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Extended Convergence of Three Step Iterative Methods for Solving Equations in Banach Space with Applications

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Abstract: Symmetries are vital in the study of physical phenomena such as quantum physics and the micro-world, among others. Then, these phenomena reduce to solving nonlinear equations in abstract spaces. These equations in turn are mostly solved iteratively. That is why the objective of this paper was to obtain a uniform way to study three-step iterative methods to solve equations defined on Banach spaces. The convergence is established by using information appearing in these methods. This is in contrast to earlier works which relied on derivatives of the higher order to establish the convergence. The numerical example completes this paper.

Keywords: numerical processes; Banach space; convergence condition

MSC: 65J15; 47H17; 49M15; 65G99; 41A25

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Citation: Regmi, S.; Argyros, I.K.; George, S.; Argyros, C.I. Extended Convergence of Three Step Iterative Methods for Solving Equations in Banach Space with Applications. *Symmetry* **2022**, *14*, 1484. https://doi.org/10.3390/sym14071484

Academic Editor: Lorentz Jäntschi

Received: 17 June 2022
Accepted: 17 July 2022
Published: 20 July 2022

1. Introduction

The objective in this paper was to locate a simple solution $x^* \in \Omega$ of

$$\mathcal{G}(x) = 0, \tag{1}$$

given that $\mathcal{G} : \Omega \subset X \longrightarrow X_1$ is a continuous operator, X, X_1 are Banach spaces and the set $\Omega \neq \emptyset$. Numerous methods can be represented for all $m = 0, 1, 2, \dots,$ by

$$\begin{aligned} y_m &= x_m - a\mathcal{G}'(x_m)^{-1}\mathcal{G}(x_m) \\ z_m &= x_m - A_m\mathcal{G}(x_m) \\ x_{m+1} &= z_m - B_m\mathcal{G}(z_m), \end{aligned} \tag{2}$$

where $A_m = A(x_m, y_m)$, $A : \Omega \times \Omega \longrightarrow L(X, X_1)$, $A^{-1} \in L(X_1, X)$, $B_m = B(x_m, y_m)$, $B : \Omega \times \Omega \longrightarrow L(X, X_1)$ and $B^{-1} \in L(X_1, X)$.

Special Cases:

Newton's method (second order) [1–10]: Set $a = 1$ and $A_m = B_m = O$,

$$y_m = x_m - \mathcal{G}'(x_m)^{-1}\mathcal{G}(x_m).$$

This method is of the order of two.

Jarrat's method (second order) [10]: Set $a = \frac{2}{3}$ and $A_m = B_m = O$ to obtain

$$y_m = x_m - \frac{2}{3}\mathcal{G}'(x_m)^{-1}\mathcal{G}(x_m).$$

Traub-like method (fifth order) [10]: Let $a = 1$ and $A_m = B_m = \mathcal{G}'(x_m)^{-1}$ to get

$$\begin{aligned} y_m &= x_m - \mathcal{G}'(x_m)^{-1}\mathcal{G}(x_m) \\ z_m &= x_m - \mathcal{G}'(x_m)^{-1}\mathcal{G}(x_m) \\ x_{m+1} &= x_m - \mathcal{G}'(x_m))^{-1}\mathcal{G}(z_m). \end{aligned}$$

Homeier method (third order) [11]: Set $a = \frac{1}{2}$, $A_m = \mathcal{G}'(y_m)^{-1})$ and $B_m = O$, to obtain

$$\begin{aligned} y_m &= x_m - \frac{1}{2}\mathcal{G}'(x_m)^{-1}\mathcal{G}(x_m) \\ x_{m+1} &= y_m - \mathcal{G}'(y_m)^{-1})\mathcal{G}(y_m). \end{aligned}$$

Corodero–Torregrosa method (third order) [12]: Set $a = 1$, $A_m = 6[\mathcal{G}'(x_m) + 4\mathcal{G}'(\frac{x_m+y_m}{2}) + \mathcal{G}'(y_m)]^{-1}$ and $B_m = O$, to obtain

$$y_m = x_m - A_m\mathcal{G}(x_m)$$

or $A_m = 2[2\mathcal{G}'(\frac{3x_m+y_m}{4}) - \mathcal{G}'(\frac{x_m+y_m}{2}) + 2\mathcal{G}'(\frac{x_m+3y_m}{4})]^{-1}$.
Noor–Wasseem method (third order) [13]: $a = 1, A_m = 4[3\mathcal{G}'(\frac{2x_m+y_m}{3} + \mathcal{G}'(y_m)]^{-1}$, and $B_0 = O$.
Xiao–Yin method (third order) [14]: $a = 1, A_m = \frac{2}{3}[(3\mathcal{G}'(y_m) - \mathcal{G}'(x_m))^{-1} + \mathcal{G}'(x_m)^{-1}]$ and $B_m = O$.
Cordero method (fifth order) [12]: $a = \frac{2}{3}, A_m = \frac{1}{2}(3\mathcal{G}'(y_m) - \mathcal{G}'(x_m))^{-1}(3\mathcal{G}'(y_m) + \mathcal{G}'(x_m))$ $\mathcal{G}'(x_m)^{-1}$ and $B_m = (\frac{1}{2}\mathcal{G}'(y_m) + \frac{1}{2}\mathcal{G}'(x_m))^{-1}$ or $a = 1, A_m = 2(\mathcal{G}'(y_m) + \mathcal{G}'(x_m))^{-1}$ and $B_m = \mathcal{G}'(y_m)^{-1}$.
Sharma–Arora method (fifth order) [15]: $a = 1, A_m = \mathcal{G}'(y_m)^{-1}$ and $B_m = 2\mathcal{G}'(y_m)^{-1} - \mathcal{G}'(x_m)^{-1}$.
Xiao–Yinmethod (fifth order) [16]: $a = \frac{2}{3}, A_m = \frac{1}{4}(3\mathcal{G}'(y_m)^{-1} + \mathcal{G}'(x_m)^{-1})$ and $B_m = \frac{1}{3}(3\mathcal{G}'(y_m)^{-1} - \mathcal{G}'(x_m)^{-1})$.

Other choices are also possible [1–3,8,9,14,15,17–19]. Therefore, it is interesting to consider the semilocal convergence of these methods not given in earlier papers under the same convergence criteria in the Banach space setting using the method (2). In earlier papers, only the local convergence was given in the finite-dimensional Euclidean space requiring the existence of derivatives one more than the order. Moreover, these derivatives do not appear on the methods but are only used to show the convergence order.

For example, let $X = X_1 = \mathbb{R}$, $\Omega = [-0.5, 1.5]$. Define function ξ on Ω by

$$\xi(x) = \begin{cases} 0, & if\ x = 0 \\ 2x^3 \log x + x^5 - x^4, & if\ x \neq 0. \end{cases}$$

Notice that $x^* = 1$. The definition of function ξ gives

$$\xi'''(x) = \log x^{12} + 60x^2 - 24x + 22.$$

However, $\xi'''(x)$ is unbounded on Ω. Thus, the convergence of method (2) is not verified by the earlier analyses. This paper extends the usage of these methods because no conditions on derivatives of high order are used to show convergence. This is the novelty of the paper. The study also includes the semilocal analysis not given in earlier research. Notice that the branching of the solutions cannot be handled using the iterative method (2) since the first step required that $G'(x_m)^{-1}$ exists. The paper contains seven sections, including a numerical and a concluding section.

2. Real Sequences

Let $\{p_m\}$ and $\{q_m\}$ be nonnegative sequences and $\eta \geq 0$ be a given parameter. Set $S = [0, \infty)$. Consider functions φ_0, $\varphi : S \longrightarrow S$ to be nondecreasing, continuous, and sequence $\{t_m\}$, $\{s_m\}$ and $\{u_m\}$ are defined by

$$\begin{aligned} t_0 &= 0,\, s_0 = \eta, \\ u_m &= s_m + p_m(s_m - t_m), \\ t_{m+1} &= u_m + q_m(u_m - s_m), \\ s_{m+1} &= t_{m+1} + \frac{\bar{\psi}(t_m, s_m, u_m)}{1 - \psi_0(t_{m+1})}, \end{aligned} \tag{3}$$

where $\bar{\psi}(t_m, s_m, u_m) = \int_0^1 \varphi(\theta(t_{m+1} - t_m))d\theta(t_{m+1} - t_m) + (1 + \varphi_0(t_m))(t_{m+1} - s_m) + |1 - a|(1 + \varphi_0(t_m))(s_m - t_m)$.

Next, three auxiliary results are given on the convergence of the majorizing sequence (3).

Lemma 1. *Suppose there exists minimal zero τ of function $\varphi_0(t) - 1$ and*

$$t_m \leq \tau_0 \quad \textit{for all } m = 0, 1, 2, \ldots. \tag{4}$$

Then, the sequence $\{t_m\}$ is nondecreasing and convergent to some $\tau^* \in [0, \tau_0]$. The limit point τ^* is the least upper bound of the sequence $\{t_m\}$ and it is unique.

Proof. The result followed by (3) and (4), since the sequence $\{t_m\}$ is bounded from τ and nondecreasing. □

A stronger result follows.

Lemma 2. *Sequence $\{t_m\}$ is strictly increasing and*

$$t_m \leq \varphi_0^{-1}(1). \tag{5}$$

Then, it holds $\lim_{m \longrightarrow +\infty} t_m = \tau^* \leq \varphi_0^{-1}(1)$.

Proof. Set $\tau = \varphi_0^{-1}(1)$ in Lemma 1. □

Next, we define sequences $\{b_m\}$ and $\{c_m\}$ for all $m = 0, 1, 2, \ldots$ by

$$b_m = (1 + p_m)q_m$$

and

$$c_m = \frac{c_m^1}{1 - \varphi_0(t_{m+1})},$$

where $c_m^1 = \int_0^1 \varphi(\theta(1 + p_m)(1 + q_m))d\theta(s_m - t_m)((1 + p_m)(1 + q_m)) + |1 - a|(1 + \varphi_0(t_m)) + (p_m + (1 + p_m)q_m)(1 + \varphi_0(t_m))$ and functions g_i, $i = 1, 2, 3$ by

$$g_1(t) = -t + c^1(t) + t\varphi_0(\frac{\eta}{1 - t}),$$

$$g_2(t) = -t + b(t),\ g_3(t) = -t + c(t)$$

and $c^1(t) = \int_0^1 \varphi(\theta(1 + p)(1 + q))\lambda\eta d\theta(1 + p)(1 + q) + |1 - a|(1 + \varphi_0(\frac{\eta}{1 - t})) + (p + (1 + p)q)(1 + \varphi_0(t))$, provided that these exist $p, q \geq 0$ such that

$$p_m \leq p \quad \text{and } q_m \leq q. \tag{6}$$

The convergence criteria given so far are very general. However, we can consider stronger ones.

Suppose functions g_i have minimal zeros $\lambda_i \in (0,1)$ and set

$$\lambda = \min\{\lambda_i\} \text{ and } \bar{\lambda} = \max\{p_0, b_0, \frac{c_0^1}{1-\varphi_0(t_1)}\}.$$

Next, we present the third result.

Lemma 3. *Suppose:*

$$p_0 \leq \lambda, \tag{7}$$

$$b_0 \leq \lambda \tag{8}$$

and

$$\frac{c_0^1}{1-\varphi_0(t_1)} \leq \lambda. \tag{9}$$

Then, the following items hold for all $m = 0, 1, 2, \ldots$

$$0 \leq u_m - s_m \leq \lambda^{m+1}\eta, \tag{10}$$

$$0 \leq t_{m+1} - u_m \leq \lambda^{m+1}\eta, \tag{11}$$

$$0 \leq s_{m+1} - t_{m=1} \leq \lambda^{m+1}\eta, \tag{12}$$

and

$$\tau^* = \lim_{m \longrightarrow +\infty} t_m \leq \frac{\eta}{1-\lambda}. \tag{13}$$

Proof. Items (10)–(12) are shown using induction on m. Using (7)–(9), and the definition of the sequence $\{t_m\}$

$$u_0 - s_0 = p_0(s_0 - t_0) \leq \lambda\eta, \tag{14}$$

$$\begin{aligned} t_1 - u_0 &= q_0(u_0 - s_0 + s_0 - t_0) \leq q_0(p_0(s_0 - t_0) + (s_0 - t_0)) \\ &\leq q_0(1 + p_0)(s_0 - t_0) = b_0\eta \leq \lambda\eta, \end{aligned} \tag{15}$$

$$\begin{aligned} t_1 - t_0 &= t_1 - u_0 + u_0 - s_0 + s_0 - t_0 \\ &\leq (q_0(1 + p_0) + (1 + p_0))(s_0 - t_0), \\ t_1 - s_0 &= t_1 - u_0 + u_0 - s_0 \leq [q_0(1 + p_0) + p_0](s_0 - t_0). \end{aligned}$$

Thus,

$$s_1 - t_1 = \frac{c_0^1}{1-\varphi_0(t_1)} \leq \lambda\eta. \tag{16}$$

By (14)–(16), estimates (10)–(12) hold for $m = 0$. Assume they are true for all integers m smaller than n. Using the induction

$$\begin{aligned} s_m &\leq t_m + \lambda^m\eta \\ &\leq s_{m-1} + \lambda^{m-1}\eta + \lambda^m\eta \\ &\vdots \\ &= \frac{1-\lambda^{m+1}}{1-\lambda}\eta < \frac{\eta}{1-\lambda} \end{aligned}$$

and

$$t_{m+1} \leq s_m + \lambda^{m+1}\eta \leq \frac{1-\lambda^{m+2}}{1-\lambda}\eta < \frac{\eta}{1-\lambda}.$$

Moreover, estimates (14)–(16) shall hold for m replacing 0 if $g_1(\lambda_i) \leq 0$, which is true by the definition of parameters λ_i and functions g_i. Hence, the induction for estimates (10)–(12) is terminated. Consequently, it follows $\lim_{m \longrightarrow +\infty} t_m = \tau^* < \frac{\eta}{1-\lambda}$. □

3. Semilocal Convergence

The convergence requires conditions:
Assume:

(C1) There exist $x_0 \in \Omega$, $\eta \geq 0$ such that $\mathcal{G}'(x_0)^{-1} \in L(X_1, X)$ and

$$|a| \|\mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0)\| \leq \eta.$$

(C2) $\|\mathcal{G}'(x_0)^{-1}(\mathcal{G}'(v) - \mathcal{G}'(x_0)) \leq \varphi_0(\|v - x_0\|)$ for all $v \in \Omega$.
(C3) Equation $\varphi_0(t) - 1 = 0$ has a minimal positive solution τ_0. Let $\Omega_0 = U(x_0, \tau_0)$.
(C4) There exist nondecreasing and continuous functions $\varphi : \Omega_0 \longrightarrow [0, \infty)$, $p : \Omega_0 \times \Omega_0 \longrightarrow [0, \infty)$, $q : \Omega_0 \times \Omega_0 \times \Omega_0 \longrightarrow [0, \infty)$ such that

$$\|\mathcal{G}'(x_0)^{-1}(\mathcal{G}'(v_1) - \mathcal{G}'(v))\| \leq \varphi(\|v_1 - v\|),$$

$$\|aI - A(v, v_1)\mathcal{G}'(v)\| \leq p(v, v_1),$$

$$\|A(v, v_1)\mathcal{G}(v) - U(v, v_1)\mathcal{G}(v_2)\| \leq q(v, v_1, v_2)$$

for all $v_1, v_2, v \in \Omega_0$.
(C5) Conditions of any of the Lemmas in Section 2 hold.
(C6) $U[x_0, \tau^*] \subset \Omega$.

It is worth noticing that if $v_1 = v - a\mathcal{G}'(v)^{-1}\mathcal{G}(v)$, the resulting (C4) conditions will have a tighter function $\bar{\varphi}$ than φ. Moreover, the same proof as that of Theorem 2 follows through (see the numerical Section).

Next, we provide the semilocal convergence.

Theorem 1. *Suppose conditions (C1)–(C6) hold. Then, sequence* $\{x_m\}$ *exists,* $\{x_m\} \in U[x_0, \tau^*]$ *and there exists* $x^* \in U[x_0, \tau^*]$ *so that* $\mathcal{G}(x^*) = 0$ *and*

$$\|x_m - x^*\| \leq \tau^* - t_m. \quad (17)$$

Proof. The iterates y_0, z_0, x_1 exist by (C1) and (2) for $m = 0$. Then, the estimate is derived by (C1)

$$\|y_0 - x_0\| = |a| \|\mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0)\| \leq \eta = s_0 - t_0 = \eta.$$

Thus, the iterate $y_0 \in U[x_0, \tau^*]$. Let $v \in U[x_0, \tau^*]$. Then, by (C2) and (C6)

$$\begin{aligned} &\|\mathcal{G}'(x_0)^{-1}(\mathcal{G}(v) - \mathcal{G}(x_0))\| \\ &\qquad \leq \varphi_0(\|v - x_0\|) \\ &\qquad \leq \varphi_0(\tau^*) < 1, \end{aligned}$$

leading to $\mathcal{G}'(v)^{-1} \in L(E_1, E)$ and

$$\|\mathcal{G}'(v)^{-1}\mathcal{G}'(x_0)\| \leq \frac{1}{1 - \varphi_0(\|v - x_0\|)} \quad (18)$$

by the Banach lemma on the linear operator with inverses [6]. Moreover, by (C4) and method (2), the following estimates are obtained in turn

$$\begin{aligned}
z_0 &= x_0 - a\mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0) \\
&\quad + a\mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0) - A_0\mathcal{G}(x_0) \\
&= y_0 + (a\mathcal{G}'(x_0)^{-1} - A_0)\mathcal{G}(x_0) \\
&= y_0 + (a\mathcal{G}'(x_0)^{-1} - A_0)\mathcal{G}'(x_0)\mathcal{G}'(x_0)^{1}\mathcal{G}(x_0) \\
&= y_0 - (aI - A_0\mathcal{G}'(x_0))(y_0 - x_0),
\end{aligned}$$

$$\begin{aligned}
\|z_0 - y_0\| &= \|(aI - A_0\mathcal{G}'(x_0))(_0 - x_0)\| \\
&\leq \|aI - A_0\mathcal{G}'(x_0)\|\|y_0 - x_0\| \\
&\leq p(s_0 - t_0) = u_0 - s_0,
\end{aligned} \tag{19}$$

$$\begin{aligned}
x_1 &= x_0 - B_0\mathcal{G}(x_0) \\
&= x_0 - A_0\mathcal{G}(x_0) + A_0\mathcal{G}(x_0) - B_0\mathcal{G}(z_0) \\
&= z_0 + A_0\mathcal{G}(x_0) - B_0\mathcal{G}(z_0), \\
\|x_0 - z_0\| &= \|A_0\mathcal{G}(x_0) - B_0\mathcal{G}(z_0)\| \\
&\leq q_0\|z_0 - x_0\| \\
&\leq q_0(u_0 - t_0) = t_1 - u_0,
\end{aligned} \tag{20}$$

$$\begin{aligned}
\mathcal{G}(x_1) &= \mathcal{G}(x_1) - \mathcal{G}(x_0) - a\mathcal{G}'(x_0)(y_0 - x_0) \\
&= \mathcal{G}(x_1) - \mathcal{G}(x_0) - \mathcal{G}'(x_0)(x_1 - x_0) \\
&\quad + \mathcal{G}'(x_0)((x_1 - y_0) + (y_0 - x_0)) - a\mathcal{G}'(x_0)(y_0 - x_0),
\end{aligned}$$

$$\begin{aligned}
\|\mathcal{G}'(x_0)^{-1}\mathcal{G}(x_1)\| &\leq \int_0^1 \varphi(\theta\|x_1 - x_0\|)d\theta\|x_1 - x_0\| \\
&\quad + (1 + \varphi_0(\|x_0 - x_0\|))\|x_1 - x_0\| \\
&\quad + |1 - a|(1 + \varphi_0(\|x_0 - x_0\|))\|y_0 - x_0\|.
\end{aligned} \tag{21}$$

Hence,

$$\begin{aligned}
\|y_1 - x_1\| &\leq \|\mathcal{G}'(x_1)^{-1}\mathcal{G}'(x_0)\|\|\mathcal{G}'(x_0)^{-1}\mathcal{G}(x_1)\| \\
&\leq \frac{\Theta}{1 - \varphi_0(t_1)} \\
&= s_1 - t_1,
\end{aligned} \tag{22}$$

where $\Theta = \int_0^1 \varphi(\theta(t_1 - t_0))d\theta(t_1 - t_0) + (1 + \varphi_0(t_1 - t_0))(t_1 - s_0) + |1 - a|(1 + \varphi_0(t_0 - t_0))(s_0 - t_0)$,

$$\begin{aligned}
\|z_0 - x_0\| &= \|z_0 - y_0 + y_0 - x_0\| \\
&\leq \|z_0 - y_0\| + \|y_0 - x_0\| \leq u_0 - s_0 + s_0 - t_0 \\
&= u_0 - t_0 \leq u_0 < \tau^*,
\end{aligned}$$

and

$$\|x_1 - x_0\| \leq \|x_1 - z_0\| + \|z_0 - x_0\| \leq t_1 - u_0 + u_0 - t_0 = t_1 \leq \tau^*.$$

Therefore,

$$\|y_m - x_m\| \leq s_m - t_m, \tag{23}$$

$$\|z_m - y_m\| \leq u_m - s_m, \tag{24}$$

$$\|x_{m+1} - z_m\| \leq t_{m+1} - u_m, \tag{25}$$

$$\|y_{m+1} - x_{m+1}\| \leq s_{m+1} - t_{m+1} \tag{26}$$

and

$$x_m, y_m, z_m, x_{m+1} \in U(x_0, \tau^*) \tag{27}$$

hold for $m = 0$. Estimates preceding (23) hold with indices $m, m+1$, replacing $0, 1$, respectively. Thus, the induction for estimates (23)–(27) is terminated.

It follows sequence $\{t_m\}$ is fundamental in X which is a Banach space, so $x^* = \lim_{m \longrightarrow +\infty} x_m$ exists and $x^* \in U[x_0, \tau^*]$. Then, considering the estimate (see (21))

$$\begin{aligned} \|\mathcal{G}'(x_0)^{-1}\mathcal{G}'(x_{m+1})\| &\leq \int_0^1 \varphi(\theta t_{m+1})d\theta(t_{m+1} - t_m) \\ &+(1 + \varphi_0(t_{m+1}))(t_{m+1} - s_m) \\ &+|1 - a|(1 + \varphi_0(t_m))(s_m - t_m). \end{aligned} \tag{28}$$

Therefore, $\mathcal{G}(x^*) = 0$ follows if $m \longrightarrow +\infty$ in (28). □

Proposition 1. *Suppose:*
(i) Point $x^* \in U(x_0, \tau_1)$ for some $\tau_1 > 0$ solves the equation $\mathcal{G}(x) = 0$.
(ii) Condition (C2) holds.
(iii) There exists $\tau_2 \geq \tau_1$ so that

$$\int_0^1 \varphi_0((1-\theta)\tau_2 + \theta\tau_1)d\theta < 1. \tag{29}$$

Let $\Omega_1 = U[x_0, \tau_2] \cap \Omega$. Then, x^* solves Equation (1) uniquely in Ω_1.

Proof. Let $y^* \in \Omega_1$ satisfy $\mathcal{G}(y^*) = 0$. Set $T = \int_0^1 \mathcal{G}'(x^* + \theta(y^* - x^*))d\theta$. By applying (29) and (C2), one obtains

$$\begin{aligned} \|\mathcal{G}'(x_0)^{-1}(T - \mathcal{G}'(x_0))\| &\leq \int_0^1 \varphi_0((1-\theta)\|\|y^* - x_0\| + \theta\|x^* - x_0\|)d\theta \\ &\leq \int_0^1 \varphi_0((1-\theta)\tau_2 + \theta\tau_1)d\theta < 1. \end{aligned}$$

Then, it follows that $y^* = x^*$ by $0 = \mathcal{G}(y^*) - \mathcal{G}(x^*) = T(y^* - x^*)$ and the implication $T^{-1} \in L(X_1, X)$. □

Remark 1. *(i) The point $\frac{\eta}{1-\lambda}$ which is in closed form may replace τ^* in the condition (C6).*
(ii) Proposition 2 is not using all conditions of Theorem 2. However, if all conditions are assumed then, set $\tau_1 = \tau^*$.

4. Local Convergence Analysis

Some auxiliary scalar functions and parameters are first introduced based on which the local convergence analysis of method (2) shall be given. Set $S = [0, +\infty)$. Let function $\psi_0 : S \longrightarrow \mathbb{R}$ be continuous and nondecreasing.

Suppose:
(H1) Equation $\psi_0(t) - 1 = 0$ has a smallest solution $r_0 \in S - \{0\}$. Set $S_1 = [0, r_0)$. Let function $\psi_1 : S_1 \longrightarrow \mathbb{R}$ be continuous and nondecreasing. Define function $g_1 : S_1 \longrightarrow \mathbb{R}$ by

$$g_1(t) = \frac{1}{1 - \psi_0(t)}[\int_0^1 \psi((1-\theta)t)d\theta + |1 - a|(1 + \int_0^1 \psi_0((1-\theta)t)d\theta)].$$

(H2) Equation

$$g_1(t) - 1 = 0$$

has a smallest solution $r_1 \in S - \{0\}$. Let $H_1 \geq 0$ be a parameter. Define function $g_2 : S_1 \longrightarrow \mathbb{R}$ by

$$g_2(t) = \frac{1}{1-\psi_0(t)}[\int_0^1 \psi((1-\theta)t)d\theta + H_1(1+\int_0^1 \psi_0((1-\theta)t)d\theta)].$$

(H3) Equation

$$g_2(t) - 1 = 0$$

has a smallest solution $r_2 \in S - \{0\}$. Let $H_2 \geq 0$ be a parameter. Define function $g_3 : S_1 \longrightarrow \mathbb{R}$ by

$$g_3(t) = \frac{1}{1-\psi_0(t)}[\int_0^1 \psi((1-\theta)t)d\theta + H_2(1+\int_0^1 \psi_0((1-\theta)t)d\theta)g_2(t)].$$

(H4) Equation

$$g_3(t) - 1 = 0$$

has a smallest solution $r_3 \in S - \{0\}$. The parameter r defined for $j = 1,2,3$ as

$$r = \min\{r_j\}. \tag{30}$$

is proven to be a radius of convergence for method (2) in Theorem 2. Set $S_1 = [0, r)$. In view of these definitions, we have that for all $t \in S_1$

$$0 \leq \psi_0(t) < 1, \tag{31}$$

and

$$0 \leq g_1(t) < 1. \tag{32}$$

Next, the relationship is given between the aforementioned functions and the operators appearing on the method (2). Consider the conditions.

Suppose:

(A1) There exists a solution $x^* \in \Omega$ such that $\mathcal{G}'(x^*)$ is invertible.

(A2) $\|\mathcal{G}'(x^*)^{-1}(\mathcal{G}(x) - \mathcal{G}'(x^*))\| \leq \psi_0(\|x - x^*\|)$ for all $x \in \Omega$. Set $U_0 = U(x^*, r_0) \cap \Omega$.

(A3) $\|\mathcal{G}'(x^*)^{-1}(\mathcal{G}'(x) - \mathcal{G}'(y))\| \leq \psi(\|x - y\|)$

$$\|\mathcal{G}'(x^*)^{-1}(I - \mathcal{G}'(x)A(x,y))\| \leq h_1(x,y),$$

$$\|\mathcal{G}'(x^*)^{-1}(I - \mathcal{G}'(x)B(x,y,z))\| \leq h_2(x,y,z),$$

$$h_1(x,y) \leq H_1$$

and

$$h_2(x,y,z) \leq H_2$$

for all $x, y \in U_0$, where functions $h_1 : U_0 \times U_0 \longrightarrow \mathbb{R}$ and $h_2 : U_0 \times U_0 \times U_0 \longrightarrow \mathbb{R}$ are continuous.

(A4) The parameter given by the Formula (30) exists and

(A5) $U[x^*, r] \subset \Omega$.

The main local convergence result follows for the method (2).

Theorem 2. *Suppose conditions (A1)–(A5) hold. Then, sequence $\{x_m\}$ produced by method (2) for $x_0 \in U(x^*, r) - \{x^*\}$ exists in $U(x^*, r)$, remains in $U(x^*, r)$ for all $m = 0, 1, 2, \ldots$ and converges to x^*. Moreover, the following items hold for all $m = 0, 1, 2, \ldots$*

$$\|y_m - x^*\| \leq \psi_1(\|x_m - x^*\|)\|x_m - x^*\| \leq \|x_m - x^*\| < r, \tag{33}$$

$$\|y_m - x^*\| \leq \psi_2(\|x_m - x^*\|)\|x_m - x^*\| \leq \|x_m - x^*\| \tag{34}$$

and

$$\|y_m - x^*\| \leq \psi_3(\|x_m - x^*\|)\|x_m - x^*\| \leq \|x_m - x^*\|, \tag{35}$$

where the functions $\varphi_j, j = 1,2,3$ are previously defined and the radius r is given by (3).

Proof. Mathematical induction is utilized to prove items (8)–(10). Let $v \in U(x^*, r) - \{x^*\}$. Let $w \in U(x^*, r)$ be arbitrary. By applying conditions (A_1) and (A_2)

$$\|\mathcal{G}'(x^*)^{-1}(\mathcal{G}'(x^*) - \mathcal{G}'(w))\| \leq \psi_0(\|x^* - w)\| \leq \psi_0(r) < 1. \tag{36}$$

Then, the linear operator $\mathcal{G}'(w)^{-1}$ exists and

$$\|\mathcal{G}'(w)^{-1}\mathcal{G}'(x^*)\| \leq \frac{1}{1 - \psi_0(\|x^* - w\|)}. \tag{37}$$

If $w = x_0$, then the iterative y_0 exists by method (2). It follows

$$y_0 - x^* = x_0 - x^* - \mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0) + (1-a)\mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0).$$

Then, by applying (A3) and (37) (for $w = x_0$)

$$\begin{aligned}\|y_0 - x^*\| &\leq (\frac{\int_0^1 \psi((1-\theta)\|x_0 - x^*\|)}{1 - \psi_0(\|x_0 - x^*\|)}d\theta \\ &\quad + |1-a|(1 + \int_0^1 \psi_0((1-\theta)\|x_0 - x^*\|)d\theta))\|x_0 - x^*\| \\ &\leq g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r. \end{aligned} \tag{38}$$

Hence, the iterate $y_0 \in U(x^*, r)$ and (33) is true for $m = 0$, where we also use the estimate

$$\mathcal{G}(x_0) = \mathcal{G}(x_0) - \mathcal{G}(x^*) = (\int_0^1 \mathcal{G}'(x^* + \theta(x_0 - x^*)) - \mathcal{G}'(x^*) + \mathcal{G}'(x^*)(x_0 - x^*))(x_0 - x^*)$$

so,

$$\begin{aligned}\|\mathcal{G}'(x^*)^{-1}\mathcal{G}(x_0)\| &\leq \|\mathcal{G}'(x^*)^{-1}(\mathcal{G}'(x^* + \int_0^1 \mathcal{G}'(x^*) + \theta(x_0 - x^*)d\theta - \mathcal{G}'(x^*))(x_0 - x^*)\| \\ &\leq (1 + \int_0^1 \psi_0(\theta\|x_0 - x^*\|)d\theta)\|x_0 - x^*\|\end{aligned}$$

Similarly, by the second substep of method (2), we can write

$$\begin{aligned} z_0 - x^* &= x_0 - x^* - A_0\mathcal{G}(x_0) = x_0 - x^* - \mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0) \\ &\quad + (\mathcal{G}'(x_0)^{-1} - A_0)\mathcal{G}(x_0) \\ &= x_0 - x^* - \mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0) + \mathcal{G}'(x_0)^{-1}(I - \mathcal{G}'(x_0)A_0)\mathcal{G}(x_0). \end{aligned}$$

By applying (A3), and (37) (for $w = x_0$), we obtain in turn that

$$\begin{aligned}\|z_0 - x^*\| &\leq \frac{\int_0^1 \psi((1-\theta)\|x_0 - x^*\|)d\theta\|x_0 - x^*\|}{1 - \psi_0(\|x_0 - x^*\|)} \\ &+ \frac{h_1(x_0, y_0)}{1 - \psi_0(\|x_0 - x^*\|)}(1 + \int_0^1 \psi_0((1-\theta)\|x_0 - x^*\|)d\theta)\|x_0 - x^*\| \\ &\leq g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\|. \end{aligned} \tag{39}$$

Thus, the iterate $z_0 \in U(x^*, r)$ and estimate (34) hold for $n = 0$. Then, again, by the third substep of method (2), we obtain:

$$\begin{aligned} x_1 - x^* &= x_0 - x^* - \mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0) + (\mathcal{G}'(x_0)^{-1} - B_0)\mathcal{G}(z_0) \\ &= x_0 - x^* - \mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0) + \mathcal{G}'(x_0)^{-1}(I - \mathcal{G}'(x_0)B_0)\mathcal{G}(z_0). \end{aligned}$$

Consequently,

$$\begin{aligned} \|x_1 - x^*\| &\leq \frac{\int_0^1 \psi((1-\theta)\|x_0 - x^*\|)d\theta\|x_0 - x^*\|}{1 - \psi_0(\|x_0 - x^*\|)} \\ &+ \frac{h_2(x_0, y_0, z_0)}{1 - \psi_0(\|x_0 - x^*\|)}(1 + \int_0^1 \psi_0(\theta\|z_0 - x^*\|d\theta)\|z_0 - x^*\| \\ &\leq g_3(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\|, \end{aligned} \quad (40)$$

where we also used (39) and

$$\begin{aligned} \mathcal{G}(z_0) &= \mathcal{G}(z_0) - \mathcal{G}(x^*) = \int_0^1 \mathcal{G}'(x^* + \theta(z_0 - x^*))d\theta(z_0 - x^*) \\ &= (\int_0^1 \mathcal{G}'(x^* + \theta(z_0 - x^*))d\theta - \mathcal{G}'(x^*) + \mathcal{G}'(x^*))(z_0 - x^*). \end{aligned}$$

Hence,

$$\|\mathcal{G}'(x^*)^{-1}\mathcal{G}(z_0)\| \leq (1 + \int_0^1 \psi_0(\theta\|z_0 - x^*\|)d\theta)\|z_0 - x^*\|.$$

It follows from estimate (40) that iterate x_1 is well defined and (35) holds for $m = 0$. Therefore, the induction for assertions (33)–(35) is completed if the iterates x_i, y_i, z_i, x_{i+1} are exchanged with the iterates x_0, y_0, z_0, x_1, respectively, in the previous calculations. Finally, from the calculation

$$\|x_{i+1} - x^*\| \leq u\|x_i - x^*\| < r, \quad (41)$$

where $u = \varphi_3(\|x_0 - x^*\| \in [0, 1)$, we obtain that $\lim_{i \longrightarrow +\infty} x_i = x^*$ and the iterate $x_{i+1} \in U(x^*, r)$. □

The uniqueness of the solution result follows.

Proposition 2. *Suppose: there exists a simple solution $x^* \in U(x^*, \rho) \subset \Omega$ of equation $\mathcal{G}(x) = 0$ for some $\rho > 0$ and (A2) holds. Furthermore, suppose equation $\psi_0(t) - 1 = 0$ has a smallest positive solution ρ_1. Set $\Omega_2 = U[x^*, \rho_1] \cap U(x^*, \rho)$. Then, the point x^* is the only solution of equation $\mathcal{G}(x) = 0$ in the region Ω_2.*

Proof. Let $y^* \in \Omega_2$ with $\mathcal{G}(y^*) = 0$. Let the linear operator $T = \int_0^1 \mathcal{G}'(x^* + \theta(y^* - x^*))d\theta$. Then, by applying condition (A3)

$$\|\mathcal{G}'(x^*)^{-1}(T - \mathcal{G}'(x^*))\| \leq \int_0^1 \psi_0(\theta\|y^* - x^*\|)d\theta < 1.$$

Hence, $y^* = x^*$ is implied by the inverse of T and the application $T(x^* - y^*) = \mathcal{G}(x^*) - \mathcal{G}(y^*) = 0 - 0 = 0$. gives $x^* - y^* = T^{-1}(0) = 0$. Therefore, we conclude that $y^* = x^*$. □

5. A Specialization of Method

Set $a = 1, A_m = \mathcal{G}'(x_m)$ and $B_m = \mathcal{G}'(x_m)$ for all $m = 0, 1, 2, \ldots$. Then, method (2) reduces to

$$\begin{aligned} y_m &= x_m - \mathcal{G}'(x_m)^{-1}\mathcal{G}(x_m) \\ z_m &= y_m - \mathcal{G}'(y_m)^{-1}\mathcal{G}(x_m) \\ x_{m+1} &= z_m - \mathcal{G}'(z_m)^{-1}\mathcal{G}(z_m). \end{aligned} \tag{42}$$

This is Newton's three-step fifth-order method, also called Traub's extended three-step method. It seems to be the most interesting special case of method (2) to study as an application.

Consider the popular choices:

Semilocal case:

$\psi_0(t) = L_0 t$ and $\psi(t) = Lt$. We can also set $p_m = p = 0$. However, for determining q_m and q, let us start with

$$\begin{aligned} \|\mathcal{G}'(x)^{-1}(\mathcal{G}(x) - \mathcal{G}(y))\| &= \|\mathcal{G}'(x)^{-1}\int_0^1 \mathcal{G}'(y + \theta(x-y))d\theta(x-y)\| \\ &\leq \|\mathcal{G}'(x)^{-1}\mathcal{G}(x)\| \\ &\quad \|\int_0^1 \mathcal{G}'(x)^{-1}(\mathcal{G}'(y+\theta(x-y))d\theta - \mathcal{G}'(x*) + \mathcal{G}'(x*))(x-y) \\ &\leq (1 + \frac{L\|y-x\|^2}{2(1 - L_0\|x - x_0\|)})\|y - x\| \end{aligned}$$

It follows that we can set

$$q_m = (1 + \frac{L\|u_m - t_m\|^2}{2(1 - L_0 t_m)})\|u_m - t_m\|).$$

Local case: $\psi_0(t) = l_0(t)$ and $\psi(t) = lt$. Then, we obtain $h_1 = h_2 = H_1 = H_2 = 0$.

These choices are used in the examples of the numerical section.

6. Numerical Examples

We verify the convergence criteria using method (42). Moreover, we compare the Lipschitz constants L_0, L, L_1, and m.

In particular, we used the first example to show that the ratio $\frac{L_0}{L_1}$ can be arbitrarily small.

Example 1. *Let $X = X_1 = \Omega = \mathbb{R}$. Define the function*

$$\lambda_1(x) = \gamma_0 x + \gamma_1 + \gamma_2 \sin \gamma_3 x,\ x_0 = 0,$$

where γ_j, $j = 0, 1, 2, 3$ are fixed parameters. It follows that for γ_3 large and γ_2 small, $\frac{L_0}{L_1}$ can be small (arbitrarily), so that $\frac{L_0}{L_1} \longrightarrow 0$.

The parameters L_0, L, K and L_1 are computed in the next example, where L_1 is the Lipschitz parameter on Ω used by Kantorovich [6], whereas K is the parameter replacing L if, as noted in Section 3, for $v \in \Omega$, we choose $v_1 = v - a\mathcal{G}'(v)^{-1}\mathcal{G}(v)$ in (C4). Moreover, the convergence conditions by Kantorovich [6] are compared to those of Lemma 1.

Example 2. *Let $X = X_1 = \mathbb{R}$. Define scalar function $\mathcal{G}$ on the interval $\Omega = U[v_0, 1 - \alpha]$ for $\alpha \in (0, \frac{1}{2})$ by*

$$\mathcal{G}(v) = v^3 - \alpha.$$

Pick $v_0 = 1$. Then, the estimates are $\eta = \frac{1-\alpha}{3}$,

$$\begin{aligned} |\mathcal{G}'(v_0)^{-1}(\mathcal{G}'(v) - \mathcal{G}'(v_0))| &= |v^2 - v_0^2| \\ &\leq |v + v_0||v - v_0| \leq (|v - v_0| + 2|v_0|)|v - v_0| \\ &= (1 - \alpha + 2)|v - v_0| = (3 - \alpha)|v - v_0|, \end{aligned}$$

for all $v \in \Omega$, so $L_0 = 3 - \alpha$, $\Omega_0 = U(v_0, \frac{1}{L_0}) \cap \Omega = U(v_0, \frac{1}{L_0})$,

$$\begin{aligned} |\mathcal{G}'(v_0)^{-1}(\mathcal{G}'(w) - \mathcal{G}'(v)| &= |w^2 - v^2| \\ &\leq |w + v||w - v| \leq (|w - v_0 + v - v_0 + 2v_0|)|w - v| \\ &= (|w - v_0| + |v - v_0| + 2|v_0|)|w - v| \\ &\leq (\frac{1}{L_0} + \frac{1}{L_0} + 2)|w - v| = 2(1 + \frac{1}{L_0})|w - v|, \end{aligned}$$

for all $v, w \in \Omega$ and so $K = 2(1 + \frac{1}{L_0})$.

$$\begin{aligned} |\mathcal{G}'(v_0)^{-1}(\mathcal{G}'(w) - \mathcal{G}'(v)| &= (|w - v_0| + |v - v_0| + 2|v_0|)|w - v| \\ &\leq (1 - \alpha + 1 - \alpha + 2)|w - v| = 2(2 - \alpha)|w - v|, \end{aligned}$$

for all $v, w \in \Omega$ and $L_1 = 2(2 - \alpha)$.

Notice that for all $\alpha \in (0, \frac{1}{2})$

$$L_0 < K < L_1.$$

Next, set $w = v - \mathcal{G}'(v)^{-1}\mathcal{G}(v), v \in \Omega$. Then, we have

$$w + v = v - \mathcal{G}'(v)^{-1}\mathcal{G}(v) + v = \frac{5v^3 + \alpha}{3v^2}.$$

Define function $\bar{\mathcal{G}}$ on the set $\Omega = [\alpha, 2 - \alpha]$ by

$$\bar{\mathcal{G}}(v) = \frac{5v^3 + \alpha}{3v^2}.$$

Then, we obtain by this definition that

$$\begin{aligned} \bar{\mathcal{G}}'(v) &= \frac{15v^4 - 6v\alpha}{9v^4\kappa} \\ &= \frac{5(v - \alpha)(v^2 + v\alpha + \alpha^2)}{3v^3}, \end{aligned}$$

with $s = \sqrt[3]{\frac{2\alpha}{5}}$ being the critical point of function $\bar{\mathcal{G}}$. Notice that $\alpha < s < 2 - \alpha$. It follows that this function is decreasing on the interval (α, p) and increasing on the interval $(\alpha, 2 - \alpha)$, since $v^2 + v\alpha + \alpha^2 > 0$ and $v^3 > 0$. Hence, we can set

$$K_2 = \frac{5(2 - \alpha)^2 + \alpha}{9(2 - \alpha)^2}$$

and

$$K_2 < L_0.$$

However, if $v \in \Omega_0 = [1 - \frac{1}{L_0}, 1 + \frac{1}{L_0}]$, then

$$L = \frac{5\varpi^3 + \alpha}{9\varpi^2},$$

where $\varpi = \frac{4-\alpha}{3-\alpha}$ and $K < K_1$ for all $\alpha \in (0, \frac{1}{2})$. Then, the Kantorovich criterion $2L_1\eta \le 1$ [6] is not satisfied for all $\alpha \in (0, \frac{1}{2})$. Therefore, there is no assurance that method (2) is convergent to $v^* = \sqrt[3]{\alpha}$.

Let us test the convergence criteria of Lemma 1 by selecting $\alpha = 0.4$. Then, we have the following Table 1, verifying the convergence condition (6) for $\tau_0 = \frac{1}{L_0}$.

Table 1. Real sequence (42).

n	1	2	3	4	5	6
u_i	0.2330	0.2945	0.3008	0.3009	0.3009	0.3009
s_i	0.2000	0.2896	0.3008	0.3009	0.3009	0.3009
t_{n+1}	0.2341	0.2946	0.3008	0.3009	0.3009	0.3009
$L_0 s_i$	0.5200	0.7530	0.7820	0.7824	0.7824	0.7824
$L_0 u_i$	0.6058	0.7658	0.7822	0.7824	0.7824	0.7824
$L_0 t_{i+1}$	0.6087	0.7659	0.7822	0.7824	0.7824	0.7824

Example 3. *Let $\Omega = U[0,1]$ for $X = X_1 = C[0,1]$. Then, the boundary value problem [4]*

$$\mu(0) = 0, \mu(1) = 1,$$

$$\mu'' = -\mu - \gamma\mu^2$$

is transformed as the integral equation

$$\mu(s) = s + \int_0^1 G(s, s_1)(\mu^3(s_1) + \gamma\mu^2(s_1))ds_1$$

where γ is a constant and $G(s, s_1)$ is due to Green's function given by

$$G(s, s_1) = \begin{cases} s_1(1-s), & s_1 \le s \\ s(1-s_1), & s < s_1. \end{cases}$$

Consider $\mathcal{G} : \Omega \longrightarrow X_1$ as

$$[\mathcal{G}(x)](s) = x(s) - s - \int_0^1 G(s, s_1)(x^3(s_1) + \gamma x^2(s_1))ds_1.$$

Let us pick $\mu_0(s) = s$ and $\Omega = U(\mu_0, \kappa_0)$. Then, clearly $U(\mu_0, \kappa_0) \subset U(0, \kappa_0 + 1)$, since $\|\mu_0\| = 1$. If $2\gamma < 5$. Then, conditions (H1)–(H4) are satisfied for

$$L_0 = \frac{2\gamma + 3\kappa_0 + 6}{8}, \; L = \frac{\gamma + 6\kappa_0 + 3}{4}.$$

Hence, $L_0 < L_1$.

The next two examples concern the local convergence of the method (2) and radii r_j, r computed using Formula (30) and the functions φ_j.

Example 4. *Consider $X = X_1 = C[0,1]$ and $\Omega = U[0,1]$. Consider $\mathcal{G} : \Omega \longrightarrow X_1$ given as*

$$\mathcal{G}(f)(x) = \varphi(x) - 5\int_0^1 x\tau f(\tau)^3 d\tau. \tag{43}$$

This definition gives

$$\mathcal{G}'(f(\xi))(x) = \xi(x) - 15\int_0^1 x\tau f(\tau)^2 \xi(\tau)d\tau, \text{ for all } \xi \in \Omega.$$

The max-norm is used. Then, since $x^* = 0$, conditions (A1)–(A5) hold, provided that $\ell_0 = 7.5$ and $\ell = 15$. Then, the radii are:

$$r = r_1 = 0.0533,\ r_2 = 0.1499, \text{ and } r_3 = 0.1660.$$

Example 5. *Let the system of differential equations*

$$\mathcal{G}_1'(\mu_1) = e^{\mu_1},\ \mathcal{G}_2'(\mu_2) = (e-1)\mu_2 + 1,\ \mathcal{G}_3'(\mu_3) = 1$$

with $\mathcal{G}_1(0) = \mathcal{G}_2(0) = \mathcal{G}_3(0) = 0$. *Let* $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3)$. *Let* $X = X_1 = \mathbb{R}^3, \Omega = U[0,1]$. *Then,* $x^* = (0,0,0)^{cr}$. *Let function* $\mathcal{G}$ *on* Ω *for* $\mu = (\mu_1, \mu_2, \mu_3)^{cr}$ *given as*

$$\mathcal{G}(\mu) = (e^{\mu_1} - 1, \frac{e-1}{2}\mu_2^2 + \mu_2, \mu_3)^{cr}.$$

Then, the derivative due to Fréchet is given by

$$\mathcal{G}'(\mu) = \begin{bmatrix} e^{\mu_1} & 0 & 0 \\ 0 & (e-1)\mu_2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This definition implies that $\mathcal{G}'(x^*) = I$. *Let* $\mu \in \mathbb{R}^3$ *with* $\mu = (\mu_1, \mu_2, \mu_3)^{cr}$. *Moreover, the norm for* $\Delta \in \mathbb{R}^3 \times \mathbb{R}^3$ *is*

$$\|\Delta\| = \max_{1 \le k \le 3} \sum_{i=1}^{3} \|\delta_{k,i}\|.$$

We need to verify the conditions (A1)–(A5). To achieve this, we study $\mathcal{G}(c) = e^c - 1$ *on* $\Omega = [-1,1]$, *so* $c^* = 1$, *hence* $\mathcal{G}'(c^*) = 1$, *and*

$$\begin{aligned} |\mathcal{G}'(c) - \mathcal{G}'(c^*)| &= |c + \frac{c^2}{2} + \ldots + \frac{c^j}{j!} + \ldots| \\ &= |1 + \frac{c-0}{2!} + \ldots + \frac{c-0)^{j-1}}{j!} + \ldots||c-0|. \end{aligned}$$

It follows that $\ell_1 = e - 1$. *Then,* $\Omega_1 = U(x^*, \frac{1}{e-1}) \cap \Omega = U(x^*, \frac{1}{e-1})$. *This time, we obtain*

$$|\mathcal{G}'(c) - \mathcal{G}'(c^*)| \le \ell_0 |c - 0|,$$

where

$$\ell_0 = 1 + \frac{1}{(e-1)2!} + \ldots + \frac{1}{(e-1)^{j-1} j!} + \ldots \approx 1.43 < \ell_1.$$

Then, we obtain for all $c \in \Omega_1$

$$\begin{aligned} |s| &= |c - \mathcal{G}'(c)^{-1}\mathcal{G}(c)| = |c - 1 + e^{-c}| \\ &= |\frac{(-c)^2}{2!} + \ldots + \frac{(-c)^j}{j!} + \ldots| \\ &= |c|(\frac{|c|}{2!} + \ldots + \frac{|c|^{j-1}}{j!} + \ldots) \le \frac{\ell_0 - 1}{e-1}. \end{aligned}$$

Moreover,

$$\begin{aligned} |\mathcal{G}'(s) - \mathcal{G}'(c^*)| &= |e^s - 1| \\ &\le |s|(1 + \frac{|s|}{2!} + \ldots + \frac{|s|^{j-1}}{j!} + \ldots) \\ &\le |c|\frac{\ell_0 - 1}{e-1}(1 + \frac{\ell_0 - 1}{(e-1)2!} + \ldots + \left(\frac{\ell_0 - 1}{e-1}\right)^{j-1}\frac{1}{j!} + \ldots) \\ &= \ell_2(c - 0), \end{aligned}$$

where $\ell_2 \approx 0.49 < \ell_1$. *We can set* $\ell_3 = \ell_2$.

Therefore, the computed radii are $r = r_1 = 0.2409 = r$, $r_2 = 0.3101$, and $r_3 = 0.3588$.

Discussion: It is important to mention some more applications. Notice that the branching of the solutions cannot be handled using the iterative method (2) since the existence of $G'(x_m)^{-1}$ is required in the first step. It is worth noticing that the present results can also apply to notable references by Singh et al. [10] and Vijayakumar et al. [20,21] involving the solution of differential equations. This is provided that the Banach space $X_1 = X$ is specialized to be the space of all Bockner integrable functions and the involved operator is defined as a Riemann–Liouville integral, Riemann–Liouville fractional derivative, or Caputo fractional derivative of a certain order in the interval $(1, 2]$ [10].

In the references [20,21], the evolution differential inclusions should be in Banach space. In particular, the control function should belong in $L^2(I, X)$, which is the Banach space of admissible functions with $X_1 = X$.

7. Conclusions

Sufficient conditions unify the convergence of generalized three-step methods. Their specializations provide a finer convergence analysis since smaller Lipschitz parameters and tighter real majorizing sequences are used than in [3,4,6,11,12,17,18].

More areas of application can be found in [3,4,6,9,19] and the references therein. These ideas can be immediately extended to include multistep as well as multipoint iterative methods along the same lines. This is the topic of future work.

Author Contributions: Conceptualization, S.R., I.K.A., S.G. and C.I.A.; methodology, S.R., I.K.A., S.G. and C.I.A.; software, S.R., I.K.A., S.G. and C.I.A.; validation, S.R., I.K.A., S.G. and C.I.A.; formal analysis, S.R., I.K.A., S.G. and C.I.A.; investigation, S.R., I.K.A., S.G. and C.I.A.; resources, S.R., I.K.A., S.G. and C.I.A.; data curation, S.R., I.K.A., S.G. and C.I.A.; writing—original draft preparation, S.R., I.K.A., S.G. and C.I.A.; writing—review and editing, S.R., I.K.A., S.G. and C.I.A.; visualization, S.R., I.K.A., S.G. and C.I.A.; supervision, S.R., I.K.A., S.G. and C.I.A.; project administration, S.R., I.K.A., S.G. and C.I.A.; funding acquisition, S.R., I.K.A., S.G. and C.I.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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