(-242299+8\ell(-70727+\ell(693+88234\ell(4+\ell)))))+276\delta+4\ell(-43+4\ell(-86+77\ell))\delta+17\delta^2)\cosh[2\ell(\chi-\gamma)]+8(6624+4\ell(-471+4\ell(-3313+\ell(-5503+2\ell(1071+3652\ell(4+\ell)))))-198\delta+16\ell(-1+\ell(-8+119\ell))\delta-\delta^2)\cosh[4\ell(\chi-\gamma)]-2(1404+4\ell(-246+\ell(-2917+8\ell(-329+\ell(531+502\ell(4+\ell)))))-228\delta+4\ell(11+4\ell(22+59\ell))\delta-\delta^2)\cosh[6\ell(\chi-\gamma)]+(-6+2\ell(1+4\ell(2+\ell))+\delta)^2\cosh[8\ell(\chi-\gamma)]+2(4\ell(-3558+\ell(-157321+8\ell(-87713+\ell(50967+45304\ell+504046\ell^2))))-4\ell(-23+868\ell(2+\ell))\delta+23\delta^2+36(3223+59\delta))\sinh[2\ell(\chi-\gamma)]-2(4(-1+2\ell)(-6507+\ell(-11136+\ell(34637+2\ell(77025+81222\ell+97708\ell^2))))+4(-1+2\ell)(159+67\ell(5+2\ell))\delta-17\delta^2)\times\sinh[4\ell(\chi-\gamma)]+6(516+4\ell(-66+\ell(-1223+8\ell(-99+\ell(369+328\ell+338\ell^2))))-92\delta+4\ell(1+36\ell(2+\ell))\delta+\delta^2)\sinh[6\ell(\chi-\gamma)]-(-6+2\ell(1+4\ell(2+\ell))+\delta)^2\times\sinh[8\ell(\chi-\gamma)]-43200(5-208\ell^3+1344\ell^6)\tanh[\ell(\chi-\gamma)])\left[\frac{\rho^2\tau^{2\rho}}{\rho(2\rho+1)}+2\rho(1-\rho)\frac{\tau^\rho}{\rho(\rho+1)}+(1-\rho)^2\right]+\cdots. \tag{77}$$

*By taking $\rho=1$, we obtain the exact solution as*

$$\xi(\chi,\tau)=\delta+9-15(\tanh[\ell(\chi-\delta\tau-\gamma)]+\tanh^2[\ell(\chi-\delta\tau-\gamma)]-\tanh^3[\ell(\chi-\delta\tau-\gamma)]). \tag{78}$$

*Result and Discussion*

In this part of the paper, we discuss the numerical study of nonlinear Fractional Kuramoto–Sivashinsky equations by using the natural transform decomposition method. The tabular and graphical views for the given problems are obtained through Maple. In Figure 1, we present the exact and analytical behavior of the FKS equation by means of our proposed scheme, while Figure 2 shows the nature of analytical results at various fractional orders. Figure 3 illustrates the exact and proposed method scheme behavior, whereas Figure 4 shows the fractional layout of the proposed scheme. It is clear from Figure 4 that the solution becomes closer towards the exact solution as the value of $\rho$ approaches integer-order. In the same manner, Figures 5 show the graphical behaviour of the exact solution and our method's solution, while Figure 6 gives the layout of the analytical solution in terms of different fractional orders. Similarly, in Tables 1–3, we presented the absolute error analysis of the FKS equation obtained with the help of the proposed method at different values of $\chi$ and $\tau$. The results in the tables reveal that our method is highly promising and accurate.

**Table 1.** The comparison on the basis of error at various fractional orders of $\rho$ for problem 1.

| $\tau$ | $\chi$ | $\rho = 0.4$ | $\rho = 0.6$ | $\rho = 0.8$ | $\rho = 1(NTDM_{CF})$ | $\rho = 1(NTDM_{ABC})$ |
|---|---|---|---|---|---|---|
| 0.1 | 0.2 | $3.8763631750 \times 10^{-5}$ | $2.5603465160 \times 10^{-5}$ | $1.2452127610 \times 10^{-5}$ | $5.2357898000 \times 10^{-9}$ | $5.2357898000 \times 10^{-9}$ |
| | 0.4 | $3.5383874710 \times 10^{-5}$ | $2.3370987630 \times 10^{-5}$ | $1.1366159900 \times 10^{-5}$ | $5.1930879000 \times 10^{-9}$ | $5.1930879000 \times 10^{-9}$ |
| | 0.6 | $3.2298917490 \times 10^{-5}$ | $2.1333562800 \times 10^{-5}$ | $1.0375564680 \times 10^{-5}$ | $4.1981581000 \times 10^{-9}$ | $4.1981581000 \times 10^{-9}$ |
| | 0.8 | $2.9481307140 \times 10^{-5}$ | $1.9472375180 \times 10^{-5}$ | $9.4701581360 \times 10^{-6}$ | $4.2586011000 \times 10^{-9}$ | $4.2586011000 \times 10^{-9}$ |
| | 1 | $2.6909407330 \times 10^{-5}$ | $1.7773682470 \times 10^{-5}$ | $8.6440866950 \times 10^{-6}$ | $3.7555413000 \times 10^{-9}$ | $3.7555413000 \times 10^{-9}$ |
| 0.2 | 0.2 | $3.9051956730 \times 10^{-5}$ | $2.6213609880 \times 10^{-5}$ | $1.2791603100 \times 10^{-5}$ | $2.1671579400 \times 10^{-8}$ | $2.1671579400 \times 10^{-8}$ |
| | 0.4 | $3.5647273900 \times 10^{-5}$ | $2.3928150910 \times 10^{-5}$ | $1.1676250400 \times 10^{-5}$ | $1.9986175500 \times 10^{-8}$ | $1.9986175500 \times 10^{-8}$ |
| | 0.6 | $3.2539153040 \times 10^{-5}$ | $2.1841946000 \times 10^{-5}$ | $1.0658420020 \times 10^{-5}$ | $1.7896316100 \times 10^{-8}$ | $1.7896316100 \times 10^{-8}$ |
| | 0.8 | $2.9700833590 \times 10^{-5}$ | $1.9936660840 \times 10^{-5}$ | $9.7285869800 \times 10^{-6}$ | $1.6517202000 \times 10^{-8}$ | $1.6517202000 \times 10^{-8}$ |
| | 1 | $2.7109715270 \times 10^{-5}$ | $1.8197396140 \times 10^{-5}$ | $8.8799030700 \times 10^{-6}$ | $1.5011082500 \times 10^{-8}$ | $1.5011082500 \times 10^{-8}$ |
| 0.3 | 0.2 | $3.9074452580 \times 10^{-5}$ | $2.6557920700 \times 10^{-5}$ | $1.3018392060 \times 10^{-5}$ | $4.9107369000 \times 10^{-8}$ | $4.9107369000 \times 10^{-8}$ |
| | 0.4 | $3.5667759500 \times 10^{-5}$ | $2.4242396230 \times 10^{-5}$ | $1.1883219200 \times 10^{-5}$ | $4.5079263000 \times 10^{-8}$ | $4.5079263000 \times 10^{-8}$ |
| | 0.6 | $3.2557659070 \times 10^{-5}$ | $2.2128595690 \times 10^{-5}$ | $1.0847147790 \times 10^{-5}$ | $4.0994474000 \times 10^{-8}$ | $4.0994474000 \times 10^{-8}$ |
| | 0.8 | $2.9717650370 \times 10^{-5}$ | $2.0198233190 \times 10^{-5}$ | $9.9007783700 \times 10^{-6}$ | $3.7675803000 \times 10^{-8}$ | $3.7675803000 \times 10^{-8}$ |
| | 1 | $2.7125122030 \times 10^{-5}$ | $1.8436205320 \times 10^{-5}$ | $9.0371291300 \times 10^{-6}$ | $3.4266623600 \times 10^{-8}$ | $3.4266623600 \times 10^{-8}$ |
| 0.4 | 0.2 | $3.8968119170 \times 10^{-5}$ | $2.6758919850 \times 10^{-5}$ | $1.3174768270 \times 10^{-5}$ | $8.7943159000 \times 10^{-8}$ | $8.7943159000 \times 10^{-8}$ |
| | 0.4 | $3.5570453110 \times 10^{-5}$ | $2.4425629700 \times 10^{-5}$ | $1.2025719880 \times 10^{-5}$ | $8.0772351000 \times 10^{-8}$ | $8.0772351000 \times 10^{-8}$ |
| | 0.6 | $3.2469220340 \times 10^{-5}$ | $2.2296233550 \times 10^{-5}$ | $1.0977604780 \times 10^{-5}$ | $7.3192632000 \times 10^{-8}$ | $7.3192632000 \times 10^{-8}$ |
| | 0.8 | $2.9636656620 \times 10^{-5}$ | $2.0350980460 \times 10^{-5}$ | $1.0019587770 \times 10^{-5}$ | $6.7334404000 \times 10^{-8}$ | $6.7334404000 \times 10^{-8}$ |
| | 1 | $2.7051309960 \times 10^{-5}$ | $1.8575742050 \times 10^{-5}$ | $9.1456888300 \times 10^{-6}$ | $6.1222165000 \times 10^{-8}$ | $6.1222165000 \times 10^{-8}$ |
| 0.5 | 0.2 | $3.8780476890 \times 10^{-5}$ | $2.6863820070 \times 10^{-5}$ | $1.3278652660 \times 10^{-5}$ | $1.3857894800 \times 10^{-7}$ | $1.3857894800 \times 10^{-7}$ |
| | 0.4 | $3.5399297510 \times 10^{-5}$ | $2.4521513270 \times 10^{-5}$ | $1.2120676190 \times 10^{-5}$ | $1.2686543800 \times 10^{-7}$ | $1.2686543800 \times 10^{-7}$ |
| | 0.6 | $3.2312565220 \times 10^{-5}$ | $2.2383331580 \times 10^{-5}$ | $1.1063856400 \times 10^{-5}$ | $1.1569079000 \times 10^{-7}$ | $1.1569079000 \times 10^{-7}$ |
| | 0.8 | $2.9494198080 \times 10^{-5}$ | $2.0431014380 \times 10^{-5}$ | $1.0098849110 \times 10^{-5}$ | $1.0559300400 \times 10^{-7}$ | $1.0559300400 \times 10^{-7}$ |
| | 1 | $2.6920845170 \times 10^{-5}$ | $1.8648358860 \times 10^{-5}$ | $9.2176004600 \times 10^{-6}$ | $9.6577706000 \times 10^{-8}$ | $9.6577706000 \times 10^{-8}$ |

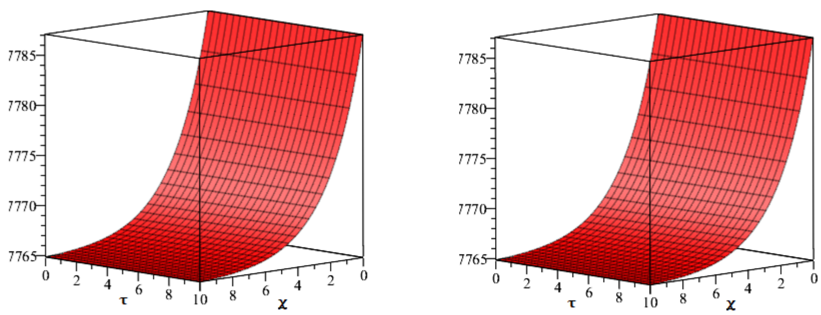


**Figure 1.** The exact and approximate solution at $\rho = 1$ of Example 1.

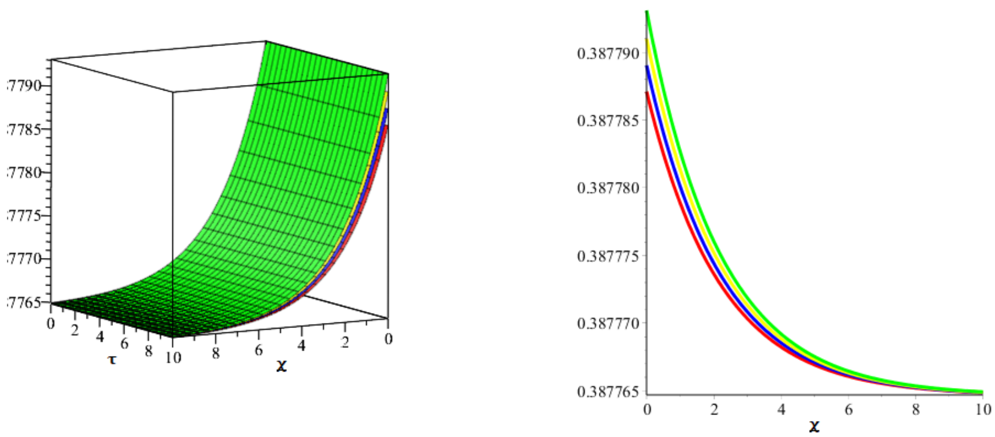


**Figure 2.** The approximate solution at various fractional orders of $\rho$ and $\tau = 0.5$; for Example 1.

**Table 2.** The comparison on the basis of error at various fractional orders of $\rho$ for problem 2.

| $\tau$ | $\chi$ | $\rho = 0.4$ | $\rho = 0.6$ | $\rho = 0.8$ | $\rho = 1(NTDM_{CF})$ | $\rho = 1(NTDM_{ABC})$ |
|---|---|---|---|---|---|---|
| 0.1 | 0.2 | $1.1612667280 \times 10^{-6}$ | $8.5363755450 \times 10^{-7}$ | $4.6919140870 \times 10^{-7}$ | $6.8704703900 \times 10^{-8}$ | $6.8704703900 \times 10^{-8}$ |
| | 0.4 | $1.0028798100 \times 10^{-6}$ | $7.3867676970 \times 10^{-7}$ | $4.0850054000 \times 10^{-7}$ | $6.4548098400 \times 10^{-8}$ | $6.4548098400 \times 10^{-8}$ |
| | 0.6 | $8.5796886930 \times 10^{-7}$ | $6.3106175850 \times 10^{-7}$ | $3.4749449090 \times 10^{-7}$ | $5.2095715600 \times 10^{-8}$ | $5.2095715600 \times 10^{-8}$ |
| | 0.8 | $7.3807877630 \times 10^{-7}$ | $5.4320275680 \times 10^{-7}$ | $2.9966494870 \times 10^{-7}$ | $4.5965813960 \times 10^{-8}$ | $4.5965813960 \times 10^{-8}$ |
| | 1 | $6.3324213410 \times 10^{-7}$ | $4.6587557540 \times 10^{-7}$ | $2.5671651650 \times 10^{-7}$ | $3.8830543490 \times 10^{-8}$ | $3.8830543490 \times 10^{-8}$ |
| 0.2 | 0.2 | $1.1160913160 \times 10^{-6}$ | $8.6892447100 \times 10^{-7}$ | $5.1845663050 \times 10^{-7}$ | $1.1940940770 \times 10^{-7}$ | $1.1940940770 \times 10^{-7}$ |
| | 0.4 | $9.5808260400 \times 10^{-7}$ | $7.4580679400 \times 10^{-7}$ | $4.4481235990 \times 10^{-7}$ | $1.0209619810 \times 10^{-7}$ | $1.0209619810 \times 10^{-7}$ |
| | 0.6 | $8.2234346360 \times 10^{-7}$ | $6.4003333330 \times 10^{-7}$ | $3.8152844530 \times 10^{-7}$ | $8.7191431200 \times 10^{-8}$ | $8.7191431200 \times 10^{-8}$ |
| | 0.8 | $7.0730673850 \times 10^{-7}$ | $5.5073221600 \times 10^{-7}$ | $3.2871888450 \times 10^{-7}$ | $7.5931629400 \times 10^{-8}$ | $7.5931629400 \times 10^{-8}$ |
| | 1 | $6.0790881850 \times 10^{-7}$ | $4.7343696780 \times 10^{-7}$ | $2.8276390620 \times 10^{-7}$ | $6.5661087000 \times 10^{-8}$ | $6.5661087000 \times 10^{-8}$ |
| 0.3 | 0.2 | $1.0067944450 \times 10^{-6}$ | $8.0987732900 \times 10^{-7}$ | $5.0016428380 \times 10^{-7}$ | $1.1911411150 \times 10^{-7}$ | $1.1911411150 \times 10^{-7}$ |
| | 0.4 | $8.6901621500 \times 10^{-7}$ | $6.9989668720 \times 10^{-7}$ | $4.3390394210 \times 10^{-7}$ | $1.0664429710 \times 10^{-7}$ | $1.0664429710 \times 10^{-7}$ |
| | 0.6 | $7.4103967680 \times 10^{-7}$ | $5.9579371980 \times 10^{-7}$ | $3.6734954620 \times 10^{-7}$ | $8.6287146800 \times 10^{-8}$ | $8.6287146800 \times 10^{-8}$ |
| | 0.8 | $6.3622255760 \times 10^{-7}$ | $5.1148008530 \times 10^{-7}$ | $3.1528398080 \times 10^{-7}$ | $7.3897444000 \times 10^{-8}$ | $7.3897444000 \times 10^{-8}$ |
| | 1 | $5.4643673440 \times 10^{-7}$ | $4.3930339730 \times 10^{-7}$ | $2.7080310190 \times 10^{-7}$ | $6.3491630400 \times 10^{-8}$ | $6.3491630400 \times 10^{-8}$ |
| 0.4 | 0.2 | $8.3045099900 \times 10^{-7}$ | $6.7775158900 \times 10^{-7}$ | $4.1126291900 \times 10^{-7}$ | $5.7818811000 \times 10^{-8}$ | $5.7818811000 \times 10^{-8}$ |
| | 0.4 | $7.1575681000 \times 10^{-7}$ | $5.8461304340 \times 10^{-7}$ | $3.5574295280 \times 10^{-7}$ | $5.2192393400 \times 10^{-8}$ | $5.2192393400 \times 10^{-8}$ |
| | 0.6 | $6.1227594040 \times 10^{-7}$ | $4.9964493820 \times 10^{-7}$ | $3.0308304070 \times 10^{-7}$ | $4.2382858300 \times 10^{-8}$ | $4.2382858300 \times 10^{-8}$ |
| | 0.8 | $5.2830801080 \times 10^{-7}$ | $4.3157643950 \times 10^{-7}$ | $2.6276198470 \times 10^{-7}$ | $3.8863255800 \times 10^{-8}$ | $3.8863255800 \times 10^{-8}$ |
| | 1 | $4.4867499390 \times 10^{-7}$ | $3.6559842940 \times 10^{-7}$ | $2.2061448310 \times 10^{-7}$ | $2.8322174000 \times 10^{-8}$ | $2.8322174000 \times 10^{-8}$ |
| 0.5 | 0.2 | $5.5952845800 \times 10^{-7}$ | $4.4690695900 \times 10^{-7}$ | $2.2473572590 \times 10^{-7}$ | $9.4476480800 \times 10^{-8}$ | $9.4476480800 \times 10^{-8}$ |
| | 0.4 | $4.8042354800 \times 10^{-7}$ | $3.8370013900 \times 10^{-7}$ | $1.9289146280 \times 10^{-7}$ | $8.1259504900 \times 10^{-8}$ | $8.1259504900 \times 10^{-8}$ |
| | 0.6 | $4.1587226470 \times 10^{-7}$ | $3.3280270810 \times 10^{-7}$ | $1.6892932770 \times 10^{-7}$ | $6.6521422000 \times 10^{-8}$ | $6.6521422000 \times 10^{-8}$ |
| | 0.8 | $3.5512618890 \times 10^{-7}$ | $2.8378305440 \times 10^{-7}$ | $1.4304267840 \times 10^{-7}$ | $5.9170930300 \times 10^{-8}$ | $5.9170930300 \times 10^{-8}$ |
| | 1 | $3.0596603570 \times 10^{-7}$ | $2.4469397700 \times 10^{-7}$ | $1.2382106280 \times 10^{-7}$ | $4.9847282600 \times 10^{-8}$ | $4.9847282600 \times 10^{-8}$ |

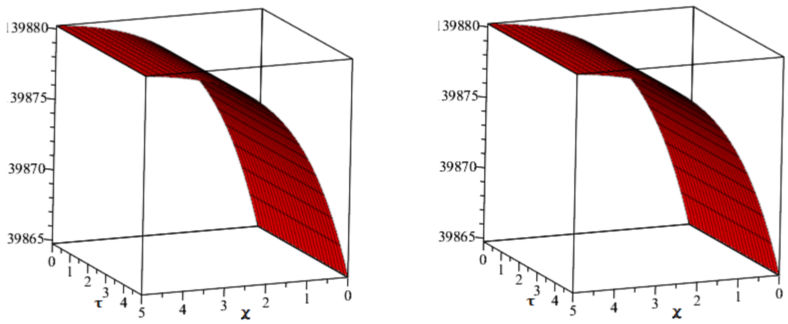


**Figure 3.** The exact and approximate solution at $\rho = 1$ of Example 2.

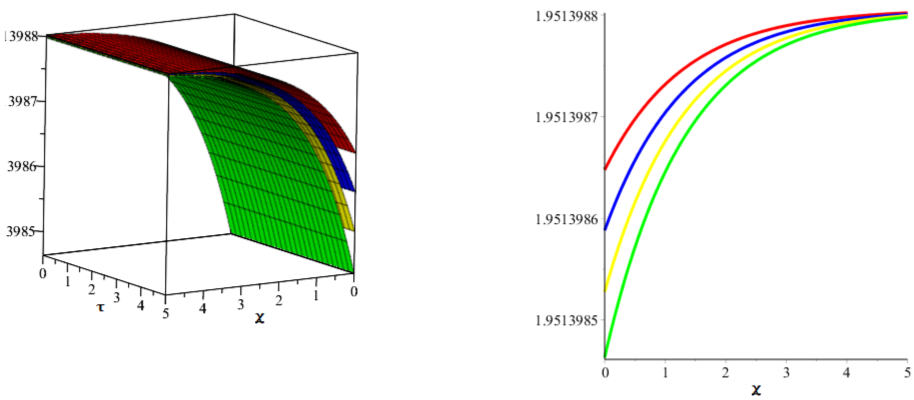


**Figure 4.** The approximate solution at various orders of $\rho$ and $\tau = 0.5$ for Example 2.

**Table 3.** The comparison on the basis of error at various fractional orders of $\rho$ for problem 3.

| $\tau$ | $\chi$ | $\rho = 0.4$ | $\rho = 0.6$ | $\rho = 0.8$ | $\rho = 1(NTDM_{CF})$ | $\rho = 1(NTDM_{ABC})$ |
|---|---|---|---|---|---|---|
| 0.1 | 0.2 | $7.60000000 \times 10^{-7}$ | $7.10000000 \times 10^{-8}$ | $3.40000000 \times 10^{-9}$ | $1.10000000 \times 10^{-8}$ | $1.10000000 \times 10^{-8}$ |
| | 0.4 | $5.13000000 \times 10^{-7}$ | $5.10000000 \times 10^{-8}$ | $6.10000000 \times 10^{-9}$ | $1.20000000 \times 10^{-8}$ | $1.20000000 \times 10^{-8}$ |
| | 0.6 | $3.35000000 \times 10^{-7}$ | $4.10000000 \times 10^{-8}$ | $3.00000000 \times 10^{-9}$ | $0.00000000 \times 10^{00}$ | $0.0000000 \times 10^{00}$ |
| | 0.8 | $2.31000000 \times 10^{-7}$ | $3.80000000 \times 10^{-8}$ | $8.20000000 \times 10^{-9}$ | $1.00000000 \times 10^{-9}$ | $1.00000000 \times 10^{-9}$ |
| | 1 | $1.52000000 \times 10^{-7}$ | $1.80000000 \times 10^{-8}$ | $1.00000000 \times 10^{-9}$ | $1.10000000 \times 10^{-9}$ | $1.10000000 \times 10^{-9}$ |
| 0.2 | 0.2 | $5.49000000 \times 10^{-7}$ | $5.10000000 \times 10^{-8}$ | $1.00000000 \times 10^{-8}$ | $1.40000000 \times 10^{-8}$ | $1.40000000 \times 10^{-8}$ |
| | 0.4 | $3.63000000 \times 10^{-7}$ | $2.80000000 \times 10^{-8}$ | $1.20000000 \times 10^{-8}$ | $1.30000000 \times 10^{-8}$ | $1.30000000 \times 10^{-8}$ |
| | 0.6 | $2.38000000 \times 10^{-7}$ | $3.60000000 \times 10^{-8}$ | $4.20000000 \times 10^{-9}$ | $1.50000000 \times 10^{-8}$ | $1.50000000 \times 10^{-8}$ |
| | 0.8 | $1.70000000 \times 10^{-7}$ | $3.10000000 \times 10^{-8}$ | $7.50000000 \times 10^{-9}$ | $1.10000000 \times 10^{-8}$ | $1.10000000 \times 10^{-8}$ |
| | 1 | $9.80000000 \times 10^{-8}$ | $1.00000000 \times 10^{-8}$ | $2.00000000 \times 10^{-9}$ | $0.00000000 \times 10^{00}$ | $0.00000000 \times 10^{00}$ |
| 0.3 | 0.2 | $9.90000000 \times 10^{-8}$ | $1.10000000 \times 10^{-8}$ | $9.10000000 \times 10^{-9}$ | $1.30000000 \times 10^{-8}$ | $1.30000000 \times 10^{-8}$ |
| | 0.4 | $7.60000000 \times 10^{-8}$ | $7.00000000 \times 10^{-9}$ | $3.40000000 \times 10^{-9}$ | $0.00000000 \times 10^{00}$ | $0.00000000 \times 10^{00}$ |
| | 0.6 | $4.60000000 \times 10^{-8}$ | $9.00000000 \times 10^{-9}$ | $5.20000000 \times 10^{-9}$ | $1.30000000 \times 10^{-8}$ | $1.30000000 \times 10^{-8}$ |
| | 0.8 | $3.60000000 \times 10^{-8}$ | $1.30000000 \times 10^{-8}$ | $1.90000000 \times 10^{-9}$ | $0.00000000 \times 10^{00}$ | $0.00000000 \times 10^{00}$ |
| | 1 | $1.20000000 \times 10^{-8}$ | $2.00000000 \times 10^{-9}$ | $2.00000000 \times 10^{-9}$ | $1.10000000 \times 10^{-8}$ | $1.10000000 \times 10^{-8}$ |
| 0.4 | 0.2 | $7.06000000 \times 10^{-7}$ | $1.15000000 \times 10^{-7}$ | $1.80000000 \times 10^{-8}$ | $1.20000000 \times 10^{-8}$ | $1.20000000 \times 10^{-8}$ |
| | 0.4 | $4.64000000 \times 10^{-7}$ | $7.30000000 \times 10^{-8}$ | $1.20000000 \times 10^{-8}$ | $1.50000000 \times 10^{-9}$ | $1.50000000 \times 10^{-9}$ |
| | 0.6 | $3.17000000 \times 10^{-7}$ | $4.80000000 \times 10^{-8}$ | $5.00000000 \times 10^{-9}$ | $1.30000000 \times 10^{-8}$ | $1.30000000 \times 10^{-8}$ |
| | 0.8 | $2.00000000 \times 10^{-7}$ | $1.50000000 \times 10^{-8}$ | $2.60000000 \times 10^{-8}$ | $0.00000000 \times 10^{00}$ | $0.00000000 \times 10^{00}$ |
| | 1 | $1.55000000 \times 10^{-7}$ | $2.70000000 \times 10^{-8}$ | $1.80000000 \times 10^{-8}$ | $1.10000000 \times 10^{-8}$ | $1.10000000 \times 10^{-8}$ |
| 0.5 | 0.2 | $2.19300000 \times 10^{-6}$ | $3.09000000 \times 10^{-7}$ | $4.70000000 \times 10^{-8}$ | $1.20000000 \times 10^{-8}$ | $1.20000000 \times 10^{-8}$ |
| | 0.4 | $1.45500000 \times 10^{-6}$ | $2.09000000 \times 10^{-7}$ | $3.10000000 \times 10^{-8}$ | $8.10000000 \times 10^{-9}$ | $8.10000000 \times 10^{-9}$ |
| | 0.6 | $9.81000000 \times 10^{-7}$ | $1.36000000 \times 10^{-7}$ | $1.40000000 \times 10^{-8}$ | $1.30000000 \times 10^{-8}$ | $1.30000000 \times 10^{-8}$ |
| | 0.8 | $6.66000000 \times 10^{-7}$ | $8.40000000 \times 10^{-8}$ | $2.60000000 \times 10^{-8}$ | $0.00000000 \times 10^{00}$ | $0.00000000 \times 10^{00}$ |
| | 1 | $4.53000000 \times 10^{-7}$ | $6.60000000 \times 10^{-8}$ | $7.00000000 \times 10^{-9}$ | $1.40000000 \times 10^{-8}$ | $1.40000000 \times 10^{-8}$ |

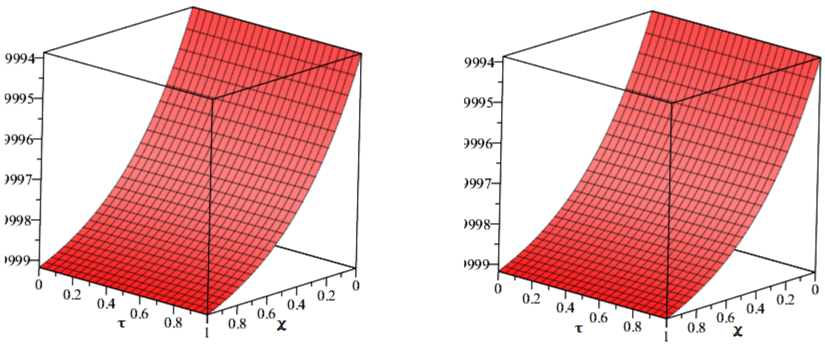


**Figure 5.** The exact and approximate solution at $\rho = 1$ of Example 3.

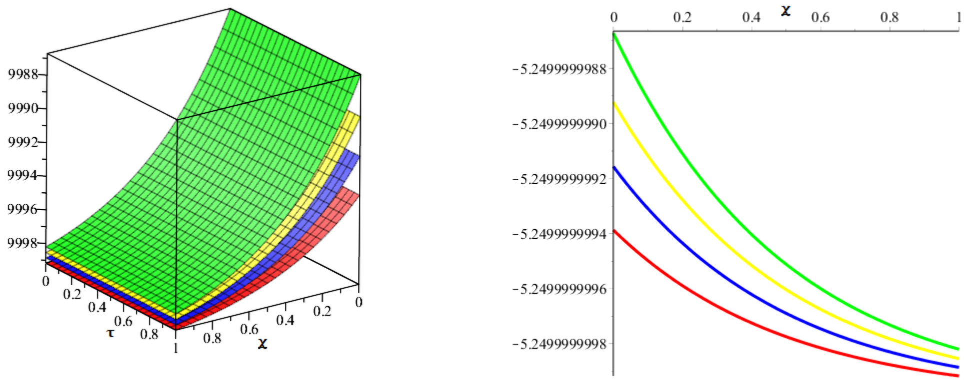


**Figure 6.** The approximate solution at various orders of $\rho$ and $\tau = 0.5$; for Example 3.

## 5. Conclusions

In this article, we have successfully employed the natural decomposition method in connection with the two different fractional derivatives to obtain the analytical solution of the fractional Kuramoto–Sivashinsky equation. To illustrate the validity of the suggested technique, we examined the FKS equation in three different cases. The numerical simulations confirm that our method results are in good agreement with the exact solution. The present scheme is very simple, effective, and appropriate for obtaining numerical solutions ofthe FKS equation. The suggested scheme's main benefit is the series form solution, which quickly converges to the exact solution. As a result, we can conclude that the proposed approach is highly systematic and powerful for analysing fractional-order mathematical models in a systematic and better manner.

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