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Graded Weakly 2-Absorbing Ideals over Non-Commutative Graded Rings

Azzh Saad Alshehry [1], Jebrel M. Habeb [2], Rashid Abu-Dawwas [2,*] and Ahmad Alrawabdeh [2]

1 Department of Mathematical Sciences, Faculty of Sciences, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia; asalshihry@pnu.edu.sa
2 Department of Mathematics, Yarmouk University, Irbid 21163, Jordan; jhabeb@yu.edu.jo (J.M.H.); 2019105023@ses.yu.edu.jo (A.A.)
* Correspondence: rrashid@yu.edu.jo

Abstract: Let G be a group and R be a G-graded ring. In this paper, we present and examine the concept of graded weakly 2-absorbing ideals as in generality of graded weakly prime ideals in a graded ring which is not commutative, and demonstrates that the symmetry is obtained as a lot of the outcomes in commutative graded rings remain in graded rings that are not commutative.

Keywords: graded prime ideals; graded weakly prime ideals; graded 2-absorbing ideals; graded weakly 2-absorbing ideals

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1. Introduction

During the whole of this article, the rings are not certainly expected to have unity except pointed out alternatively. Likewise, an ideal in a ring means a two-sided ideal. Let G be a group with identity e and R be a ring. Then R is called graded ring which denoted by 'GR-ring' if $R = \bigoplus_{g\in G} R_g$ where $R_g R_h \subseteq R_{gh}$ for $g, h \in G$. The additive subgroup stood for R_g where $g \in G$. We call the homogeneous of degree g for the components of R_g. If $a \in R$, then a can be represented by $\sum_{g\in G} a_g$, with a_g being the element of a in R_g. In fact the additive subgroup R_e is a sub-ring of R, if R has a unity 1, then $1 \in R_e$. Let $\bigcup_{g\in G} R_g$ be the collection of all homogeneous elements of R which is denoted by $h(R)$. Assume P is an ideal of a graded ring R. If $P = \bigoplus_{g\in G}(P \cap R_g)$, so, P is announced for a graded ideal, and denoted by 'GR-I', i.e., for $a \in P$, $a = \sum_{g\in G} a_g$ where $a_g \in P$ and $g \in G$. It is not necessary for every 'GR-I' to be a GR-ring ([1], Example 1.1). For more details and terminology, see [2,3].

The following abbreviations are used towards the end of this paper: 'CGR-ring' stand for commutative graded rings, 'NCGR-ring' for non-commutative graded rings, 'GR-P' for graded prime, 'GR-PI' for graded prime ideals, 'PGR-PI' for proper graded prime ideals, 'PGR-I' for proper graded ideals, 'GR-WPI' for graded weakly prime ideals, 'GR-2-AI' for graded 2-Absorbing ideals, 'GR-W-2-AI' for a graded weakly 2-Absorbing ideals, and 'GR-CW-2-AI' for a graded completely weakly 2-Absorbing ideals.

For 'CGR-ring', 'GR-2-AI', generalized from 'GR-PI', which were presented as well as examined within [4]. Remember from [5] that a 'PGR-I' P of a 'CGR-ring' R is estimated to be a 'GR-WPI' of R if $x, y \in h(R)$ and $0 \neq xy \in P$, then either $x \in P$ or $y \in P$. Also from [4] a 'PGR-I' P of a 'CGR-ring' R is announced for a 'GR-2-AI' of R, where, $x, y, z \in h(R)$ along with $xyz \in P$, therefore, either $xy \in P$, $xz \in P$ or $yz \in P$. The idea of a 'GR-W-2-AI' of a 'CGR-ring' R was presented in [4]. A 'PGR-I' P of a 'CGR-ring' R is called a 'GR-W-2-AI' of R if given $x, y, z \in h(R)$ and $0 \neq xyz \in P$, so one of xy, xz or yz be in P.

The 'GR-PI' over 'NCGR-rings' have been put in place and examined by Abu-Dawwas, Bataineh, and Al-Muanger in [6]. A 'PGR-I' P of R is expressed to be 'GR-P' for both of I and J were 'GR-I' of R where, $IJ \subseteq P$, therefore $I \subseteq P$ or $J \subseteq P$. As a summarization of 'GR-PI' over 'NCGR-ring', the concept of 'GR-2-AI' over 'NCGR-ring' has been reported and investigated by Abu-Dawwas, Shashan and Dagher in [7]. A 'PGR-I' P of R is said to be 'GR-2-AI' where $x, y, z \in h(R)$ so that $xRyRz \subseteq P$, then $xy \in P$, $yz \in P$ or $xz \in P$. Recently, 'GR-WPI' over 'NCGR-rings' have been brought up and served by Alshehry and Abu-Dawwas in [1]. A 'PGR-I' P of R is said to be 'GR-WP' if once I and J are 'GR-I' of R such that $0 \neq IJ \subseteq P$, then $I \subseteq P$ or $J \subseteq P$.

Within this article, we are following [8] to introduce and investigate the concept of 'GR-W-2-AI' as a generalization of 'GR-WPI' in a 'GR-ring' which is non-commutative, and demonstrates that the symmetry is obtained as a lot of the outcomes in 'CGR-ring' still remain in 'NCGR-ring'.

2. Graded Weakly 2-Absorbing Ideals

This section consists of an examination and studies of 'GR-W-2-AI'. During the whole of this section, we are dealing with a ring R, that is an 'NCGR-ring', having unity except pointed out alternatively.

Definition 1. *Let R be a 'GR-ring'. Assume that P is a 'PGR-I' of R. Then we call P being a 'GR-W-2-AI' when $0 \neq xRyRz \subseteq P$ gives $xy \in P$, $yz \in P$ or $xz \in P$ for each $x, y, z \in h(R)$. If $0 \neq xyz \in P$ implies $xy \in P$, $yz \in P$ or $xz \in P$ for all $x, y, z \in h(R)$, we call P to be 'GR-CW-2-AI'.*

Apparently, when R is a 'CGR-rings' having unity, then the concepts of 'GR-W-2-A' and 'GR-CW-2-AI' coincide. The following example demonstrates that this will not be the case for 'NCGR-ring'.

Example 1. *Consider $R = M_2(\mathbb{Z})$ (the ring of all 2×2 matrices with integer entries) and $G = \mathbb{Z}_4$. Then R is graded by $R_0 = \begin{pmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Z} \end{pmatrix}$, $R_2 = \begin{pmatrix} 0 & \mathbb{Z} \\ \mathbb{Z} & 0 \end{pmatrix}$ and $R_1 = R_3 = 0$.*

Deal with 'GR-I' $P = M_2(2\mathbb{Z})$ of R. P is Clearly a 'GR-PI' of R and so a 'GR-W-2-AI' of R. On the other side, P is not a 'GR-CW-2-AI' of R since $n, m \in \mathbb{Z}$, $A = \begin{pmatrix} 2n+1 & 0 \\ 0 & 2m \end{pmatrix}$, $B = \begin{pmatrix} 0 & 2n+1 \\ 2n+1 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 2n+1 & 0 \\ 0 & 4m \end{pmatrix} \in h(R)$ where $0 \neq ABC \in P$, for each of AB, AC and $BC \notin P$.

Undoubtedly, every 'GR-2-AI' of a 'GR-ring' is a 'GR-W-2-AI'. In any 'GR-ring', $P = \{0\}$ is 'GR-W-2-AI'.

Individually, it is not necessary for $P = \{0\}$ to be 'GR-2-AI', check the next example.

Example 2. *Suppose that $R = M_2(\mathbb{Z}_8)$ along with $G = \mathbb{Z}_4$. Hence R will be 'GR-ring' by $R_0 = \begin{pmatrix} \mathbb{Z}_8 & 0 \\ 0 & \mathbb{Z}_8 \end{pmatrix}$, $R_2 = \begin{pmatrix} 0 & \mathbb{Z}_8 \\ \mathbb{Z}_8 & 0 \end{pmatrix}$ and $R_1 = R_3 = 0$. Undeniably, $P = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ is not a 'GR-2-AI' of R since $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \in h(R)$ with $ARARA \subseteq P$ but $A.A \notin P$.*

Lemma 1. *For a 'GR-ring' R. Assume that P is a 'GR-WPI' of R.*

1. *If for both I and J are graded right (left) ideals of R where, $0 \neq IJ \subseteq P$. Then it is either $I \subseteq P$ or $J \subseteq P$.*
2. *If $0 \neq xRyRz \subseteq P$ such that $x, y, z \in h(R)$, therefore each of x, y or $z \in P$.*

Proof.

1. Assume that both I and J are graded right (left) ideals of R in order that $0 \neq IJ \subseteq P$. Let (I) and (J) be the 'GR-I' generated by I and J respectively. Then $0 \neq (I)(J) \subseteq P$, whence $I \subseteq (I) \subseteq P$ or $J \subseteq (J) \subseteq P$.
2. Suppose that $x, y, z \in h(R)$ where $0 \neq xRyRz \subseteq P$. That being $0 \neq (Rx)RyRz \subseteq P$ which it comes from (1) that $x \in Rx \subseteq P$ or $0 \neq RyRz \subseteq P$. By reiterating this, the result follows.

□

Proposition 1. *In the 'GR-ring' R. P is a 'GR-W-2-AI' of R, if it is a 'GR-WPI' of R.*

Proof. Let $x, y, z \in h(R)$ where $0 \neq xRyRz \subseteq P$. By Lemma 1, $x \in P$ or $y \in P$ or $z \in P$. Accordingly, $xy \in P$ or $yz \in P$ or $xz \in P$, and the result holds. □

Proposition 2. *If P and K are two distinct 'GR-WPI' of a 'GR-ring' R, then $P \bigcap K$ is a 'GR-W-2-AI' of R.*

Proof. Assume that $P \bigcap K = \{0\}$, it seems that $P \bigcap K$ is a 'GR-W-2-AI' of R. Let $x_1, x_2, x_3 \in h(R)$ where $0 \neq x_1Rx_2Rx_3 \subseteq P \bigcap K$. Then $0 \neq x_1Rx_2Rx_3 \subseteq P$ and $0 \neq x_1Rx_2Rx_3 \subseteq K$. By Lemma 1 we have $x_i \in P$ and $x_j \in K$ for some i and j, then $x_ix_j \in P \bigcap K$. As a result, $P \bigcap K$ is a 'GR-W-2-AI' of R. □

Consider the two 'GR-rings' R and T. For all $g \in G$, $R \times T$ is a graded by $(R \times T)_g = R_g \times T_g$. $P \times K$ is a 'GR-I' of $R \times T$ if and only if P is a 'GR-I' of R and K is a 'GR-I' of T. The following example reveals that one can find 'GR-W-2-AI' which is not 'GR-WPI'. Unfortunately, these rings that are used are commutative, indeed, we could not find such an example consisting of a non-commutative ring.

Example 3. *Let $R = \mathbb{Z}_2[i]$, $T = \mathbb{Z}_4[i]$, and $G = \mathbb{Z}_2$. Then $R_0 = \mathbb{Z}_2$ and $R_1 = i\mathbb{Z}_2$ are the grades at that point of R. As well, T is a graded by $T_0 = \mathbb{Z}_4$ and $T_1 = i\mathbb{Z}_4$. In order that, $R \times T$ is a graded by $(R \times T)_j = R_j \times T_j$ for all $j = 0, 1$. Therefore, $\{0\}$ is a 'GR-I' of R and $2T$ is a 'GR-I' of T as $2 \in h(T)$, so $P = \{0\} \times 2T$ is a 'GR-I' of $R \times T$. Since $x = (0, 1), y = (1, 2) \in h(R \times T)$ with $(0, 0) \neq xy = (0, 2) \in P$, $x \notin P$ and $y \notin P$. Then P is not a 'GR-WPI' of $R \times T$. Individually, P is 'GR-2-AI' and hence a 'GR-W-2-AI' of $R \times T$.*

Theorem 1. *Let R be a 'GR-ring'. Suppose that P is a 'PGR-I' of R. Assume that for graded left ideals E, F and G of R such that $0 \neq EFG \subseteq P$, since $EG \subseteq P$, $FG \subseteq P$ or $EF \subseteq P$. Then P is a 'GR-W-2-AI' of R.*

Proof. Suppose that $x, y, z \in h(R)$ where $0 \neq xRyRz \subseteq P$. therefore, $RxRyRzR \subseteq P$, and as a consequence, since R has a unity, $0 \neq xRyRz = 1.xR.1.yR.1.z.1 \subseteq (RxR)(RyR)(RzR) \subseteq P$. By assumption, we have $xy \in (RxR)(RyR) \subseteq P$ or $yz \in (RyR)(RzR) \subseteq P$ or $xz \in (RxR)(RzR) \subseteq P$. Accordingly, P will be 'GR-W-2-AI'. □

Theorem 2. *Theorem 1 still true if graded left ideals are replaced by graded right ideals.*

Let R be a 'GR-ring' and K is a 'GR-I' of R, then R/K is a graded by $(R/K)_g = (R_g + K)/K$ for any $g \in G$. For P as an ideal of R and K is a 'GR-I' of R such that $K \subseteq P$, then P is a 'GR-I' of R if and only if P/K is a 'GR-I' of R/K.

Proposition 3. *For a graded ring R. Assume that P is a 'GR-W-2-AI' of R. Let $K \subseteq P$, if K is a 'GR-I' of R, then P/K is a 'GR-W-2-AI' of R/K.*

Proof. Let $x + K, y + K, z + K \in h(R/K)$ with $0 + K \neq (x + K)(R/K)(y + K)(R/K)(z + K) \subseteq P/K$. Hence $x, y, z \in h(R)$ with $0 \neq xRyRz \subseteq P$. Because P is 'GR-W-2-AI', for

$xy \in P$, $yz \in P$ or $xz \in P$, therefore, $(x+K)(y+K) \in P/K$ or $(y+K)(z+K) \in P/K$ or $(x+K)(z+K) \in P/K$. So, P/K is a 'GR-W-2-AI' of R/K. □

Proposition 4. *For a graded ring R. Let $K \subseteq P$ is a 'PGR-I' of a 'GR-ring' R. Then P is a 'GR-W-2-AI' of R, if K is a 'GR-W-2-AI' of R and P/K is a 'GR-W-2-AI' of R/K.*

Proof. Suppose that $x, y, z \in h(R)$ with $0 \neq xRyRz \subseteq P$. Therefore, $x+K, y+K, z+K \in h(R/K)$ such that $(x+K)(R/K)(y+K)(R/K)(z+K) \subseteq P/K$. If $0 \neq xRyRz \subseteq K$, then $xy \in K \subseteq P$ or $yz \in K \subseteq P$ or $xz \in K \subseteq P$ since K is a 'GR-W-2-AI' of R. If $xRyRz \nsubseteq K$, then $0+K \neq (x+K)(R/K)(y+K)(R/K)(z+K) \subseteq P/K$. Since P/K is a 'GR-W-2-AI' of R/K, $(x+K)(y+K) \in P/K$ or $(y+K)(z+K) \in P/K$ or $(x+K)(z+K) \in P/K$, that yields that $xy \in P$, $yz \in P$ or $xz \in P$. Therefore, P is a 'GR-W-2-AI' of R. □

For two 'GR-rings' S and T. We call $f : S \to T$ to be graded homomorphism f is ring homomorphism and $f(S_g) \subseteq T_g$ for every $g \in G$.

Proposition 5. *Let S and T be two 'GR-rings' and $f : S \to T$ be graded homomorphism. Then $Ker(f)$ is a 'GR-I' of S.*

Proof. Apparently, $Ker(f)$ is an ideal of S. Assume that $x \in Ker(f)$. Hence $x \in S$ such that $f(x) = 0$. Now, $x = \sum_{g \in G} x_g$, with $x_g \in S_g$ for all $g \in G$, which lead to $f(x_g) \in f(S_g) \subseteq T_g$ for all $g \in G$. As a result, for $g \in G$, $f(x_g) \in h(T)$ with $0 = f(x) = f\left(\sum_{g \in G} x_g\right) = \sum_{g \in G} f(x_g)$, which yields that $f(x_g) = 0$ for all $g \in G$ along with $\{0\}$ is a 'GR-I'. Therefore, $x_g \in Ker(f)$ for any $g \in G$, and then $Ker(f)$ is a 'GR-I' of S. □

Theorem 3. *For the two 'GR-rings' S and T and $f : S \to T$ be surjective graded homomorphism.*

1. *$f(P)$ will be a 'GR-W-2-AI' of T, if P is a 'GR-W-2-AI' of S and $Ker(f) \subseteq P$.*
2. *$f^{-1}(I)$ will be a 'GR-W-2-AI' of S, if I is a 'GR-W-2-AI' of T and $Ker(f)$ is a 'GR-W-2-AI' of R.*

Proof.

1. Let $f(P)$ be a 'GR-I' of T. Because P is a 'GR-W-2-AI' of R and $Ker(f) \subseteq P$, Proposition 3 shows that $P/Ker(f)$ is a 'GR-W-2-AI' of $S/Ker(f)$. The result holds Since $S/Ker(f)$ is isomorphic to T.
2. Assume that $f^{-1}(I)$ is a 'GR-I' of S. Let $K = f^{-1}(I)$. Then $Ker(f) \subseteq K$. We observe that $K/Ker(f)$ is a 'GW-2-AI' of $S/Ker(f)$, since $S/Ker(f)$ is isomorphic to T. Because $Ker(f)$ is a 'GR-W-2-AI' of S and $K/Ker(f)$ is a 'GR-W-2-AI' of $S/Ker(f)$, Proposition 4 states that $K = f^{-1}(I)$ is a 'GR-W-2-AI' of S.

□

Motivated by Theorem 1, we observe the next question.

Question 1. *If P is a 'GR-W-2-AI' of R that is not a 'GR-2-AI' and $0 \neq EFK \subseteq P$ for some 'GR-I' E, F and K of R. Does it indicate that $EF \subseteq P$ or $EK \subseteq P$ or $FK \subseteq P$?*

We will give a partial answer through the coming discussions. Motivated by ([4], Definition 3.3), we introduce the following:

Definition 2. *Assume that R is a 'GR-ring', $g \in G$ and P is a 'GR-I' of R with $P_g \neq R_g$.*

1. *If for each $x, y, z \in R_g$ where $xR_eyR_ez \subseteq P$, then P is said to be a 'GR-2-AI' of R, therefore, $xy \in P$, $yz \in P$ or $xz \in P$.*

2. *If for each $x, y, z \in R_g$ where $0 \neq xR_eyR_ez \subseteq P$, then P is said to be a 'GR-W-2-AI' of R, therefore, $xy \in P$, $yz \in P$ or $xz \in P$.*
3. *For $x, y, z \in R_g$, let P is a 'GR-W-2-AI' of R and. We denote 'GR-3-Z' for (x, y, z) which is the graded-triple-zero of P if $xR_eyR_ez = 0$, such that $xy \notin P$, $yz \notin P$ and $xz \notin P$.*

Note that if P is 'GR-W-2-AI' which is not 'GR-2-AI', then P involves a 'GR-3-Z' (x, y, z) for $x, y, z \in R_g$.

Proposition 6. *Assume that $xR_ey, K_g \subseteq P$ for any $x, y \in R_g$ and some graded left ideal K of R, and that P is a 'GR-W-2-AI' of R. Let (x, y, z) is not a 'GR-3-z' of P for every $z \in K_g$. If $xy \notin P$, then $xK_g \subseteq P$ or $yK_g \subseteq P$.*

Proof. Consider that $xK_g \nsubseteq P$ along with $yK_g \nsubseteq P$. Then there exist $r, s \in K_g$ such that $xr \notin P$ and $ys \notin P$. Since $xR_eyR_er \subseteq xR_eyK_g \subseteq P$ and since (x, y, r) is not a GR-3-Z of P and $xy \notin P$, $xr \notin P$, we obtain that $yr \in P$. Also, since $xR_eyR_es \subseteq xR_eyK_g \subseteq P$ and since (x, y, s) is not a GR-3-Z of P and $xy \notin P$, $ys \notin P$, we obtain that $xs \in P$. Now, since $xR_eyR_e(r+s) \subseteq xR_eyK_g \subseteq P$ and since $(x, y, r+s)$ is not a GR-3-Z of P and $xy \notin P$, we get $x(r+s) \in P$ or $y(r+s) \in P$. If $x(r+s) \in P$, then since $xs \in P$, $xr \in P$, a contradiction. If $y(r+s) \in P$, then since $yr \in P$, $ys \in P$, a contradiction. Hence, $xK_g \subseteq P$ or $yK_g \subseteq P$. □

Definition 3. *Let R be a 'GR-ring' $g \in G$ and P be a 'GR-W-2-AI' of R. Assume that $A_gB_gK_g \subseteq P$ for some 'GR-I' A, B and K of R. If (x, y, z) is not a 'GR-3-Z' of P for every $x \in A_g, y \in B_g$ and $z \in K_g$. We can state P as being a free 'GR-3-Z' respecting ABK. The next proposition is clear.*

Proposition 7. *Let P is a 'GR-W-2-AI' of R. Presume that $A_gB_gK_g \subseteq P$ and P to be a free 'GR-3-Z' in respect to ABK, for some 'GR-I' A, B and K of R. If $x \in A_g, y \in B_g$ and $z \in K_g$, then $xy \in P$, $xz \in P$ or $yz \in P$.*

Theorem 4. *Infer that P is a 'GR-W-2-AI' of R. Lets take $0 \neq A_gB_gK_g \subseteq P$ and P to be a free 'GR-3-Z' in respect to ABK, for some 'GR-I' A, B and K of R. Then $A_gK_g \subseteq P$, $B_gK_g \subseteq P$ or $A_gB_g \subseteq P$.*

Proof. Suppose that $A_gK_g \nsubseteq P$, $B_gK_g \nsubseteq P$ and $A_gB_g \nsubseteq P$. There exist $x \in A_g$ and $y \in B_g$ where $xK_g \nsubseteq P$ and $yK_g \nsubseteq P$. Now, $xR_eyK_g \subseteq A_gB_gK_g \subseteq P$. Since $xK_g \nsubseteq P$ and $yK_g \nsubseteq P$, it comes from Proposition 6 that $xy \in P$. Because $A_gB_g \nsubseteq P$, there are $a \in A_g$ and $b \in B_g$ where $ab \notin P$. Since $aR_ebK_g \subseteq A_gB_gK_g \subseteq P$ and $ab \notin P$, it comes from Proposition 6 that $aK_g \subseteq P$ or $bK_g \subseteq P$.

Case (1): $aK_g \subseteq P$ and $bK_g \nsubseteq P$. Since $xR_ebK_g \subseteq A_gB_gK_g \subseteq P$ and $xK_g \nsubseteq P$ and $bK_g \nsubseteq P$, it follows from Proposition 6 that $xb \in P$. Since $aK_g \subseteq P$ and $xK_g \nsubseteq P$, we obtain that $(x+a)K_g \nsubseteq P$. On the other hand, since $(x+a)R_ebK_g \subseteq P$ and neither $(x+a)K_g \subseteq P$ nor $bK_g \subseteq P$, we have that $(x+a)b \in P$ by Proposition 6, and hence $ab \in P$, which is not true.

Case (2): $bK_g \subseteq P$ and $aK_g \nsubseteq P$. Using an analogous assertion to case (1), we will have an inconsistency.

Case (3): $aK_g \subseteq P$ and $bK_g \subseteq P$. Since $bK_g \subseteq P$ and $yK_g \nsubseteq P$, $(y+b)K_g \nsubseteq P$. But $xR_e(y+b)K_g \subseteq P$ and neither $xK_g \subseteq P$ nor $(y+b)K_g \subseteq P$, and hence $x(y+b) \subseteq P$ by Proposition 6. Since $xy \in P$ and $(xy+xb) \in P$, we have that $xb \in P$. Since $(x+a)R_eyK_g \subseteq P$ and neither $yK_g \subseteq P$ nor $(x+a)K_g \subseteq P$, we conclude that $(x+a)y \in P$ by Proposition 6, and hence $ax \in P$. Since $(x+a)R_e(y+b)K_g \subseteq P$ and neither $(x+a)K_g \subseteq P$ nor $(y+b)K_g \subseteq P$, we have $(x+a)(y+b) \in P$ by Proposition 6. But $xy, xb, ay \in P$, so $ab \in P$, a contradiction. Consequently, $A_gK_g \subseteq P$ or $B_gK_g \subseteq P$ or $A_gB_g \subseteq P$. □

Lemma 2. *For a 'GR-ring' R. Assume that P is a 'GR-W-2-AI' and (x, y, z) is a 'GR-3-Z' of P for some $x, y, z \in R_g$. Then*

1. $xR_eyP_g = \{0\}$,
2. $P_gyR_ez = \{0\}$,
3. $xP_gz = \{0\}$,
4. $P_g^2z = \{0\}$,
5. $xP_g^2 = \{0\}$,
6. $P_gyP_g = \{0\}$.

Proof.

1. Assume that $xR_eyP_g \neq \{0\}$. Then there exist $r \in R_e$ and $p \in P_g$ such that $0 \neq xryp$. Now, $xry(p+z) = xryp + xryz = xryp \neq 0$. Hence, $0 \neq xR_eyR_e(p+z) \subseteq P$. We have $x(p+z) \in P$ or $y(p+z) \in P$, since P is 'GR-W-2-AI'. Thus $xz \in P$ or $yz \in P$ is a contradiction.
2. Suppose that $P_gyR_ez \neq \{0\}$. Then there exist $r \in R_e$ and $p \in P_g$ such that $0 \neq pyrz$. Now, $(x+p)yrz = xyrz + pyrz = pyrz \neq 0$. Hence, $0 \neq (x+p)R_eyR_ez \subseteq P$. If P is 'GR-W-2-AI' We have $(x+p)y \in P$ or $(x+p)z \in P$. As a result, $xy \in P$ or $xz \in P$ is a contradiction.
3. Suppose that $xP_gz \neq \{0\}$. However, there exists $p \in P_g$ for which $0 \neq xpz$. Now, $x(y+p)z = xyz + xpz = xpz \neq 0$. Hence, $0 \neq xR_e(y+p)R_ez \subseteq P$. We have $x(y+p) \in P$ or $(y+p)z \in P$. Because P is 'GR-W-2-AI'. Hence, $xy \in P$ or $yz \in P$ is a contradiction.
4. Suppose that $P_g^2z \neq \{0\}$. Moreover, there exist $p,q \in P_g$ in which $0 \neq pqz$. Now, $(x+p)(y+q)z = xyz + xqz + pyz + pqz = pqz \neq 0$ by (2) and (3). Hence, $0 \neq (x+p)R_e(y+q)R_ez \subseteq P$. We have $(x+p)z \in P$ or $(y+q)z \in P$ or $(x+p)(y+q) \in P$. Because P is 'GR-W-2-AI'. Hence, $xz \in P$ or $yz \in P$ or $xy \in P$ is a contradiction.
5. Suppose that $xP_g^2 \neq \{0\}$. Moreover, there exist $p,q \in P_g$, where, $0 \neq xpq$. Now, by (1) and (3), $x(y+p)(z+q) = xyz + xyq + xpz + xpq = xpq \neq 0$. As a result, $0 \neq xR_e(y+p)R_e(z+q) \subseteq P$. We have $x(y+p) \in P$, $x(z+q) \in P$ or $(y+p)(z+q) \in P$. Because, P is 'GR-W-2-AI'. Hence, $xy \in P$, $xz \in P$ or $yz \in P$ is a contradiction.
6. Suppose that $P_gyP_g \neq \{0\}$. Then there exist $p,q \in P_g$ such that $0 \neq pyq$. Now, by (1) and (2), $(x+p)y(z+q) = xyz + xyq + pyz + pyq = pyq \neq 0$. Hence, $0 \neq (x+p)R_eyR_e(z+q) \subseteq P$. We have $(x+p)y \in P$ or $y(z+q) \in P$ or $(x+p)(z+q) \in P$. Because P is 'GR-W-2-AI'. As a result, $xy \in P$, $yz \in P$ or $xz \in P$ is a contradiction.

□

The following theorem is a consequence result from Lemma 2.

Theorem 5. *Let R be a 'GR-ring', $g \in G$ and P be a 'GR-I' of R such that $P_g^3 \neq \{0\}$. Then P is 'GR-W-2-AI' if and only if P is 'GR-2-AI'.*

Proof. Assume that P is a 'GR-W-2-AI' that is not the same as a 'GR-2-AI' of R. For some $x,y,z \in R_g$. Let P has a 'GR-3-Z', say (x,y,z) . Therefore, if $P_g^3 \neq \{0\}$, there exist $p,q,r \in P_g$ where $pqr \neq 0$, and then $(x+p)(y+q)(z+r) = pqr \neq 0$. As a result, $0 \neq (x+p)R_e(y+q)R_e(z+r) \subseteq P$. We have either $(x+p)(y+q) \in P$, $(x+p)(z+r) \in P$ or $(y+q)(z+r) \in P$. Because P is 'GR-W-2-AI', , and thus either $xy \in P$, $xz \in P$ or $yz \in P$ which is a contradiction. Hence, P is a 'GR-2-AI' of R. The contrary is self-evident. □

Corollary 1. *Assume R to be a 'GR-ring'. If P is a 'GR-W-2-AI' of R and it is not 'GR-2-AI', then $P_g^3 = \{0\}$.*

Allow R to be a 'GR-ring' and M to be an R-module. Then M is considered to be a graded if for any $g \in G$, $M = \bigoplus_{g \in G} M_g$ with $R_gM_h \subseteq M_{gh}$, where M_g is an additive subgroup of M . The components of M_g are known as homogeneous of degree g.

For any $g \in G$ It is obvious that M_g is an R_e-submodule of M . The set of all homogeneous components of M is $\bigcup_{g\in G} M_g$ and is denoted by $h(M)$. Let N be an R-submodule which is a graded R-module M, and denoted by 'GR-R'-submodule.

If $N = \bigoplus_{g\in G}(N \cap M_g)$, or equivalently, $x = \sum_{g\in G} x_g \in N$, i.e., $x_g \in N$ for any $g \in G$. Then N is said to be graded R-submodule.

It is well known that an R-submodule of a 'GR-R'-module does not need to be graded. For more terminology see [2,3].

Assume M to be an *bi*-R-module. The idealization (trivial extension) $R \ltimes M = \{(r, m) : r \in R, m \in M\}$ of M is a ring with component wise addition defined by: $(x, m_1) + (y, m_2) = (x + y, m_1 + m_2)$ and multiplication is defined by: $(x, m_1)(y, m_2) = (xy, xm_2 + m_1y)$ for each $x, y \in R$ and $m_1, m_2 \in M$. Let G be an Abelian group and M be a 'GR-R'-module. Then for any $g \in G$, $X = R \ltimes M$ is a graded by $X_g = R_g \bigoplus M_g$ [9].

Theorem 6. *Let R be a GR-ring with unity, M be a GR-bi-R-module and P be a 'P-GR-I' of R. Hence, $P \ltimes M$ is a 'GR-2-AI' of $R \ltimes M$ if and only if P is a 'GR-2-AI' of R.*

Proof. For some $x, y, z \in h(R)$. Assume that $P \ltimes M$ is a 'GR-2-AI' of $R \ltimes M$ and $xRyRz \subseteq P$. Then $(x, 0), (y, 0), (z, 0) \in h(R \ltimes M)$ with $(x, 0)R \ltimes M(y, 0)R \ltimes M(z, 0) \subseteq P \ltimes M$, and then $(x, 0)(y, 0) = (xy, 0) \subseteq P \ltimes M$, $(x, 0)(z, 0) = (xz, 0) \subseteq P \ltimes M$ or $(y, 0)(z, 0) = (yz, 0) \subseteq P \ltimes M$. A a result, $xy \in P$, $xz \in P$ or $yz \in P$, as required. In the opposite case, let $(x, m)R \ltimes M(y, n)R \ltimes M(z, p) \subseteq P \ltimes M$ for some $(x, m), (y, n), (z, p) \in h(R \ltimes M)$. Therefore, $x, y, z \in h(R)$ with $xRyRz \subseteq P$, we obtain $xy \in P$, $xz \in P$ or $yz \in P$. If $xy \in P$ true, then $(x, m)(y, n) = (xy, xn + ym) \subseteq P \ltimes M$. Similarly, if $xz \in P$, then $(x, m)(z, p) \in P \ltimes M$, and if $yz \in P$, then $(y, n)(z, p) \in P \ltimes M$, and so on, this completes the proof. □

Theorem 7. *Let R be a 'GR-ring' with unity, M to be a 'GR-bi-R'-module and P to be a 'PGR-I' of R. If $P \ltimes M$ is a 'GR-W-2-AI' of $R \ltimes M$, then P is a 'GR-W-2-AI' of R.*

Proof. For $x, y, z \in h(R)$, let $0 \neq xRyRz \subseteq P$, . Then $(0, 0) \neq (x, 0)R \ltimes M(y, 0)R \ltimes M(z, 0) \subseteq P \ltimes M$, and then $(xy, 0) \in P \ltimes M$, $(xz, 0) \in P \ltimes M$ or $(yz, 0) \in P \ltimes M$. As a result, $xy \in P$, $xz \in P$ or $yz \in P$. So, P is 'GR-W-2-AI'. □

Theorem 8. *Let R be a 'GR-ring' with unity, M be a 'GR-bi-R'-module, $g \in G$ and P to be a 'GR-I' of R with $P_g \neq R_g$. Hence $P \ltimes M$ is a 'GR-W-2-AI' of $R \ltimes M$ if and only if P is a 'GR-W-2-AI' of R and for every 'GR-3-Z', (x, y, z) of P we got $xR_eyR_eM_g = M_gR_eyR_ez = xM_gz = 0$.*

Proof. Assume that $P \ltimes M$ is a 'GR-W-2-AI' of $R \ltimes M$. Let $0 \neq xR_eyR_ez \subseteq P$, with $x, y, z \in R_g$. Then $(0, 0) \neq (x, 0)R_e \ltimes M_e(y, 0)R_e \ltimes M_e(z, 0) \subseteq P \ltimes M$, and then $(xy, 0) \in P \ltimes M$ or $(xz, 0) \in P \ltimes M$ or $(yz, 0) \in P \ltimes M$. As a result, $xy \in P$, $xz \in P$ or $yz \in P$. So, P is 'GR-W-2-AI'. Preduse that (x, y, z) is a 'GR-3-Z' of P. Assume that $xR_eyR_eM_g \neq 0$. Hence there exist $r, s \in R_e$ and $m \in M_g$ such that $xrysm \neq 0$, and then $(0, 0) \neq (xrysz, xrysm) = (x, 0)(r, 0)(y, 0)(s, 0)(z, m) \in (x, 0)R_e \ltimes M_e(y, 0)R_e \ltimes M_e(z, m) \subseteq xR_eyR_ez \ltimes M_g = 0 \ltimes M_g \subseteq P \ltimes M$. However, $(x, 0)(y, 0) \notin P \ltimes M$ and $(x, 0)(z, m) \notin P \ltimes M$ and $(y, 0)(z, m) \notin P \ltimes M$, which contradicting the statement that $P \ltimes M$ is a 'GR-W-2-AI'. If $M_gR_eyR_ez \neq 0$, hence, there exist $n \in M_g$ and $r, s \in R_e$ such that $nrysz \neq 0$. As above, we have $(0, 0) \neq (xrysz, nrysz) = (x, n)(r, 0)(y, 0)(s, 0)(z, 0) \in (x, n)R_e \ltimes M_e(y, 0)R_e \ltimes M_e(z, 0) \subseteq xR_eyR_ez \ltimes M_g = 0 \ltimes M_g \subseteq P \ltimes M$. however, there is a contradiction between $(x, n)(y, 0) \notin P \ltimes M$, $(x, n)(z, 0) \notin P \ltimes M$ and $(y, 0)(z, 0) \notin P \ltimes M$. If $xM_gz \neq 0$, then there exists $t \in M_g$ where, $xtz \neq 0$. At the present, $(0, 0) \neq (xyz, xtz) = (x, 0)(1, 0)(y, t)(1, 0)(z, 0) \in (x, 0)R_e \ltimes M_e(y, t)R_e \ltimes M_e(z, 0) \subseteq xR_eyR_ez \ltimes M_g = 0 \ltimes M_g \subseteq P \ltimes M$. However, there is a contradiction between $(x, 0)(y, t) \notin P \ltimes M$ and $(x, 0)(z, 0) \notin P \ltimes M$ and $(y, t)(z, 0) \notin P \ltimes M$. Conversely, suppose that $(0, 0) \neq (x, n)R_e \ltimes M_e(y, m)R_e \ltimes M_e(z, t) \subseteq P \ltimes M$ for $(x, n), (y, m), (z, t) \in R_g \ltimes M_g$. Then $x, y, z \in R_g$ with $xR_eyR_ez \subseteq P$.

Case (1): $xR_eyR_ez \neq 0$. Since P is GR-W-2-AI, it might be $xy \in P$, $xz \in P$ or $yz \in P$. Hence, $(x,n)(y,m) \in P \ltimes M$, $(x,n)(z,t) \in P \ltimes M$ or $(y,m)(z,t) \in P \ltimes M$, as desired.

Case (2): $xR_eyR_ez \neq 0$. If $xy \notin P$, $xz \notin P$ and $yz \notin P$, then (x,y,z) is a 'GR-3-Z' of P and by assumption $xR_eyR_eM_g = M_gR_eyR_ez = xM_gz = 0$. Now, $(x,n)R_e \ltimes M_e(y,m)R_e \ltimes M_e(z,t) \subseteq \big(xR_eyR_ez, M_gR_eyR_ez + xM_gz + xR_eyR_eM_g\big) = (0,0)$, a contradiction. □

Question 2. *As a proposal for future work, we think it will be worthy to study non-commutative graded rings such that every 'GR-I' is 'GR-W-2-AI'. What kind of results will be achieved?*

The following abbreviations are used throw this Article: 'GR-SW-2-AI' for the graded strongly weakly 2-absorbing ideals.

On the other hand, we present the idea of 'GR-SW-2-AI', and examine 'GR-rings' in which every 'GR-I' is 'GR-SW-2-AI'.

Definition 4. *Let R be a 'GR-ring' and P to be a 'PGR-I' of R. If A, B and C are 'GR-I' of R where $0 \neq ABC \subseteq P$. So, $AC \subseteq P$, $BC \subseteq P$ or $AB \subseteq P$. Then P is said to be a 'GR-SW-2-AI' of R.*

Proposition 8. *Let P be a 'PGR-I' of R. Then P is a 'GR-SW-2-AI' of R if and only if for any 'GR-I' A, B and C of R such that $P \subseteq A$ (or $P \subseteq B$ or $P \subseteq C$), $0 \neq ABC \subseteq P$ implies that $AB \subseteq P$, $AC \subseteq P$ or $BC \subseteq P$.*

Proof. The result holds by the above definition If P is a 'GR-SW-2-AI' of R. Conversely, let K, B and C be 'GR-I' of R where, $0 \neq KBC \subseteq P$. Hence $A = K + P$ is a GR-I of R such that $0 \neq ABC \subseteq P$, and then by assumption, $AB \subseteq P$ or $AC \subseteq P$ or $BC \subseteq P$. As a result, $KB \subseteq P$, $KC \subseteq P$ or $BC \subseteq P$. Hence, P becomes a 'GR-SW-2-AI' of R. □

Proposition 9. *Let R be a 'GR-ring'. Then every 'GR-I' of R is 'GR-SW-2-AI' if and only if for any 'GR-I' I, J and K of R, $IJ = IJK$, $IK = IJK$, $JK = IJK$ or $IJK = 0$.*

Proof. Suppose that every 'GR-I' of R is 'GR-SW-2-AI'. Let I, J and K be 'GR-I' of R. If $IJK \neq R$, then IJK is 'GR-SW-2-AI'. Suppose that $IJK \neq 0$. Then $0 \neq IJK \subseteq IJK$ and $IJ \subseteq IJK$, $IK \subseteq IJK$ or $JK \subseteq IJK$ and hence $IJ = IJK$, $IK = IJK$ or $JK = IJK$. If $IJK = R$, then $I = J = K = R$. Conversely, let P be a PGR-I of R, $0 \neq IJK \subseteq P$ for some 'GR-I' I, J and K of R. Then $IJ = IJK \subseteq P$ or $IK = IJK \subseteq P$ or $JK = IJK \subseteq P$. Hence, P is a 'GR-SW-2-AI' of R. □

Corollary 2. *Assume R to be a 'GR-ring' where every 'GR-I' of R is 'GR-SW-2-AI'. Then $I^3 = I^2$ or $I^3 = 0$ for every 'GR-I' of R.*

3. Conclusions

In this study, we introduced and examined the concept of Gr-W-2-AI over non-commutative graded rings, several results were achieved. As a proposal for further work on the topic, we are going to examine the concept of Gr-W-1-AI over non-commutative graded rings.

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