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Implementation of Two-Mode Gaussian States Whose Covariance Matrix Has the Standard Form

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Abstract: This paper deals with the covariance matrix (CM) of two-mode Gaussian states, which, together with the mean vector, fully describes these states. In the two-mode states, the (ordinary) CM is a real symmetric matrix of order 4; therefore, it depends on 10 real variables. However, there is a very efficient representation of the CM called the standard form (SF) that reduces the degrees of freedom to four real variables, while preserving all the relevant information on the state. The SF can be easily evaluated using a set of symplectic invariants. The paper starts from the SF, introducing an architecture that implements with primitive components the given two-mode Gaussian state having the CM with the SF. The architecture consists of a beam splitter, followed by the parallel set of two single–mode real squeezers, followed by another beam splitter. The advantage of this architecture is that it gives a precise non-redundant physical meaning of the generation of the Gaussian state. Essentially, all the relevant information is contained in this simple architecture.

Keywords: continuous quantum variables; Gaussian states; covariance matrix; standard form

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1. Introduction

In the last few years the development of quantum information has given a great attention to continuous-variable systems [1–3]. In particular, multimode states received great interest, since they may exhibit the entanglement, which represents a key resource in quantum computing and quantum protocols such as teleportation and cryptography. Among the theoretical work devoted to multimode quantum systems, Gaussian states and Gaussian transformations have attracted a lot of interest, because of their easy implementation and manipulation. Gaussian states cover a wide range applications in quantum information [2], among which is quantum key distribution with continuous variables [4] and protocols based on entanglement sharing [5,6]. Experiments [7] show also the robustness of the quantum coherence of Gaussian states in disturbed quantum channels.

Several tools can be used to efficiently describe Gaussian states. In the phase space, a Gaussian state is completely represented by the *covariance matrix* (CM) and the mean vector (often neglected in the analysis). For two-mode states the CM is a real symmetric matrix of order four; therefore, it depends on 10 real variables. These variables turn out to be somehow redundant, since a fundamental result on two-mode Gaussian states shows that the degrees of freedom of the CM can be reduced from 10 to 4 real variables [8–10]. This compact representation, called the *standard form (SF) of the covariance matrix*, is given by

$$\mathbf{V}_{sf} = \begin{bmatrix} a & 0 & c_+ & 0 \\ 0 & a & 0 & c_- \\ c_+ & 0 & b & 0 \\ 0 & c_- & 0 & b \end{bmatrix} \tag{1}$$

The SF is easily obtained from the ordinary CM by elementary symplectic operations and contains all the relevant information on the given two-mode Gaussian state—in particular, the information on entanglement.

Our approach starts from the SF of the covariance matrix, introducing an architecture that implements with primitive components any desired two-mode Gaussian which CM is expressed in the SF. The architecture consists of a beam splitter, followed by two real single–mode (local) squeezers, followed by another beam splitter. This set of primitive components is driven by two thermal states, as shown in Figure 1. Squeezing and quantum nonclassical correlations like the entanglement are fundamental characteristics of physics and show their effects even in fields very different form the framework of quantum information and communications, such as the quantum dynamics of cosmological perturbations [11]. In particular, the entanglement, for pure states, can arise from a simple device such as a beam-splitter by the nonclassical behavior of the input fields [12], obtained, for example, by squeezing. On the other hand, entanglement can be obtained in a beam-splitter by the non-monochromaticity of photons [13–15]. Here the most general case is considered, since at the beam splitter input we consider mixed states (thermal states). The physical origin of entanglement is not very relevant in our scheme, since the entanglement characterisics can be referred directly to the properties of the covariance matrix (see, for example, [9]) and we focus on the implementation with minimal resources of any desired covariance matrix in standard form.

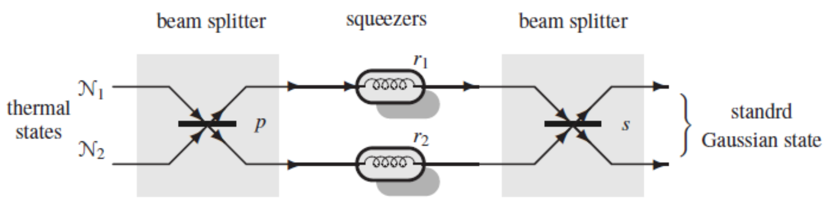

Figure 1. Implementation of a two-mode Gaussian state with the covariance matrix in *standard form*.

Note that the four variables of the CM in standard form have entropic and statistical meanings—namely, a and b are auto-correlations, and c_+ and c_- are cross–correlations—but not physical meanings. The implementation with primitive components adds a physical meaning to the SF. In fact, by varying the parameters of the physical devices, i.e., p, r_1, r_2 and s in Figure 1, and the thermal states, one can cover the whole class of two-mode Gaussian states. The main contribution of this work is the formulation of the fundamental features of Gaussian states through a universal architecture, which consists of the connection of a few elementary physical components (called *primitive components*) with the main goal of finding a two-mode Gaussian state whose covariance matrix has the standard form given by (1). By this procedure, one obtains easy formulas for the four values of the SF, for any choices of the architecture parameters. The other way around, given some desired properties of the state in terms of CM, we give easy formulas to set the parameters of the experimental implementation to build the desired two-mode Gaussian state.

The paper is organized as follows. The first part deals with the descriptions of Gaussian unitaries and Gaussian states arriving at the general implementation with primitive components. In the second part, the SF of the CM and its implementation anticipated in Figure 1 is presented. Specifically, in Section 2 we formulate the *Gaussian unitaries* and their decomposition into elementary unitaries and the derivation of *Gaussian states* according to Williamson's theorem. In Section 3, we show the implementation of Gaussian unitaries with primitive components based on the Bloch–Messiah reduction, where the Takagi factorization [16] is applied to the decomposition of the squeeze matrix. Although this implementation could be carried out for the general multimode case [17], here for simplicity we detail the architecture for two-mode states, and in Section 4 we evaluate the corresponding *ordinary* covariance matrix from the architecture with primitive components. As we will realize, the derivation based on this architecture leads to very simple formulas. Sections 5 and 6 deal with the SF of the CM and the symplectic algebra involved in the evaluation of the SF from the ordinary CM. Additionally, starting from the SF, the physical parameters $\mathcal{N}_1, \mathcal{N}_2, p, r_1, r_2, s$ appearing in the implementation of Figure 1 are evaluated. In Section 7, and in the Appendix B, we suggest several paths of the usage of the theory formulated in this paper.

2. Gaussian Unitaries and Gaussian States

A quantum transformation is *Gaussian* when *it transforms Gaussian states into Gaussian states* and it is called *Gaussian unitary* when it is performed according to a unitary map.

Any Gaussian unitary in the N-mode can be expressed as a combination of three fundamental unitary operators: displacement, rotation, and squeezer operators. Here we follow the representation based on the Bloch–Messiah (BM) reduction [1,18] that was recently reconsidered in [16,19] in terms of the Takagi factorization [20].

We remind the reader that the most general Gaussian unitary can be decomposed as the cascade of a rotation operator $R(\boldsymbol{\psi})$, a squeeze operator $S(\mathbf{r}_D)$ characterized by a diagonal matrix $\mathbf{r}_D$ with real entries, a rotation operator $R(\gamma)$, and finally, a displacement operator $D(\boldsymbol{\alpha})$, as illustrated in Figure 2.

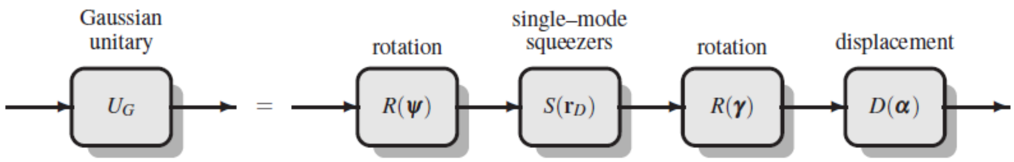

Figure 2. Decomposition of a Gaussian unitary according to the Bloch–Messiah (BM) reduction.

2.1. Degrees of Freedom

The specification of an arbitrary N–mode Gaussian unitary is provided by the matrices $\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{r}_D, \boldsymbol{\phi}$. Owing to their symmetry, the degrees of freedom are

$$2N^2 + 3N \text{ real variables} \qquad (\text{Gaussian unitary}) \tag{2}$$

2.2. Gaussian States

Gaussian states can be obtained from a Gaussian unitary driven by *thermal states*: we remind the reader that a thermal state corresponds to a state in thermal equilibrium and can be characterized by a mixture of Fock states [21], although not maximally mixed, thereby not with the maximum Von Neumann entropy.

In particular, we obtain pure Gaussian states when the thermal states degenerate to vacuum states, that is, the quantum states characterized by the lowest possible energy [22] and by zero photons.

A Gaussian state is completely characterized by the covariance matrix and the mean vector. We recall that, according to Williamson's theorem, the *covariance matrix* can always be written in the form

$$\mathbf{V} = \mathbf{S}\, \mathbf{V}^{\oplus}\, \mathbf{S}^{\mathrm{T}} \tag{3}$$

where $\mathbf{S}$ is an N-mode symplectic matrix and

$$\mathbf{V}^{\oplus} = \operatorname{diag}\,[n_1, n_1, \cdots, n_N, n_N] \tag{4}$$

corresponding to the tensor product of N thermal states, with average thermal photons $\mathcal{N}_k = \frac{1}{2}(n_k - 1)\,, \quad k = 1, 2, \ldots, N$. The quantities $\{n_k\}$ are referred to as the symplectic eigenvalues of the CM $\mathbf{V}$, and the matrix $\mathbf{S}$ performs the symplectic diagonalization of $\mathbf{V}$.

With reference to the decomposition of the Gaussian unitaries shown in Figure 2, we have that, when the architecture is driven by N input thermal states, at the output we obtain the most general N-mode Gaussian.

2.3. Gaussian States in the Two-Mode

Since in this work we focus on two-mode Gaussian states, we review here their specification, given by the complex matrices of the architecture of Figure 2

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \; \boldsymbol{\psi} = \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{12}^* & \psi_{22} \end{bmatrix}, \; \boldsymbol{\gamma} = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{12}^* & \gamma_{22} \end{bmatrix}, \; \mathbf{r}_D = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \tag{5}$$

and by two thermal noises $\mathcal{N}_1$, $\mathcal{N}_2$.

3. Implementation with Primitive Components

In order to evaluate the quantities involved in the two-mode CM from the parameters (5) we followed the implementation with primitive components that has been detailed in [23] and that could be carried out in the general N-mode [17]. Note that the displacement vector does not enter in the evaluation, and therefore, it will not be further considered. In the following, we summarize the main steps.

The *primitive components* are: (1) single–mode displacement, (2) single–mode rotation operators, briefly *shifters*, (3) single–mode real squeezers, and (4) beam splitters (BSs).

A shifter is specified by a phase $\beta \in [0, 2\pi)$, leading to the 1×1 exponential matrix $e^{i\beta}$. A single-mode squeezer is specified by the squeeze factor $r \in \mathbb{R}$. A free-phase BS is specified by the *rotation matrix*

$$\mathbf{U}_{\text{bs}} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \tag{6}$$

where $\tau = c^2$ determines the transmissivity and $s = \sqrt{1-\tau}$ the reflectivity.

Implementation in the Two-Mode

The objective is to implement the architecture of Figure 2 with a primitive component in the two-mode case. Note that the squeezer is already decomposed into primitive components, and the two-mode displacement operator $D(\boldsymbol{\alpha})$ is trivial, since it is given by two parallel single-mode displacement operators $D(\alpha_1)$ and $D(\alpha_2)$. For the rotation operators, we remind the reader that an arbitrary two-mode rotation operator with unitary matrix $\mathbf{U} = e^{i\boldsymbol{\Psi}} = [\rho_{hk} e^{i\psi_{hk}}]$ can be implemented by: (1) two phase shifters with phase γ_{11} and γ_{12}, (2) a BS with reflectivity $s = \rho_{12}$, and (3) a phase shifter with phase $\mu = \gamma_{22} - \gamma_{12}$. For the proof see [17,23].

Therefore, any Gaussian unitary can be implemented with primitive components as in Figure 3.

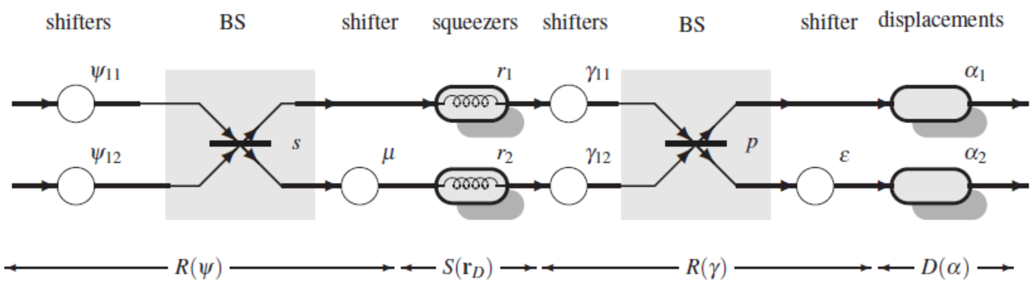

Figure 3. Implementation of a general two-mode Gaussian unitary with primitive components.

Note that this architecture generates the whole class of Gaussian unitaries in the two-mode. It is composed of 6 shifters, 2 BSs, 2 real squeezers, and 2 displacements, corresponding to a degrees of freedom of 14 real variables, as in (2).

The following objective is the generation of two-mode Gaussian states, which are obtained when the architecture of Figure 3 is driven by two thermal noises, as shown in Figure 4. Then the phase shifters ψ_{11} and ψ_{12} can be removed, since they are irrelevant when driven by thermal states. Finally, the final displacements are removed, because they do not contribute to the covariance matrix **V** (whose evaluation is the fundamental objective of this paper). Note that the number of real parameters in the architecture is 10.

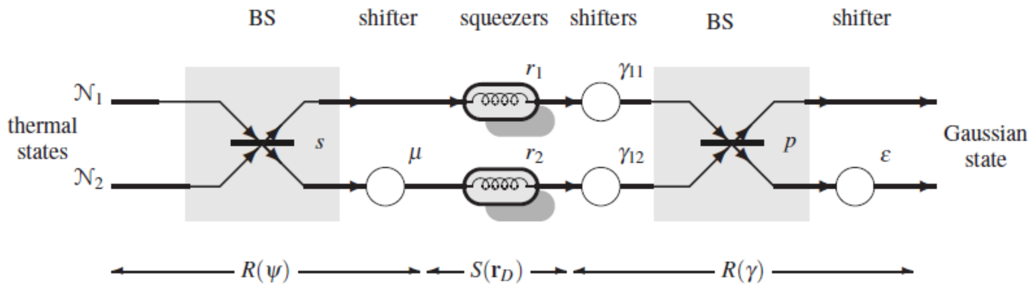

Figure 4. Scheme with primitive components for the generation of a general two-mode Gaussian state, starting from two thermal states, $\mathcal{N}_1$ and $\mathcal{N}_2$. The architecture does not contain the irrelevant initial rotations ψ_{11} and ψ_{12} and the final displacements α_1 and α_2, which do not influence the covariance matrix.

4. Evaluation of the Covariance Matrix

The architectures of Figure 3 and of Figure 4 represent the basis for the derivation of the CM for two-mode Gaussian states.

4.1. The Symplectic Matrix

A Gaussian unitary is fully described by the symplectic matrix (SM), neglecting the displacement. Here we consider the *real SM* **S**, where the phase-space variables are arranged in the form $\mathbf{X} := [q_1, p_1, q_2, p_2]^{\mathsf{T}}$. A *symplectic transformation* has the form

$$\mathbf{X} \to \mathbf{S}\mathbf{X} + \mathbf{d} \tag{7}$$

where **S** is a $2N \times 2N$ real matrix and $\mathbf{d} \in \mathbb{R}^{2N}$. The condition for preserving the commutation relations is

$$\mathbf{S}\,\mathbf{\Omega}\,\mathbf{S}^{\mathsf{T}} = \mathbf{\Omega} \tag{8}$$

where

$$\mathbf{\Omega} = \bigoplus_{i}^{N} \mathbf{\Omega} \quad \text{with} \quad \mathbf{\Omega} = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix} \tag{9}$$

a matrix **S** that verifies this condition is called *symplectic*.

Following the architecture of Figure 3, we found the global SM

$$\mathbf{S}_g = \mathbf{S}_{\text{rot}}(\boldsymbol{\gamma})\mathbf{S}_{\text{sq}}(\mathbf{r}_D)\mathbf{S}_{\text{rot}}(\boldsymbol{\psi}) \tag{10}$$

For the evaluation of the trigonometric matrices in the two-mode, we start from the exponential

$$e^{i\boldsymbol{\psi}} = \begin{bmatrix} c e^{i\,\psi_{11}} & s e^{i\,\psi_{12}} \\ -s e^{i\,(\psi_{11}+\mu)} & c e^{i\,(\psi_{12}+\mu)} \end{bmatrix} \tag{11}$$

then

$$\begin{aligned} \cos(\boldsymbol{\psi}) &= \begin{bmatrix} c\cos(\psi_{11}) & s\cos(\psi_{12}) \\ -s\cos(\psi_{11}+\mu) & c\cos(\psi_{12}+\mu) \end{bmatrix} \\ \sin(\boldsymbol{\psi}) &= \begin{bmatrix} c\sin(\psi_{11}) & s\sin(\psi_{12}) \\ -s\sin(\psi_{11}+\mu) & c\sin(\psi_{12}+\mu) \end{bmatrix} \end{aligned} \tag{12}$$

analogously

$$\cos(\gamma) = \begin{bmatrix} q\cos(\gamma_{11}) & p\cos(\gamma_{12}) \\ -p\cos(\gamma_{11}+\epsilon) & q\cos(\gamma_{12}+\epsilon) \end{bmatrix}$$
$$\sin(\gamma) = \begin{bmatrix} q\sin(\gamma_{11}) & p\sin(\gamma_{12}) \\ -p\sin(\gamma_{11}+\epsilon) & q\sin(\gamma_{12}+\epsilon) \end{bmatrix} \tag{13}$$

where p is the reflectivity of the second BS and $q = \sqrt{1-p^2}$.

For the central squeezer, considering that it is real and diagonal, we find

$$\begin{bmatrix} \cosh \mathbf{r} + \sinh \mathbf{r} & 0 \\ 0 & \cosh \mathbf{r} - \sinh \mathbf{r} \end{bmatrix} = \begin{bmatrix} e^{\mathbf{r}} & 0 \\ 0 & e^{-\mathbf{r}} \end{bmatrix} \text{ with } e^{\mathbf{r}} = \begin{bmatrix} e^{r_1} & 0 \\ 0 & e^{r_2} \end{bmatrix}; \tag{14}$$

then,

$$\mathbf{S}_{\text{sq}} = \mathbf{\Pi} \begin{bmatrix} e^{r} & 0 \\ 0 & e^{-r} \end{bmatrix} \mathbf{\Pi}^{\mathrm{T}} = \begin{bmatrix} e^{r} & 0 \\ 0 & e^{-r} \end{bmatrix} = \begin{bmatrix} e^{r_1} & 0 & 0 & 0 \\ 0 & e^{-r_1} & 0 & 0 \\ 0 & 0 & e^{r_2} & 0 \\ 0 & 0 & 0 & e^{-r_2} \end{bmatrix} \tag{15}$$

This completes the evaluation of the global symplectic matrix $\mathbf{S}$. Note that $\mathbf{S}$ depends on the 10 real variables ψ_{11}, $\psi_{1,2}$, s, μ, r_1, r_2, γ_{11}, γ_{12}, p, and ϵ. Note also that all the formulas are "radical free".

4.2. The Covariance Matrix (cm)

The covariance matrix $\mathbf{V}$ is evaluated from the global SM by adding the information on thermal noise (see (3))

$$\mathbf{V} = \mathbf{S}\mathbf{V}^{\oplus}\mathbf{S}^{\mathrm{T}}, \quad \mathbf{V}^{\oplus} = \text{diag}\,[n_1, n_1, n_2, n_2] \tag{16}$$

It is convenient to express the result in partitioned form of 2×2 blocks. Letting

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12}\, \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^{\mathrm{T}} & \mathbf{B} \end{bmatrix}, \tag{17}$$

one finds

$$\begin{aligned} \mathbf{A} &= n_1\mathbf{S}_{11}\mathbf{S}_{11}^{\mathrm{T}} + n_2\mathbf{S}_{12}\mathbf{S}_{12}^{\mathrm{T}} \\ \mathbf{B} &= n_1\mathbf{S}_{21}\mathbf{S}_{21}^{\mathrm{T}} + n_2\mathbf{S}_{22}\mathbf{S}_{22}^{\mathrm{T}} \\ \mathbf{C} &= n_1\mathbf{S}_{11}\mathbf{S}_{21}^{\mathrm{T}} + n_2\mathbf{S}_{12}\mathbf{S}_{22}^{\mathrm{T}} \end{aligned} \tag{18}$$

As evidenced by Figure 4, the ordinary CM $\mathbf{V}$ depends on 10 real parameters, namely,

$$n_1\,,\, n_2\,,\, s\,,\, \mu\,,\, r_1\,,\, r_2\,,\, \gamma_{11}\,,\, \gamma_{12}\,,\, p\,,\, \epsilon \tag{19}$$

The $\mathbf{S}_{ij}$ depend also on the phases ψ_{11}, ψ_{21}.

Remark 1. *Here the CM is evaluated from the implementation of the Gaussian states with primitive components. This approach, discussed in [23], has the advantage of a simple algebra, and it is completely radical-free. Other methods of evaluation start from the polar decomposition of the squeeze matrix, which leads to radicals of radicals.*

5. The Standard Form of the Covariance Matrix

Hereafter we deal with the *standard form of the CM* given by (1), a form of *symplectic invariant* of a two-mode Gaussian state, which depends only on four real parameters (recall that in the general case the CM depends on 10 real parameters). Hereafter the general form evaluated in the previous sections will be called *ordinary CM*, symbolized by $\mathbf{V}$.

It is important to state the following:

- For every two-mode Gaussian state having the ordinary CM $\mathbf{V}$, it is possible to obtain the corresponding standard form $\mathbf{V}_{sf}$ from $\mathbf{V}$ with a local symplectic transformation $\mathbf{S}_l$.
- The standard form $\mathbf{V}_{sf}$ contains all the relevant information on the Gaussian state, so that the transformation $\mathbf{V} \longrightarrow \mathbf{V}_{sf}$ may be considered as the removal of the redundancy in $\mathbf{V}$.

5.1. Properties of Symplectic Invariants

The correlations a, b, c_+, c_- are determined by the four local symplectic invariants

$$\det\mathbf{V} = (ab - c_+^2)(ab - c_-^2)\ , \ \ \det\mathbf{A} = a^2\ , \ \ \det\mathbf{B} = b^2\ , \ \ \det\mathbf{C} = c_+c_- \tag{20}$$

Therefore, the SF of any CM is unique (up to a common flip of the signs of c_- and c_+). For two-mode states, the uncertainty principle [2] can be recast as a constraint on the $Sp_{4,\mathbb{R}}$ invariants $\det\mathbf{V}$ and $\Delta(\mathbf{V}) = \det\mathbf{A} + \det\mathbf{B} + 2\det\mathbf{C}$, namely, $\Delta(\mathbf{V}) \leq 1 + \det\mathbf{V}$. For a two-mode Gaussian state, the symplectic eigenvalues will be named n_1 and n_2 with $n_2 \leq n_1$, where the Heisenberg uncertainty relation imposes $n_2 \geq 1$. The values of $n_{1,2}$ are related by a simple expression to the $Sp_{4,\mathbb{R}}$ invariants (invariants under global, two-mode symplectic operations) [24,25]:

$$2n_{1,2}^2 = \Delta(\mathbf{V}) \pm \sqrt{\Delta(\mathbf{V})^2 - 4\det\mathbf{V}} \tag{21}$$

The determinantal invariants $\det(\mathbf{V})$ and $\Delta(\mathbf{V})$ are simply related to the thermal noises by

$$\det(\mathbf{V}) = n_1^2 n_2^2\ , \quad \Delta(\mathbf{V}) = n_1^2 + n_2^2 \tag{22}$$

Meaning of the CM Entries According to Probability Theory

For the interpretation of the CM (ordinary or standard), it is convenient to recall the properties of the covariance matrix of two random variables x, y:

$$\mathbf{V}_{xy} = \begin{bmatrix} v_{xx} & v_{xy} \\ v_{xy} & v_{yy} \end{bmatrix}$$

The diagonal entries v_{xx} and v_{yy} represent, respectively, the *variances* of x and y, usually denoted by σ_x^2 and σ_y^2. The nondiagonal entry $v_{xy} = v_{yx}$ represents the *cross–covariance*, or simply the *covariance* between the two random variables. The CM entries verify the important inequality $0 \leq v_{xy}^2 \leq \sigma_x\sigma_y$. Then, the *normalized correlation* is introduced $c_{xy} := v_{xy}/(\sigma_x\sigma_y)$, with $|c_{xy}| \leq 1$ having the limit cases: (1) $c_{xy} = 0 \quad \rightarrow \quad v_{xy} = 0$: *uncorrelated variables* and (2) $c_{xy} = 1$: *completely correlated variables*. In this second case, the random variables are deterministically related in the form $y = a\,x + b$, with a and b real quantities ($a \neq 0$).

We now use the above ideas for the interpretation of the standard CM:

$$\mathbf{V}_{sf} = \begin{array}{c} \\ q_1 \\ p_1 \\ q_2 \\ p_2 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} q_1 & p_1 & q_2 & p_2 \end{array} \\ \begin{bmatrix} a & 0 & c_+ & 0 \\ 0 & a & 0 & c_- \\ c_+ & 0 & b & 0 \\ 0 & c_- & 0 & b \end{bmatrix} \end{array}$$

where q_1, p_1, q_2, p_2 are considered as random variables. We find that

1. q_1, p_1 are uncorrelated with the same variance $\sigma_{q_1}^2 = \sigma_{p_1}^2 = a$;
2. q_2, p_2 are uncorrelated with the same variance $\sigma_{q_2}^2 = \sigma_{p_2}^2 = b$;
3. q_1, q_2 have cross–covariance $v_{q_1q_2} = c_+$ and then normalized covariance $c_{q_1q_2} = \frac{c_+}{\sqrt{ab}} \quad \rightarrow \quad 0 \leq |c_+| \leq \sqrt{ab}$;

4. p_1, p_2 have cross–covariance $v_{p_1p_2} = c_-$ and then normalized covariance $c_{p_1p_2} = \frac{c_-}{\sqrt{ab}} \quad \to \quad 0 \le |c_-| \le \sqrt{ab}$;
5. (q_1, p_1), (q_1, p_2), (q_2, p_1) and (q_2, p_2) are uncorrelated pairs.

5.2. The Correlations $(A, B, C_\pm)$ from the Ordinary Cm $\mathbf{V}$

The standard variables $(a, b, c_\pm)$ can be obtained from the invariants of the ordinary CM.

Proposition 1. *From the blocks of the ordinary covariance matrix* $\mathbf{V}$ *(see (17)), one obtains*

$$\begin{aligned} a &= \sqrt{\det\mathbf{A}}\,, \quad b = \sqrt{\det\mathbf{B}} \\ c_+ &= \mp\frac{\sqrt{Z^2 - 4a^2b^2\det\mathbf{C}} - Z}{2ab} \\ c_- &= \pm\frac{\sqrt{Z^2 - 4a^2b^2\det\mathbf{C}} + Z}{2ab} \end{aligned} \tag{23}$$

where $Z = a^2b^2 - \det\mathbf{V} + \det\mathbf{C}$.

Indeed, c_+ and c_- are obtained by solving the equations $\det\mathbf{C} = c_+c_-$ and $\det\mathbf{V} = (ab - c_+^2)(ab - c_-^2)$.

5.3. The Standard Form Ii (Sf–Ii)

Another form of CM sometimes considered in the literature [9] is the *standard form II*

$$\mathbf{V}_{sf,II} = \begin{bmatrix} a_1 & 0 & c_1 & 0 \\ 0 & a_2 & 0 & c_2 \\ c_1 & 0 & b_1 & 0 \\ 0 & c_2 & 0 & b_2 \end{bmatrix} \tag{24}$$

which depends on six real variables. This form will be very useful in our investigation.

It is easy to obtain the SF from the SF-II with a local symplectic matrix, as illustrated in Figure 5. In this context, $\mathbf{V}_{sf}$ is denoted by $\mathbf{V}_{sf,I}$

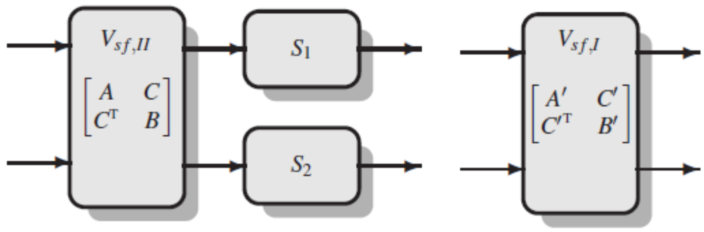

Figure 5. The local symplectic matrices $\mathbf{S}_1$ and $\mathbf{S}_2$ that provide the transformation of the SF-II to the SF: $\mathbf{V}_{sf,II} \to \mathbf{V}_{sf,I}$.

Proposition 2. *The equalization of the blocks* $\mathbf{A}$ *and* $\mathbf{B}$ *of the SF-II is obtained with the symplectic matrices*

$$\mathbf{S}_1 = \begin{bmatrix} e^{R_1} & 0 \\ 0 & e^{-R_1} \end{bmatrix}, \quad \mathbf{S}_2 = \begin{bmatrix} e^{R_2} & 0 \\ 0 & e^{-R_2} \end{bmatrix} \tag{25}$$

where R_1 *and* R_2 *are determined by*

$$e^{4R_1} = a_2/a_1 \text{with } a = (a_1a_2)^{1/2}, \quad e^{4R_2} = b_2/b_1 \text{with } b = (b_1b_2)^{1/2}$$

and leads to the equalized blocks $\mathbf{A}' = a\,\mathbf{I}_2$, $\mathbf{B}' = b\,\mathbf{I}_2$. *The block* $\mathbf{C}$ *becomes*

$$\mathbf{C}' = \mathbf{S}_1\mathbf{C}\mathbf{S}_2 = \begin{bmatrix} c_1e^{R_1+R_2} & 0 \\ 0 & c_2e^{-R_1-R_2} \end{bmatrix} := \begin{bmatrix} c_+ & 0 \\ 0 & c_- \end{bmatrix} \tag{26}$$

6. Gallery of Covariance Matrices and Classification

We recall that our main target is *finding a two-mode Gaussian state whose ordinary CM has the standard form*. A state having this property will be called the **standard Gaussian state**. Before we discuss the forms of interest encountered in the solution of our problem, we believe it will be convenient to discuss several forms of CMs, which are collected in Table 1.

Table 1. Forms of covariance matrices related to the standard form (SF).

Type	Covariance Matrix	Degrees of Fredom
general	$\mathbf{V} = \begin{bmatrix} a_{11} & a_{12} & c_{11} & c_{12} \\ a_{12} & a_{22} & c_{12} & c_{22} \\ c_{11} & c_{12} & b_{11} & b_{12} \\ c_{12} & c_{22} & b_{12} & b_{22} \end{bmatrix}$	10 real variables
standard form II	$\mathbf{V}_{sf}^{II} = \begin{bmatrix} a_1 & 0 & c_1 & 0 \\ 0 & a_2 & 0 & c_2 \\ c_1 & 0 & b_1 & 0 \\ 0 & c_2 & 0 & b_2 \end{bmatrix}$	6 real variables
standard form (SF)	$\mathbf{V}_{sf} = \begin{bmatrix} a & 0 & c_+ & 0 \\ 0 & a & 0 & c_- \\ c_+ & 0 & b & 0 \\ 0 & c_- & 0 & b \end{bmatrix}$	4 real variables
SF lateral symmetric	$\mathbf{V}_{sf}^{LS} = \begin{bmatrix} a & 0 & c & 0 \\ 0 & a & 0 & c \\ c & 0 & b & 0 \\ 0 & c & 0 & b \end{bmatrix}$	3 real variables
SF lateral antisymmetric	$\mathbf{V}_{sf}^{LA} = \begin{bmatrix} a & 0 & c & 0 \\ 0 & a & 0 & -c \\ c & 0 & b & 0 \\ 0 & -c & 0 & b \end{bmatrix}$	3 real variables

The first form is the ordinary CM, where the submatrices $\mathbf{A}$ and $\mathbf{B}$ are symmetric, and since the matrix $\mathbf{V}$ itself is real symmetric, the degrees of freedom are 10 real variables. In the second form, called standard form II in [9], the 2×2 submatrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are diagonal and the degrees of freedom are six real variables. The central form is the standard covariance with degrees of freedom of four real variables. The last two forms represent special cases of the SF in which $c_- = c_+$ or $c_- = -c_-$ with a reduction in the degrees of freedom to three real variables.

A more stringent classification will be useful for the SF:

- *Full SF*: is the class obtained by imposing the conditions $a \neq b$, $|c_-| \neq |c_+|$.
- *Lateral–symmetric SF*: is the class in which $a \neq b$, $c_- = c_+$.
- *Lateral–antiymmetric SF*: is the class in which $a \neq b$, $c_= - c_+$.

The classification is transferred to Gaussian states, e.g., *lateral–symmetric Gaussian state*. We need also suitable terms for the variables:

- *standard variables*: (a, b, c_+, c_-)
- *standard II variables*: $(a_1, a_2, b_1, b_2, c_1, c_2)$
- *physical variables*: $(n_1, n_2, s, \mu, r_1, r_2, \gamma_{11}, \gamma_{12}, p, \epsilon)$

We also recall from [10] that a lateral antisymmetric state is called *symmetric* and that pure states are always symmetric.

7. Two Fundamental Cases

In this section, we develop two fundamental cases. The first one (EPR) is important mainly for historical reasons. The second one, where all phases are set to zero, represents the starting point to solve our main task. In both cases we evaluate the ordinary CM and the standard form.

7.1. EPR State with Noise

The EPR unitary is a squeezing with the following matrix:

$$\mathbf{z} = \begin{bmatrix} 0 & e^{i\theta} r_0 \\ e^{i\theta} r_0 & 0 \end{bmatrix}$$

where we set the squeeze phase to zero: $\theta = 0$. To get this unitary, the architecture must have the following physical variables $r_1 = r_2 = r_0$, balanced BSs $p = q = s = c = 1/\sqrt{2}$ and the phases $\mu = \pi/2$, $\gamma_{11} = -\pi/4$, $\gamma_{12} = \pi/4$, $\epsilon = -\pi/2$.

The blocks of the CM result in

$$\mathbf{A} = \begin{bmatrix} n_1 \cosh^2(r_0) + n_2 \sinh^2(r_0) & 0 \\ 0 & n_1 \cosh^2(r_0) + n_2 \sinh^2(r_0) \end{bmatrix}$$
$$\mathbf{B} = \begin{bmatrix} n_2 \cosh^2(r_0) + n_1 \sinh^2(r_0) & 0 \\ 0 & n_2 \cosh^2(r_0) + n_1 \sinh^2(r_0) \end{bmatrix}$$
$$\mathbf{C} = \begin{bmatrix} \cosh(r_0)(n_1 + n_2) \sinh(r_0) & 0 \\ 0 & -\cosh(r_0)(n_1 + n_2) \sinh(r_0) \end{bmatrix}$$

in agreement with the result of [10].

The form is *lateral antisymmetric* without the introduction of equalization. The physical variables had degrees of freedom of three real variables given by n_1, n_2, r_0.

The implementation is a special case of the general architecture of Figure 4, obtained with the values of the physical variables indicated above. With free variables n_1, n_2, r_0 running in their ranges, this architecture *generates the whole class of antisymmetric standard states*. This means that all two-mode Gaussian states having the invariants that verify the condition $c_- = -c_+$ can be studied as "EPR state with noise".

From a practical viewpoint, one may proceed as follows: given a two-mode Gaussian state, one evaluates from the symplectic invariant the standard variables a, b, c_+, c_-. If the antisymmetric condition $c_- = -c_+$ is verified, the study can proceed with the present architecture, but the corresponding physical variables n_1, n_2, r_0 remain to be found. To solve this problem, we use the equations

$$\begin{aligned} a &= n_1 \cosh^2(r_0) + n_2 \sinh^2(r_0) \\ b &= n_2 \cosh^2(r_0) + n_1 \sinh^2(r_0) \\ c_+ &= \cosh(r_0)(n_1 + n_2) \sinh(r_0) \end{aligned}$$

whose solution is

$$\begin{aligned} n_{1,2} &= \frac{1}{2}\left(\sqrt{(a+b)^2 - 4c_+^2} \pm (a-b)\right) \\ e^{2r_0} &= \frac{\sqrt{a+b+2c_+}}{\sqrt{a+b-2c_+}} \end{aligned}$$

7.2. Cases Obtained by Setting All the Phases to Zero

In the general architecture of Figure 4, we set all the phases to zero: $\mu = \gamma_{11} = \gamma_{12} = \epsilon = 0$. Then, we find that the CM has the SF II given by (24), where the six variables in the diagonal of each sub-block result in

$$
\begin{aligned}
a_1 &= n_1(e^{r_1}cq - e^{r_2}ps)^2 + (e^{r_2}cp + e^{r_1}qs)^2 n_2 \\
a_2 &= n_1\left(e^{-r_1}cq - e^{-r_2}ps\right)^2 + \left(e^{-r_2}cp + e^{-r_1}qs\right)^2 n_2 \\
b_1 &= n_1(e^{r_1}cp + e^{r_2}qs)^2 + n_2(e^{r_2}cq - e^{r_1}ps)^2 \\
b_2 &= n_1\left(e^{-r_1}cp + e^{-r_2}qs\right)^2 + n_2\left(e^{-r_2}cq - e^{-r_1}ps\right)^2 \\
c_1 &= n_1(qce^{r_1} - pse^{r_2})(-pce^{r_1} - qse^{r_2}) + n_2(qse^{r_1} + pce^{r_2})(-pse^{r_1} + qce^{r_2}) \\
c_2 &= n_1\left(qce^{-r_1} - pse^{-r_2}\right)\left(-pce^{-r_1} - qse^{-r_2}\right) + n_2\left(qse^{-r_1} + pce^{-r_2}\right)\left(-pse^{-r_1} + qce^{-r_2}\right)
\end{aligned}
\tag{27}
$$

To get the standard form, an equalization according to Proposition 2 is needed with the symplectic matrices given by (25), where

$$
e^{4R_1} = a_2/a_1\,, \quad e^{4R_2} = b_2/b_1 \tag{28}
$$

the standard variables result in (see (20) and (26))

$$
\begin{aligned}
a &= \sqrt{a_1 a_2}\,, & b &= \sqrt{b_1 b_2} \\
c_+ &= c_1\, e^{R_1+R_2}\,, & c_- &= c_2\, e^{-(R_1+R_2)}
\end{aligned}
\tag{29}
$$

The implementation has been anticipated in Figure 1.

To evaluate the physical variables n_1, n_2, r_1, r_2 from the standard variables a, b, c_+, c_-, the four equations are given by (29). Perhaps it is impossible to solve the system in a closed form, due to the complication coming from the equalization. However, we introduce a procedure that avoids the equalization and solves the problem in a closed form. Note that the solution is not unique, but we will give a minimal solution, indeed with four degrees of freedom.

We proceed in two steps: first we get the physical variables from the standard variables II and then the physical variables from the standard variables I, a procedure echoed from [10].

7.3. Physical Variables from the Standard Variables II

We work on the SF-II relations (27) in order to evaluate the physical variables n_1, n_2, r_1, r_2, p, s from the SF-II variables $a_1, a_2, b_1, b_2, c_1, c_2$. We first evaluate the number of thermal photons using (21); that is,

$$
n_{1,2} = \sqrt{\frac{\Delta(\mathbf{V}) \pm \sqrt{\Delta(\mathbf{V})^2 - 4\det V}}{2}} \tag{30}
$$

where

$$
\det\mathbf{V} = -a_2 b_2 c_1^2 - a_1 b_1 c_2^2 + a_1 a_2 b_1 b_2 + c_2^2 c_1^2\,, \quad \Delta\mathbf{V} = a_1 a_2 + b_1 b_2 + 2 c_1 c_2
$$

Hence, the symplectic eigenvalues corresponding to the number of photons of the input thermal states are obtained independently of the other physical variables. The ambiguity between n_1 and n_2 can be solved in the following.

Next, we introduce the ancillary variables:

$$
X = c^2\, n_1 + s^2\, n_2 \qquad\qquad Y = s^2\, n_1 + c^2\, n_2 \tag{31}
$$

and Equations (27) become

$$\begin{aligned}
a_1 &= q^2\, e^{2r_1}\, X + p^2\, e^{2r_2}\, Y - 2\frac{cs}{1-2s^2}\, pq\, e^{r_1+r_2}(X-Y) && (32)\\
a_2 &= q^2\, e^{-2r_1}\, X + p^2\, e^{-2r_2}\, Y - 2\frac{cs}{1-2s^2}\, pq\, e^{-r_1-r_2}(X-Y) && (33)\\
b_1 &= p^2\, e^{2r_1}\, X + q^2\, e^{2r_2}\, Y + 2\frac{cs}{1-2s^2}\, pq\, e^{r_1+r_2}(X-Y) && (34)\\
b_2 &= p^2\, e^{-2r_1}\, X + q^2\, e^{-2r_2}\, Y + 2\frac{cs}{1-2s^2}\, pq\, e^{-r_1-r_2}(X-Y) && (35)\\
c_1 &= -pq\left(X\, e^{2r_1} - Y\, e^{2r_2}\right) + \frac{cs}{1-2s^2}(p^2-q^2)e^{r_1+r_2}(X-Y) && (36)\\
c_2 &= -pq\left(X\, e^{-2r_1} - Y\, e^{-2r_2}\right) + \frac{cs}{1-2s^2}(p^2-q^2)e^{-r_1-r_2}(X-Y) && (37)
\end{aligned}$$

Note that from (31), one gets

$$X+Y = n_1+n_2\,, \quad X-Y = (c^2-s^2)(n_1-n_2) \tag{38}$$

Considering (38), easy algebra from (32)–(37) leads to

$$\begin{aligned}
(a_1+b_1)\, e^{-R} + (a_2+b_2)\, e^{R} &= 2(n_1+n_2)\cosh(\Delta r) && (39)\\
(a_1-b_1)\, e^{-R} - (a_2-b_2)\, e^{R} &= 2(q^2-p^2)(n_1+n_2)\sinh(\Delta r) && (40)\\
c_1\, e^{-R} - c_2\, e^{R} &= -2pq(n_1+n_2)\sinh(\Delta r) && (41)\\
(a_1+b_1)\, e^{-R} - (a_2+b_2)\, e^{R} &= 2(c^2-s^2)(n_1-n_2)\sinh(\Delta r) && (42)
\end{aligned}$$

where

$$R = r_1+r_2\,, \quad \Delta r = r_1 - r_2$$

Now the unknown variables are R, Δr, p and s. In Appendix A we solve the system of Equations (39)–(42).

7.4. Physical Variables from the Standard Variables

The previous procedure to obtain the physical parameters from the CM SF-II works also for the SF, with the setting

$$a_1 = a_2 = a\,, \quad b_1 = b_2 = b \tag{43}$$

Then, Equations (39)–(42) reduce to

$$\begin{aligned}
(a+b)\, \cosh(R) &= \cosh(\Delta r)(n_1+n_2) && (44)\\
(a-b)\, \sinh(R) &= (p^2-q^2)\sinh(\Delta r)(n_1+n_2) && (45)\\
c_1\, e^{-R} - c_2\, e^{R} &= -2pq(n_1+n_2)\sinh(\Delta r) && (46)\\
(a+b)\, \sinh(R) &= -\sinh(\Delta r)(c^2-s^2)(n_1-n_2) && (47)
\end{aligned}$$

Now the degrees of freedom are reduced to four, instead of six as in the case of SF-II, and the equations become redundant. In fact, it is easy to show that in the case (43) from (39)–(42), some algebra gives

$$-(c^2-s^2)\frac{n_1-n_2}{n_1+n_2} = \frac{\tanh(R)}{\tanh(\Delta r)}\,, \quad \frac{a-b}{a+b} = (q^2-p^2)(c^2-s^2)\frac{n_1+n_2}{(n_1-n_2)} \tag{48}$$

Below we can see the solutions in a convenient form, where the subsequent requires the knowledge of the former.

7.4.1. Thermal Photon Numbers

The photon numbers were evaluated separately in the previous subsection. Now they are given by

$$n_{1,2} = \frac{1}{\sqrt{2}}\sqrt{\Delta(\mathbf{V}) \pm \sqrt{\Delta(\mathbf{V})^2 - 4\det V}} \tag{49}$$

where

$$\det\mathbf{V} = \left(ab - c_+^2\right)\left(ab - c_-^2\right), \quad \Delta(\mathbf{V}) = a^2 + b^2 + 2c_+c_-$$

7.4.2. Squeeze Parameters

The combination of (44) to (47) gives (see Appendix A)

$$e^{2R} = \frac{n_1 n_2}{ab - c_-^2} \quad \to \quad R = \frac{1}{2}\log\left(\frac{n_1 n_2}{ab - c_-^2}\right)$$

and then from (44)

$$\cosh(\Delta r) = \frac{(a+b)\cosh(R)}{n_1 + n_2}$$

and

$$r_1 = \frac{1}{2}(R + \Delta r), \quad r_2 = \frac{1}{2}(R - \Delta r)$$

7.4.3. BS Parameters

From (45) and (47)

$$p = \frac{\sqrt{(a-b)\sinh(R) + (n_1+n_2)\sinh(T)}}{\sqrt{2}\sqrt{(n_1+n_2)\sinh(\Delta r)}} \tag{50}$$

$$s = \frac{\sqrt{(a+b)\sinh(R) + (n_1-n_2)\sinh(T)}}{\sqrt{2}\sqrt{(n_1-n_2)\sinh(\Delta r)}} \tag{51}$$

A plot of the standard variables as a function of the physical variables is shown in Figure 6.

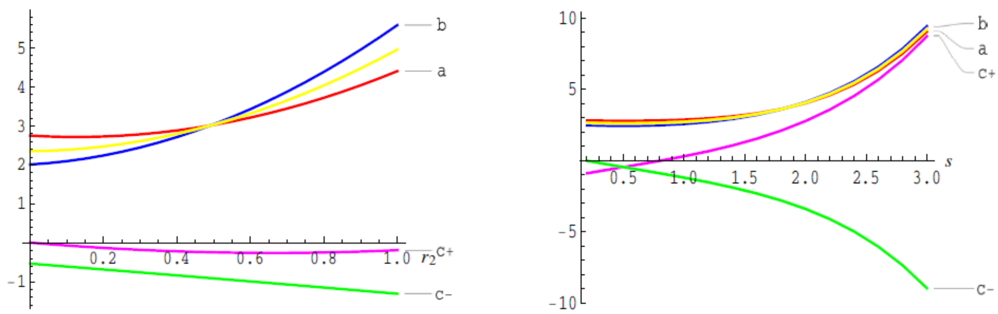

Figure 6. Left: The standard variables (a, b, c_+, c_-) as a function of r_2, for $s = 0.3$, $n_1 = 3.1$, $n_2 = 2.1$. Right: The standard variables (a, b, c_+, c_-) as a function of s, for $r_2 = 0.7$, $n_1 = 3.1$, $n_2 = 2.1$.

Examples of plots of the physical variables as functions of the standard variables are shown in Figure 7 and in Figure 8. In Figure 7, the physical variables (n_1, n_2) and (r_1, r_2) are presented as functions of the standard variables, for $a = 2.5$, $b = 2.8$, $c_- = 1.35$.

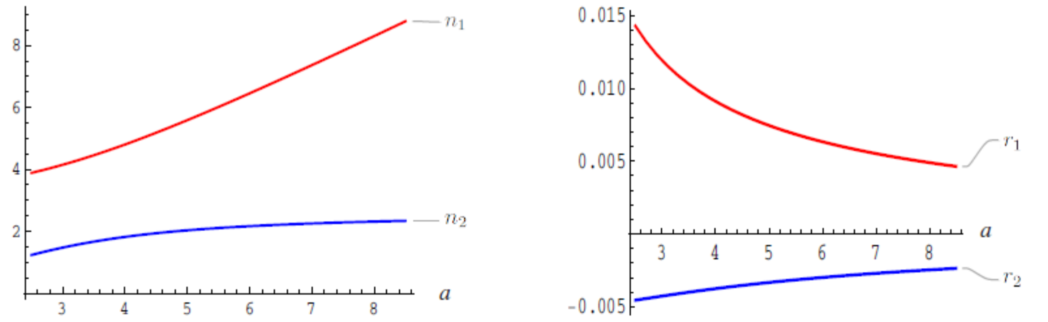

Figure 7. The physical variables (n_1, n_2) **(left)** and (r_1, r_2) **(right)** as functions of a for $b = 2.62$, $c_+ = 1.29$, $c_- = 1.36$.

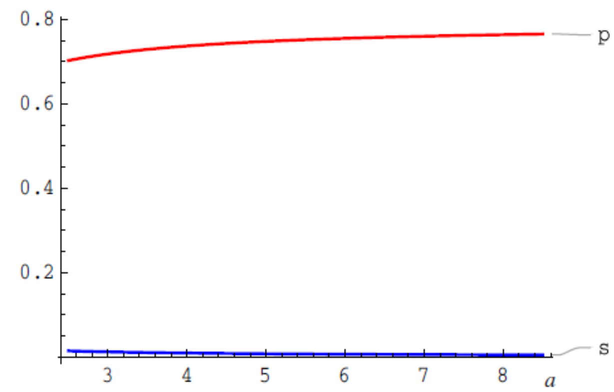

Figure 8. The physical variables (p, s) (as functions of a) for $b = 2.62$, $c_+ = 1.29$, $c_- = 1.36$.

In Figure 8, the physical variables (p, s) are shown as a function of a for $b = 2.62$, $c_+ = 1.29$, $c_- = 1.36$.

Another plot of the physical variables (n_1, n_2) and (r_1, r_2) as functions of the standard variables are shown in Figure 9.

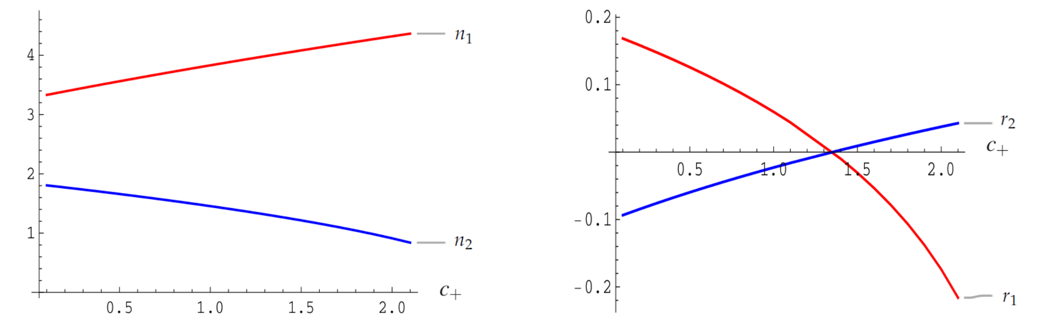

Figure 9. The physical variables (n_1, n_2) (left) and (r_1, r_2) (right) as functions of c_+ for $a = 2.5$, $b = 2.8$, $c_- = 1.35$.

8. Conclusions

Gaussian states and transformations are fundamental for continuous-variable systems and in general for quantum information. However, the characterization of Gaussian states is usually developed by employing heavy algebra (*algebraic approach*), which often removes the attention from the physical meanings of parameters in the the quantities involved, such as the covariance matrix.

Here we have developed the details of a *structural approach* in which the algebra is reduced to the minimum, while the attention is focused on the implementation architecture, which can serve also as the basis for experimental setups. This architecture provides the way to generate the whole class of two-mode Gaussian states having a covariance matrix in standard form, which retains all the characteristics of the two-mode Gaussian state, for example, about the entanglement. Moreover, this architecture to derive all the classes of two–mode Gaussian states is *minimal*, in that it consists of only two beam splitter and two local single-mode squeezers.

The expression of all the physical parameters of the architecture to obtain any desired CM was presented for both SF-I and SF-II. Given the parameters of the structural approach, the expression of the elements of the CM was given by simple expressions.

Note that the proposed architecture technique can be extended to general multimode Gaussian functions, with the unavoidable complication of an increased order. This paper gives the complete basis for this extension.

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Appendix A. Solution of the System (39) to (42)

We first note that:

- it is not a limitation to assume that in the beam-splitters $p, q, c, s \geq 0$
- $(q^2 - p^2)^2 = 1 - (2pq)^2$
- $\cosh^2(\Delta r) - \sinh^2(\Delta r) = 1$

Step 1—Find R from (39)–(41)

From (39)–(41) one can find

$$\begin{aligned} (a_1 + b_1)\, e^{-R} + (a_2 + b_2)\, e^R &= \sqrt{4(X+Y)^2 + Z^2} && \text{(A1)} \\ (a_1 - b_1)\, e^{-R} - (a_2 - b_2)\, e^R &= (q^2 - p^2) Z && \text{(A2)} \\ c_1\, e^{-R} - c_2\, e^R &= -pqZ && \text{(A3)} \\ c_1\, e^{-R} + c_2\, e^R &= -2pq(X-Y)\cosh(\Delta r) - 2(q^2 - p^2)\, W && \text{(A4)} \end{aligned}$$

where we introduced the variables

$$Z = 2(X+Y)\sinh(\Delta r)\,, \quad W = \frac{cs}{1-2s^2}(X-Y) \tag{A5}$$

Then, combining the squares of (A2) and (A3) with (A1), one gets

$$\begin{aligned} Z^2 &= \left[(a_2 - b_2)\, e^R\right]^2 + 4\left(c_1\, e^{-R} - c_2\, e^R\right)^2 && \text{(A6)} \\ 4(X+Y)^2 + Z^2 &= \left[(a_1 + b_1)\, e^{-R} + (a_2 + b_2)\, e^R\right]^2 && \text{(A7)} \end{aligned}$$

which gives the equation in the variable R

$$\left(a_1b_1 - c_1^2\right) e^{-2R} + \left(a_2b_2 - c_2^2\right) e^{2R} = (X+Y)^2 - (a_1a_2 + b_1b_2 - 2c_1c_2) \tag{A8}$$

First, note that from (22) we have

$$(X+Y)^2 - (a_1a_2 + b_1b_2 - 2c_1c_2) = (n_1+n_2)^2 - (n_1^2 + n_2^2) = 2n_1n_2 \tag{A9}$$

Moreover,

$$(2n_1n_2)^2 - 4\left(a_1b_1 - c_1^2\right)\left(a_2b_2 - c_2^2\right) = 0 \tag{A10}$$

so that the only solution to (A8) is the value

$$e^{2R} = \frac{n_1n_2}{a_2b_2 - c_2^2} \tag{A11}$$

Step 2—Find p, q from (A2) and (A3)

From (A2) and (A3), once R is known, one gets

$$\frac{p^2 - q^2}{pq} = -\frac{(a_1-b_1)e^{-R} - (a_2-b_2)e^{R}}{c_1\, e^{-R} - c_2\, e^{R}} \tag{A12}$$

which can be solved in terms of p, with complementary solutions

$$p_1 = \sqrt{\frac{e^{2R}(a_2-b_2) - (a_1-b_1) + M}{2M}} \qquad p_2 = \sqrt{\frac{e^{2R}(b_2-a_2) + (a_1-b_1) + M}{2M}} \tag{A13}$$

where we denote with M the expression

$$M = \sqrt{e^{2R}[(b_2-a_2) + a_1 - b_1]^2 + 4c_2^2e^{4R} - 8c_1c_2e^{2R} + 4c_1^2}\,. \tag{A14}$$

Note that from (A2) and (A3), we can find

$$\mathrm{sign}(p^2 - q^2) = \mathrm{sign}\left\{\frac{(a_1-b_1) - (a_2-b_2)e^{2R}}{c_1 - c_2e^{2R}}\right\} \tag{A15}$$

Therefore,

$$p = \begin{cases} \max(p_1, p_2) & \text{if } \mathrm{sign}\,(p^2-q^2) \geq 0 \\ \min(p_1, p_2) & \text{if } \mathrm{sign}\,(p^2-q^2) < 0 \end{cases}$$

Step 3—Find Δr from (39) and (41)

From (39) and (41) we have

$$e^{-2\Delta r} = \frac{pq\left[e^{-R}(a_1+b_1) + e^{R}(a_2+b_2)\right] + c_1e^{-R} - c_2e^{R}}{pq[e^{-R}(a_1+b_1) + e^{R}(a_2+b_2)] - c_1e^{-R} + c_2e^{R}} \tag{A16}$$

which leads immediately to Δr and finally to $r_1 = (R + \Delta r)/2$, $r_2 = (R - \Delta r)/2$.

Step 4—Find n_1 and n_2

The ambiguity between n_1 and n_2 is removed by observing that from (A5) and (38) one gets $\mathrm{sign}(n_1 - n_2) = \mathrm{sign}(W)$. On the other hand, we can find first $(X - Y)$ as

$$(X - Y) = \frac{(a_1+b_1)e^{-R} - (a_2+b_2)e^{R}}{e^{\Delta r} - e^{-\Delta r}} \tag{A17}$$

and W can be calculated from (A4) as

$$W = \frac{c_1 e^{-R} + c_2 e^{R} + pq(X - Y)\left(e^{\Delta r} + e^{-\Delta r}\right)}{2(p^2 - q^2)} \tag{A18}$$

Therefore,

$$n_1 = \begin{cases} \max(n_1, n_2) & \text{if } \operatorname{sign} W \geq 0 \\ \min(n_1, n_2) & \text{if } \operatorname{sign} W < 0 \end{cases} \tag{A19}$$

Step 5—Find s from (38)

From the second of (38), one can find directly

$$s = \sqrt{\frac{n_1 - n_2 - (X - Y)}{2(n_1 - n_2)}} \tag{A20}$$

where $(X - Y)$ has already been obtained in (A17). Note that the solution with the negative sign can be discarded according to the hypothesis that the coefficients of the beam-splitters are nonnegative.

Appendix B. Possible Approaches for the Use of This Theory

The usage of the theory developed in this paper can be manifold.

Reader I.

If one wants to study a specific two-mode Gaussian state starting from the "physical" specification, based on rotation, squeeze, and displacement operators, one can easily obtain the *full architecture* of Figure 4 and evaluate the ordinary CM $\mathbf{V}$ using the procedure of Section IV. Then, from the symplectic invariants of $\mathbf{V}$, one can evaluate the standard CM $\mathbf{V}_{sf}$ using the procedure of Section V-B arriving at the "minimal" architecture of Figure 1 with the advantage of the simplification therein.

Reader II.

If one wants to obtain any standard CM, which contains all the information on the Gaussian state, the physical variables of the architecture are obtained by the procedure of Section VIII-B, arriving at the *minimal architecture*.

Reader III.

To study the class of two-mode Gaussian states, one can start directly from the *minimal architecture* by managing the six primitive components. For the evaluation of the standard CM, relations (27) can be used with $a_1 = a$, $b_1 = b$.

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