symmetry

MDPI

Article

Confidence Intervals for Comparing the Variances of Two Independent Birnbaum–Saunders Distributions

Wisunee Puggard, Sa-Aat Niwitpong and Suparat Niwitpong *

Department of Applied Statistics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand; s6204051910056@kmutnb.ac.th (W.P.); sa-aat.n@sci.kmutnb.ac.th (S.-A.N.)

* Correspondence: suparat.n@sci.kmutnb.ac.th

Abstract: Fatigue in a material occurs when it is subjected to fluctuating stress and strain, which usually results in failure due to the accumulated damage. In statistics, asymmetric distribution, which is commonly used for describing the fatigue life of materials, is the Birnbaum–Saunders (BS) distribution. This distribution can be transform to the normal distribution, which is symmetrical. Furthermore, variance is used to examine the dispersion of the fatigue life data. However, comparing the variances of two independent samples that follow BS distributions has not previously been reported. To accomplish this, we propose methods for providing the confidence interval for the ratio of variances of two independent BS distributions based on the generalized fiducial confidence interval (GFCI), a Bayesian credible interval (BCI), and the highest posterior density (HPD) intervals based on a prior distribution with partial information (HPD-PI) and a proper prior with known hyperparameters (HPD-KH). A Monte Carlo simulation study was carried out to examine the efficacies of the methods in terms of their coverage probabilities and average lengths. The simulation results indicate that the HPD-PI performed satisfactorily for all sample sizes investigated. To illustrate the efficacies of the proposed methods with real data, they were also applied to study the confidence interval for the ratio of the variances of two 6061-T6 aluminum coupon fatigue-life datasets.

Keywords: Birnbaum–Saunders distribution; confidence interval; variance; fiducial inference; Bayesian; fatigue life

check for updates

Citation: Puggard, W.; Niwitpong, S.-A.; Niwitpong, S. Confidence Intervals for Comparing the Variances of Two Independent Birnbaum–Saunders Distributions. *Symmetry* **2022**, *14*, 1492. https://doi.org/10.3390/sym14071492

Academic Editor: Jinyu Li

Received: 22 June 2022
Accepted: 19 July 2022
Published: 21 July 2022

1. Introduction

Fatigue, defined as the degradation of the mechanical properties of a material under loading that change over time, is one of the leading causes of machine and structural failure. A critical characteristic of fatigue is that the load is not sufficiently large to cause instantaneous failure. Instead, failure occurs after a particular number of load fluctuations have been encountered (i.e., after the cumulative damage has reached a critical threshold) [1]. As a result, having a good understanding of the fatigue life of materials is critical for preventing damage caused by their failure, predicting the consequences of changes in operational conditions, identifying the cause of fatigue failure, and taking effective mitigating measures. In order to evaluate the fatigue life of materials, statistical distributions of the fatigue life can be considered. These distributions are often positive asymmetry or skewness (non-normality) and start from zero, since the fatigue life is always non-negative. Therefore, the fatigue life of materials cannot be described by either the normal or symmetrical distributions. In recent decades, asymmetric distribution that has received considerable attention for describing the fatigue life of materials is the Birnbaum–Saunders (BS) distribution. It was originally developed in response to a material fatigue problem and has been extensively used in reliability and fatigue research [2]. The BS distribution explains the total amount of time that will pass until a dominant crack develops and grows to a point where the cumulative damage exceeds the threshold and causes failure. Desmond [3] presented a more generalized extension of the BS distribution based on a biological model and also contributed to generalizing the actual reasons for using this distribution by relaxing the assumptions stated by Birnbaum and Saunders [2]. In addition, Desmond [4] deduced that the BS distribution is a mixture of the inverse Gaussian distribution with 0.5 as the mixing probability.

Although the BS distribution has its origins in materials science, it has subsequently been employed in a variety of other fields, including engineering, environmental studies, agriculture, and finance [5–8]. Many researchers have made significant contributions to the development of and parameter inference for the BS distribution. For example, Birnbaum and Saunders [5] solved a nonlinear equation to obtain the maximum likelihood estimators (MLEs) for shape parameter α and scale parameter β. Engelhardt et al. [9] investigated the asymptotic joint distribution of the MLEs and demonstrated that they are asymptotically independent. Based on this asymptotic joint distribution, they calculated the asymptotic confidence intervals for α and β. Approximations of the posterior marginal distributions of α and β were used by Achcar [10] to produce Bayesian estimates. Ng et al. [11] provided modified moment estimators (MMEs) for α and β, and then devised a bias reduction approach for the MLEs and MMEs. Lemonte et al. [12] examined various bias correction strategies for the MLEs by using bootstrap methods (both parametric and nonparametric). Wang [13] proposed a generalized confidence interval for α, as well as certain critical reliability quantities, such as the mean, quantiles, and a reliability function. Xu and Tang [14] considered the Bayesian estimators for α and β under the reference prior and obtained Bayesian estimators by using the idea of Lindley's approximation and the Gibbs sampling procedure. Niu et al. [15] proposed two test statistics based on the exact generalized p-value approach and the delta method for comparing the characteristic quantities of several BS distributions, including the mean, quantiles, and a reliability function. Wang et al. [16] applied inverse-gamma priors for α and β and presented an efficient sampling algorithm via the generalized ratio-of-uniforms method to calculate the Bayesian estimates and credible intervals. Li and Xu [17] utilized fiducial inference for the parameters of a BS distribution. Guo et al. [18] presented approaches that are hybrids between the generalized inference method and the large sample theory for interval estimation and hypothesis testing for the common mean of several BS populations. Recently, Puggard et al. [19] proposed the confidence intervals for the coefficient of variation (CV) and the difference between the CVs of BS distributions based on the concept of generalized confidence interval (GCI), a bootstrapped confidence interval (BCI), a Bayesian credible interval (BayCI), and the highest posterior density (HPD) interval.

In statistics, variance is used to describe the deviation from the average (mean). It is determined by squaring the differences between each value in the dataset and the mean, then dividing the sum of the squares by the total number of values in the dataset. Moreover, variance is defined as the second central moment, while the square root of the variance is called the standard deviation [20]. In the case of two independently collected datasets, determining whether the variance of the first one is significantly different from that of the second one is a critical statistical problem. To this end, the confidence interval for the ratio of the variances of two independent datasets can be used to compare the variance between them. If the confidence interval contains 1, it can be concluded that the variance of the first and second datasets is not significantly different. Many authors have focused on the construction of the confidence interval for the ratio of variances of two datasets by using different methods for various distributions. For example, Bonett [21] proposed an approximate confidence interval for the ratio of variances of bivariate non-normal distributions. Bebu and Mathew [22] applied the GCI approach and a modified signed log-likelihood ratio approach to construct the confidence interval for the ratio of variances of bivariate lognormal distributions. Paksaranuwat and Niwitpong [23] compared the efficacies of adaptive and classical confidence intervals for the variance and the ratio of variances of non-normal distributions with missing data. Niwitpong [24] examined the GCI approach for the ratio of variances of lognormal distributions. Wongyai and Suwan [25] developed the confidence interval for the ratio of variances of bivariate non-normal distributions by using a kurtosis estimator. Recently, Maneerat et al. [26] presented the HPD interval based on the normal-gamma prior and the method of variance estimates recovery (MOVER) to compute the confidence interval for the ratio of variances of delta-lognormal distributions. Nevertheless, the construction of the confidence interval for the ratio of variances of two independent BS distributions has not yet been reported. Therefore, the goal of the present study is to propose methods for constructing the confidence interval for the ratio of the variances of two BS distributions based on the generalized fiducial confidence interval (GFCI), a Bayesian credible interval (BCI), and the HPD intervals based on a prior with partial information (HPD-PI) and a proper prior with known hyperparameters (HPD-KH).

The rest of this article is structured as follows. The background on the BS distribution and the concepts of each of the methods for constructing the confidence interval for the ratio of variances of two BS distributions are described in Section 2. The simulation studies and results are presented in Section 3. Section 4 provides an illustration of the proposed methods with real fatigue datasets from Birnbaum and Saunders [5]. The final section contains conclusions on the study.

2. Methods

Let $X_{ij} = (X_{i1}, X_{i2}, \ldots, X_{in_i})$, $i = 1,2$ and $j = 1,2,\ldots,n_i$ be non-negative random samples drawn from BS distributions denoted by $X_{ij} \sim BS(\alpha_i, \beta_i)$, where α_i and β_i are the shape and scale parameters, respectively. The cumulative distribution function (cdf) can be written as

$$F(x_{ij}) = \Phi\left[\frac{1}{\alpha_i}\left(\sqrt{\frac{x_{ij}}{\beta_i}} - \sqrt{\frac{\beta_i}{x_{ij}}}\right)\right], \tag{1}$$

where $x_{ij} > 0, \alpha_i > 0, \beta_i > 0$, and $\Phi(\cdot)$ is the standard normal cdf. Note that β_i is also the median of the distribution. The corresponding probability density function (pdf) of this cdf is given by

$$f(x_{ij}, \alpha_i, \beta_i) = \frac{1}{2\alpha_i\beta_i\sqrt{2\pi}} \times \left\{\left(\frac{\beta_i}{x_{ij}}\right)^{\frac{1}{2}} + \left(\frac{\beta_i}{x_{ij}}\right)^{\frac{3}{2}}\right\} exp\left[-\frac{1}{2\alpha_i^2}\left(\frac{x_{ij}}{\beta} + \frac{\beta_i}{x_{ij}} - 2\right)\right]. \tag{2}$$

The following transformation was applied to generate samples from the BS distributions and to enable the derivation of some of their other properties, including various moments. If $X_{ij} \sim BS(\alpha_i, \beta_i)$, then

$$Z_{ij} = \frac{1}{2}\left(\sqrt{\frac{X_{ij}}{\beta_i}} - \sqrt{\frac{\beta_i}{X_{ij}}}\right) \sim N(0, \alpha_i^2/4). \tag{3}$$

Thus,

$$X_{ij} = \beta_i(1 + 2Z_{ij}^2 + 2Z_{ij}\sqrt{1 + Z_{ij}^2}). \tag{4}$$

By applying the above transformation, the expected value and variance of X_{ij} are $E(X_{ij}) = \beta_i(1 + \frac{1}{2}\alpha_i^2)$ and $V(X_{ij}) = (\alpha_i\beta_i)^2(1 + \frac{5}{4}\alpha_i^2)$, respectively. Since X_{ij} are independent, the ratio of the variances simply becomes

$$\theta = \frac{(\alpha_1\beta_1)^2(1 + \frac{5}{4}\alpha_1^2)}{(\alpha_2\beta_2)^2(1 + \frac{5}{4}\alpha_2^2)}. \tag{5}$$

2.1. Generalized Fiducial Inference

Generalized fiducial inference can be used to transform the original data into other distributions that are known. According to the rules of that distribution, the transformed data are manipulated, and the results are transferred back to the original via an inverse transformation [27]. This idea brings us to construct the confidence interval for the ratio of the variances of two BS distributions. Let

$$\boldsymbol{Y} = \boldsymbol{G}(\boldsymbol{\eta}, \boldsymbol{U}) \tag{6}$$

be the relationship between $\boldsymbol{Y}$ and parameter $\boldsymbol{\eta} \in \Xi$, where $\boldsymbol{G}(\cdot,\cdot)$ is a structural equation and $\boldsymbol{U}$ is a random variable for which the distribution is definitively known and independent of any parameters. For any given realization $\boldsymbol{y}$ of $\boldsymbol{Y}$, inverse $\boldsymbol{\eta} = \boldsymbol{H}(\boldsymbol{y}, \boldsymbol{u})$ always exists for any realization $\boldsymbol{u}$ of $\boldsymbol{U}$. Since the distribution of $\boldsymbol{U}$ is definitively known, random sample $\tilde{\boldsymbol{u}}_1, \tilde{\boldsymbol{u}}_2, \ldots, \tilde{\boldsymbol{u}}_M$ can be generated from it. This random sample of $\boldsymbol{U}$ can be transformed into a random sample of $\boldsymbol{\eta}$ via the inverse $\tilde{\boldsymbol{\eta}}_1 = \boldsymbol{H}(\boldsymbol{y}, \tilde{\boldsymbol{u}}_1), \tilde{\boldsymbol{\eta}}_2 = \boldsymbol{H}(\boldsymbol{y}, \tilde{\boldsymbol{u}}_2), \ldots, \tilde{\boldsymbol{\eta}}_M = \boldsymbol{H}(\boldsymbol{y}, \tilde{\boldsymbol{u}}_M)$, such that a random sample of $\boldsymbol{\eta}$ (i.e., a fiducial sample) can be obtained. However, in some situations, the inverse does not exist, for which Hannig [27,28] proposed the following solutions.

If $\boldsymbol{G} = (G_1, G_2, \ldots, G_n)$ is a structural equation, $Y_i = G_i(\eta, U)$ for $i = 1, 2, \ldots, n$. Suppose that parameter $\boldsymbol{\eta} \in \Xi \subseteq \mathbb{R}^p$ is p-dimensional and $\boldsymbol{U} = (U_1, U_2, \ldots, U_n)$ are independent identically distributed (i.i.d.) samples from Uniform (0,1). Under certain differentiability conditions, Hannig [28] illustrated that the generalized fiducial distribution for $\boldsymbol{\eta}$ is definitively continuous with

$$r(\boldsymbol{\eta}) = \frac{L(\boldsymbol{y}, \boldsymbol{\eta}) J(\boldsymbol{y}, \boldsymbol{\eta})}{\int_{\Xi} L(\boldsymbol{y}, \boldsymbol{\eta}') J(\boldsymbol{y}, \boldsymbol{\eta}') d\eta'}, \tag{7}$$

where $L(\boldsymbol{y}, \boldsymbol{\eta})$ denotes the likelihood function of the data and function

$$J(\boldsymbol{y}, \boldsymbol{\eta}) = \sum_{\substack{i=(i_1,\ldots,i_p) \\ 1 \leqslant i_1 < \ldots < i_p \leqslant n}} \left| det\left(\left(\frac{\boldsymbol{d}}{\boldsymbol{dy}} \boldsymbol{G}^{-1}(\boldsymbol{y}, \boldsymbol{\eta}) \right)^{-1} \frac{\boldsymbol{d}}{\boldsymbol{d\eta}} \boldsymbol{G}^{-1}(\boldsymbol{y}, \boldsymbol{\eta}) \right)_i \right|. \tag{8}$$

The above sum covers all possible p-tuples of indexes $i = (1 \leqslant i_1 < \ldots < i_p \leqslant n) \subset \{1, \ldots, n\}$ and $d\boldsymbol{G}^{-1}(\boldsymbol{y}, \boldsymbol{\eta})/d\boldsymbol{\eta}$ and $d\boldsymbol{G}^{-1}(\boldsymbol{y}, \boldsymbol{\eta})/d\boldsymbol{y}$ are $n \times p$ and $n \times n$ Jacobian matrixes, respectively. For any $n \times p$ matrix B, submatrix $(B)_i$ is a $p \times p$ matrix containing rows $i = (i_1, i_2, \ldots, i_p)$ of B. In addition, if observation $\boldsymbol{y}$ from a definitively continuous distribution is i.i.d. with cdf $F_\eta(y)$, then $\boldsymbol{H}^{-1} = (F_\eta(y_1), F_\eta(y_2), \ldots, F_\eta(y_n))$.

For a BS distribution, the generalized fiducial distribution of (α_i, β_i) derived by Li and Xu [17] is in the form

$$f(\alpha_i, \beta_i | \mathbf{x}_{ij}) \propto J(\mathbf{x}_{ij}, (\alpha_i, \beta_i)) L(\mathbf{x}_{ij} | \alpha_i, \beta_i), \tag{9}$$

where

$$L(\boldsymbol{x}_{ij} | \alpha_i, \beta_i) \propto \frac{1}{\alpha_i^{n_i} \beta_i^{n_i}} \prod_{j=1}^{n_i} \left[\left(\frac{\beta_i}{x_{ij}} \right)^{\frac{1}{2}} + \left(\frac{\beta_i}{x_{ij}} \right)^{\frac{3}{2}} \right] exp\left[-\sum_{j=1}^{n_i} \frac{1}{2\alpha_i^2} \left(\frac{x_{ij}}{\beta_i} + \frac{\beta_i}{x_{ij}} - 2 \right) \right] \tag{10}$$

and

$$J(\boldsymbol{x}_{ij}, (\alpha_i, \beta_i)) = \sum_{1 \leqslant j < k \leqslant n_i} \frac{4|x_{ij} - x_{ik}|}{\alpha_i (1 + \frac{\beta_i}{x_{ij}})(1 + \frac{\beta_i}{x_{ik}})}. \tag{11}$$

Note that the symbol "∝" means "is proportional to." In brief, if a is proportional to b, then the only difference between a and b is a multiplicative constant. By applying Equation (11), Li and Xu [17] showed that the priors of α_i and β_i can be denoted as

$$\begin{aligned} \pi(\alpha_i) &\propto \frac{1}{\alpha_i} \\ \pi(\beta_i) &\propto \sum_{1 \leqslant j < k \leqslant n_i} \frac{|x_{ij} - x_{ik}|}{(1 + \beta_i / x_{ij})(1 + \beta_i / x_{ik})}. \end{aligned} \tag{12}$$

Thus, $f(\alpha_i, \beta_i | \boldsymbol{x}_{ij})$ is proper for the particular case of a prior with partial information given by the priors of α_i and β_i (12). Therefore, $\hat{\alpha}_i$ and $\hat{\beta}_i$, which are the generalized fiducial samples of α_i and β_i, can be obtained from the generalized fiducial distribution in the same way as the Bayesian posterior. The adaptive rejection Metropolis sampling (ARMS), which origins to adaptive rejection sampling (ARS), was used to generate the fiducial samples ($\hat{\alpha}_i$ and $\hat{\beta}_i$) from the generalized fiducial distribution (9). The ARS was proposed by Gilks and Wild [29]. It was only suitable for log-concave target densities. In order to address the limitations of ARS, Gilks et al. [30] improved ARS to handle multivariate distributions and non-log-concave densities by permitting the proposal distribution to remain lower than the target in some regions and adding a Metropolis–Hastings step to guarantee that the accepted samples are properly distributed. This method was called ARMS, which can be easily implemented via the function *arms* in package *dlm* of R software suite (version 3.5.1). Note that $\hat{\alpha}_i$ and $\hat{\beta}_i$ are treated as random variables. Therefore, α_i and β_i are substituted by $\hat{\alpha}_i$ and $\hat{\beta}_i$, respectively, resulting in the generalized fiducial estimates of θ being derived as

$$\hat{\theta} = \frac{(\hat{\alpha}_1 \hat{\beta}_1)^2 (1 + \frac{5}{4}\hat{\alpha}_1^2)}{(\hat{\alpha}_2 \hat{\beta}_2)^2 (1 + \frac{5}{4}\hat{\alpha}_2^2)}. \tag{13}$$

Finally, the $100(1-\gamma)\%$ GFCI for θ is $[\hat{\theta}(\gamma/2), \hat{\theta}(1-\gamma/2)]$, where $\hat{\theta}(\nu)$ is the $100\nu\%$ percentile of $\hat{\theta}$. Algorithm 1 summarizes the steps for constructing GFCI for θ, as seen below.

Algorithm 1 : GFCI

1. Generate datasets $x_{ij}, i = 1, 2, j = 1, 2, \ldots, n_i$ from a BS distribution.
2. Generate K samples of α_i and β_i by applying the *arms* function in the *dlm* package of the R software suite.
3. Burn-in B samples (the number of remaining samples is $K - B$).
4. Thin the samples by applying sampling lag $L > 1$ (the final number of samples is $K' = (K - B)/L$). Note that the generated samples are not independent, and so we need to reduce the autocorrelation by thinning the samples.
5. Calculate $\hat{\theta}$ by applying Equation (13) and obtain $\hat{\theta}_{(1)}, \hat{\theta}_{(2)}, \ldots, \hat{\theta}_{(K')}$.
6. Calculate the $100(1-\gamma)\%$ GFCI.

2.2. Bayesian Inference

For this method, Xu and Tang [14] illustrated that the reference prior of a BS distribution (a type of Jeffreys' prior) results in an improper posterior distribution. Thus, to guarantee its propriety, proper priors with known hyperparameters are obtained by assuming that an inverse-gamma distribution with parameters a_i and b_i is the prior of β_i and an inverse-gamma distribution with parameters c_i and d_i is the prior of $\lambda_i = \alpha_i^2$ [16].

In accordance with Bayes' theorem, the joint posterior density function of (α_i, β_i) can be written as

$$\begin{aligned} p(\lambda_i, \beta_i | \boldsymbol{x}_{ij}) &\propto L(\boldsymbol{x}_{ij} | \alpha_i, \beta_i) \pi(\beta_i | a_i, b_i) \pi(\lambda_i | c_i, d_i) \\ &\propto \frac{1}{(\lambda_i)^{\frac{n_i}{2}} \beta_i^{n_i}} \prod_{j=1}^{n_i} \left[\left(\frac{\beta_i}{x_{ij}} \right)^{\frac{1}{2}} + \left(\frac{\beta_i}{x_{ij}} \right)^{\frac{3}{2}} \right] exp \left[- \sum_{j=1}^{n_i} \frac{1}{2\lambda_i} \left(\frac{x_{ij}}{\beta_i} + \frac{\beta_i}{x_{ij}} - 2 \right) \right] \\ &\times \beta_i^{-a_i - 1} exp \left(- \frac{b_i}{\beta_i} \right) (\lambda_i)^{-c_i - 1} exp \left(- \frac{d_i}{\lambda_i} \right). \end{aligned} \tag{14}$$

Integrating the joint posterior density function (14) with respect to α_i yields the marginal posterior distribution of β_i as follows:

$$\pi(\beta_i|\boldsymbol{x}_{ij}) \propto \beta_i^{-(n_i+a_i+1)} exp\left(-\frac{b_i}{\beta_i}\right) \prod_{j=1}^{n_i} \left[\left(\frac{\beta_i}{x_{ij}}\right)^{\frac{1}{2}} + \left(\frac{\beta_i}{x_{ij}}\right)^{\frac{3}{2}}\right] \times \left[\sum_{j=1}^{n_i} \frac{1}{2}\left(\frac{x_{ij}}{\beta_i} + \frac{\beta_i}{x_{ij}} - 2\right) + d_i\right]^{\frac{-(n_i+1)}{2-c_i}}. \tag{15}$$

From the joint posterior density function (14), it is clear that the fully conditional posterior distribution of λ_i given β_i is given by

$$\pi(\lambda_i|\boldsymbol{x}_{ij}, \beta_i) \propto IG\left(\frac{n_i}{2} + c_i, \frac{1}{2}\sum_{j=1}^{n_i}\left(\frac{x_{ij}}{\beta_i} + \frac{\beta_i}{x_{ij}} - 2\right) + d_i\right). \tag{16}$$

Posterior samples are drawn by adopting the Markov chain Monte Carlo technique. Since the marginal posterior distribution (15) cannot be written as if it were known, generating posterior samples of β_i from this density is impossible using the usual methods. There are three common approaches, such as the random-walk Metropolis procedure, the Metropolis–Hastings algorithm and the slice sampler by introducing an auxiliary variable to simplify the sampling problem that might be considered to sample from the marginal posterior distribution (15). However, all three approaches are susceptible to serially correlated draws, indicating that a very large sample size is frequently required to produce a reasonable estimate of any desired attribute of the posterior distribution. To avoid these potential problems when generating the posterior samples, the generalized ratio-of-uniforms method of Wakefield et al. [31] is used to generate posterior samples of β_i (denoted as $\tilde{\beta}_i$). The concept of the generalized ratio-of-uniforms method is as follows.

Suppose that a pair of variables (u_i, v_i) is uniformly distributed inside region

$$A(r_i) = \left\{(u_i, v_i) : 0 < u_i \leq \left[\pi\left(\frac{v_i}{u_i^{r_i}}|\boldsymbol{x}_{ij}\right)\right]^{1/(r_i+1)}\right\}, \tag{17}$$

where $r_i \geq 0$ is a constant term and $\pi(\cdot|\boldsymbol{x}_{ij})$ is specified by using the marginal posterior distribution (15). Subsequently, the pdf of $\beta_i = v_i/u_i^{r_i}$ becomes $\pi(\beta_i|\boldsymbol{x}_{ij})/\int \pi(\beta_i\,|\boldsymbol{x}_{ij})d\beta_i$. For generating random points uniformly distributed in $A(r_i)$, the accept–reject method from a convenient enveloping region (usually from the minimal bounding rectangle) is applied. According to Wakefield et al. [31], the minimal bounding rectangle for $A(r_i)$ is given by

$$[0, a(r_i)] \times [b^-(r_i), b^+(r_i)], \tag{18}$$

where

$$a(r_i) = \sup_{\beta_i>0}\{[\pi(\beta_i|\boldsymbol{x}_{ij})]^{1/(r_i+1)}\}, \tag{19}$$

$$b^-(r_i) = \inf_{\beta_i>0}\{\beta_i[\pi(\beta_i|\boldsymbol{x}_{ij})]^{r_i/(r_i+1)}\}, \tag{20}$$

and

$$b^+(r_i) = \sup_{\beta_i>0}\{\beta_i[\pi(\beta_i|\boldsymbol{x}_{ij})]^{r_i/(r_i+1)}\}. \tag{21}$$

Note that $\pi(\beta_i|\boldsymbol{x}_{ij}) \to 0$ as $\beta_i \to 0^+$ and $\pi(\beta_i|\boldsymbol{x}_{ij}) \to O(\beta_i^{-(a_i+c_i+3/2)})$ as $\beta_i \to +\infty$. Hence, $b^-(r_i) = 0$, $a(r_i)$ is finite, and $b^+(r_i)$ is also finite when choosing an appropriate value for r_i [16]. The generalized ratio-of-uniforms method consists of the following three steps.

1. Compute $a(r_i)$ and $b^+(r_i)$.

2. Draw u_i and v_i from $U(0, a(r_i))$ and $U(0, b^+(r_i))$, where $U(v, w)$ refers to a uniform distribution with parameter v and w, and compute $\rho_i = v_i / u_i^{r_i}$.
3. Set $\tilde{\beta}_i = \rho_i$ if $u_i \leq [\pi(\rho_i|\boldsymbol{x}_{ij})]^{1/(r_i+1)}$; otherwise, the process is repeated.

Meanwhile, $\tilde{\lambda}_i$, which are the posterior samples of λ_i, can be obtained from the conditional posterior distribution (16) by applying the *LearnBayes* package from the R software suite. Subsequently, the posterior samples of α_i (denoted as $\tilde{\alpha}_i$) comprise the square roots of $\tilde{\lambda}_i$. Note that $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ are also treated as random variables. Hence, the Bayesian estimates for θ can be written as

$$\tilde{\theta} = \frac{(\tilde{\alpha}_1\tilde{\beta}_1)^2\left(1 + \frac{5}{4}\tilde{\alpha}_1^2\right)}{(\tilde{\alpha}_2\tilde{\beta}_2)^2\left(1 + \frac{5}{4}\tilde{\alpha}_2^2\right)}. \tag{22}$$

Finally, the $100(1-\gamma)\%$ BCI for θ is $[\tilde{\theta}(\gamma/2), \tilde{\theta}(1-\gamma/2)]$, where $\tilde{\theta}(\nu)$ is the $100\nu\%$ percentile of $\tilde{\theta}$. In conclusion, BCI for θ can be obtained by using Algorithm 2.

Algorithm 2 : BCI

1. Set a_i, b_i, c_i and d_i, where $i = 1, 2$.
2. Compute $a(r_i)$ and $b^+(r_i)$.
3. At the m^{th} steps,
   (a) Generate $u_i \sim U(0, a(r_i))$ and $v_i \sim U(0, b^+(r_i))$, and then compute $\rho_i = v_i / u_i^{r_i}$.
   (b) If $u_i \leq [\pi(\rho_i|\boldsymbol{x}_{ij})]^{1/(r_i+1)}$, set $\tilde{\beta}_{i,(m)} = \rho_i$; otherwise, repeat Step (a).
   (c) Generate $\tilde{\lambda}_{i,(m)} \sim IG\left(\frac{n_i}{2} + c_i, \frac{1}{2}\sum_{j=1}^{n_i}\left(\frac{x_{ij}}{\beta_{i,(m)}} + \frac{\beta_{i,(m)}}{x_{ij}} - 2\right) + d_i\right)$, and then $\tilde{\alpha}_{i,(m)} = \sqrt{\tilde{\lambda}_{i,(m)}}$.
   (d) Compute the Bayesian estimates for θ by applying Equation (22).
4. Repeat Step (3), M times.
5. Calculate the $100(1-\gamma)\%$ BCI.

2.3. The Highest Posterior Density (HPD) Interval

The HPD interval is where the posterior density for every point within the interval is higher than the posterior densities of the points outside of it, indicating that the interval contains the more likely values of the parameter while excluding the less likely ones. According to Box and Tiao [32], the HPD interval has two main properties:

1. Every point within the interval has a higher probability density than the points outside of it.
2. For given probability level $(1-\gamma)$, the interval has the narrowest length.

By applying Equation (11), Li and Xu [17] showed that $J(\mathbf{x}_{ij}, (\alpha_i, \beta_i))$ is a special case of a prior with partial information, and the generalized fiducial estimates of α_i and β_i can be obtained by using the same method as for the Bayesian posterior. Therefore, at Step (6) in Algorithm 1, we applied the *HDInterval* package (version 0.2.2) from the R software suite to compute the HPD interval based on a prior with partial information (HPD-PI). Moreover, we also applied the *HDInterval* package at Step (5) in Algorithm 2 to compute the HPD interval based on a proper prior with known hyperparameters (HPD-KH).

3. Simulation Studies

To compare the performance of the proposed methods, a Monte Carlo simulation study was conducted with various sample sizes and parameter values. Equal sample sizes were set as (n_1, n_2) = (10, 10), (20, 20), (30, 30), (50, 50), or (100, 100) and unequal sample sizes as (10, 20), (30, 20), (30, 50), or (100, 50) while the values for the shape parameters (α_1, α_2) were set as (0.25, 0.25), (0.25, 0.50), (0.25, 1.00), (0.50, 0.50), (0.50, 1.00), or (1.00, 1.00).

Without loss of generality, scale parameters β_1 and β_2 were set as 1 in all scenarios. The confidence intervals were calculated at the nominal level of 0.95. All simulation results were obtained by running 1000 replications with $K = 3000$ and $B = 1000$ for GFCI and HPD-PI while $M = 1000$ for BCI and HPD-KH. According to Wang et al. [16], BCI and HPD-KH were considered with $r_1 = r_2 = 2$ and hyperparameters $a_i = b_i = c_i = d_i = 10^{-4}$. The criteria for evaluating the performances of the proposed methods are their coverage probabilities and average lengths. The method with a coverage probability greater than or close to the nominal level 0.95 and with the narrowest average length was chosen as the best performing method for a particular scenario.

Tables 1 and 2 report the simulation results, while Figures 1 and 2 summarize the coverage probabilities and average lengths in Tables 1 and 2. The simulation results from Tables 1 and 2 indicate that the coverage probabilities of the four methods were greater than or close to 0.95 under all configurations. In addition, it was found that the differences in coverage probability among the four methods were very small. Moreover, HPD-PI provided the narrowest average lengths while BCI provided the longest ones under all circumstances. The average lengths of HPD-IP were mostly narrower than GFCI, while the average lengths of HPD-KH were mostly narrower than those of BCI. Moreover, the average length of the four methods decreased as the sample sizes (n_1, n_2) increased.

Table 1. Coverage probabilities and average lengths of the 95% confidence interval for the ratio of variances of two BS distributions with equal sample sizes $(n_1 = n_2)$ constructed via the various methods.

		Coverage Probability				**Average Length**			
(n_1, n_2)	(α_1, α_2)	**GFCI**	**BCI**	**HPD-PI**	**HPD-KH**	**GFCI**	**BCI**	**HPD-PI**	**HPD-KH**
(10, 10)	(0.25, 0.25)	0.945	0.950	0.941	0.943	6.0636	6.6395	4.7856	5.1260
	(0.25, 0.50)	0.922	0.939	0.935	0.941	1.4846	1.6160	1.1602	1.2418
	(0.25, 1.00)	0.923	0.925	0.936	0.946	0.3996	0.4311	0.3032	0.3231
	(0.50, 0.50)	0.932	0.944	0.932	0.938	11.447	12.746	8.0329	8.7739
	(0.50, 1.00)	0.942	0.949	0.943	0.949	2.4825	2.7433	1.7235	1.8892
	(1.00, 1.00)	0.931	0.935	0.949	0.951	46.848	50.345	26.871	28.443
(20, 20)	(0.25, 0.25)	0.940	0.947	0.948	0.948	2.8954	3.0005	2.5508	2.6280
	(0.25, 0.50)	0.925	0.923	0.937	0.941	0.6826	0.7011	0.5969	0.6111
	(0.25,1.00)	0.942	0.944	0.943	0.950	0.1414	0.1450	0.1201	0.1227
	(0.50, 0.50)	0.939	0.940	0.935	0.939	4.3582	4.5050	3.6126	3.7198
	(0.50, 1.00)	0.932	0.942	0.935	0.939	0.8850	0.9149	0.7180	0.7383
	(1.00, 1.00)	0.941	0.946	0.944	0.952	9.2748	9.4142	6.8587	6.9256
(30, 30)	(0.25, 0.25)	0.938	0.939	0.942	0.941	2.0301	2.0760	1.8599	1.8937
	(0.25, 0.50)	0.932	0.932	0.936	0.940	0.4906	0.4988	0.4469	0.4524
	(0.25, 1.00)	0.942	0.945	0.934	0.934	0.0945	0.0960	0.0844	0.0851
	(0.50, 0.50)	0.954	0.958	0.941	0.945	2.7533	2.8141	2.4221	2.4720
	(0.50, 1.00)	0.946	0.953	0.948	0.952	0.5621	0.5713	0.4841	0.4910
	(1.00, 1.00)	0.946	0.945	0.948	0.947	5.5838	5.6471	4.4982	4.5328
(50, 50)	(0.25, 0.25)	0.936	0.936	0.939	0.941	1.4400	1.4564	1.3603	1.3747
	(0.25, 0.50)	0.943	0.947	0.939	0.938	0.3322	0.3354	0.3127	0.3156
	(0.25, 1.00)	0.954	0.957	0.953	0.956	0.0658	0.0663	0.0610	0.0613
	(0.50, 0.50)	0.953	0.955	0.945	0.951	1.9450	1.9674	1.7917	1.8075
	(0.50, 1.00)	0.954	0.953	0.946	0.946	0.3509	0.3548	0.3187	0.3212
	(1.00, 1.00)	0.948	0.952	0.944	0.944	3.2806	3.2875	2.8341	2.8475
(100, 100)	(0.25,0.25)	0.945	0.949	0.948	0.936	0.9518	0.9558	0.9197	0.9242
	(0.25, 0.50)	0.949	0.953	0.937	0.943	0.2175	0.2183	0.2102	0.2106
	(0.25, 1.00)	0.964	0.959	0.964	0.962	0.0423	0.0424	0.0405	0.0406
	(0.50, 0.50)	0.938	0.934	0.943	0.951	1.2308	1.2399	1.1757	1.1813
	(0.50, 1.00)	0.941	0.947	0.954	0.956	0.2265	0.2270	0.2144	0.2151
	(1.00, 1.00)	0.953	0.952	0.953	0.956	1.8445	1.8532	1.7034	1.7082

GFCI, the generalized fiducial confidence interval; BCI, the Bayesian credible interval; HPD-PI, the highest posterior density interval based on a prior with partial information; HPD-KH, the highest posterior density interval based on the proper prior with known hyperparameters.

Table 2. Coverage probabilities and average lengths of the 95% confidence interval for the ratio of variances of two BS distributions with unequal sample sizes ($n_1 \neq n_2$) constructed via the various methods.

		Coverage Probability				Average Length			
(n_1, n_2)	(α_1, α_2)	GFCI	BCI	HPD-PI	HPD-KH	GFCI	BCI	HPD-PI	HPD-KH
(10,20)	(0.25,0.25)	0.943	0.953	0.927	0.941	4.2072	4.8670	3.3413	3.7992
	(0.25,0.50)	0.930	0.938	0.924	0.938	1.0368	1.1881	0.8214	0.9220
	(0.25,1.00)	0.933	0.948	0.931	0.946	0.1935	0.2184	0.1506	0.1679
	(0.50,0.50)	0.926	0.940	0.922	0.937	8.1924	9.6088	5.7758	6.7158
	(0.50,1.00)	0.945	0.952	0.938	0.954	1.4323	1.6745	1.0112	1.1602
	(1.00,1.00)	0.935	0.948	0.930	0.949	27.601	30.801	15.728	17.728
(30,20)	(0.25,0.25)	0.948	0.951	0.946	0.953	2.4309	2.4642	2.2196	2.2377
	(0.25,0.50)	0.939	0.944	0.958	0.954	0.5910	0.5920	0.5342	0.5325
	(0.25,1.00)	0.945	0.948	0.940	0.947	0.1273	0.1273	0.1113	0.1107
	(0.50,0.50)	0.936	0.937	0.939	0.946	3.4294	3.4549	2.9905	3.0062
	(0.50,1.00)	0.940	0.947	0.952	0.952	0.6915	0.6953	0.5865	0.5876
	(1.00,1.00)	0.946	0.947	0.949	0.949	7.1486	7.0966	5.6707	5.6205
(30,50)	(0.25,0.25)	0.951	0.952	0.949	0.952	1.8153	1.8767	1.6645	1.7200
	(0.25,0.50)	0.936	0.938	0.936	0.945	0.4030	0.4144	0.3683	0.3779
	(0.25,1.00)	0.943	0.946	0.944	0.950	0.0743	0.0764	0.0672	0.0689
	(0.50,0.50)	0.952	0.954	0.945	0.955	2.4422	2.5389	2.1544	2.2279
	(0.50,1.00)	0.954	0.953	0.950	0.955	0.4475	0.4633	0.3915	0.4032
	(1.00,1.00)	0.945	0.947	0.942	0.945	4.4938	4.6107	3.6559	3.7389
(100,50)	(0.25,0.25)	0.936	0.937	0.936	0.933	1.1848	1.1830	1.1426	1.1422
	(0.25,0.50)	0.942	0.947	0.947	0.945	0.2873	0.2858	0.2759	0.2741
	(0.25,1.00)	0.945	0.944	0.943	0.938	0.0581	0.0577	0.0547	0.0544
	(0.50,0.50)	0.939	0.942	0.941	0.936	1.5533	1.5555	1.4779	1.4747
	(0.50,1.00)	0.947	0.951	0.949	0.948	0.3059	0.3039	0.2854	0.2832
	(1.00,1.00)	0.959	0.957	0.956	0.954	2.4564	2.4478	2.2328	2.2283

GFCI, the generalized fiducial confidence interval; BCI, the Bayesian credible interval; HPD-PI, the highest posterior density interval based on a prior with partial information; HPD-KH, the highest posterior density interval based on the proper prior with known hyperparameters.

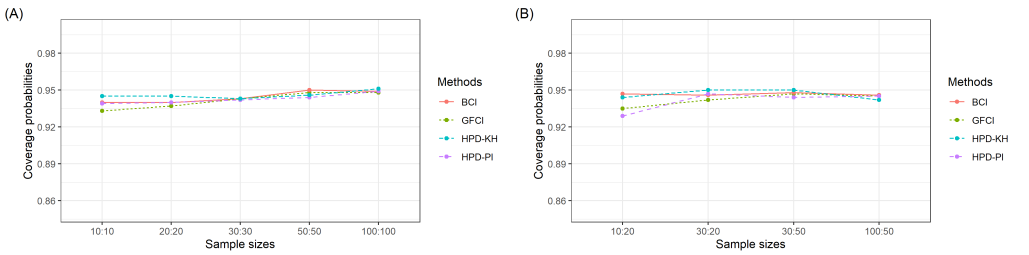

Figure 1. A summary of the coverage probabilities of the methods in Tables 1 and 2. (**A**) Equal sample sizes and (**B**) unequal sample sizes.

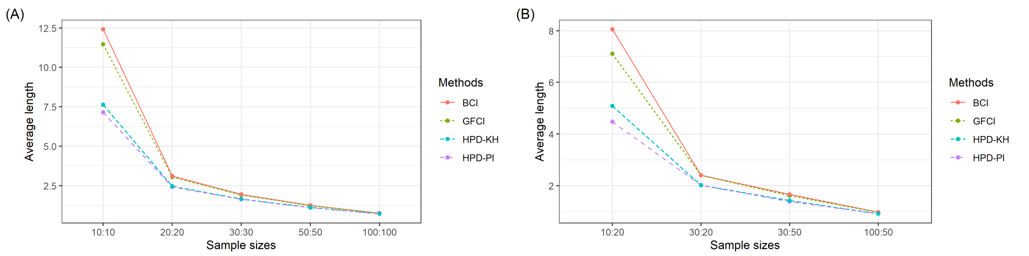

Figure 2. A summary of the average lengths of the methods in Tables 1 and 2. (**A**) Equal sample sizes and (**B**) unequal sample sizes.

4. Application of the Methods to Real Fatigue Life Data

To illustrate the effectiveness of the confidence interval construction methods proposed in this study in a real-life scenario, we used real datasets concerning the fatigue life of

6061-T6 aluminum coupons that were cut parallel to the rolling direction and oscillated at 18 cycles per second [5]. As reported in Table 3, there are two groups consisting of 101 and 102 observations with maximum stress levels per cycle of 21,000 and 26,000 psi, respectively (the summary statistics of each group are provided in Table 4). Hence, the ratio of variances was 0.0254. We chose $r_i = 2$ and $a_i = b_i = c_i = d_i = 10^{-4}$, where $i = 1, 2$, for the Bayesian credible interval and HPD-KH.

Table 3. Fatigue lifetime data of 6061-T6 aluminum coupons.

Group 1	370	706	716	746	785	797	844	855	858	886	886
	930	960	988	990	1000	1010	1016	1018	1020	1055	1085
	1102	1102	1108	1115	1120	1134	1140	1199	1200	1200	1203
	1222	1235	1238	1252	1258	1262	1269	1270	1290	1293	1300
	1310	1313	1315	1330	1355	1390	1416	1419	1420	1420	1450
	1452	1475	1478	1481	1485	1502	1505	1513	1522	1522	1530
	1540	1560	1567	1578	1594	1602	1604	1608	1630	1642	1674
	1730	1750	1750	1763	1768	1781	1782	1792	1820	1868	1881
	1890	1893	1895	1910	1923	1940	1945	2023	2100	2130	2215
	2268	2440									
Group 2	233	258	268	276	290	310	312	315	318	321	321
	329	335	336	338	338	342	342	342	344	349	350
	350	351	351	352	352	356	358	358	360	362	363
	366	367	370	370	372	372	374	375	376	379	379
	380	382	389	389	395	396	400	400	400	403	404
	406	408	408	410	412	414	416	416	416	420	422
	423	426	428	432	432	433	433	437	438	439	439
	443	445	445	452	456	456	460	464	466	468	470
	470	473	474	476	476	486	488	489	490	491	503
	517	540	560								

Table 4. Summary statistics for the fatigue lifetime data of 6061-T6 aluminum coupons.

Group	n	Min.	Median	Mean	Max.	Variance
1	101	370	1416	1400.9110	2440	153,134.5
2	102	233	400	397.8824	560	3834.3030

The results for the 95% confidence interval for the ratio of variances reported in Table 5 indicate that the length provided by HPD-PI was the narrowest while that of the Bayesian credible interval was the longest. These results are in accordance with those from the simulation studies where $(n_1, n_2) = (100, 100)$. In addition, the confidence intervals constructed by using the various methods did not contain 1, and so it can be concluded that there is no significant difference in terms of variance for the fatigue lifetime of 6061-T6 aluminum coupons with maximum stress per cycle of 21,000 and 26,000 psi, respectively.

Table 5. The 95% confidence interval for the ratio of variances of the fatigue lifetime data of 6061-T6 aluminum coupons with maximum stress levels per cycle of 21,000 and 26,000 psi.

Methods	Interval	Length
GFCI	0.0138–0.0332	0.0193
BCI	0.0138–0.0337	0.0199
HPD-PI	0.0132–0.0315	0.0184
HPD-KH	0.0126–0.0315	0.0189

GFCI, the generalized fiducial confidence interval; BCI, the Bayesian credible interval; HPD-PI, the highest posterior density interval based on a prior with partial information; HPD-KH, the highest posterior density interval based on the proper prior with known hyperparameters.

5. Conclusions

Four methods, namely GFCI, BCI, HPD-PI, and HPD-KH, were proposed for constructing the confidence interval for the ratio of variances of two BS distributions. A Monte Carlo simulation study was conducted to assess their performances based on their coverage probabilities and average lengths. The simulation study results indicate that the coverage probabilities of all of the methods were greater than or close to the nominal level of 0.95. However, HPD-PI outperformed the others by providing the narrowest average lengths in all of the scenarios studied. In addition, the results of using fatigue lifetime data of 6061-T6 aluminum coupons coincided with those from the simulation study. Therefore, HPD-PI can be recommended for constructing the confidence interval for the ratio of variances of two BS distributions.

Author Contributions: Conceptualization, S.N.; Data curation, W.P.; Formal analysis, W.P. and S.N.; Funding acquisition, S.N.; Investigation, S.-A.N. and S.N.; Methodology, S.-A.N. and S.N.; Project administration, S.-A.N.; Resources, S.-A.N.; Software, W.P.; Supervision, S.-A.N. and S.N.; Visualization, S.-A.N.; Writing original draft, W.P.; Writing–review & editing, S.-A.N. and S.N. All authors have read and agreed to the published version of the manuscript.

Funding: This research received financial support from the National Science, Research, and Innovation Fund (NSRF), and King Mongkut's University of Technology North Bangkok (Grant No. KMUTNB–FF–65–23).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The real data sets of the fatigue life of 6061-T6 aluminum coupons are available in the work of Birnbaum and Saunder [5].

Acknowledgments: The first author wishes to express gratitude for financial support provided by the Thailand Science Achievement Scholarship (SAST).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Vassilopoulos, A.P. Fatigue life modeling and prediction methods for composite materials and structures—Past, present, and future prospects. In *Fatigue Life Prediction of Composites and Composite Structures*; Vassilopoulos, A.P., Ed.; Woodhead Publishing: Sawston, UK, 2020; pp. 1–43.
2. Birnbaum, Z.W.; Saunders S.C. A new family of life distributions. *J. Appl. Probab.* **1969**, *6*, 319–327. [CrossRef]
3. Desmond, A.F. Stochastic models of failure in random environments. *Can. J. Stat.* **1985**, *13*, 171–183. [CrossRef]
4. Desmond, A.F. On the relationship between two fatigue-life model. *IEEE Trans. Reliab.* **1986**, *35*, 167–169. [CrossRef]
5. Birnbaum, Z.W.; Saunders S.C. Estimation for a family of life distributions with applications to fatigue. *J. Appl. Probab.* **1969**, *6*, 328–347. [CrossRef]
6. Aslam, M.; Jun, C.H.; Ahmad, M. New acceptance sampling plans based on life tests for Birnbaum-Saunders distributions. *J. Stat. Comput. Simul.* **2011**, *81*, 461–470. [CrossRef]
7. Leiva, V.; Marchant, C.; Ruggeri, F.; Saulo, H. A criterion for environmental assessment using Birnbaum–Saunders attribute control charts. *Environmetrics* **2015**, *26*, 463–476. [CrossRef]
8. Marchant, C.; Bertin, K.; Leiva, V.; Saulo, H. Generalized Birnbaum-Saunders kernel density estimators and an analysis of financial data. *Comput. Stat. Data Anal.* **2013**, *63*, 1–15. [CrossRef]
9. Engelhardt, M.; Bain, L.J.; Wright, F.T. Inference on the parameters of the Birnbaum–Saunders fatigue life distribution based on maximum likelihood estimation. *Technometrics* **1981**, *23*, 251–255. [CrossRef]
10. Achcar, J. Inferences for the Birnbaum-Saunders fatigue life model using Bayesian methods. *Comput. Stat. Data Anal.* **1993**, *15*, 367–380. [CrossRef]
11. Ng, H.K.T.; Kundu, D.; Balakrishnan, N. Modified moment estimation for the two-parameter Birnbaum–Saunders distribution. *Comput. Stat. Data Anal.* **2003**, *43*, 283–298. [CrossRef]
12. Lemonte, A.J.; Simas, A.B.; Cribari-Neto, F. Bootstrap-based improved estimators for the two-parameter Birnbaum–Saunders distribution. *J. Stat. Comput. Simul.* **2008**, *78*, 37–49. [CrossRef]
13. Wang, B.X. Generalized interval estimation for the Birnbaum-Saunders distribution. *Comput. Stat. Data Anal.* **2012**, *56*, 4320–4326. [CrossRef]
14. Xu, A.; Tang, Y. Reference analysis for Birnbaum-Saunders distribution. *Comput. Stat. Data Anal.* **2010**, *54*, 185–192. [CrossRef]
15. Niu, C.; Guo, X.; Zhu, L. Comparison of several Birnbaum–Saunders distributions. *J. Stat. Comput. Simul.* **2014**, *84*, 2721–2733. [CrossRef]
16. Wang, M.; Sun, X.; Park, C. Bayesian analysis of Birnbaum-Saunders distribution via the generalized ratio-of-uniforms method. *Comput. Stat.* **2016**, *31*, 207–225. [CrossRef]
17. Li, Y.; Xu, A. Fiducial inference for Birnbaum-Saunders distribution. *J. Stat. Comput. Simul.* **2016**, *86*, 1673–1685. [CrossRef]

18. Guo, X.; Wu, H.; Li, G.; Li, Q. Inference for the common mean of several Birnbaum–Saunders populations.*J. Appl. Stat.* **2017** *44*, 941–954. [CrossRef]
19. Puggard, W.; Niwitpong, S.A.; Niwitpong, S. Bayesian estimation for the coefficients of variation of Birnbaum-Saunders distributions. *Symmetry* **2021**, *13*, 2130. [CrossRef]
20. Casella, G.; Berger, R.L. *Statistical Inference*; Thomson Learning: Boston, MA, USA, 2002.
21. Bonett, D.G. Confidence interval for a ratio of variances in bivariate nonnormal distributions. *J. Stat. Comput. Simul.* **2006**, *76*, 637–644. [CrossRef]
22. Bebu, I.; Mathew, T. Comparing the means and variances of a bivariate log-normal distribution. *Stat. Med.* **2008**, *27*, 2684–2696. [CrossRef]
23. Paksaranuwat, P.; Niwitpong, S.A. Confidence intervals for the variance and the ratio of two variances of non-normal distributions with missing data *Thail. Stat.* **2010**, *8*, 81–92.
24. Niwitpong, S.A. Generalized confidence intervals for function of variances of lognormal distribution. *Adv. Appl. Stat.* **2017**, *51*, 151–163. [CrossRef]
25. Wongyai, C.; Suwan, S. Comparisons of confidence interval for a ratio of non-normal variances using a kurtosis estimator. In Proceedings of the 12th International Conference of the Thailand Econometric Society (TES2019), Chiang Mai, Thailand, 9–11 January 2019.
26. Maneerat, P.; Niwitpong, S.A.; Niwitpong, S. A Bayesian approach to construct confidence intervals for comparing the rainfall dispersion in Thailand. *PeerJ* **2020**, *8*, e8502. [CrossRef] [PubMed]
27. Hannig, J. On generalized fiducial inference *Stat. Sin.* **2009** *19*, 491–544.
28. Hannig, J. Generalized fiducial inference via discretization. *Stat. Sin.* **2013** *23*, 489–514. [CrossRef]
29. Gilks, W.R.; Wild, P. Adaptive rejection sampling for Gibbs sampling. *J. Appl. Stat.* **1992**, *41*, 337–348. [CrossRef]
30. Gilks, W.R.; Best, N.G.; Tan, K.K.C. Adaptive rejection metropolis sampling within Gibbs sampling. *J. Appl. Stat.* **1995**, *44*, 455–472. [CrossRef]
31. Wakefield, J.C.; Gelfand, A.E.; Smith, A.F.M. Efficient generation of random variates via the ratio-of-uniforms method. *Stat. Comput.* **1991**, *1*, 129–133. [CrossRef]
32. Box, G.E.P.; Tiao, G.C. *Bayesian Inference in Statistical Analysis*; Wiley: New York, NY, USA, 1992.