symmetry

MDPI

Article

Reconstructing the Unknown Source Function of a Fractional Parabolic Equation from the Final Data with the Conformable Derivative

Omid Nikan [1,†], Ho Duy Binh [2,†], Zakieh Avazzadeh [3,†] and Le Dinh Long [4,5,*,†]

1 School of Mathematics, Iran University of Science and Technology, Narmak, Tehran 16846-13114, Iran; omidnikan77@yahoo.com
2 Division of Applied Mathematics, Thu Dau Mot University, Thu Dau Mot City 75000, Vietnam; hoduybinh@tdmu.edu.vn
3 Department of Applied Mathematics, Xi'an Jiaotong-Liverpool University, Suzhou 215123, China; zakieh.avazzadeh@xjtlu.edu.cn or z.avazzadeh@yahoo.com
4 Division of Applied Mathematics, Science and Technology Advanced Institute, Van Lang University, Ho Chi Minh City 700000, Vietnam
5 Faculty of Applied Technology, School of Engineering and Technology, Van Lang University, Ho Chi Minh City 700000, Vietnam
* Correspondence: ledinhlong@vlu.edu.vn
† These authors contributed equally to this work.

Abstract: The paper's main purpose is to find the unknown source function for the conformable heat equation. In the case of $(\Phi, g) \in \mathcal{L}^2(0,T) \times \mathcal{L}^2(\Omega)$, we give a modified Fractional Landweber solution and explore the error between the approximate solution and the desired solution under a priori and a posteriori parameter choice rules. The error between the regularized and exact solution is then calculated in $\mathcal{L}^q(\mathcal{D})$, with $q \neq 2$ under some reasonable Cauchy data assumptions.

Keywords: parabolic equations; conformable derivative; Fourier truncation method; inverse source problem; inverse initial problem; regularization; Sobolev embeddings

Citation: Nikan, O.; Binh, H.D.; Avazzadeh, Z.; Long, L.D. Reconstructing the Unknown Source Function of a Fractional Parabolic Equation from the Final Data with the Conformable Derivative. *Symmetry* **2022**, *14*, 1490. https://doi.org/10.3390/sym14071490

Academic Editor: Ahmet Ocak Akdemir

Received: 24 May 2022
Accepted: 27 June 2022
Published: 21 July 2022

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

1. Introduction

In this paper, we consider the initial value problem for the conformable heat equation (or called parabolic equation with conformable operator):

$$\begin{cases} \dfrac{{}^{C}\partial^{\beta}}{\partial t^{\beta}}(u - k\Delta u) - \Delta u(x,t) = \Phi(t) f(x), & x \in \mathcal{D}, \ \ t \in (0,T), \\ u(x,t) = 0, & x \in \partial\mathcal{D}, t \in (0,T). \end{cases} \tag{1}$$

Here, $\mathcal{D} \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with the smooth boundary $\partial\mathcal{D}$, and $T > 0$. It is an obvious fact that a conformable operator has many practical applications, branches of science and engineering; see for example [1–10]. The applications of conformable derivative models in the harmonic oscillator include the damped oscillator, and the forced oscillator (see [11]), electrical circuits (see [12]), chaotic systems in dynamics (see [13]), and many more applications, (see [14–21]) .

Conformable derivative model: Let us take B as a Banach space, and f as a B-valued function on $[0,\infty)$. Let $\frac{{}^{C}\partial^{\beta}}{\partial t^{\beta}}$ be the conformable derivative of order $0 < \beta \leq 1$ locally defined by:

$$\frac{{}^{C}\partial^{\beta} f(t)}{\partial t^{\beta}} := \lim_{\tau \to 0} \frac{f(t + \tau t^{1-\beta}) - f(t)}{\tau} \quad \text{in } B,$$

for each $t > 0$. For more knowledge about the above definition, we refer the reader to [22–24]. There are two interesting points regarding the relationship between conformable and classical derivatives:

- Let us assume that $B \equiv \mathbb{R}$, if f is a real function and $s > 0$, then f has a conformable fractional derivative of order β, and $\frac{{}^{C}\partial^{\beta} f(s)}{\partial s^{\beta}} = s^{1-\beta}\frac{\partial f(s)}{\partial s}$;
- If B is not $\mathbb{R}$, for example B are Sobolev spaces. There are not many conformable related results in Banach spaces, see [25].

The inverse source problem for (1) is described as follows. The final time condition $u(x,T) = g(x)$, together with the additional condition $u(x,0) = 0, \ \ x \in \mathcal{D}$. The inverse source problem for (1) is understood as finding the function f when the input data g, Φ is given. As we know, the Problem (1) is ill-posed, and according to our experience and understanding, the frequent infringement is the continuity of the solution according to data. Therefore, to provide a good approximation, we need to regularize these problems. Before going into adjustment, we would like to review a bit of the history of the Problem (1).

- In case $\beta = 1$, the above equation becomes the classical pseudo-parabolic equation; this type of equation has received much attention from mathematicians, see [26–28];
- In case $\beta \neq 1$, with Caputo derivative model, we find the following documents, see [4]. Luc and co-authors studied the existence and uniqueness of a class of mild solutions of these equations. In [29], the authors considered the non-local Problem (1) for a pseudo-parabolic equation with fractional time and space. In [30], Tuan and his group considered a class of pseudoparabolic equations with the nonlocal condition in two cases: the nonlinear source function ad linear source function. For the first case, by using the Sobolev embeddings, they established the existence, the uniqueness, and some regularity results for the mild solution of Problem (1). For the second case, using the Banach fixed-point theorem, they proved the existence and the uniqueness of the mild solution for (1). In [31], the authors considered two problems. For the first problem with the source term satisfying the globally Lipschitz condition, we establish the local well-posedness theory, and the further local existence theory related to the finite time blow-up are also obtained for the problem with logarithmic nonlinearity. For the second problem, they proved the global existence theorem.
- We have not seen any findings for this kind in case $\beta \neq 1$ with the conformable derivative model, and the source function survey problem is much more sparse, thus our study concentrates on this topic.

The regularization problem is a very interesting problem, with common regularization methods such as the Tikhonov method [32], Quasi Boundary Value method [33], Fractional Tikhonov method [34] and mollification method [35]. In [36], T. Wei with the Tikhonov regularization method, in [37], Ting Wei and her group considered a time-dependent source term by using a boundary element method combined with a generalized Tikhonov regularization. However, in this article, we use a modified fractional Landweber method to solve the unknown source Problem (1). Besides, there is a new point in this paper; we evaluate the error of exact and normalized solutions in $\mathcal{L}^q(\mathcal{D})$ space, with $q \neq 2$. In this case, the equality Parseval was not used. One way to overcome this weakness is to use the embedding between $\mathcal{L}^q(\mathcal{D})$ and Hilbert scales spaces $\mathbb{X}^r(\mathcal{D})$. The main analytical technique in our paper is to use some embeddings and some analysis estimators related to Hölder inequality. To complete our proofs, we learn many interesting techniques from N.H. Tuan [38]. For the reader's convenience, we would like to outline the main results of the paper.

- We give the ill-posedness of Problem (1);
- Showing the regularization of Problem (1), with the two subsections;
  - Uncertainty of Problem (1) of determining the source function;
  - The conditional stability of Problem (1);

- Using the modified fractional Landweber method to solve the Problem (1). We obtain the convergence rate as follows:
  - In Section 4.1, under a priori parameter choice rule;
  - In Section 4.2, under a posteriori parameter choice rule.
- It gives the error estimate in $\mathcal{L}^q$ space, with $q \neq 2$.

This paper is organized as follows. Section 2 introduces some function spaces and embeddings. In Section 4, we deal with the regularized solution for the inverse source problem for (1) by the Modified Fractional Landweber method under a priori and a posteriori parameter choice rules. In Section 5, we solve the Problem (1) in the case of observed data in $\mathcal{L}^q$ space. Finally, we present a numerical experiment.

2. Preliminary Results

Let us recall that the spectral problem:

$$\begin{cases} \Delta e_n(x) = -\lambda_n e_n(x), & x \in \mathcal{D}, \\ e_n(x) = 0, & x \in \partial D, \end{cases}$$

admits the eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \ldots$ with $\lambda_n \to \infty$ as $n \to \infty$. The corresponding eigenfunctions are $\psi_n \in H_0^1(\Omega)$.

Definition 1. *(Hilbert scale space). We recall the Hilbert scale space, which is given as follows:*

$$\mathbb{X}^r(\mathcal{D}) = \left\{ f \in L^2(\mathcal{D}), \quad \sum_{n=1}^{\infty} \lambda_n^{2r} \Big(\int_{\mathcal{D}} f(x) e_n(x) dx \Big)^2 < \infty \right\},$$

for any $r \geq 0$. It is well-known that $\mathbb{X}^r(\mathcal{D})$ is a Hilbert space corresponding to the norm,

$$\|f\|_{\mathbb{X}^r(\mathcal{D})} = \left(\sum_{n=1}^{\infty} \lambda_n^{2r} \Big(\int_{\mathcal{D}} f(x) e_n(x) dx \Big)^2 \right)^{1/2}, \quad f \in \mathbb{X}^r(\mathcal{D}). \tag{2}$$

Lemma 1. *Let $\Phi : [0, T] \to \mathbb{R}$ such that $\Phi_0 \leq \Phi(t) \leq \Phi_1$ where Φ_0 and Φ_1 are positive numbers. Then the following estimates are true:*

$$\frac{1}{1 + k\lambda_n} \int_0^T s^{\beta-1} \exp\Big(\frac{\lambda_n}{1 + k\lambda_n} \frac{s^\beta - T^\beta}{\beta} \Big) \Phi(s) ds \leq \frac{T^\beta}{\beta} \frac{\Phi_1}{\lambda_n}, \tag{3}$$

and

$$\frac{\Phi_0}{\lambda_n} \Big[1 - \exp\Big(- \frac{\lambda_1}{1 + k\lambda_1} \frac{T^\beta}{\beta} \Big) \Big] \leq \frac{1}{1 + k\lambda_n} \int_0^T s^{\beta-1} \exp\Big(\frac{\lambda_n}{1 + k\lambda_n} \frac{s^\beta - T^\beta}{\beta} \Big) \Phi(s) ds. \tag{4}$$

Proof. First of all, we have the estimate $\int_0^T s^{\beta-1} \exp\big(\frac{\lambda_n}{1+k\lambda_n} \frac{s^\beta - T^\beta}{\beta} \big) ds = \exp\big(- \frac{\lambda_n}{1+k\lambda_n} \frac{T^\beta}{\beta} \big) \int_0^T s^{\beta-1} \exp\big(\frac{\lambda_n}{1+k\lambda_n} \frac{s^\beta}{\beta} \big) ds$, putting $z = \frac{\lambda_n}{1+k\lambda_n} \frac{s^\beta}{\beta}$, and through basic calculations, and applying the inequality $(1 - e^{-x}) \leq x$, for $x \geq 0$, since $\Phi(t) \leq \Phi_1$, we get:

$$\frac{1}{1 + k\lambda_n} \int_0^T s^{\beta-1} \exp\Big(\frac{\lambda_n}{1 + k\lambda_n} \frac{s^\beta - T^\beta}{\beta} \Big) \Phi(s) ds \leq \frac{T^\beta}{\beta} \frac{\Phi_1}{\lambda_n} \cdot . \tag{5}$$

Since $\Phi(t) \geq \Phi_0 > 0$, we have:

$$\frac{1}{1+k\lambda_n}\int_0^T s^{\beta-1}\exp\Big(\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta-T^\beta}{\beta}\Big)\Phi(s)ds \geq \Phi_0\int_0^T s^{\beta-1}\exp\Big(\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta-T^\beta}{\beta}\Big)ds$$
$$= \frac{\Phi_0}{\lambda_n}\Big[1-\exp\Big(-\frac{\lambda_1}{1+k\lambda_1}\frac{T^\beta}{\beta}\Big)\Big]. \tag{6}$$

Let us consider the following function: $G(x) = \dfrac{x}{1+kx}$, $x > 0$. The derivative of which is equal to: $G'(x) = \dfrac{1}{(1+kx)^2} > 0$. This implies that G is an increasing function on $(0, +\infty)$. Therefore, we get:

$$\lambda_n(1+k\lambda_n)^{-1} \geq \lambda_1(1+k\lambda_1)^{-1}.$$

It follows from (6) that:

$$\frac{1}{1+k\lambda_n}\int_0^T s^{\beta-1}\exp\Big(\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta-T^\beta}{\beta}\Big)\Phi(s)ds \geq \frac{\Phi_0}{\lambda_n}\Big[1-\exp\Big(-\frac{\lambda_1}{1+k\lambda_1}\frac{T^\beta}{\beta}\Big)\Big]. \tag{7}$$

The Lemma is proven. □

Lemma 2. *Let Φ_0, Φ_1 be positive constants such that $\Phi_0 < \Phi(t) < \Phi_1$. By choosing $\delta \in \Big(0, \dfrac{\Phi_1}{4}\Big)$, and $\mathcal{B}(\Phi_0, \Phi_1) = \Phi_1 + \dfrac{\Phi_0}{4}$, we obtain*

$$4^{-1}\Phi_0 \leq |\Phi_\delta(t)| \leq \mathcal{B}\big(\Phi_0, \Phi_1\big)\cdot \tag{8}$$

Proof. See in [39]. □

Lemma 3. *(See [40]) The following statements are true:*

$$\left.\begin{array}{lll} L^p(\mathcal{D}) \hookrightarrow \mathbb{X}^\mu(\mathcal{D}), & \text{if} & -\dfrac{N}{4} < \mu \leq 0, \quad p \geq \dfrac{2N}{N-4\mu}, \\ \mathbb{X}^s(\mathcal{D}) \hookrightarrow L^p(\mathcal{D}), & \text{if} & 0 \leq s < \dfrac{N}{4}, \quad p \leq \dfrac{2N}{N-4s}. \end{array}\right\} \tag{9}$$

3. Regularization of Inverse Source Problem

We consider the mild solution in Fourier series, $u(x,t) = \sum_{n=1}^{\infty} u_n(t)e_n(x)$, with $u_n(t) = \int_{\mathcal{D}} u(\cdot,t)e_n(x)dx$. Taking the inner product of the equations of Problem (1) with e_n gives:

$$\left\{\begin{array}{ll} \dfrac{{}^C\partial^\beta}{\partial t^\beta}\langle u(.,t), e_n\rangle + k\lambda_n\dfrac{{}^C\partial^\beta}{\partial t^\beta}\langle u(.,t), e_n\rangle - \lambda_n\langle u(.,t), e_n\rangle = \langle F(.,t), e_n\rangle, & t \in (0,T), \\ \langle u(.,0), e_n\rangle = \langle u_0, e_n\rangle \quad . \end{array}\right. \tag{10}$$

The first equation of (10) is a differential equation with a conformable derivative as follows:

$$\frac{{}^C\partial^\beta}{\partial t^\beta}u_n(t) - \frac{\lambda_n}{1+k\lambda_n}u_n(t) = \frac{1}{1+k\lambda_n}F_n(t). \tag{11}$$

In view of the result in (Theorem 5, "[41], p. 318") and (Theorem 3.3, "[42], p. 318"), the solution of Problem (1) is:

$$u_n(t) = \exp\Big(-\frac{\lambda_n}{1+k\lambda_n}\frac{t^\beta}{\beta}\Big)u_{0,n} + \frac{1}{1+k\lambda_n}\int_0^t s^{\beta-1}\exp\Big(\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta - t^\beta}{\beta}\Big)F_n(s)ds.$$

To find the formula of the mild solution to Problem (1), with $u_n(0) = 0$ and $F_n(s) = \Phi(s)f(x)$. Letting $t = T$, we know that:

$$\Big(\int_{\mathcal{D}} g(x)e_n(x)dx\Big) = \frac{1}{1+k\lambda_n}\Big(\int_{\mathcal{D}} f(x)e_n(x)dx\Big)\int_0^T s^{\beta-1}\exp\big(\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta - T^\beta}{\beta}\big)\Phi(s)ds. \tag{12}$$

After a simple transformation, we get:

$$\Big(\int_{\mathcal{D}} f(x)e_n(x)dx\Big) = \Big[\int_0^T s^{\beta-1}\exp\big(\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta - T^\beta}{\beta}\big)\Phi(s)ds\Big]^{-1}(1+k\lambda_n)\Big(\int_{\mathcal{D}} g(x)e_n(x)dx\Big). \tag{13}$$

This leads to:

$$f(x) = \sum_{n=1}^{+\infty}\Big[\int_0^T s^{\beta-1}\exp\Big(\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta - T^\beta}{\beta}\Big)\Phi(s)ds\Big]^{-1}(1+k\lambda_n)\Big(\int_{\mathcal{D}} g(x)e_n(x)dx\Big)e_n(x). \tag{14}$$

3.1. Uncertainty of Source Problem

Theorem 1. *The inverse source problem* (1) *is ill-posed.*

Proof of Theorem 1. We defined a linear operator $\mathcal{S} : \mathcal{L}^2(\mathcal{D}) \to \mathcal{L}^2(\mathcal{D})$ as follows.

$$\mathcal{S}f(x) = \sum_{n=1}^{+\infty}(1+k\lambda_n)^{-1}\Big[\int_0^T s^{\beta-1}\exp\Big(\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta - T^\beta}{\beta}\Big)\Phi(s)ds\Big] = \int_{\mathcal{D}} k(x,\omega)f(\omega)d\omega, \tag{15}$$

$$\text{where } k(x,\omega) = \sum_{n=1}^{+\infty}(1+k\lambda_n)^{-1}\Big[\int_0^T s^{\beta-1}\exp\Big(\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta - T^\beta}{\beta}\Big)\Phi(s)ds\Big]e_n(x)e_n(\omega).$$

Due to $k(x,\omega) = k(\omega,x)$, $\mathcal{S}$ is a self-adjoint operator. We defined the finite rank operators $\mathcal{S}_\mathcal{N}$ and considered its compactness :

$$\mathcal{S}_\mathcal{N}f(x) = \sum_{n=1}^{\mathcal{N}}(1+k\lambda_n)^{-1}\Big[\int_0^T s^{\beta-1}\exp\Big(\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta - T^\beta}{\beta}\Big)\Phi(s)ds\Big]\Big(\int_{\mathcal{D}} f(x)e_n(x)dx\Big)e_n(x). \tag{16}$$

From (16), through some basic calculations and using the Lemma 1, we have:

$$\|\mathcal{S}_\mathcal{N}f - \mathcal{S}f\|^2_{\mathcal{L}^2(\mathcal{D})} \le \frac{T^\beta\Phi_1}{\beta}\sum_{n=\mathcal{N}+1}^{+\infty}\frac{1}{\lambda_n^2}\Big(\int_{\mathcal{D}} f(x)e_n(x)dx\Big)^2. \tag{17}$$

From (17), we have:

$$\|\mathcal{S}_\mathcal{N}f - \mathcal{S}f\|^2_{\mathcal{L}^2(\mathcal{D})} \le \Big(\frac{T^\beta\Phi_1}{\beta}\Big)^2\frac{1}{\mathcal{N}^2}\|f\|^2_{\mathcal{L}^2(\mathcal{D})} \text{ to } 0 \text{ in } \mathcal{L}(\mathcal{L}^2(\mathcal{D});\mathcal{L}^2(\mathcal{D})) \text{ as } \mathcal{N}\to\infty. \tag{18}$$

Additionally, $\mathcal{S}$ is a compact operator. The SVDs for the linear self-adjoint compact operator $\mathcal{S}$ are:

$$\mathcal{S} = (1+k\lambda_n)^{-1}\Big[\int_0^T s^{\beta-1}\exp\Big(\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta - T^\beta}{\beta}\Big)\Phi(s)ds\Big], \tag{19}$$

and corresponding eigenvectors are e_n is an orthonormal basis in $\mathcal{L}_2(\mathcal{D})$. Therefore, the inverse source Problem (1) can be formulated as an operator equation $\mathcal{S}f(x) = g(x)$ where, by $g(x)$, and by Kirsch, it is ill-posed. Next, we show an example, with final time data $g_i = \lambda_i^{-\frac{1}{2}} e_i$ and $u_0 = 0$. By (14), the source term corresponding to g_i is:

$$f_i(x) = \Big[\int_0^T s^{\beta-1} \exp\left(\frac{\lambda_i}{1+k\lambda_i}\frac{s^\beta - T^\beta}{\beta}\right)\Phi(s)ds\Big]^{-1} \times (1+k\lambda_i)\Big(\int_{\mathcal{D}} g(x)e_i(x)dx\Big)e_i(x) \geq \frac{\beta\,\lambda_i^{\frac{1}{2}}}{T^\beta \Phi_1}. \tag{20}$$

The input final data $g = 0$ then the source term corresponding to g is $f = 0$. We have error in $\mathcal{L}^2$ norm between g_i and g,

$$\lim_{i\to+\infty} \|g_i - g\|_{\mathcal{L}^2(\mathcal{D})} = \lim_{i\to+\infty} \lambda_i^{-\frac{1}{2}} = 0. \tag{21}$$

Then the error in $\mathcal{L}^2$ norm between f_i and f

$$\|f_i - f\|_{\mathcal{L}^2(\mathcal{D})} \geq \frac{\beta}{T^\beta \Phi_1}\lambda_i^{\frac{1}{2}} \longrightarrow \lim_{i\to+\infty} \|f_i - f\|_{\mathcal{L}^2(\mathcal{D})} \geq +\infty. \tag{22}$$

From (21) and (22), we deduce that the solution to Problem (1) is unstable in $\mathcal{L}^2(\mathcal{D})$. □

3.2. The Conditional Stability

Theorem 2. *Assume that $f \in \mathbb{X}^r(\mathcal{D})$, $g \in \mathcal{L}^2(\mathcal{D})$, and $\|f\|_{\mathbb{X}^r(\mathcal{D})} \leq \mathcal{B}$ then we have:*

$$\|f\|_{\mathcal{L}^2(\mathcal{D})} \leq \overline{C}(m,\mathcal{B})\|g\|_{\mathcal{L}^2(\mathcal{D})}^{\frac{r}{r+1}}, \tag{23}$$

whereby $\overline{C}(r,\mathcal{B}) = \mathcal{B}^{\frac{1}{r+1}}\left|\Phi_0\Big(1 - \exp\big(-\lambda_1(1+k\lambda_1)^{-1}\frac{T^\beta}{\beta}\big)\Big)\right|^{-\frac{r}{r+1}}$.

Proof of Theorem 2. The proof of this theorem can be conducted similar to the articles [39]. We omit this here. □

4. A Modified Fractional Landweber Method and Convergent Rate

Based on a modified Fractional Landweber method, we show the error estimate under the a priori regularization parameter choice rule and the a-posteriori regularization parameter choice rule, respectively. Now, we use the modified Fractional Landweber iterative method to obtain the regularization solution for Problem (1). We give the following iterative form:

$$f_0(x) := 0, \quad f_\tau(x) = f_{\tau-1}(x) - \xi\Big((\mathcal{S}^*\mathcal{S})^{\frac{\alpha+1}{2}} f_{\tau-1}(x) - (\mathcal{S}^*\mathcal{S})^{\frac{\alpha-1}{2}}\mathcal{S}^* g(x)\Big), \tau = 1,2,3,\ldots, \tag{24}$$

where τ is the iterative step number and the regularization parameter is τ^{-1}. The coefficient ξ is called the relaxation factor and satisfies $0 < \xi < \|\mathcal{S}\|^{-(\alpha+1)}$, by denoting an operator $R_\tau : g \to f$ such that

$$f_m(x) = \xi\sum_{k=0}^{m-1}\big(I - \xi(\mathcal{S}^*\mathcal{S})^{\frac{\alpha+1}{2}}\big)^k(\mathcal{S}^*\mathcal{S})^{\frac{\alpha-1}{2}}\mathcal{S}^* g(x), \tag{25}$$

with (19), it gives :

$$f_{m,\delta}(x) = \mathcal{R}_m g_\delta(x) = \sum_{n=1}^{+\infty} \frac{1-\left(1-\xi\left|Q\lambda_n^{-1}\right|^{\alpha+1}\right)^m}{\mathcal{P}_n(k,\beta,T,\Phi_\delta)} \Big(\int_{\mathcal{D}} g_\delta(x)e_n(x)dx\Big)e_n(x), \tag{26}$$

whereby

$$\mathcal{P}_n(k,\beta,T,\Phi_\delta) = (1+k\lambda_n)^{-1}\Big[\int_0^T s^{\beta-1}\exp\left(\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta - T^\beta}{\beta}\right)\Phi(s)ds\Big]. \tag{27}$$

Lemma 4. *For $0<\alpha<1$, if we choose $\xi \in \left(0, \left|\frac{\lambda_1}{Q}\right|^{\alpha+1}\right)$ then $0<\xi\left|\frac{Q}{\lambda_1}\right|^{\alpha+1}<1$, inwhich Q depends on $Q = T^\beta\Phi_1\beta^{-1}$. With how to choose ξ in above, by denoting $z = \xi\left|Q\lambda_1\right|^{\alpha+1}$, we have*

$$\left(1-(1-z)^m\right)\left(z\xi^{-1}\right)^{-\frac{1}{\alpha+1}} \leq m^{\frac{1}{\alpha+1}}\xi^{\frac{1}{\alpha+1}}. \tag{28}$$

Proof of Lemma 4. A complete demonstration of this inequality can be found in [9]. □

4.1. A Priori Parameter Choice Rule

Before we go into proving the main theorem, we need the following lemmas:

Lemma 5. *For $0<\alpha\leq 1$, then we get:*

$$\left(1-\xi\left|\frac{Q}{\lambda_n}\right|^{\alpha+1}\right)^m \lambda_n^{-1} \leq |Q|^{-1}\xi^{-\frac{1}{\alpha+1}}m^{-\frac{1}{\alpha+1}}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{\alpha+1}}. \tag{29}$$

Proof of Lemma 5. In here, we denote $\lambda_n = z$, this implies that:

$$\mathcal{G}_1(z) = \left(1-\xi|Q|^{\alpha+1}z^{-(\alpha+1)}\right)^m z^{-1}.$$

Taking the derivative of the variable z and solving the $\mathcal{G}_1'(z_0)=0$, we can find that:

$$z_0 = \xi^{\frac{1}{\alpha+1}}Q\left[1+m(\alpha+1)\right]^{\frac{1}{\alpha+1}}, \tag{30}$$

maximuizes the function $\mathcal{G}(z)$. Hence, we get:

$$\mathcal{G}_1(z) \leq \mathcal{G}_1(z_0) = \left(\frac{m\alpha+m}{m\alpha+m+1}\right)^m\left(\frac{1}{\xi Q^{\alpha+1}(1+m\alpha+m)}\right)^{\frac{1}{\alpha+1}}. \tag{31}$$

Finally, one has:

$$\mathcal{G}_1(z) \leq Q^{-1}\xi^{-\frac{1}{\alpha+1}}\, m^{-\frac{1}{\alpha+1}}(\alpha+1)^{-\frac{1}{\alpha+1}}. \tag{32}$$

□

Theorem 3. *Let $f \in \mathbb{X}^r(\mathcal{D})$, for any $g, g_\delta \in \mathcal{L}^2(\mathcal{D})$ such that $\left\|g_\delta - g\right\|_{\mathcal{L}^2(\mathcal{D})} \leq \delta$. The regularization parameter m is chosen*

$$m = \left[\left(\frac{\mathcal{B}}{\delta}\right)^{\frac{\alpha+1}{2}}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{2}}\frac{1}{\xi}\right], \tag{33}$$

then we get:

$$\left\|f_{m,\delta}-f\right\|_{\mathcal{L}^2(\mathcal{D})} \text{ is of order } \delta^{\frac{1}{2}}. \tag{34}$$

whereby $[m]$ *denotes the largest integer less than or equal to* m.

Proof of Theorem 3. We have:

$$\left\|f_{m,\delta}-f\right\|_{\mathcal{L}^2(\mathcal{D})} \leq \left\|f_{m,\delta}-f_m\right\|_{\mathcal{L}^2(\mathcal{D})} + \left\|f_m-f\right\|_{\mathcal{L}^2(\mathcal{D})}. \tag{35}$$

Applying the inequality $(a+b)^2 \leq 2(a^2+b^2)$, we estimate at $\left\|f_{m,\delta}-f_m\right\|_{\mathcal{L}^2(\mathcal{D})}$ as follows:

$$\begin{aligned}
&\left\|f_{m,\delta}-f_m\right\|^2_{\mathcal{L}^2(\mathcal{D})}\\
&=\sum_{n=1}^{+\infty}\Big[1-\big(1-\xi\big|Q\lambda_n^{-1}\big|^{\alpha+1}\big)^m\Big]^2\left(\frac{\big(\int_{\mathcal{D}} g_\delta(x)e_n(x)dx\big)e_n(\cdot)}{\mathcal{P}_n(k,\beta,T,\Phi_\delta)}-\frac{\big(\int_{\mathcal{D}} g(x)e_n(x)dx\big)e_n(\cdot)}{\mathcal{P}_n(k,\beta,T,\Phi)}\right)^2\\
&\leq 2\Big[1-\big(1-\xi\big|Q\lambda_n^{-1}\big|^{\alpha+1}\big)^m\Big]^2\left(\frac{\big(\int_{\mathcal{D}} \big(g_\delta(x)-g(x)\big)e_n(x)dx\big)e_n(\cdot)}{\mathcal{P}_n(k,\beta,T,\Phi_\delta)}\right)^2\\
&\quad+2\sum_{n=1}^{+\infty}\Big[1-\big(1-\xi\big|Q\lambda_n^{-1}\big|^{\alpha+1}\big)^m\Big]^2\left(\frac{\Big(\int_{\mathcal{D}} g(x)e_n(x)dx\Big)e_n(\cdot)}{\mathcal{P}_n(k,\beta,T,\Phi_\delta)}-\frac{\big(\int_{\mathcal{D}} g(x)e_n(x)dx\big)e_n(x)}{\mathcal{P}_n(k,\beta,T,\Phi)}\right)^2\\
&\leq \underbrace{2\sum_{n=1}^{+\infty}\Big[1-\big(1-\xi\big|Q\lambda_n^{-1}\big|^{\alpha+1}\big)^m\Big]^2\big|Q\lambda_n^{-1}\big|^{-\frac{2}{\alpha+1}}\big|Q\lambda_n^{-1}\big|^{\frac{2}{\alpha+1}}\left(\frac{\Big(\int_{\mathcal{D}} \big(g_\delta(x)-g(x)\big)e_n(x)dx\Big)e_n(\cdot)}{\mathcal{P}_n(k,\beta,T,\Phi_\delta)}\right)^2}_{\mathcal{H}_1}\\
&\quad\underbrace{+2\sum_{n=1}^{+\infty}\Big[1-\big(1-\xi\big|Q\lambda_n^{-1}\big|^{\alpha+1}\big)^m\Big]^2\left(\frac{\mathcal{P}_n(k,\beta,T,\Phi_\delta-\Phi)}{\mathcal{P}_n(k,\beta,T,\Phi_\delta)}\frac{\big(\int_{\mathcal{D}} g(x)e_n(x)dx\big)e_n(\cdot)}{\mathcal{P}_n(k,\beta,T,\Phi)}\right)^2}_{\mathcal{H}_2}.
\end{aligned}\tag{36}$$

We will divide the evaluation (36) into two steps as follows:
Step 1: By means of the Lemma 4, we have estimate of $\mathcal{H}_1$ as follows:

$$\begin{aligned}
\mathcal{H}_1 &\leq \frac{2Q^{\frac{2}{\alpha+1}}\lambda_1^{\frac{\alpha-1}{\alpha+1}}m^{\frac{2}{\alpha+1}}\xi^{\frac{2}{\alpha+1}}}{\Phi_0^2\Big[1-\exp\big(-\frac{\lambda_1}{1+k\lambda_1}\frac{T^\beta}{\beta}\big)\Big]^2}\sum_{n=1}^{+\infty}\Big(\int_{\mathcal{D}}\big(g_\delta(x)-g(x)\big)e_n(x)dx\Big)^2\\
&\leq \delta^2 m^{\frac{2}{\alpha+1}}\xi^{\frac{2}{\alpha+1}}\Xi^2_{\alpha,\beta}\big(Q,\lambda_1,T,\Phi_0\big),
\end{aligned}\tag{37}$$

whereby $\Xi^2_{\alpha,\beta}\big(Q,\lambda_1,T,\Phi_0\big)=2Q^{\frac{2}{\alpha+1}}\lambda_1^{\frac{\alpha-1}{\alpha+1}}\,\Phi_0^{-2}\left[1-\exp\Big(-\frac{\lambda_1}{1+k\lambda_1}\frac{T^\beta}{\beta}\Big)\right]^{-2}$.
Step 2: Next, $\mathcal{H}_2$ can be bounded as follows:

$$\begin{aligned}
\mathcal{H}_2 &\leq 2\sum_{n=1}^{+\infty}\frac{\big[1-\big(1-\xi\big|Q\lambda_n^{-1}\big|^{\alpha+1}\big)^m\big]^2}{\big|Q\lambda_n^{-1}\big|^{\frac{2}{\alpha+1}}}\big|Q\lambda_n^{-1}\big|^{\frac{2}{\alpha+1}}\left(\frac{\mathcal{P}_n(k,\beta,T,\Phi_\delta-\Phi)}{\mathcal{P}_n(k,\beta,T,\Phi_\delta)}\right)^2\Big(\int_{\mathcal{D}} f(x)e_n(x)dx\Big)^2,\\
&\leq \frac{32Q^{\frac{2}{\alpha+1}}}{\Phi_0^2\lambda_1^{\frac{2}{\alpha+1}}}\,\delta^2\sum_{n=1}^{+\infty}\frac{\big[1-\big(1-\xi\big|Q\lambda_n^{-1}\big|^{\alpha+1}\big)^m\big]^2}{\big|Q\lambda_n^{-1}\big|^{\frac{2}{\alpha+1}}}\Big(\int_{\mathcal{D}} f(x)e_n(x)dx\Big)^2\\
&\leq \frac{32}{\Phi_0^2}\Big(\frac{Q}{\lambda_1}\Big)^{\frac{2}{\alpha+1}}\delta^2\,m^{\frac{2}{\alpha+1}}\xi^{\frac{2}{\alpha+1}}\|f\|^2_{\mathcal{L}^2(\mathcal{D})}.
\end{aligned}\tag{38}$$

In here, we have:

$$\begin{aligned}\left\|f_{m,\delta}-f_m\right\|_{\mathcal{L}^2(\mathcal{D})} &\le \delta m^{\frac{1}{\alpha+1}}\xi^{\frac{1}{\alpha+1}}\Xi_{\alpha,\beta}(Q,\lambda_1,T,\Phi_0)+\frac{4\sqrt{2}}{\Phi_0}\Big(\frac{Q}{\lambda_1}\Big)^{\frac{1}{\alpha+1}}\delta m^{\frac{1}{\alpha+1}}\xi^{\frac{1}{\alpha+1}}\|f\|_{L^2(\mathcal{D})}\\ &\le \delta m^{\frac{1}{\alpha+1}}\xi^{\frac{1}{\alpha+1}}\left[\mathcal{V}_{\alpha,\beta}(Q,\lambda_1,T,\Phi_0,f)\right],\end{aligned} \tag{39}$$

whereby

$$\mathcal{V}_{\alpha,\beta}(Q,\lambda_1,T,\Phi_0,f)=\sqrt{2}\Xi_{\alpha,\beta}(Q,\lambda_1,T,\Phi_0)+\frac{4\sqrt{2}}{\Phi_0}\Big(\frac{Q}{\lambda_1}\Big)^{\frac{1}{\alpha+1}}\|f\|_{L^2(\mathcal{D})}. \tag{40}$$

Next, we give the estimate of the approximation error $\left\|f_m-f\right\|_{\mathcal{L}^2(\mathcal{D})}$ as follows:

$$\begin{aligned}\left\|f_m-f\right\|_{\mathcal{L}^2(\mathcal{D})} &\le \sum_{n=1}^{+\infty}\left[1-\xi\Big|\frac{Q}{\lambda_n}\Big|^{\alpha+1}\right]^m\lambda_n^{-1}\frac{\lambda_n\Big(\int\limits_{\mathcal{D}}g(x)e_n(x)dx\Big)}{\mathcal{P}_n(k,\beta,T,\Phi)}\\ &\le \sum_{n=1}^{+\infty}\left[1-\xi\Big|\frac{Q}{\lambda_n}\Big|^{\alpha+1}\right]^m\lambda_n^{-1}\lambda_n\Big(\int\limits_{\mathcal{D}}f(x)e_n(x)dx\Big).\end{aligned} \tag{41}$$

From (41), using the Lemma 5, we can know that:

$$\left\|f_m-f\right\|_{\mathcal{L}^2(\mathcal{D})}\le Q^{-1}\xi^{-\frac{1}{\alpha+1}}m^{-\frac{1}{\alpha+1}}(\alpha+1)^{-\frac{1}{\alpha+1}}\|f\|_{\mathbb{X}^1(\mathcal{D})}. \tag{42}$$

Combining (36) to (42), we get:

$$\begin{aligned}\left\|f_{m,\delta}-f\right\|_{\mathcal{L}^2(\mathcal{D})} &\le \delta m^{\frac{1}{\alpha+1}}\xi^{\frac{1}{\alpha+1}}\left[\mathcal{V}_{\alpha,\beta}(Q,\lambda_1,T,\Phi_0,f)\right]\\ &\quad+Q^{-1}\xi^{-\frac{1}{\alpha+1}}m^{-\frac{1}{\alpha+1}}(\alpha+1)^{-\frac{1}{\alpha+1}}\left\|f\right\|_{\mathbb{X}^1(\mathcal{D})}.\end{aligned} \tag{43}$$

Let by choose m as follows:

$$m=\left[\Big(\frac{\mathcal{B}}{\delta}\Big)^{\frac{\alpha+1}{2}}\Big(\frac{1}{\alpha+1}\Big)^{\frac{1}{2}}\frac{1}{\xi}\right], \tag{44}$$

This leads to

$$\begin{aligned}\left\|f_{m,\delta}-f\right\|_{\mathcal{L}^2(\mathcal{D})} &\le \delta^{\frac{1}{2}}\mathcal{B}^{\frac{1}{2}}(\alpha+1)^{-\frac{1}{2(\alpha+1)}}\left[\mathcal{V}_{\alpha,\beta}(Q,\lambda_1,T,\Phi_0,f)\right]\\ &\quad+\delta^{\frac{1}{2}}\left(\mathcal{B}^{\frac{1}{2}}(\alpha+1)^{\frac{1}{2(\alpha+1)}}Q\right)^{-1}\|f\|_{\mathbb{X}^1(\mathcal{D})}.\end{aligned} \tag{45}$$

□

4.2. A Posteriori Parameter Choice Rule

In this section, we look at the following regulatory parameter choices in Morozov's difference principle. We construct the regular solution sequence $g_\delta, f_{m,\delta}$ equals the Landweber iteration method. Stop algorithm at the first occurrence of $m=m(\delta)$.

$$\left\|\mathcal{S}f_{m,\delta}-g_\delta\right\|_{\mathcal{L}^2(\mathcal{D})}\le\sigma\delta. \tag{46}$$

where $\sigma>1$ be a fixed constant an $\left\|g_\delta\right\|\ge\sigma>0$.

Lemma 6. *Let* $\mathcal{Y}(m)=\left\|\mathcal{S}f_{m,\delta}-g_\delta\right\|_{\mathcal{L}^2(\mathcal{D})}$. *Then we declare that:*

a. $\mathcal{Y}(m)$ *is a continuous function.*

b. $\mathcal{Y}(m) \to 0$ *as* $m \to 0$.

c. $\mathcal{Y}(m) \to \|g_\delta\|_{\mathcal{L}^2(\mathcal{D})}$ *as* $m \to \infty$.

d. $\mathcal{Y}(m)$ *is a strictly increasing function, for any* $m \in (0, +\infty)$.

The proof of the above lemma is simple and completely similar to that in [39].

Proof of Lemma 6. From (15) and (26), our proof starts with the observation that:

$$\mathcal{Y}(m) = \Big\| \sum_{n=1}^{+\infty} \big(1 - \xi |Q\lambda_n^{-1}|^{\alpha+1}\big)^m \Big(\int_{\mathcal{D}} g_\delta(x) e_n(x) dx \Big) e_n(x) \Big\|_{L^2(\mathcal{D})}. \tag{47}$$

□

Lemma 7. *Assume that* (47) *holds, the regularization parameter* m *satisfies*

$$m \le \big(T^\beta \Phi_1 \beta^{-1}\big)^{\frac{\alpha+1}{1+r}} Q^{-(\alpha+1)} \Big(\frac{1+r}{\xi}\Big)(\sigma-1)^{-\frac{\alpha+1}{1+r}} \Big(\frac{E}{\delta}\Big)^{\frac{\alpha+1}{1+r}}. \tag{48}$$

Proof of Lemma 7. From (26), we have:

$$R_m g(x) = \sum_{n=1}^{+\infty} \frac{1 - \big(1 - \xi |Q\lambda_n^{-1}|^{\alpha+1}\big)^m}{\mathcal{P}_n(k, \beta, T, \Phi)} \Big(\int_{\mathcal{D}} g(x) e_n(x) dx \Big) e_n(x), \tag{49}$$

and

$$\|\mathcal{S}R_m g - g\|_{\mathcal{L}^2(\mathcal{D})} = \Big\| \sum_{n=1}^{+\infty} \big(1 - \xi |Q\lambda_n^{-1}|^{\alpha+1}\big)^m \Big(\int_{\mathcal{D}} g(x) e_n(x) dx \Big) \Big\|_{\mathcal{L}^2(\mathcal{D})}. \tag{50}$$

It is easy to see that $\Big(1 - \xi |Q\lambda_n^{-1}|^{\alpha+1}\Big) \le 1$, by using the (46), we get :

$$\begin{aligned} \|\mathcal{S}R_{m-1} g - g\|_{\mathcal{L}^2(\mathcal{D})} &\ge \|\mathcal{S}R_{m-1} g_\delta - g_\delta\|_{\mathcal{L}^2(\mathcal{D})} - \|(\mathcal{S}R_{m-1} - I)(g - g_\delta)\|_{\mathcal{L}^2(\mathcal{D})} \\ &\ge \sigma\delta - \|(\mathcal{S}R_{m-1} - I)\|_{\mathcal{L}^2(\mathcal{D})} \ge \sigma\delta - \delta \ge (\sigma - 1)\delta. \end{aligned} \tag{51}$$

On the other hand, from the reviews above, add the Lemma 6, we receive:

$$\begin{aligned} \|\mathcal{S}R_{m-1} g - g\|_{\mathcal{L}^2(\mathcal{D})} &= \Big\| \sum_{n=1}^{+\infty} \big(1 - \xi |Q\lambda_n^{-1}|^{\alpha+1}\big)^{m-1} \Big(\int_{\mathcal{D}} g(x) e_n(x) dx \Big) \Big\|_{\mathcal{L}^2(\mathcal{D})} \\ &\le \Big\| \sum_{n=1}^{+\infty} \big(1 - \xi |Q\lambda_n^{-1}|^{\alpha+1}\big)^{m-1} \lambda_n^{-r} \, \mathcal{P}_n(k, \beta, T, \Phi) \frac{\lambda_n^r \Big(\int_{\mathcal{D}} g(x) e_n(x) dx \Big)}{\mathcal{P}_n(k, \beta, T, \Phi)} \Big\|_{L^2(\mathcal{D})} \\ &\le \Big\| \sum_{n=1}^{+\infty} \big(1 - \xi |Q\lambda_n^{-1}|^{\alpha+1}\big)^{m-1} \lambda_n^{-1-r} \, \big(T^\beta \Phi_1 \beta^{-1}\big) \lambda_n^r \Big(\int_{\mathcal{D}} f(x) e_n(x) dx \Big) \Big\|_{\mathcal{L}^2(\mathcal{D})} \\ &\le \big(T^\beta \Phi_1 \beta^{-1}\big) \mathcal{B} Q^{-(1+r)} \xi^{-\frac{1+r}{\alpha+1}} (1+r)^{\frac{1+r}{\alpha+1}} m^{-\frac{1+r}{\alpha+1}}, \end{aligned} \tag{52}$$

whereby

$$(\sigma - 1)\, \delta \le \big(T^\beta \Phi_1 \beta^{-1}\big) \mathcal{B} Q^{-(1+r)} \xi^{-\frac{1+r}{\alpha+1}} (1+r)^{\frac{1+r}{\alpha+1}} m^{-\frac{1+r}{\alpha+1}}. \tag{53}$$

From (53), we conclude that:

$$m \le \big(T^\beta \Phi_1 \beta^{-1}\big)^{\frac{\alpha+1}{1+r}} Q^{-(\alpha+1)} \Big(\frac{1+r}{\xi}\Big)(\sigma-1)^{-\frac{\alpha+1}{1+r}} \Big(\frac{E}{\delta}\Big)^{\frac{\alpha+1}{1+r}}. \tag{54}$$

□

Theorem 4. *Let the condition* $\|g_\delta - g\|_{\mathcal{L}^2(\mathcal{D})} \leq \delta$ *and* $f \in \mathbb{X}^r(\mathcal{D})$ *hold, and the parameter regularization m is found in the Formula* (46)*, then it gives:*

$$\|f_{m,\delta} - f\|_{\mathcal{L}^2(\mathcal{D})} \leq \text{ is of order } \delta^{\frac{r}{r+1}}. \tag{55}$$

Proof of Theorem 4. Using the triangle inequality, we have:

$$\|f_{m,\delta} - f\|_{\mathcal{L}^2(\mathcal{D})} \leq \|f_{m,\delta} - f_m\|_{\mathcal{L}^2(\mathcal{D})} + \|f_m - f\|_{\mathcal{L}^2(\mathcal{D})}. \tag{56}$$

The proof in the Formula (39) has given us:

$$\|f_{m,\delta} - f_m\|_{\mathcal{L}^2(\mathcal{D})} \leq \delta m^{\frac{1}{\alpha+1}} \xi^{\frac{1}{\alpha+1}} \mathcal{V}_{\alpha,\beta}(Q, \lambda_1, T, \Phi_0, f). \tag{57}$$

Applying the Lemma 7, we get:

$$\begin{aligned}\|f_{m,\delta} - f_m\|_{\mathcal{L}^2(\mathcal{D})} \leq{}& \delta^{\frac{r}{r+1}} \mathcal{B}^{\frac{1}{r+1}} \xi^{\frac{1}{\alpha+1}} \left(T^\beta \Phi_1 \beta^{-1}\right)^{\frac{1}{1+r}} \\ &\times \left(Q^{r+1}(\sigma-1)\right)^{-\frac{1}{1+r}} \left(\frac{1+r}{\xi}\right)^{\frac{1}{\alpha+1}} \mathcal{V}_{\alpha,\beta}(Q, \lambda_1, T, \Phi_0, f).\end{aligned} \tag{58}$$

Now, using the Holder's inequality and the Formula (46), and results obtained from the Lemma 2, we get the estimate of error $\|f_m - f\|_{\mathcal{L}^2(\Omega)}$ as follows:

$$\begin{aligned}\|f_m - f\|_{\mathcal{L}^2(\mathcal{D})} \leq{}& \left\| \sum_{n=1}^{+\infty} \left(\left[1 - \xi |Q\lambda_n^{-1}|^{\alpha+1}\right]^m \right)^{\frac{1}{r+1}} \frac{1}{|\mathcal{P}_n(k,\beta,T,\Phi)|} \left(\int_{\mathcal{D}} g(x) e_n(x) dx \right)^{\frac{1}{r+1}} \right\|_{\mathcal{L}^2(\mathcal{D})} \\ &\times \left\| \sum_{n=1}^{+\infty} \left(\left[1 - \xi |Q\lambda_n^{-1}|^{\alpha+1}\right]^m \right)^{\frac{r}{r+1}} \left(\int_{\mathcal{D}} g(x) e_n(x) dx \right)^{\frac{r}{r+1}} \right\|_{\mathcal{L}^2(\mathcal{D})} \\ \leq{}& \left\| \sum_{n=1}^{+\infty} \left(\left[1 - \xi |Q\lambda_n^{-1}|^{\alpha+1}\right]^m \right) \frac{1}{|\mathcal{P}_n(k,\beta,T,\Phi)|} \left(\int_{\mathcal{D}} g(x) e_n(x) dx \right) \right\|_{\mathcal{L}^2(\mathcal{D})}^{\frac{1}{r+1}} \\ &\times \left\| \sum_{n=1}^{+\infty} \left(\left[1 - \xi |Q\lambda_n^{-1}|^{\alpha+1}\right]^m \right) \left(\int_{\mathcal{D}} g(x) e_n(x) dx \right) \right\|_{L^2(\mathcal{D})}^{\frac{r}{r+1}} \\ \leq{}& \underbrace{\left\| \sum_{n=1}^{+\infty} \left(\left[1 - \xi |Q\lambda_n^{-1}|^{\alpha+1}\right]^m \right) \frac{\lambda_n^{-r}}{|\mathcal{P}_n(k,\beta,T,\Phi)|^r} \frac{\lambda_n^r \left(\int_{\mathcal{D}} g(x) e_n(x) dx \right)}{|\mathcal{P}_n(k,\beta,T,\Phi)|} \right\|_{\mathcal{L}^2(\mathcal{D})}^{\frac{1}{r+1}}}_{\mathcal{H}_3} \\ &\times \underbrace{\left\| \sum_{n=1}^{+\infty} \left(\left[1 - \xi |Q\lambda_n^{-1}|^{\alpha+1}\right]^m \right) \left(\int_{\mathcal{D}} (g(x) - g_\delta(x) + g_\delta(x)) e_n(x) dx \right) \right\|_{L^2(\mathcal{D})}^{\frac{r}{r+1}}}_{\mathcal{H}_4},\end{aligned} \tag{59}$$

From (59), we have estimate of $\mathcal{H}_3$ and $\mathcal{H}_4$ as follows:

$$\begin{aligned}
\mathcal{H}_3 &\leq \Big\| \sum_{j=1}^{+\infty} \Big(\Big[1 - \xi \big| Q\lambda_n^{-1} \big|^{\alpha+1} \Big]^m \Big) \frac{\lambda_n^{-r}}{\big|\mathcal{P}_n(k,\beta,T,\Phi)\big|^r} \lambda_n^r \Big(\int_{\mathcal{D}} f(x) e_n(x) dx \Big) \Big\|_{\mathcal{L}^2(\mathcal{D})}^{\frac{1}{r+1}} \\
&\leq \Big\| \sum_{n=1}^{+\infty} \frac{\lambda_n^{-r}}{\big|\mathcal{P}_n(k,\beta,T,\Phi)\big|^r} \lambda_n^r \Big(\int_{\mathcal{D}} f(x) e_n(x) dx \Big) \Big\|_{\mathcal{L}^2(\Omega)}^{\frac{1}{r+1}} \\
&\leq \Big\| \sum_{n=1}^{+\infty} \frac{1}{\big|\Phi_0 \big[1 - \exp\big(\frac{\lambda_1}{1+k\lambda_1} \frac{T^\beta}{\beta}\big)\big]\big|^r} \lambda_n^r \Big(\int_{\mathcal{D}} f(x) e_n(x) dx \Big) \Big\|_{\mathcal{L}^2(\mathcal{D})}^{\frac{1}{r+1}} \\
&\leq \mathcal{B}^{\frac{1}{r+1}} \Big| \Phi_0 \big[1 - \exp\big(\frac{\lambda_1}{1+k\lambda_1} \frac{T^\beta}{\beta}\big)\big] \Big|^{-\frac{r}{r+1}}. \qquad (60)
\end{aligned}$$

Next, $\mathcal{H}_4$ can be bounded.

$$\begin{aligned}
\mathcal{H}_4 &\leq \Big\| \sum_{n=1}^{+\infty} \Big(\Big[1 - \xi \big| Q\lambda_n^{-1} \big|^{\alpha+1} \Big]^m \Big) \Big(\int_{\mathcal{D}} \big(g(x) - g_\delta(x)\big) e_n(x) dx \Big) \\
&\qquad + \sum_{n=1}^{+\infty} \Big(\Big[1 - \xi \big| Q\lambda_n^{-1} \big|^{\alpha+1} \Big]^m \Big) \Big(\int_{\mathcal{D}} g_\delta(x) e_n(x) dx \Big) \Big\|_{\mathcal{L}^2(\mathcal{D})}^{\frac{r}{r+1}} \\
&\leq \Big\| \sum_{n=1}^{+\infty} \Big(\Big[1 - \xi \big| Q\lambda_n^{-1} \big|^{\alpha+1} \Big]^m \Big) \Big(\int_{\mathcal{D}} \big(g(x) - g_\delta(x)\big) e_n(x) dx \Big) + \sigma\delta \Big\|_{\mathcal{L}^2(\mathcal{D})}^{\frac{r}{r+1}} \\
&\leq \delta^{\frac{r}{r+1}} (1+\sigma)^{\frac{r}{r+1}}. \qquad (61)
\end{aligned}$$

Combining (59) to (61), it gives :

$$\big\| f_{m,\delta} - f \big\|_{\mathcal{L}^2(\mathcal{D})} \leq \delta^{\frac{r}{r+1}} \, \mathcal{B}^{\frac{1}{r+1}} \Big| \Phi_0 \big[1 - \exp\big(\frac{\lambda_1}{1+k\lambda_1} \frac{T^\beta}{\beta}\big)\big] \Big|^{-\frac{r}{r+1}} (1+\sigma)^{\frac{r}{r+1}}. \qquad (62)$$

From (58) and (62), we conclude that:

$$\big\| f_{m,\delta} - f \big\|_{\mathcal{L}^2(\mathcal{D})} \leq \delta^{\frac{r}{r+1}} \mathcal{B}^{\frac{1}{r+1}} \big(\widetilde{\mathcal{J}_1} + \widetilde{\mathcal{J}_2} \big), \qquad (63)$$

whereby

$$\begin{aligned}
\widetilde{\mathcal{J}_1} &= \Big| \Phi_0 \big[1 - \exp\big(\frac{\lambda_1}{1+k\lambda_1} \frac{T^\beta}{\beta}\big)\big] \Big|^{-\frac{r}{r+1}} (1+\sigma)^{\frac{r}{r+1}}, \\
\widetilde{\mathcal{J}_2} &= \xi^{\frac{1}{\alpha+1}} \big(T^\beta \Phi_1 \beta^{-1} \big)^{\frac{1}{1+r}} \big(Q^{r+1}(\sigma - 1) \big)^{-\frac{1}{1+r}} \Big(\frac{1+r}{\xi} \Big)^{\frac{1}{\alpha+1}} \mathcal{V}_{\alpha,\beta}\big(Q, \lambda_1, T, \Phi_0, f\big). \qquad (64)
\end{aligned}$$

□

In the next section, we provide the error estimation between the exact solution and the regularized solution by the Fourier truncation method.

5. Regularization of Inverse Source in $\mathcal{L}^q(\mathcal{D})$ Space

Theorem 5. *Let us take* $(\Phi_\delta, g_\delta) \in \mathcal{L}^q(0,T) \times \mathcal{L}^q(\mathcal{D})$ *such that* $\Phi_\delta > \Phi_0 > 0$ *for any* $0 \leq t \leq T$ *for any* $\frac{1}{\beta} < q < 2$ *and*

$$\big\| \Phi_\delta - \Phi \big\|_{\mathcal{L}^q(0,T)} + \big\| g_\delta - g \big\|_{\mathcal{L}^q(\mathcal{D})} \leq \delta. \qquad (65)$$

Let us assume that $\Phi_\delta \in \mathbb{X}^{r+\gamma}(\mathcal{D})$ *for* $r > 0$ *and* $0 < \gamma < \frac{N}{4}$. *Constructing a regularized solution as follows:*

$$f_\delta(x)=\sum_{\lambda_n<\mathcal{N}_\delta}\left[\int_0^T s^{\beta-1}\exp\left(\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta-T^\beta}{\beta}\right)\Phi_\delta(s)ds\right]^{-1}(1+k\lambda_n)\Big(\int_{\mathcal{D}} g_\delta(x)e_n(x)dx\Big)e_n(x). \tag{66}$$

Then the error estimate is bounded by:

$$\left\|f_\delta-f\right\|_{\mathcal{L}^{\frac{2N}{N-4\gamma}}(\mathcal{D})}\lesssim|\mathcal{N}_\delta|^{-r}\|f\|_{\mathbb{X}^{\gamma+r}(\mathcal{D})}+\mathcal{P}_1\mathcal{N}_\delta\delta\|f\|_{\mathbb{X}^{\gamma}(\mathcal{D})}+\mathcal{P}_2(\mathcal{N}_\delta)^{2+2\gamma+\frac{N}{q}-\frac{N}{2}}\delta. \tag{67}$$

here $\mathcal{N}_\delta$ satisfies that:

$$\lim_{\delta\to0}\mathcal{N}_\delta\delta=\lim_{\delta\to0}\left((\mathcal{N}_\delta)^{\gamma+1+\frac{N}{2q}-\frac{N}{4}}\delta\right)=0,\quad\lim_{\delta\to0}\mathcal{N}_\delta=+\infty. \tag{68}$$

Remark 1. *If $\mathcal{N}_\delta=\delta^{\frac{\varepsilon-1}{\gamma+1+\frac{N}{2q}-\frac{N}{4}}}$, for $0<\varepsilon<1$. then $\left\|f_\delta-f\right\|_{\mathcal{L}^{\frac{2N}{N-4\gamma}}(\mathcal{D})}\to0$ as $\delta\to0$.*

Proof of Theorem 5. It is clear that:

$$\|f_\delta-f\|_{\mathbb{X}^\gamma(\mathcal{D})}\le\|f_{2,\delta}-f\|_{\mathbb{X}^\gamma(\mathcal{D})}+\|f_{2,\delta}-f_{1,\delta}\|_{\mathbb{X}^\gamma(\mathcal{D})}+\|f_{1,\delta}-f_\delta\|_{\mathbb{X}^\gamma(\mathcal{D})}. \tag{69}$$

where we denote some following functions:

$$\begin{aligned}f_{1,\delta}(x)=&\sum_{\lambda_n\le\mathcal{N}_\delta}\left[\int_0^T s^{\beta-1}\exp\left(\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta-T^\beta}{\beta}\right)\Phi_\delta(s)ds\right]^{-1}\\&\times(1+k\lambda_n)\Big(\int_{\mathcal{D}}g(x)e_n(x)dx\Big)e_n(x),\end{aligned} \tag{70}$$

and

$$\begin{aligned}f_{2,\delta}(x)=&\sum_{\lambda_n\le\mathcal{N}_\delta}\left[\int_0^T s^{\beta-1}\exp\left(\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta-T^\beta}{\beta}\right)\Phi(s)ds\right]^{-1}\\&\times(1+k\lambda_n)\Big(\int_{\mathcal{D}}g(x)e_n(x)dx\Big)e_n(x).\end{aligned} \tag{71}$$

Now, we need to establish the upper bound of the expressions on the right of (13). For convenience, we consider the following step.

Step 1: Estimate of $\|f_{2,\delta}-f\|_{\mathbb{X}^\gamma(\mathcal{D})}$, let us recall the function f as (14). This expression together with the Formula (71) gives us the claim of the following difference:

$$\begin{aligned}&f(x)-f_{2,\delta}(x)\\&=\sum_{\lambda_n>\mathcal{N}_\delta}\left[\int_0^T s^{\beta-1}\exp\left(\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta-T^\beta}{\beta}\right)\Phi_\delta(s)ds\right]^{-1}(1+k\lambda_n)\Big(\int_{\mathcal{D}}g(x)e_n(x)dx\Big)e_n(x)\\&=\sum_{\lambda_n>\mathcal{N}_\delta}\Big(\int_{\mathcal{D}}f(x)e_n(x)dx\Big)e_n(x).\end{aligned} \tag{72}$$

Using the Parseval equality, the left hand side of (72) is calculated as follows:

$$\begin{aligned}\left\|f_{2,\delta}-f\right\|^2_{\mathbb{X}^\gamma(\mathcal{D})} &= \sum_{\lambda_n>\mathcal{N}_\delta}\lambda_n^{2\gamma}\Big(\int_{\mathcal{D}} f(x)e_n(x)dx\Big)^2\\ &= \sum_{\lambda_n>\mathcal{N}_\delta}\lambda_n^{-2r}\lambda_n^{2\gamma+2r}\Big(\int_{\mathcal{D}} f(x)e_n(x)dx\Big)^2.\end{aligned} \tag{73}$$

It is obvious to see that $\lambda_n^{-2r}\leq|\mathcal{N}_\delta|^{-2r}$ if $\lambda_n>\mathcal{N}_\delta$ and $m>0$. Therefore, we have:

$$\left\|f_{2,\delta}-f\right\|^2_{\mathbb{X}^\gamma(\mathcal{D})}\leq|\mathcal{N}_\delta|^{-2r}\sum_{\lambda_n>\mathcal{N}_\delta}\lambda_j^{2\gamma+2r}\Big(\int_{\mathcal{D}} f(x)e_n(x)dx\Big)^2=|\mathcal{N}_\delta|^{-2r}\|f\|^2_{\mathbb{X}^{\gamma+r}(\mathcal{D})}, \tag{74}$$

which leads to

$$\left\|f_{2,\delta}-f\right\|_{\mathbb{X}^\gamma(\mathcal{D})}\leq|\mathcal{N}_\delta|^{-r}\|f\|_{\mathbb{X}^{\gamma+r}(\mathcal{D})}. \tag{75}$$

Step 2: Estimate of $\left\|f_{2,\delta}-f_{1,\delta}\right\|_{\mathbb{X}^\gamma(\mathcal{D})}$, we get

$$\begin{aligned}&f_{2,\delta}(x)-f_{1,\delta}(x)\\ &=\sum_{\lambda_n\leq\mathcal{N}_\delta}\frac{\int_0^T s^{\beta-1}\exp\Big(\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta-T^\beta}{\beta}\Big)\big(\Phi_\delta(s)-\Phi(s)\big)ds\ \big(1+k\lambda_n\big)\Big(\int_{\mathcal{D}} g(x)e_n(x)dx\Big)e_n(x)}{\Big[\int_0^T s^{\beta-1}\exp\Big(\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta-T^\beta}{\beta}\Big)\Phi_\delta(s)ds\Big]\ \Big[\int_0^T s^{\beta-1}\exp\Big(\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta-T^\beta}{\beta}\Big)\Phi(s)ds\Big]}.\end{aligned} \tag{76}$$

From (76), we know that:

$$\begin{aligned}&f_{2,\delta}(x)-f_{1,\delta}(x)\\ &=\sum_{\lambda_n\leq\mathcal{N}_\delta}\frac{\int_0^T s^{\beta-1}\exp\Big(\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta-T^\beta}{\beta}\Big)\big(\Phi_\delta(s)-\Phi(s)\big)ds\ \Big(\int_{\mathcal{D}} f(x)e_n(x)dx\Big)e_n(x)}{\int_0^T s^{\beta-1}\exp\Big(\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta-T^\beta}{\beta}\Big)\Phi_\delta(s)ds}.\end{aligned} \tag{77}$$

By applying the Holder inequality, we receive:

$$\begin{aligned}&\left\|f_{2.\delta}-f_{1,\delta}\right\|^2_{\mathbb{X}^\gamma(\mathcal{D})}\\ &=\sum_{\lambda_n\leq\mathcal{N}_\delta}\left[\frac{\int_0^T s^{\beta-1}\exp\Big(\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta-T^\beta}{\beta}\Big)\big(\Phi_\delta(s)-\Phi(s)\big)ds}{\int_0^T s^{\beta-1}\exp\Big(\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta-T^\beta}{\beta}\Big)\Phi_\delta(s)ds}\right]^2\lambda_n^{2\gamma}\Big(\int_{\mathcal{D}} f(x)e_n(x)dx\Big)^2.\end{aligned} \tag{78}$$

Next, we have the estimate $\int_0^T s^{\beta-1}\exp\Big(\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta-T^\beta}{\beta}\Big)\big(\Phi_\delta(s)-\Phi(s)\big)ds$ as follows:

$$\begin{aligned}&\left|\int_0^T s^{\beta-1}\exp\Big(\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta-T^\beta}{\beta}\Big)\big(\Phi_\delta(s)-\Phi(s)\big)ds\right|\\ &\quad\leq\Big(\int_0^T\big|\Phi_\delta(s)-\Phi(s)\big|^q ds\Big)^{\frac{1}{q}}\Big(\int_0^T s^{q^*(\beta-1)}\exp\Big(q^*\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta-T^\beta}{\beta}\Big)ds\Big)^{\frac{1}{q^*}},\end{aligned} \tag{79}$$

where $q^* = 1 + \frac{1}{s-1}$, this implies that:

$$\Big(\int_0^T |\Phi_\delta(s) - \Phi(s)|^q ds \Big)^{\frac{1}{q}} = \big\| \Phi_\delta - \Phi \big\|_{\mathcal{L}^q(0,T)}, \tag{80}$$

and through some basic calculations, we obtain:

$$\begin{aligned} &\Big(\int_0^T s^{q^*(\beta-1)} \exp\Big(q^* \frac{\lambda_n}{1+k\lambda_n} \frac{s^\beta - T^\beta}{\beta} \Big) ds \Big) \leq \int_0^T s^{q^*(\beta-1)} ds \\ &= \frac{T^{q^*(\beta-1)+1}}{q^*(\beta-1)+1} = \frac{q-1}{\beta q - 1} T^{\frac{\beta q - 1}{q-1}}, \end{aligned} \tag{81}$$

whereby we note that $q > \frac{1}{\beta}$ and we also have used the fact that $\exp\Big(q^* \frac{\lambda_n}{1+k\lambda_n} \frac{s^\beta - T^\beta}{\beta} \Big) \leq 1$. Combining three evaluations (79), (80) and (81), we derive that the following estimate:

$$\Big| \int_0^T s^{\beta-1} \exp\Big(\frac{\lambda_n}{1+k\lambda_n} \frac{s^\beta - T^\beta}{\beta} \Big) \big(\Phi_\delta(s) - \Phi(s) \big) ds \Big| \leq \big\| \Phi_\delta - \Phi \big\|_{\mathcal{L}^q(0,T)}. \tag{82}$$

Next, applying the Lemma 1 and the Lemma 2, we have estimated

$$\begin{aligned} &\int_0^T s^{\beta-1} \exp\Big(\frac{\lambda_n}{1+k\lambda_n} \frac{s^\beta - T^\beta}{\beta} \Big) \Phi_\delta(s) ds \\ &\geq \frac{\Phi_0}{4} \Big(\frac{1+k\lambda_n}{\lambda_n} \Big) \Big[1 - \exp\Big(- \frac{\lambda_1}{1+k\lambda_1} \frac{T^\beta}{\beta} \Big) \Big] \\ &\geq \frac{\Phi_0}{4} \frac{1}{\lambda_n} \Big[1 - \exp\Big(- \frac{\lambda_1}{1+k\lambda_1} \frac{T^\beta}{\beta} \Big) \Big]. \end{aligned} \tag{83}$$

From the two observation above, we assert that :

$$\frac{\int_0^T s^{\beta-1} \exp\Big(\frac{\lambda_n}{1+k\lambda_n} \frac{s^\beta - T^\beta}{\beta} \Big) \big(\Phi_\delta(s) - \Phi(s) \big) ds}{\int_0^T s^{\beta-1} \exp\Big(\frac{\lambda_n}{1+k\lambda_n} \frac{s^\beta - T^\beta}{\beta} \Big) \Phi_\delta(s) ds} \leq \mathcal{P}_1 \lambda_n \big\| \Phi_\delta - \Phi \big\|_{\mathcal{L}^q(0,T)}. \tag{84}$$

whereby $\mathcal{P}_1$ depends on $\Phi_0, \lambda_1, T, \beta, q$ and

$$\mathcal{P}_1 = \Big(\frac{q-1}{\beta q - 1} \Big)^{\frac{q-1}{q}} T^{\beta - \frac{1}{q}} \Big[\frac{\Phi_0}{4} \Big[1 - \exp\Big(- \frac{\lambda_1}{1+k\lambda_1} \frac{T^\beta}{\beta} \Big) \Big] \Big]^{-1}. \tag{85}$$

Combining (78) to (84), it gives:

$$\big\| f_{2,\delta} - f_{1,\delta} \big\|^2_{\mathbb{X}^\gamma(\mathcal{D})} \leq |\mathcal{P}_1|^2 \big\| \Phi_\delta - \Phi \big\|^2_{\mathcal{L}^q(0,T)} \sum_{\lambda_n \leq \mathcal{N}_\delta} \lambda_n^{2\gamma+2} \Big(\int_{\mathcal{D}} f(x) e_n(x) dx \Big)^2 \tag{86}$$

We assume that the finite sum $\sum_{\lambda_n \leq \mathcal{N}_\delta} \lambda_n^{2\gamma+2} \Big(\int_{\mathcal{D}} f(x) e_n(x) dx \Big)^2$ is bounded by

$$|\mathcal{N}_\delta|^2 \sum_{\lambda_n \leq \mathcal{N}_\delta} \lambda_n^{2\gamma} \Big(\int_{\mathcal{D}} f(x) e_n(x) dx \Big)^2 \leq |\mathcal{N}_\delta|^2 \|f\|^2_{\mathbb{X}^\gamma(\mathcal{D})}. \tag{87}$$

Therefore, we conclude that:

$$\left\|f_{2.\delta}-f_{1,\delta}\right\|_{\mathbb{X}^{\gamma}(\mathcal{D})}\leq \mathcal{P}_1\delta\mathcal{N}_\delta\|f\|_{\mathbb{X}^{\gamma}(\mathcal{D})}. \tag{88}$$

where $\mathcal{P}_1$ is defined in (85).

Step 3: Estimate of $\left\|f_{1,\delta}-f\right\|_{\mathbb{X}^{\gamma}(\mathcal{D})}$, due to Formulas (14) and (70), so

$$\begin{aligned} f_{1,\delta}(x)-f(x)=&\sum_{\lambda_n\leq\mathcal{N}_\delta}\Big[\int_0^T s^{\beta-1}\exp\Big(\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta-T^\beta}{\beta}\Big)\Phi_\delta(s)ds\Big]^{-1}\\ &\times(1+k\lambda_n)\Big(\int_{\mathcal{D}}(g_\delta(x)-g(x))e_n(x)dx\Big)e_n(x). \end{aligned} \tag{89}$$

Using the Parseval's equality, and taking the norm of both sides, we obtain that:

$$\begin{aligned} \left\|f_{1,\delta}-f\right\|^2_{\mathbb{X}^{\gamma}(\mathcal{D})}=&\sum_{\lambda_n\leq\mathcal{N}_\delta}\Big[\int_0^T s^{\beta-1}\exp\Big(\frac{\lambda_n}{1+k\lambda_n}\frac{s^\beta-T^\beta}{\beta}\Big)\Phi_\delta(s)ds\Big]^{-2}\\ &\times(1+k\lambda_n)^2\Big(\int_{\mathcal{D}}(g_\delta(x)-g(x))e_n(x)dx\Big)^2. \end{aligned} \tag{90}$$

Because of the inequality (83), one has :

$$\begin{aligned} &\left\|f_{1,\delta}-f\right\|^2_{\mathbb{X}^{\gamma}(\mathcal{D})}\\ &=\left[\frac{\Phi_0}{4}\Big[1-\exp\Big(-\frac{\lambda_1}{1+k\lambda_1}\frac{T^\beta}{\beta}\Big)\Big]\right]^{-2}\sum_{\lambda_n\leq\mathcal{N}_\delta}\lambda_n^{2+2\gamma}\Big(\int_{\mathcal{D}}(g_\delta(x)-g(x))e_n(x)dx\Big)^2. \end{aligned} \tag{91}$$

Continuing to deal with the finite series on the right above, we have:

$$\begin{aligned} &\sum_{\lambda_n\leq\mathcal{N}_\delta}\lambda_n^{2+2\gamma}\Big(\int_{\mathcal{D}}(g_\delta(x)-g(x))e_n(x)dx\Big)^2\\ &=\sum_{\lambda_n\leq\mathcal{N}_\delta}\lambda_n^{2+2\gamma+\frac{N}{q}-\frac{N}{2}}\lambda_n^{\frac{Nq-2N}{2q}}\Big(\int_{\mathcal{D}}(g_\delta(x)-g(x))e_n(x)dx\Big)^2\\ &\leq(\mathcal{N}_\delta)^{2+2\gamma+\frac{N}{q}-\frac{N}{2}}\sum_{\lambda_n\leq\mathcal{N}_\delta}\lambda_n^{\frac{Nq-2N}{2q}}\Big(\int_{\mathcal{D}}(g_\delta(x)-g(x))e_n(x)dx\Big)^2\\ &\leq(\mathcal{N}_\delta)^{2+2\gamma+\frac{N}{q}-\frac{N}{2}}\left\|g_\delta-g\right\|^2_{\mathbb{X}^{\frac{Nq-2N}{4q}}(\mathcal{D})}. \end{aligned} \tag{92}$$

Since $1<q<2$, we know that $\mathcal{L}^q(\Omega)\hookrightarrow\mathbb{X}^{\frac{Nq-2N}{4q}}(\mathcal{D})$. Therefore, we get that:

$$\|g_\delta-g\|^2_{\mathbb{X}^{\frac{Nq-2N}{4q}}(\mathcal{D})}\leq C(N,q)\left\|g_\delta-g\right\|^2_{\mathcal{L}^q(\mathcal{D})}\leq C(N,q)\delta. \tag{93}$$

Combining (91) to (93), one has:

$$\|f_{1,\delta}-f\|_{\mathbb{X}^{\gamma}(\mathcal{D})}\leq\mathcal{P}_2(\mathcal{N}_\delta)^{2+2\gamma+\frac{N}{q}-\frac{N}{2}}\delta. \tag{94}$$

where $P_2=\left[\frac{\Phi_0}{4}\Big[1-\exp\Big(-\frac{\lambda_1}{1+k\lambda_1}\frac{T^\beta}{\beta}\Big)\Big]\right]^{-2}C(N,q)$. Finally, from *Step 1* to *Step3*, we can conclude that:

$$\left\|f_\delta-f\right\|_{\mathbb{X}^{\gamma}(\mathcal{D})}\leq|\mathcal{N}_\delta|^{-r}\|f\|_{\mathbb{X}^{\gamma+r}(\mathcal{D})}+\mathcal{P}_1\delta\mathcal{N}_\delta\|f\|_{\mathbb{X}^{\gamma}(\mathcal{D})}+\mathcal{P}_2(\mathcal{N}_\delta)^{2+2\gamma+\frac{N}{q}-\frac{N}{2}}\delta. \tag{95}$$

By using the Lemma 3, since $0 < \gamma < \frac{N}{4}$, with Sobolev embedding $\mathbb{X}^{\gamma}(\mathcal{D}) \hookrightarrow \mathcal{L}^{\frac{2N}{N-4\gamma}}(\mathcal{D})$, we have the results as in (67). □

6. Simulation

In this section, we present one numerical example. By choosing $\Omega = (0, \pi)$, $T = 1$, $\beta = 0.5$, $k = 0.8$, and $\alpha = 0.5$, and $r = 1$ are shown in this section, respectively. In this section, we consider the problem as follows:

$$\frac{{}^{C}\partial^{\beta}}{\partial t^{\beta}}(u - k\Delta u) - \Delta u(x,t) = \Phi(t)f(x),\ (x,t) \in (0,\pi) \times (0,1), \tag{96}$$

where $\frac{{}^{C}\partial^{\beta}}{\partial t^{\beta}}$ is the conformable derivative is given by [23]. In this calculation, we chose the operator $\Delta u = \frac{\partial^2}{\partial x^2}u$, we have chosen $\lambda_n = n^2,\ n = 1, 2, \ldots$ and $e_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$, respectively. We have the function,

$$g(x) = \sqrt{\frac{2}{\pi}}\Big(\sin(3x) + \sin(4x)\Big),\ \ \theta(t) = 1. \tag{97}$$

In general, the numerical procedure is summarized in the following steps:

Step 1: Finite difference to discretize the time and spatial variable for $x \in (0, \pi)$ as follows:

$$x_k = k\Delta x,\ \ 0 \le k \le N,\ \ \Delta x = \frac{\pi}{N}.$$

Step 2: The input data g is noised by observation data g_{δ} such that:

$$g_{\delta} = g + \frac{1}{\pi}\delta(2\text{rand}(\cdot) - 1). \tag{98}$$

From (14), we have the exact solution,

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{N}\left[\int_0^T s^{\beta-1}\exp\left(\frac{\lambda_n}{1+k\lambda_n}\frac{s^{\beta}-T^{\beta}}{\beta}\right)\Phi(s)ds\right]^{-1} \times (1+k\lambda_n)\Big(\int_0^{\pi} g(x)\sin(nx)dx\Big)\sin(nx). \tag{99}$$

From (26) and (27), by choosing the regularization parameter as in the number Formula (33), in the case of a priori parameter choice rule, and Formula (48), in the case of a posteriori parameter choice rule, where N is a large enough truncation number, we have the regularized solution with Modified Fractional Landweber as follows:

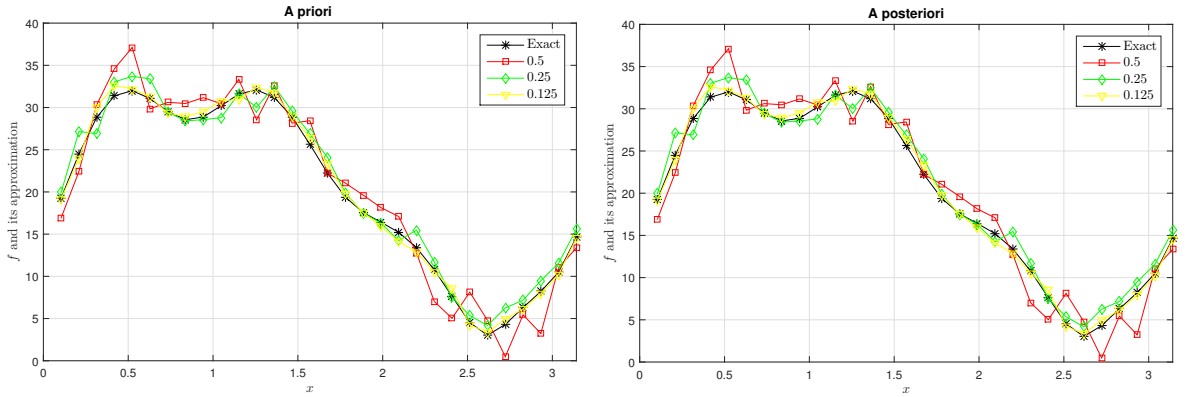
$$f_{m,\delta}(x) = \frac{2}{\pi}\sum_{n=1}^{N}\frac{1-\left(1-\xi\left|Q\lambda_n^{-1}\right|^{\alpha+1}\right)^{m}}{\mathcal{P}_n(k,\beta,T,\Phi)}\Big(\int_0^{\pi} g_{\delta}(x)\sin(nx)dx\Big)\sin(nx). \tag{100}$$

whereby

$$\mathcal{P}_n(k,\beta,T,\Phi) = (1+k\lambda_n)^{-1}\left[\int_0^T s^{\beta-1}\exp\left(\frac{\lambda_n}{1+k\lambda_n}\frac{s^{\beta}-T^{\beta}}{\beta}\right)\Phi(s)ds\right]. \tag{101}$$

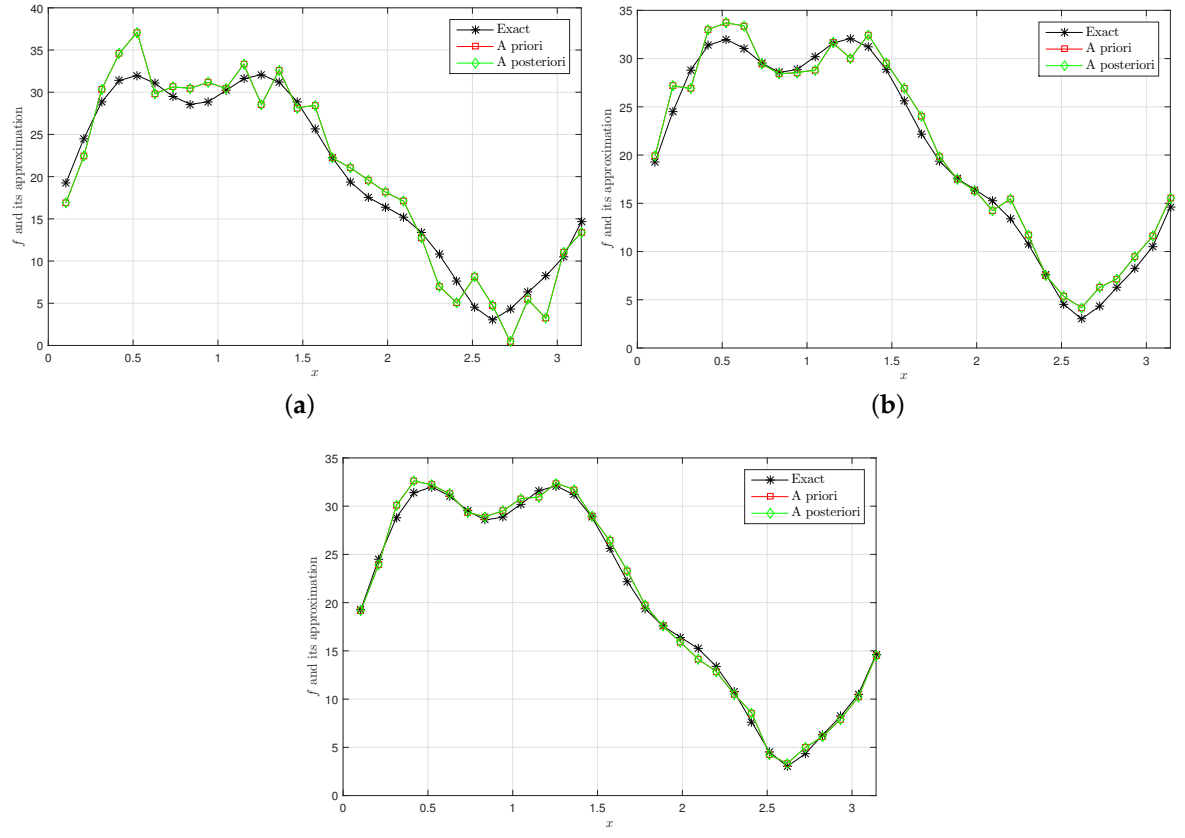
We choose $N = 50$ and $\delta = 0.5$, $\delta = 0.25$ and $\delta = 0.125$. Figure 1a shows the 2D graphs of the source function with the exact solution and its approximation for the case for the a

priori parameter choice rule. Figure 1b shows the error estimate between the exact solution and regularized solution for the a posteriori parameter choice rule. Figure 2a–c shows the 2D graphs comparing the convergent rate between the exact solution and its approximation under a priori and a posteriori parameter choice rules with noise levels $\delta = 0.5$, $\delta = 0.25$, and $\delta = 0.125$. From the observations above, the comparison with the results developed in theory (see evaluation (34) and (55)) shows that the convergence in these two cases is almost equivalent, illustrating that the proposed method is effective.



(**a**) (**b**)

Figure 1. The exact approximation for a priori (**a**) and a posteriori (**b**).



(**a**) (**b**)

(**c**)

Figure 2. A priori and a posteriori when (**a**) $\delta = 0.5$, (**b**) $\delta = 0.25$ and (**c**) $\delta = 0.125$, respectively.

Author Contributions: Methodology, Z.A.; software, H.D.B.; validation, L.D.L. and Z.A.; formal analysis, Z.A.; resources, H.D.B.; data curation, H.D.B.; writing—original draft preparation, L.D.L.; writing—review and editing, O.N.; project administration, O.N. All authors have read and agreed to the published version of the manuscript.

Funding: The author Le Dinh Long is supported by Van Lang University.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Alharbia, F.M.; Baleanu, D.; Abdelhalim, E. Physical properties of the projectile motion using the conformable derivative. *Chin. J. Phys.* **2019**, *58*, 18–28. [CrossRef]
2. Kilbas, A.A.; Marichev, O.I.; Samko, S.G. *Fractional Integrals and Derivatives (Theory and Applications)*, 1st ed.; CRC Press: Boca Raton, FL, USA, 1993.
3. Tuan, N.H.; Aghdam, Y.E.; Jafari, H.; Mesgarani, H. A novel numerical manner for two-dimensional space fractional diffusion equation arising in transport phenomena . *Numer. Methods Part. Differ. Equ.* **2021**, *37*, 1397–1406. [CrossRef]
4. Luc, N.H.; Jafari, H.; Kumam, P.; Tuan, N.H. On an initial value problem for time fractional pseudo-parabolic equation with Caputo derivarive. *Math. Methods Appl. Sci.* **2022**, 1–23. [CrossRef]
5. Au, V.V.; Jafari, H.; Hammouch, Z.; Tuan, N.H. On a final value problem for a nonlinear fractional pseudo-parabolic equation. *Electron. Res. Arch.* **2021**, *29*, 1709–1734.
6. Can, N.H.; Jafari, H.; Ncube, M.N. Fractional calculus in data fitting. *Alex. Eng. J.* **2020**, *59*, 3269–3274. [CrossRef]
7. Tuan, N.H.; Nemati, S.; Ganji, R.M.; Jafari, H. Numerical solution of multi-variable order fractional integro-differential equations using the Bernstein polynomials. *Eng. Comput.* **2022**, *38*, 139–147. [CrossRef]
8. Han, Y.; Xiong, X.; Xue, X. A fractional Landweber method for solving backward timefractional diffusion problem. *Comput. Math. Appl.* **2019**, *78*, 81–91. [CrossRef]
9. Yang, S.; Xiong, X.; Han, Y. A modified fractional Landweber method for a backward problem for the inhomogeneous time-fractional diffusion equation in a cylinder. *Int. J. Comput. Math.* **2020**, *97*, 2375–2393. [CrossRef]
10. Huynh, L.H.; Zhou, Y.; O'Regan, D.; Tuan, N.H. Fractional Landweber method for an initial inverse problem for time-fractional wave equations. *Appl. Anal.* **2021**, *100*, 860–878. [CrossRef]
11. Chung, W.S. Fractional Newton mechanics with conformable fractional derivative. *J. Comput. Appl. Math.* **2015**, *290*, 150–158. [CrossRef]
12. Morales-Delgado, V.F.; Gómez-Aguilar, J.F.; Escobar-Jiménez, R.F.; Taneco-Hernández, M.A. Fractional conformable derivatives of Liouville-Caputo type with low-fractionality. *Phys. A Stat. Mech. Appl.* **2018**, *503*, 424–438. [CrossRef]
13. He, S.; Sun, K.; Mei, X.; Yan, B.; Xu, S. Numerical analysis of a fractional-order chaotic system based on conformable fractional-order derivative. *Eur. Phys. J. Plus* **2017**, *132*, 36. [CrossRef]
14. Hama, M.F.; Rasul, R.R.Q.; Hammouch, Z.; Rasul, K.A.H.; Danane, J. Analysis of a stochastic SEIS epidemic model with the standard Brownian motion and Lévy jump. *Results Phys.* **2022**, *37*, 105477. [CrossRef]
15. Hammouch, Z.; Rasul, R.Q.R.; Ouakka, A.; Elazzouzi, A. Mathematical analysis and numerical simulation of the Ebola epidemic disease in the sense of conformable derivative. *Chaos Solitons Fractals* **2022**, *158*, 112006. [CrossRef]
16. Karthikeyan, K.; Murugapandian, G.S.; Hammouch, Z. On mild solutions of fractional impulsive differential systems of Sobolev type with fractional nonlocal conditions. *Math. Sci.* **2022**, *36*, 37. [CrossRef]
17. Hamou, A.A.; Azroul, E.H.; Hammouch, Z.; Alaoui, A.L. A monotone iterative technique combined to finite element method for solving reaction-diffusion problems pertaining to non-integer derivative. *Eng. Comput.* **2022**, *36*, 105. [CrossRef]
18. Gürbüz, M.; Akdemir, A.O.; Dokuyucu, M.A. Novel Approaches for Differentiable Convex Functions via the Proportional Caputo-Hybrid Operators. *Fractal Fract.* **2022**, *6*, 258. [CrossRef]
19. Avcı, A.M.; Akdemir, A.O.; Set, E. On New Integral Inequalities via Geometric-Arithmetic Convex Functions with Applications. *Sahand Commun. Math. Anal.* **2022**. [CrossRef]
20. Butt, S.I.; Akdemir, A.O.; Agarwal, P.; Baleanu, D. Non-conformable integral inequalities of chebyshev-polya-szeg o type. *J. Math. Inequalities* **2021**, *4*, 1391–1400. [CrossRef]
21. Kavurmacı Önalan, H.; Akdemir, A.O.; Avcı Ardıç, M.; Baleanu, D. On new general versions of Hermite–Hadamard type integral inequalities via fractional integral operators with Mittag-Leffler kernel. *J. Inequalities Appl.* **2021**, *1*, 186. [CrossRef]
22. Khalil, R.; Al Horani, M.; Yousef, A.; Sababheh, M. A new definition of fractional derivative. *J. Comput. Appl. Math.* **2014**, *264*, 65–70. [CrossRef]
23. Abdelhakim, A.A.; Machado, J.A.T. A critical analysis of the conformable derivative. *Nonlinear Dyn.* **2019**, *95*, 3063–3073. [CrossRef]

24. Abdeljawad, T. On conformable fractional calculus. *J. Comput. Appl. Math.* **2015**, *279*, 57–66. [CrossRef]
25. Tuan, N.H.; Huynh, L.N.; Ngoc, T.B.; Zhou, Y. On a backward problem for nonlinear fractional diffusion equations. *Appl. Math. Lett.* **2019**, *92*, 76–84. [CrossRef]
26. Ikehata, R.; Suzuki, T. Stable and unstable sets for evolution equations of parabolic and hyperbolic type. *Hiroshima Math. J.* **1996**, *26*, 475–491. [CrossRef]
27. Quittner, P.; Souplet, P. *Superlinear Parabolic Problems, Blow-Up, Global Existence and Steady States*; *Birkhäuser Advanced Texts*; Birkhäuser Basel: Basel, Switzerland, 2007.
28. Liu, Y.; Xu, R.; Yu, T. Global existence, nonexistence and asymptotic behavior of solutions for the Cauchy problem of semilinear heat equations. *Nonlinear Anal.* **2008**, *68*, 3332–3348. [CrossRef]
29. Can, N.H.; Kumar, D.; Vo Viet, T.; Nguyen, A.T. On time fractional pseudo-parabolic equations with non-local in time condition. *Math. Methods Appl. Sci.* **2021**. [CrossRef]
30. Nguyen, A.T.; Hammouch, Z.; Karapinar, E.; Tuan, N.H. On a nonlocal problem for a Caputo time-fractional pseudoparabolic equation. *Math. Methods Appl. Sci.* **2021**, **44**, 14791–14806. [CrossRef]
31. Nguyen, H.T.; Vo, V.A.; Runzhang, X. Semilinear Caputo time-fractional pseudo-parabolic equations. *Commun. Pure Appl. Anal.* **2021**, *20*, 583–621. [CrossRef]
32. Nguyen, H.T.; Le, D.L.; Nguyen, V.T. Regularized solution of an inverse source problem for a time fractional diffusion equation. *Appl. Math. Model.* **2016**, *40*, 8244–8264. [CrossRef]
33. Tuan, N.H.; Zhou, Y.; Long, L.D.; Can, N.H. Identifying inverse source for fractional diffusion equation with Reimann-Liouville derivetive. *Comput. Appl. Math.* **2020**, *39*, 75. [CrossRef]
34. Long, L.D.; Luc, N.H.; Zhou, Y.; Nguyen, A.C. Identification of Source term for the time-fractional duffusion-wave equation by Fractional Tikhonov method. *Mathematics* **2019**, *7*, 934. [CrossRef]
35. Long, L.D.; Zhou, Y.; Thanh Binh, T.; Can, N. A Mollification Regularization Method for the Inverse Source Problem for a Time Fractional Diffusion Equation. *Mathematics* **2019**, *7*, 1048. [CrossRef]
36. Wei, T.; Li, X.L.; Li, Y.S. An inverse time-dependent source problem for a time-fractional diffusion equation. *Inverse Probl.* **2016**, *32*, 8. [CrossRef]
37. Wang, J.G.; Zhou, Y.B.; Wei, T. Two regularization methods to identify a space-dependent source for the time-fractional diffusion equation. *Appl. Numer. Math.* **2013**, *68*, 39–57. [CrossRef]
38. Tuan, N.H. On some inverse problem for bi-parabolic equation with observed data in L_p spaces. *Opuscula Math.* **2022**, *42*, 305–335. [CrossRef]
39. Can, N.H.; Luc, N.H.; Baleanu, D.; Yong, Z.; Long, L.D. Inverse source problem for time fractional diffusion equation with Mittag-Leffler kernel. *Adv. Differ. Equ.* **2020**, *2020*, 210. [CrossRef]
40. Tuan, N.H.; Caraballo, T. On initial and terminal value problems for fractional nonclassical diffusion equations. *Proc. Am. Math. Soc.* **2021**, *149*, 143–161. [CrossRef]
41. Jaiswal, A.; Bahuguna, D. Semilinear Conformable Fractional Differential Equations in Banach Spaces. *Differ. Equ. Dyn. Syst.* **2019**, *27*, 313–325. [CrossRef]
42. Li, M.; Wang, J.R.; O'Regan, D. Existence and Ulam's stability for conformable fractional differential equations with constant coefficients. *Bull. Malays. Math. Sci. Soc.* **2019**, *42*, 1791–1812. [CrossRef]