symmetry

MDPI

Article

Abstraction of Interpolative Reich-Rus-Ćirić-Type Contractions and Simplest Proof Technique

Monairah Alansari [1,†] and Muhammad Usman Ali [2,*,†]

1 Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia; malansari@kau.edu.sa
2 Department of Mathematics, COMSATS University Islamabad, Attock Campus, Attock 43600, Pakistan
* Correspondence: musman.ali@cuiatk.edu.pk or muh_usman_ali@yahoo.com
† These authors contributed equally to this work.

Abstract: The concept of symmetry is a very vast topic that is involved in the studies of several phenomena. This concept enables us to discuss the phenomenon in some systematic pattern depending upon the type of phenomenon. Each phenomenon has its own type of symmetry. The phenomenon that is used in the discussion of this article is a symmetric distance-measuring function. This article presents the notions of abstract interpolative Reich-Rus-Ćirić-type contractions with a shrink map and examines the existence of ϕ-fixed points for such maps in complete metric space. These notions are defined through special types of simulation functions. The proof technique of the results presented in this article is easy to understand compared with the existing literature on interpolative Reich-Rus-Ćirić-type contractions.

Keywords: ϕ-fixed points; interpolative Kannan contraction; abstract interpolative Reich-Rus-Ćirić-type contractions with a shrink map

Citation: Alansari, M.; Ali, M.U. Abstraction of Interpolative Reich-Rus-Ćirić-Type Contractions and Simplest Proof Technique. *Symmetry* **2022**, *14*, 1504. https://doi.org/10.3390/sym14081504

Academic Editor: Savin Treanta

Received: 4 July 2022
Accepted: 17 July 2022
Published: 22 July 2022

1. Introduction and Preliminaries

Metric fixed point theory has a significant contribution to nonlinear analysis with its applications. This branch of fixed point theory is based on the work of the famous mathematician Banach. He proved that [1], on a complete metric space, every contraction map possesses a unique fixed point. Later on, Kannan [2] and Chatterjea [3] modified the contraction inequality to study the existence of fixed points of discontinuous self-maps on a complete metric space. Afterward, this field has flourished with several interesting results. A few results have been obtained for the following aspects:

(1) Modifying contraction inequality,
(2) Modifying distance measuring function.

Recently, Karapınar [4] derived the interpolative Kannan contraction, which can be considered a modified form of the Kannan contraction. Inspiration from this work led several researchers to extend the existing contraction type inequalities in the pattern of interpolative Kannan contraction.

A few generalizations of contraction inequality have been obtained using some special types of simulation functions, for example [5,6].

Symmetry is a very vast topic that is involved in the studies of several phenomena. Each phenomenon has its own definition of symmetry, which helps to discuss the phenomenon in a systematic pattern. Metric space is a symmetric distance measuring function, which is used in the discussion of this article. In the literature related to interpolative Kannan contractions, we have seen several results based on the symmetric distance measuring function, for example, [7,8], and the asymmetric distance measuring function, for example, [9,10].

In this article, we use special types of simulation functions to extend interpolative Reich-Rus-Ćirić-type contraction inequalities. The proof technique of the fixed point results involving interpolative contraction type inequalities is more complicated than the proof technique of the fixed point results involving contraction type inequalities. With the help of a simulation function, we have tried minimizing these complications of the proof technique, and now the presented proofs are easier to understand.

Before moving on to the next section, we will recall some basic concepts such as interpolative Kannan contraction, a few generalizations of the interpolative Kannan contraction, well-known simulation functions and some other notions that are required for the next section.

Let (V, d_V) be a metric space and let $Q : V \to V$ be a self map. Then, we have the following notions.

- A map $Q : V \to V$ is said to be an interpolative Kannan contraction [4], if

$$d_V(Qk, Ql) \leq \eta d_V(k, Qk)^{\omega_1} d_V(l, Ql)^{1-\omega_1}$$

  for all $k, l \in V$ with $k \neq Qk$, where $\eta \in [0, 1)$ and $\omega_1 \in (0, 1)$.
  Later on, it was observed by Karapinar et al. [11] that the above inequality does not ensure the existence of a unique fixed point of a map in complete metric space. Hence, to discuss the uniqueness of a fixed point, the above inequality was redefined in the following way.
- A map $Q : V \to V$ is said to be an improved interpolative Kannan contraction [11], if

$$d_V(Qk, Ql) \leq \eta d_V(k, Qk)^{\omega_1} d_V(l, Ql)^{1-\omega_1}$$

  for all $k, l \in V \backslash Fix(Q)$, where $\eta \in [0, 1)$, $\omega_1 \in (0, 1)$ and $Fix(Q) = \{k \in V : Qk = k\}$.
- A map $Q : V \to V$ is said to be an interpolative Reich-Rus-Ćirić-type contraction [12], if

$$d_V(Qk, Ql) \quad \leq \quad \eta d_V(k, l)^{\omega_1} d_V(k, Qk)^{\omega_2} d_V(l, Ql)^{1-\omega_1-\omega_2}$$

  for each $k, l \in V \setminus Fix(Q)$, where $\eta \in [0, 1)$ and $\omega_1, \omega_2 \in (0, 1)$ with $\omega_1 + \omega_2 < 1$.

In the literature, $CB(V)$ represents the collection of all nonvoid closed and bounded subsets of V and the Pompeiu–Hausdorff distance is a map $H_V : CB(V) \times CB(V) \to [0, \infty)$ defined by

$$H_V(E, F) = \max\{\sup_{e \in E} d_V(e, F), \sup_{f \in F} d_V(f, E)\}$$

where $d_V(f, E) = \inf\{d_V(f, e) : e \in E\}$.

A set-valued generalization of interpolative Reich-Rus-Ćirić-type contraction is defined in the way: A map $Q : V \to CB(V)$ is said to be a set-valued interpolative Reich-Rus-Ćirić-type contraction [13], if

$$H_V(Qk, Ql) \quad \leq \quad \eta d_V(k, l)^{\omega_1} d_V(k, Qk)^{\omega_2} d_V(l, Ql)^{1-\omega_1-\omega_2}$$

for each $k, l \in V \setminus Fix(Q)$, where $\eta \in [0, 1)$ and $\omega_1, \omega_2 \in (0, 1)$ with $\omega_1 + \omega_2 < 1$.

In the literature, we have seen many auxiliary type functions from $[0, \infty) \times [0, \infty)$ into $\mathbb{R}$, for example, simulation functions, R-functions and C-class functions. Recently, Karapinar [14] used the simulation function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ given by Khojasteh et al. [15] to define the following notion.

A map $Q : V \to V$ is said to be an interpolative Hardy–Rogers type Z-contraction, if

$$\zeta(d_V(Qk, Ql), C(k, l)) \geq 0,$$

for each $k, l \in V \setminus Fix(Q)$, where $\omega_1, \omega_2, \omega_3 \in (0, 1)$ with $\omega_1 + \omega_2 + \omega_3 < 1$, and

$$C(k, l) = d_V(k, l)^{\omega_1} d_V(k, Qk)^{\omega_2} d_V(l, Ql)^{\omega_3} \left[\frac{d_V(k, Ql) + d_V(l, Qk)}{2}\right]^{1-\omega_1-\omega_2-\omega_3}.$$

A few more studies related to interpolative type contractions are available in [16–18].

In the next section, we use the following family of functions defined in [19]:
Θ_F is the collection of functions $\theta_f : [0,\infty)^4 \to [0,\infty)$ with the given properties

θ_1: $\theta_f(d,b,c,0) = 0\ \forall d,b,c \in [0,\infty)$;
θ_2: continuous and nondecreasing.

It is well-known that for a self-map $Q : V \to V$, a point $v \in V$ with $v = Qv$ is called a fixed point of Q. If v is a fixed point of Q with $\phi(v) = 0$ for a map $\phi : V \to [0,\infty)$, then v is called a ϕ-fixed point of Q. This notion is presented in [20].

2. Results

In this section, we denote Ξ_F by the collection of functions $\xi_f : [0,\infty)^3 \to [0,\infty)$ such that

(f1) ξ_f is nondecreasing in each coordinate;
(f2) $\xi_f(g^{\omega_1}, g^{\omega_2}, g^{\omega_3}) \le g$ for each $g \in (0,\infty)$ and for each $\omega_1,\omega_2,\omega_3 \in [0,1]$ with $\omega_1 + \omega_2 + \omega_3 = 1$.

Example 1. *The following functions belong to Ξ_F.*

(E1) $\xi_f(a,b,c) = abc$;

(E2) $\xi_f(a,b,c) = \left(\frac{ac}{1+b}\right)\left(\frac{ab}{1+c}\right)\left(\frac{bc}{1+a}\right)$.

Throughout this article, ξ_f belongs to Ξ_f, θ_f belongs to Θ_F, ϕ represents a map from V into $[0,\infty)$, and (V, d_V) is a metric space.

The following definition is the first form of abstract interpolative Reich-Rus-Ćirić type contraction with a shrink map.

Definition 1. *A self-map $Q : V \to V$ is called an abstract interpolative Reich-Rus-Ćirić type-I contraction with ϕ shrink, if the below-stated inequalities hold:*

$$\begin{aligned} d_V(Qk,Ql) \le\ & \eta\xi_f\big(d_V(k,l)^{\omega_1}, d_V(k,Qk)^{\omega_2}, d_V(l,Ql)^{\omega_3}\big) \\ & + L\theta_f\big(d_V(k,l)^{\omega_1}, d_V(k,Qk)^{\omega_2}, d_V(l,Ql)^{\omega_3}, d_V(l,Qk)^{\omega_4}\big) \end{aligned} \tag{1}$$

for each $k,l \in V \setminus Fix(Q)$ with $l \ne k$, where $\omega_1,\omega_2,\omega_3 \in [0,1]$ with $\omega_1+\omega_2+\omega_3 = 1$, $\omega_4 > 0$, and $L \ge 0$;

for every $l \in V$, we have

$$\phi(Ql) \le \eta\phi(l), \tag{2}$$

where $\eta \in [0,1)$ and $Fix(Q) = \{v \in V : v = Qv\}$.

The following theorem ensures the existence of ϕ-fixed points of the map Q satisfying the above definition.

Theorem 1. *Let $Q : V \to V$ be an abstract interpolative Reich-Rus-Ćirić type-I contraction with ϕ shrink on a complete metric space (V, d_V). Then at least one ϕ-fixed point of Q exists in V.*

Proof. Take an arbitrary point $l_0 \in V$, and define an iterative sequence $l_n = Ql_{n-1} \forall n \in \mathbb{N}$. If $l_{n_0} = l_{n_0+1}$ for some n_0, then l_{n_0} is a fixed point of Q. Moreover, by (2) we get $\phi(l_{n_0}) = \phi(Ql_{n_0}) \le \lambda\phi(l_{n_0})$. This gives $\phi(l_{n_0}) = 0$. Hence, l_{n_0} is a ϕ-fixed point of Q. Now, consider $l_{n-1} \ne l_n\ \forall n \in \mathbb{N}$. By (1), for each $n \in \mathbb{N}$, we get

$$\begin{aligned} d_V(Ql_{n-1},Ql_n) \le\ & \eta\xi_f\big(d_V(l_{n-1},l_n)^{\omega_1}, d_V(l_{n-1},Ql_{n-1})^{\omega_2}, d_V(l_n,Ql_n)^{\omega_3}\big) \\ & + L\theta_f\big(d_V(l_{n-1},l_n)^{\omega_1}, d_V(l_{n-1},Ql_{n-1})^{\omega_2}, d_V(l_n,Ql_n)^{\omega_3}, d_V(l_n,Ql_{n-1})^{\omega_4}\big). \end{aligned} \tag{3}$$

That is,

$$d_V(l_n, l_{n+1}) \leq \eta\xi_f\big(d_V(l_{n-1}, l_n)^{\omega_1}, d_V(l_{n-1}, l_n)^{\omega_2}, d_V(l_n, l_{n+1})^{\omega_3}\big)\ \forall n \in \mathbb{N}. \quad (4)$$

Now, claim that $d_V(l_n, l_{n+1}) < d_V(l_{n-1}, l_n)\ \forall n \in \mathbb{N}$. If it is wrong, then we have $m_0 \in N$ with $d_V(l_{m_0}, l_{m_0+1}) \geq d_V(l_{m_0-1}, l_{m_0})$. By (4) we get

$$\begin{aligned} d_V(l_{m_0}, l_{m_0+1}) &\leq \eta\xi_f\big(d_V(l_{m_0-1}, l_{m_0})^{\omega_1}, d_V(l_{m_0-1}, l_{m_0})^{\omega_2}, d_V(l_{m_0}, l_{m_0+1})^{\omega_3}\big) \\ &\leq \eta\xi_f\big(d_V(l_{m_0}, l_{m_0+1})^{\omega_1}, d_V(l_{m_0}, l_{m_0+1})^{\omega_2}, d_V(l_{m_0}, l_{m_0+1})^{\omega_3}\big) \\ &\leq \eta d_V(l_{m_0}, l_{m_0+1}) \end{aligned}$$

which is only possible when $d_V(l_{m_0}, l_{m_0+1}) = 0$, and it contradicts our assumption. Thus, the claim is true. Since $d_V(l_n, l_{n+1}) < d_V(l_{n-1}, l_n)\ \forall n \in \mathbb{N}$, then (4) we get

$$\begin{aligned} d_V(l_n, l_{n+1}) &\leq \eta\xi_f\big(d_V(l_{n-1}, l_n)^{\omega_1}, d_V(l_{n-1}, l_n)^{\omega_2}, d_V(l_n, l_{n+1})^{\omega_3}\big) \\ &\leq \eta\xi_f\big(d_V(l_{n-1}, l_n)^{\omega_1}, d_V(l_{n-1}, l_n)^{\omega_2}, d_V(l_{n-1}, l_n)^{\omega_3}\big) \\ &\leq \eta d_V(l_{n-1}, l_n)\ \forall n \in \mathbb{N}. \end{aligned} \quad (5)$$

The above inequality implies that

$$d_V(l_n, l_{n+1}) \leq \eta^n d_V(l_0, l_1)\ \forall n \in \mathbb{N}. \quad (6)$$

To verify that the sequence $\{l_n\}$ is Cauchy. Consider $m, n \in \mathbb{N}$ with $n > m$. By triangle inequality and (6) we obtain

$$d_V(l_m, l_n) \leq \sum_{j=m}^{n-1} d_V(l_j, l_{j+1}) \leq \sum_{j=m}^{n-1} \eta^j d_V(l_0, l_1).$$

Since $\sum_{j=1}^{\infty} \eta^j$ is a convergent series, thus, by the above inequality, we get $\lim_{n,m\to\infty} d_V(l_m, l_n) = 0$. As (V, d_V) is complete and $\{l_n\}$ is Cauchy in V, then there exists an element $l^* \in V$ with $l_n \to l^*$. Now, claim that $l^* = Ql^*$. If it is wrong, then $d_V(l^*, Ql^*) > 0$. Since $\{l_n\}$ is an iterative sequence with $l_n \to l^*$, thus, we get

$$\max\{d_V(l_n, l^*), d_V(l_n, l_{n+1}), d_V(l^*, Ql^*)\} = d_V(l^*, Ql^*)\ \forall n \geq N_0 \quad (7)$$

for some $N_0 \in \mathbb{N}$. By (1), for each $n \in \mathbb{N}$, we obtain

$$\begin{aligned} d_V(Ql_n, Ql^*) &\leq \eta\xi_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, Ql_n)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}\big) \\ &\quad + L\theta_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, Ql_n)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}, d_V(l^*, Ql_n)^{\omega_4}\big). \end{aligned} \quad (8)$$

From (7) and (8), for each $n \geq N_0$, we get

$$\begin{aligned} d_V(l_{n+1}, Ql^*) &\leq \eta\xi_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, l_{n+1})^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}\big) \\ &\quad + L\theta_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, Ql_n)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}, d_V(l^*, l_{n+1})^{\omega_4}\big) \\ &\leq \eta\xi_f\big(d_V(l^*, Ql^*)^{\omega_1}, d_V(l^*, Ql^*)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}\big) \\ &\quad + L\theta_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, Ql_n)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}, d_V(l^*, l_{n+1})^{\omega_4}\big) \\ &\leq \eta d_V(l^*, Ql^*) \\ &\quad + L\theta_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, Ql_n)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}, d_V(l^*, l_{n+1})^{\omega_4}\big). \end{aligned} \quad (9)$$

By applying the limit $n \to \infty$ in (9), we get

$$d_V(l^*, Ql^*) \leq \eta d_V(l^*, Ql^*).$$

As $\eta < 1$, thus, the above inequality, only exists when $d_V(l^*, Ql^*) = 0$. Hence, the claim is correct. Since $l^* = Ql^*$, then by (2) we get

$$\phi(l^*) = \phi(Ql^*) \leq \lambda\phi(l^*).$$

This implies that $\phi(l^*) = 0$. Hence, l^* is ϕ-fixed point of Q. □

By letting $\xi_f(a, b, c) = abc$ and $\theta_f(a, b, c, d) = abcd$ in Theorem 1, we get the following result.

Corollary 1. *Let (V, d_V) be a complete metric space. Let $Q : V \to V$ and $\phi : V \to [0, \infty)$ be two maps such that*

$$\begin{aligned} d_V(Qk, Ql) &\leq \eta d_V(k, l)^{\omega_1} d_V(k, Qk)^{\omega_2} d_V(l, Ql)^{\omega_3} \\ &\quad + L d_V(k, l)^{\omega_1} d_V(k, Qk)^{\omega_2} d_V(l, Ql)^{\omega_3} d_V(l, Qk)^{\omega_4} \end{aligned}$$

for each $k, l \in V \setminus Fix(Q)$ with $l \neq k$, where $\omega_1, \omega_2, \omega_3 \in [0, 1]$ with $\omega_1 + \omega_2 + \omega_3 = 1$ and $\omega_4 > 0$; further, for every $l \in V$, we have

$$\phi(Ql) \leq \eta\phi(l),$$

where $\eta \in [0, 1)$ and $L \geq 0$. Then at least one ϕ-fixed point of Q exists in V.

By taking $\omega_1 = \omega_4 = 1$ and $\omega_2 = \omega_3 = 0$ in the above mentioned corollary, we obtain the following result.

Corollary 2. *Let (V, d_V) be a complete metric space. Let $Q : V \to V$ and $\phi : V \to [0, \infty)$ be two maps such that*

$$d_V(Qk, Ql) \leq \eta d_V(k, l) + L d_V(k, l) d_V(l, Qk)$$

for each $k, l \in V \setminus Fix(Q)$ with $l \neq k$; further, for every $l \in V$, we have

$$\phi(Ql) \leq \eta\phi(l),$$

where $\eta \in [0, 1)$ and $L \geq 0$. Then at least one ϕ-fixed point of Q exists in V.

Corollary 3. *Let (V, d_V) be a complete metric space. Let $Q : V \to V$ be a map such that*

$$d_V(Qk, Ql) \leq \eta d_V(k, l)^{\omega_1} d_V(k, Qk)^{\omega_2} d_V(l, Ql)^{\omega_3} \tag{10}$$

for each $k, l \in V \setminus Fix(Q)$ with $l \neq k$, where $\omega_1, \omega_2, \omega_3 \in [0, 1]$ with $\omega_1 + \omega_2 + \omega_3 = 1$, and $\eta \in [0, 1)$. Then a fixed point of Q exists in V.

The conclusion of the above result can be concluded from Corollary 1 by considering $L = 0$ and $\phi(k) = 0\ \forall k \in V$.

The following corollary follows from Corollary 3 by defining $\omega_1 = \tau_1$, $\omega_2 = \tau_2$ and $\omega_3 = 1 - \tau_1 - \tau_2$.

Corollary 4. *Let (V, d_V) be a complete metric space. Let $Q : V \to V$ be a map such that*

$$d_V(Qk, Ql) \leq \eta d_V(k, l)^{\tau_1} d_V(k, Qk)^{\tau_2} d_V(l, Ql)^{1-\tau_1-\tau_2} \tag{11}$$

for each $k, l \in V \setminus Fix(Q)$ with $l \neq k$, where $\tau_1, \tau_2 \in (0, 1)$ with $\tau_1 + \tau_2 < 1$, and $\eta \in [0, 1)$. Then fixed point of Q exists in V.

Inequality (12) can be considered as a rational type interpolative contraction inequality obtained through (1) by taking $\xi_f(a,b,c) = \left(\frac{ac}{1+b}\right)\left(\frac{ab}{1+c}\right)\left(\frac{bc}{1+a}\right)$ and $L = 0$. Some interesting results related to rational type contraction conditions are given in [21].

Corollary 5. *Let (V, d_V) be a complete metric space. Let $Q : V \to V$ and $\phi : V \to [0,\infty)$ be two maps such that*

$$d_V(Qk,Ql) \leq \eta\Big(\frac{d_V(k,l)^{\omega_1} d_V(l,Ql)^{\omega_3}}{1+d_V(k,Qk)^{\omega_2}}\Big)\Big(\frac{d_V(k,l)^{\omega_1} d_V(k,Qk)^{\omega_2}}{1+d_V(l,Ql)^{\omega_3}}\Big)\Big(\frac{d_V(k,Qk)^{\omega_2} d_V(l,Ql)^{\omega_3}}{1+d_V(k,l)^{\omega_1}}\Big) \quad (12)$$

for each $k,l \in V \setminus Fix(Q)$ with $k \neq l$, where $\omega_1, \omega_2, \omega_3 \in [0,1]$ with $\omega_1 + \omega_2 + \omega_3 = 1$; further, for every $l \in V$, we have

$$\phi(Ql) \leq \eta\phi(l)$$

where $\eta \in [0,1)$. Then at least one ϕ-fixed point of Q exists in V.

Consider a simulation function $\beta_\psi : [0,\infty)^2 \to \mathbb{R}$ with the properties:

(b1) $\beta_\psi(0,0) = 0$;
(b2) $\beta_\psi(t,s) \leq \psi(s) - t$;

where $\psi : [0,\infty) \to [0,\infty)$ is a nondecreasing function that fulfills that $\sum_{j=1}^{\infty} \psi^j(s)$ is convergent for each $s > 0$, moreover, $\psi(0) = 0$ and $\psi(s) < s$ if $s > 0$.

Example 2. *A function $\beta_\psi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ defined by $\beta_\psi(k,l) = \alpha l - k$ for each $k,l \in [0,\infty)$, where $\psi(l) = \alpha l$ and $\alpha \in (0,1)$, is the simplest example of the above-defined simulation function.*

Throughout the article, β_ψ represents the above simulation function. Now, we define an abstract interpolative Reich-Rus-Ćirić type-II contraction with ϕ shrink by using the simulation function β_ψ.

Definition 2. *A self-map $Q : V \to V$ is called an abstract interpolative Reich-Rus-Ćirić type-II contraction with ϕ shrink, if the below-stated inequalities hold:*

$$\beta_\psi\Big(d_V(Qk,Ql), \xi_f\big(d_V(k,l)^{\omega_1}, d_V(k,Qk)^{\omega_2}, d_V(l,Ql)^{\omega_3}\big)\Big) + L\theta_f\big(d_V(k,l)^{\omega_1}, d_V(k,Qk)^{\omega_2}, d_V(l,Ql)^{\omega_3}, d_V(l,Qk)^{\omega_4}\big) \geq 0 \quad (13)$$

for each $k,l \in V \setminus Fix(Q)$ with $l \neq k$, where $\omega_1, \omega_2, \omega_3 \in [0,1]$ with $\omega_1 + \omega_2 + \omega_3 = 1$, $\omega_4 > 0$, and $L \geq 0$;

for every $l \in V$, we have

$$\beta_\psi\big(\phi(Ql), \phi(l)\big) \geq 0. \quad (14)$$

Now, we discuss the following ϕ-fixed point result for self-maps satisfying the above definition.

Theorem 2. *Let $Q : V \to V$ be an abstract interpolative Reich-Rus-Ćirić type-II contraction with ϕ shrink on a complete metric space (V, d_V). Then at least one ϕ-fixed point of Q exists in V.*

Proof. Define an iterative sequence $\{l_n\}$, that is $l_n = Ql_{n-1} \forall n \in \mathbb{N}$, for an arbitrary point $l_0 \in V$. If $l_{n_0} = l_{n_0+1}$ for some n_0, then l_{n_0} is a fixed point of Q. Moreover, from (14) we obtain $0 \leq \beta_\psi(\phi(Ql_{n_0}), \phi(l_{n_0})) \leq \psi(\phi(l_{n_0})) - \phi(Ql_{n_0})$; that is $\phi(l_{n_0}) = \phi(Ql_{n_0}) \leq \psi(\phi(l_{n_0}))$. This gives $\phi(l_{n_0}) = 0$. Hence, l_{n_0} is a ϕ-fixed point of Q. To work with the proof, we consider $l_{n-1} \neq l_n\ \forall n \in \mathbb{N}$. By (13), for each $n \in \mathbb{N}$, we get

$$\beta_\psi\Big(d_V(Ql_{n-1}, Ql_n), \xi_f\big(d_V(l_{n-1}, l_n)^{\omega_1}, d_V(l_{n-1}, Ql_{n-1})^{\omega_2}, d_V(l_n, Ql_n)^{\omega_3}\big)\Big) + L\theta_f\big(d_V(l_{n-1}, l_n)^{\omega_1}, d_V(l_{n-1}, Ql_{n-1})^{\omega_2}, d_V(l_n, Ql_n)^{\omega_3}, d_V(l_n, Ql_{n-1})^{\omega_4}\big) \geq 0. \tag{15}$$

Using (b2) and (15), we get

$$\begin{aligned}
&\psi(\xi_f\big(d_V(l_{n-1}, l_n)^{\omega_1}, d_V(l_{n-1}, Ql_{n-1})^{\omega_2}, d_V(l_n, Ql_n)^{\omega_3}\big)) - d_V(Ql_{n-1}, Ql_n) \\
&\quad + L\theta_f\big(d_V(l_{n-1}, l_n)^{\omega_1}, d_V(l_{n-1}, Ql_{n-1})^{\omega_2}, d_V(l_n, Ql_n)^{\omega_3}, d_V(l_n, Ql_{n-1})^{\omega_4}\big) \\
&\geq \beta_\psi\Big(d_V(Ql_{n-1}, Ql_n), \xi_f\big(d_V(l_{n-1}, l_n)^{\omega_1}, d_V(l_{n-1}, Ql_{n-1})^{\omega_2}, d_V(l_n, Ql_n)^{\omega_3}\big)\Big) \\
&\quad + L\theta_f\big(d_V(l_{n-1}, l_n)^{\omega_1}, d_V(l_{n-1}, Ql_{n-1})^{\omega_2}, d_V(l_n, Ql_n)^{\omega_3}, d_V(l_n, Ql_{n-1})^{\omega_4}\big) \geq 0.
\end{aligned}$$

This implies

$$\begin{aligned}
d_V(Ql_{n-1}, Ql_n) \leq\ & \psi(\xi_f\big(d_V(l_{n-1}, l_n)^{\omega_1}, d_V(l_{n-1}, Ql_{n-1})^{\omega_2}, d_V(l_n, Ql_n)^{\omega_3}\big)) \\
& + L\theta_f\big(d_V(l_{n-1}, l_n)^{\omega_1}, d_V(l_{n-1}, Ql_{n-1})^{\omega_2}, d_V(l_n, Ql_n)^{\omega_3}, d_V(l_n, Ql_{n-1})^{\omega_4}\big).
\end{aligned} \tag{16}$$

That is,

$$d_V(l_n, l_{n+1}) \leq \psi(\xi_f\big(d_V(l_{n-1}, l_n)^{\omega_1}, d_V(l_{n-1}, Ql_{n-1})^{\omega_2}, d_V(l_n, Ql_n)^{\omega_3}\big))\ \forall n \in \mathbb{N}. \tag{17}$$

Now, let us claim that $d_V(l_n, l_{n+1}) < d_V(l_{n-1}, l_n)\ \forall n \in \mathbb{N}$. Assume that the claim is wrong, then we have $m_0 \in N$ with $d_V(l_{m_0}, l_{m_0+1}) \geq d_V(l_{m_0-1}, l_{m_0})$. By (17) we get

$$\begin{aligned}
d_V(l_{m_0}, l_{m_0+1}) &\leq \psi(\xi_f\big(d_V(l_{m_0-1}, l_{m_0})^{\omega_1}, d_V(l_{m_0-1}, l_{m_0})^{\omega_2}, d_V(l_{m_0}, l_{m_0+1})^{\omega_3}\big)) \\
&\leq \psi(\xi_f\big(d_V(l_{m_0}, l_{m_0+1})^{\omega_1}, d_V(l_{m_0}, l_{m_0+1})^{\omega_2}, d_V(l_{m_0}, l_{m_0+1})^{\omega_3}\big)) \\
&\leq \psi\big(d_V(l_{m_0}, l_{m_0+1})\big)
\end{aligned}$$

which is impossible, since $l_{m_0} \neq l_{m_0+1}$. Hence, the claim holds. As $d_V(l_n, l_{n+1}) < d_V(l_{n-1}, l_n)\ \forall n \in \mathbb{N}$, then (17) we get

$$\begin{aligned}
d_V(l_n, l_{n+1}) &\leq \psi(\xi_f\big(d_V(l_{n-1}, l_n)^{\omega_1}, d_V(l_{n-1}, l_n)^{\omega_2}, d_V(l_n, l_{n+1})^{\omega_3}\big)) \\
&\leq \psi(\xi_f\big(d_V(l_{n-1}, l_n)^{\omega_1}, d_V(l_{n-1}, l_n)^{\omega_2}, d_V(l_{n-1}, l_n)^{\omega_3}\big)) \\
&\leq \psi\big(d_V(l_{n-1}, l_n)\big)\ \forall n \in \mathbb{N}.
\end{aligned} \tag{18}$$

This yields

$$d_V(l_n, l_{n+1}) \leq \psi^n\big(d_V(l_0, l_1)\big)\ \forall n \in \mathbb{N}. \tag{19}$$

Consider $m, n \in \mathbb{N}$ with $n > m$. By triangle inequality and (19) we obtain

$$d_V(l_m, l_n) \leq \sum_{j=m}^{n-1} d_V(l_j, l_{j+1}) \leq \sum_{j=m}^{n-1} \psi^j\big(d_V(l_0, l_1)\big).$$

Since $\sum_{j=1}^{\infty} \psi^j(s)$ is a convergent series for each $s > 0$, hence, by the above inequality we get $\lim_{n,m\to\infty} d_V(l_m, l_n) = 0$. The completeness of (V, d_V) confirms the existence of an element $l^* \in V$ with $l_n \to l^*$. Now, let us claim that $l^* = Ql^*$. Let us suppose that the claim is wrong, then $d_V(l^*, Ql^*) > 0$. Since $\{l_n\}$ is an iterative sequence with $l_n \to l^*$, thus, we get

$$\max\{d_V(l_n, l^*), d_V(l_n, l_{n+1}), d_V(l^*, Ql^*)\} = d_V(l^*, Ql^*)\ \forall n \geq N_0 \tag{20}$$

for some $N_0 \in \mathbb{N}$. By (13), for each $n \in \mathbb{N}$, we obtain

$$\beta_\psi\big(d_V(Ql_n, Ql^*), \xi_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, Ql_n)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}\big)\big) \\ +L\theta_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, Ql_n)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}, d_V(l^*, Ql_n)^{\omega_4}\big) \geq 0. \tag{21}$$

This gives

$$\begin{aligned} d_V(Ql_n, Ql^*) \leq\ & \psi\big(\xi_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, Ql_n)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}\big)\big) \\ & +L\theta_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, Ql_n)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}, d_V(l^*, Ql_n)^{\omega_4}\big). \end{aligned} \tag{22}$$

By (20) and (22), for each $n \geq N_0$, we get

$$\begin{aligned} d_V(l_{n+1}, Ql^*) \leq\ & \psi\big(\xi_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, l_{n+1})^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}\big)\big) \\ & +L\theta_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, Ql_n)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}, d_V(l^*, l_{n+1})^{\omega_4}\big) \\ \leq\ & \psi\big(\xi_f\big(d_V(l^*, Ql^*)^{\omega_1}, d_V(l^*, Ql^*)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}\big)\big) \\ & +L\theta_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, Ql_n)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}, d_V(l^*, l_{n+1})^{\omega_4}\big) \\ \leq\ & \psi\big(d_V(l^*, Ql^*)\big) \\ & +L\theta_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, Ql_n)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}, d_V(l^*, l_{n+1})^{\omega_4}\big). \end{aligned} \tag{23}$$

Letting $n \to \infty$ in (23), we get

$$d_V(l^*, Ql^*) \leq \psi\big(d_V(l^*, Ql^*)\big).$$

The above inequality, only holds when $d_V(l^*, Ql^*) = 0$. Hence, the claim is correct, $l^* = Ql^*$. By (14) we get $0 \leq \beta_\psi\big(\phi(Ql^*), \phi(l^*)\big) \leq \psi(\phi(l^*)) - \phi(Ql^*)$; that is $\phi(l^*) = \phi(Ql^*) \leq \psi(\phi(l^*))$. This implies that $\phi(l^*) = 0$. Hence, l^* is a ϕ-fixed point of Q. □

We will extend the above results by considering Q as a set-valued map. In the following, $CB(V)$ represents the collection of all nonvoid closed and bounded subsets of V and $CL(V)$ represents the collection of all nonvoid closed subsets of V.

Definition 3. *A set-valued map $Q : V \to CB(V)$ is called an abstract interpolative Reich-Rus-Ćirić type-I set-valued contraction with ϕ shrink, if the below-stated inequalities hold:*

$$\begin{aligned} H_V(Qk, Ql) \leq\ & \eta\xi_f\big(d_V(k, l)^{\omega_1}, d_V(k, Qk)^{\omega_2}, d_V(l, Ql)^{\omega_3}\big) \\ & +L\theta_f\big(d_V(k, l)^{\omega_1}, d_V(k, Qk)^{\omega_2}, d_V(l, Ql)^{\omega_3}, d_V(l, Qk)^{\omega_4}\big) \end{aligned} \tag{24}$$

for each $k, l \in V \setminus Fix(Q)$ with $l \neq k$, where $\omega_1, \omega_2, \omega_3 \in [0,1]$ with $\omega_1 + \omega_2 + \omega_3 = 1$, $\omega_4 > 0$, and $L \geq 0$;

for every $k \in V$, we have

$$\sup_{l \in Qk} \phi(l) \leq \eta\phi(k), \tag{25}$$

where $\eta \in (0,1)$ and $Fix(Q) = \{v \in V : v \in Qv\}$.

The following theorem can be used to validate the existence of ϕ-fixed points for a map satisfying the above definition.

Theorem 3. *Let $Q : V \to CB(V)$ be an abstract interpolative Reich-Rus-Ćirić type-I set-valued contraction with ϕ shrink on a complete metric space (V, d_V). Then at least one ϕ-fixed point of Q exists in V; that is, there exists a point v^* in V with $v^* \in Qv^*$ and $\phi(v^*) = 0$.*

Proof. For an arbitrary point $l_0 \in V$, we get some $l_1 \in Ql_0$. If $l_0 = l_1$, then l_0 is a fixed point of Q. Moreover, by (25) we get $\phi(l_0) \leq \sup_{l \in Ql_0} \phi(l) \leq \eta\phi(l_0)$; that is $\phi(l_0) = 0$. Hence, l_0 is a ϕ-fixed point of Q. Suppose that neither l_0 nor l_1 is a fixed point of Q, then by (24) we get

$$\begin{aligned} d_V(l_1, Ql_1) &\leq H_V(Ql_0, Ql_1) \\ &\leq \eta\xi_f\big(d_V(l_0, l_1)^{\omega_1}, d_V(l_0, Ql_0)^{\omega_2}, d_V(l_1, Ql_1)^{\omega_3}\big) \\ &\quad + L\theta_f\big(d_V(l_0, l_1)^{\omega_1}, d_V(l_0, Ql_0)^{\omega_2}, d_V(l_1, Ql_1)^{\omega_3}, d_V(l_1, Ql_0)^{\omega_4}\big). \end{aligned} \tag{26}$$

That is,

$$d_V(l_1, Ql_1) \leq \eta\xi_f\big(d_V(l_0, l_1)^{\omega_1}, d_V(l_0, Ql_0)^{\omega_2}, d_V(l_1, Ql_1)^{\omega_3}\big). \tag{27}$$

Since $\eta \in (0, 1)$, thus, for $\frac{1}{\sqrt{\eta}} > 1$ we have $l_2 \in Ql_1$ satisfying the given inequality

$$d_V(l_1, l_2) \leq \frac{1}{\sqrt{\eta}} d_V(l_1, Ql_1). \tag{28}$$

To proceed with the proof, we assume that $l_1 \neq l_2$, otherwise l_2 is a ϕ-fixed point. From (27) and (28), we get

$$d_V(l_1, l_2) \leq \sqrt{\eta}\xi_f\big(d_V(l_0, l_1)^{\omega_1}, d_V(l_0, Ql_0)^{\omega_2}, d_V(l_1, Ql_1)^{\omega_3}\big). \tag{29}$$

From the facts that $l_1 \in Ql_0$, $l_2 \in Ql_1$, and nondecreasing property of ξ_f, by (29), we get

$$d_V(l_1, l_2) \leq \sqrt{\eta}\xi_f\big(d_V(l_0, l_1)^{\omega_1}, d_V(l_0, l_1)^{\omega_2}, d_V(l_1, l_2)^{\omega_3}\big). \tag{30}$$

If $d_V(l_0, l_1) \leq d_V(l_1, l_2)$, then from the above inequality we get $d_V(l_1, l_2) = 0$, which is impossible. Thus, $d_V(l_1, l_2) < d_V(l_0, l_1)$. Now, by (30), we get

$$\begin{aligned} d_V(l_1, l_2) &\leq \sqrt{\eta}\xi_f\big(d_V(l_0, l_1)^{\omega_1}, d_V(l_0, l_1)^{\omega_2}, d_V(l_1, l_2)^{\omega_3}\big) \\ &\leq \sqrt{\eta}\xi_f\big(d_V(l_0, l_1)^{\omega_1}, d_V(l_0, l_1)^{\omega_2}, d_V(l_0, l_1)^{\omega_3}\big) \\ &\leq \sqrt{\eta} d_V(l_0, l_1). \end{aligned} \tag{31}$$

Continuing the proof on the above lines we can obtain a sequence $\{l_n\}$ with $l_n \in Ql_{n-1}$ $\forall n \in \mathbb{N}$, $l_{n-1} \neq l_n$ $\forall n \in \mathbb{N}$, and

$$d_V(l_n, l_{n+1}) \leq (\sqrt{\eta})^n d_V(l_0, l_1) \ \forall n \in \mathbb{N}.$$

Moreover, it is trivial to conclude that $\{l_n\}$ is a Cauchy sequence in a complete metric space (V, d_V), thus, there is a point $l^* \in V$ with $l_n \to l^*$. Now, we claim that $l^* \in Ql^*$. If it is wrong, then $d_V(l^*, Ql^*) > 0$. Thus, we can obtain $N_0 \in \mathbb{N}$ such that

$$\max\{d_V(l_n, l^*), d_V(l_n, l_{n+1}), d_V(l^*, Ql^*)\} = d_V(l^*, Ql^*) \ \forall n \geq N_0. \tag{32}$$

By (24), for $k = l_n$ and $l = l^*$, we obtain

$$\begin{aligned} d_V(l_{n+1}, Ql^*) &\leq H_V(Ql_n, Ql^*) \\ &\leq \eta\xi_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, Ql_n)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}\big) \\ &\quad + L\theta_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, Ql_n)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}, d_V(l^*, Ql_n)^{\omega_4}\big) \ \forall n \in \mathbb{N}. \end{aligned} \tag{33}$$

From (32) and (33), for each $n \geq N_0$, we get

$$
\begin{aligned}
d_V(l_{n+1}, Ql^*) \;&\leq\; \eta\xi_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, l_{n+1})^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}\big) \\
&\quad + L\theta_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, Ql_n)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}, d_V(l^*, l_{n+1})^{\omega_4}\big) \\
&\leq\; \eta\xi_f\big(d_V(l^*, Ql^*)^{\omega_1}, d_V(l^*, Ql^*)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}\big) \\
&\quad + L\theta_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, Ql_n)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}, d_V(l^*, l_{n+1})^{\omega_4}\big) \\
&\leq\; \eta d_V(l^*, Ql^*) \\
&\quad + L\theta_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, Ql_n)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}, d_V(l^*, l_{n+1})^{\omega_4}\big).
\end{aligned} \tag{34}
$$

By applying the limit $n \to \infty$ in (34), we get

$$d_V(l^*, Ql^*) \leq \eta d_V(l^*, Ql^*).$$

The existence of the above inequality is impossible when $d_V(l^*, Ql^*) > 0$. Hence, the claim is correct, $l^* \in Ql^*$. By (25) we get

$$\phi(l^*) \leq \sup_{l \in Ql^*} \phi(l) \leq \lambda\phi(l^*).$$

This implies that $\phi(l^*) = 0$. Hence, l^* is a ϕ-fixed point of Q. □

The following result examines the existence of ϕ-fixed points for a set-valued map $Q : V \to CL(V)$.

Theorem 4. *Let (V, d_V) be a complete metric space and let $Q : V \to CL(V)$ be a set-valued map and $\phi : V \to [0, \infty)$ be another map fulfilling the following inequalities:*

$$d_V(l, Ql) \;\leq\; \eta\xi_f\big(d_V(k, l)^{\omega_1}, d_V(k, Qk)^{\omega_2}, d_V(l, Ql)^{\omega_3}\big) \tag{35}$$

for each $k, l \in V \setminus Fix(Q)$ with $l \in Qk$, where $\omega_1, \omega_2, \omega_3 \in [0, 1]$ with $\omega_1 + \omega_2 + \omega_3 = 1$, and $\omega_3 \neq 1$; further, for every $k \in V$, we have

$$\sup_{l \in Qk} \phi(l) \leq \eta\phi(k), \tag{36}$$

where $\eta \in (0, 1)$. Moreover, assume that $Graph(Q) = \{(k, l) : k \in V, l \in Qk\}$ is closed. Then at least one ϕ-fixed point of Q exists in V.

Proof. Following the proof of Theorem 3, here, one can easily obtain a Cauchy sequence $\{l_n\}$ in a complete metric space (V, d_V) with $l_n \in Ql_{n-1}\ \forall n \in \mathbb{N}$, $l_{n-1} \neq l_n\ \forall n \in \mathbb{N}$, and

$$d_V(l_n, l_{n+1}) \leq (\sqrt{\eta})^n d_V(l_0, l_1)\ \forall n \in \mathbb{N}.$$

Furthermore, there exists a point $l^* \in V$ with $l_n \to l^*$. Since $l_n \in Ql_{n-1}\ \forall n \in \mathbb{N}$, thus, $(l_{n-1}, l_n) \in Graph(Q)\ \forall n \in \mathbb{N}$. As given that $Graph(Q)$ is closed, thus, $(l^*, l^*) \in Graph(Q)$, that is $l^* \in Ql^*$. Hence, l^* is a fixed point of Q. By considering (36), we conclude that l^* is a ϕ-fixed point of Q. □

Now we present the definition of the abstract interpolative Reich-Rus-Ćirić type-II set-valued contraction with ϕ shrink.

Definition 4. *A set-valued map $Q : V \to CB(V)$ is called an abstract interpolative Reich-Rus-Ćirić type-II set-valued contraction with ϕ shrink, if the below-stated inequalities are fulfilled:*

$$
\begin{aligned}
&\beta_\psi\Big(H_V(Qk, Ql), \xi_f\big(d_V(k, l)^{\omega_1}, d_V(k, Qk)^{\omega_2}, d_V(l, Ql)^{\omega_3}\big)\Big) \\
&\quad + L\theta_f\big(d_V(k, l)^{\omega_1}, d_V(k, Qk)^{\omega_2}, d_V(l, Ql)^{\omega_3}, d_V(l, Qk)^{\omega_4}\big) \geq 0
\end{aligned} \tag{37}
$$

for each $k, l \in V \setminus Fix(Q)$ *with* $l \neq k$, *where* $\omega_1, \omega_2, \omega_3 \in [0,1]$ *with* $\omega_1 + \omega_2 + \omega_3 = 1, \omega_3 \neq 0$, $\omega_4 > 0$, *and* $L \geq 0$;

for every $k \in V$, *we have*

$$\beta_\psi\big(\sup_{l \in Qk} \phi(l), \phi(k)\big) \geq 0. \tag{38}$$

In the following theorems, we assume that ξ_f and ψ are strictly increasing instead of nondecreasing.

Theorem 5. *Let* $Q : V \to CB(V)$ *be an abstract interpolative Reich-Rus-Ćirić type-II set-valued contraction with* ϕ *shrink on a complete metric space* (V, d_V). *Then at least one* ϕ*-fixed point of* Q *exists in* V.

Proof. For an arbitrary point $l_0 \in V$, we get a point $l_1 \in Ql_0$. If $l_0 = l_1$, then l_0 is a fixed point of Q. Moreover, by (38), we get $0 \leq \beta_\psi\big(\sup_{l \in Ql_0} \phi(l), \phi(l_0)\big) \leq \psi(\phi(l_0)) - \sup_{l \in Ql_0} \phi(l)$, this implies $\phi(l_0) \leq \psi(\phi(l_0))$, hence, l_0 is a ϕ-fixed point of Q. Suppose that neither l_0 nor l_1 is a fixed point of Q, then by (37) we get

$$\begin{aligned} &\beta_\psi\Big(H_V(Ql_0, Ql_1), \xi_f\big(d_V(l_0, l_1)^{\omega_1}, d_V(l_0, Ql_0)^{\omega_2}, d_V(l_1, Ql_1)^{\omega_3}\big) \\ &+L\theta_f\big(d_V(l_0, l_1)^{\omega_1}, d_V(l_0, Ql_0)^{\omega_2}, d_V(l_1, Ql_1)^{\omega_3}, d_V(l_1, Ql_0)^{\omega_4}\big) \geq 0. \end{aligned} \tag{39}$$

This implies that

$$\begin{aligned} H_V(Ql_0, Ql_1) \leq\ & \psi\big(\xi_f\big(d_V(l_0, l_1)^{\omega_1}, d_V(l_0, Ql_0)^{\omega_2}, d_V(l_1, Ql_1)^{\omega_3}\big)\big) \\ &+L\theta_f\big(d_V(l_0, l_1)^{\omega_1}, d_V(l_0, Ql_0)^{\omega_2}, d_V(l_1, Ql_1)^{\omega_3}, d_V(l_1, Ql_0)^{\omega_4}\big). \end{aligned} \tag{40}$$

Since $l_1 \in Ql_0$, thus, by the above inequality we get

$$d_V(l_1, Ql_1) \leq \psi\big(\xi_f\big(d_V(l_0, l_1)^{\omega_1}, d_V(l_0, l_1)^{\omega_2}, d_V(l_1, Ql_1)^{\omega_3}\big)\big). \tag{41}$$

If $d_V(l_0, l_1) \leq d_V(l_1, Ql_1)$, then by (41) we get $d_V(l_1, Ql_1) \leq \psi(d_V(l_1, Ql_1)) < d_V(l_1, Ql_1)$, which is impossible. Thus, we conclude $d_V(l_0, l_1) > d_V(l_1, Ql_1)$. By considering strictly increasing behavior of ψ, ξ_f, and using (41) we get

$$\begin{aligned} d_V(l_1, Ql_1) &\leq \psi\big(\xi_f\big(d_V(l_0, l_1)^{\omega_1}, d_V(l_0, l_1)^{\omega_2}, d_V(l_1, Ql_1)^{\omega_3}\big)\big) \\ &< \psi\big(\xi_f\big(d_V(l_0, l_1)^{\omega_1}, d_V(l_0, l_1)^{\omega_2}, d_V(l_0, l_1)^{\omega_3}\big)\big) \\ &\leq \psi\big(d_V(l_0, l_1)\big). \end{aligned} \tag{42}$$

As $d_V(l_1, Ql_1) < \psi\big(d_V(l_0, l_1)\big)$, there exists some real number $\epsilon_1 > 0$ such that $d_V(l_1, Ql_1) + \epsilon_1 = \psi\big(d_V(l_0, l_1)\big)$. Thus, we get $l_2 \in Ql_1$ such that $d_V(l_1, l_2) \leq d_V(l_1, Ql_1) + \epsilon_1$. Hence, we conclude that

$$d_V(l_1, l_2) \leq \psi\big(d_V(l_0, l_1)\big). \tag{43}$$

Continuing the proof on the above lines we can obtain a sequence $\{l_n\}$ with $l_n \in Ql_{n-1}\ \forall n \in \mathbb{N}$, $l_{n-1} \neq l_n\ \forall n \in \mathbb{N}$, and

$$d_V(l_n, l_{n+1}) \leq \psi^n\big(d_V(l_0, l_1)\big)\ \forall n \in \mathbb{N}.$$

Further, it can be seen that $\{l_n\}$ is a Cauchy sequence in a complete metric space (V, d_V) and there exists $l^* \in V$ with $l_n \to l^*$. Now, we claim that $l^* \in Ql^*$. If it is wrong then $d_V(l^*, Ql^*) > 0$. Thus, we can obtain $N_0 \in \mathbb{N}$ such that

$$\max\{d_V(l_n, l^*), d_V(l_n, l_{n+1}), d_V(l^*, Ql^*)\} = d_V(l^*, Ql^*)\ \forall n \geq N_0. \tag{44}$$

By (37), for $k = l_n$ and $l = l^*$, we get

$$\beta_\psi\big(H_V(Ql_n, Ql^*), \xi_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, Ql_n)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}\big)\big) \\ +L\theta_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, Ql_n)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}, d_V(l^*, Ql_n)^{\omega_4}\big)\ \forall n \in \mathbb{N}. \quad (45)$$

From the above inequality, we obtain

$$\begin{aligned} d_V(l_{n+1}, Ql^*) &\leq H_V(Ql_n, Ql^*) \\ &\leq \psi\big(\xi_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, Ql_n)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}\big)\big) \\ &\quad +L\theta_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, Ql_n)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}, d_V(l^*, Ql_n)^{\omega_4}\big)\ \forall n \in \mathbb{N}. \end{aligned} \quad (46)$$

From (44) and (46), for each $n \geq N_0$, we get

$$\begin{aligned} d_V(l_{n+1}, Ql^*) &\leq \psi\big(\xi_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, l_{n+1})^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}\big)\big) \\ &\quad +L\theta_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, Ql_n)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}, d_V(l^*, l_{n+1})^{\omega_4}\big) \\ &\leq \psi\big(\xi_f\big(d_V(l^*, Ql^*)^{\omega_1}, d_V(l^*, Ql^*)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}\big)\big) \\ &\quad +L\theta_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, Ql_n)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}, d_V(l^*, l_{n+1})^{\omega_4}\big) \\ &\leq \psi\big(d_V(l^*, Ql^*)\big) \\ &\quad +L\theta_f\big(d_V(l_n, l^*)^{\omega_1}, d_V(l_n, Ql_n)^{\omega_2}, d_V(l^*, Ql^*)^{\omega_3}, d_V(l^*, l_{n+1})^{\omega_4}\big). \end{aligned} \quad (47)$$

By letting $n \to \infty$ in (47), we get

$$d_V(l^*, Ql^*) \leq \psi\big(d_V(l^*, Ql^*)\big)$$

which is impossible for $d_V(l^*, Ql^*) > 0$. Hence, the claim is correct, $l^* \in Ql^*$. Moreover, by (38) we get $0 \leq \beta_\psi\big(\sup_{l \in Ql^*} \phi(l), \phi(l^*)\big) \leq \psi(\phi(l^*)) - \sup_{l \in Ql^*} \phi(l)$. As $l^* \in Ql^*$, thus, $\phi(l^*) \leq \sup_{l \in Ql^*} \phi(l) \leq \psi(\phi(l^*))$. This implies that $\phi(l^*) = 0$. Hence, l^* is a ϕ-fixed point of Q. □

The following theorem can examine ϕ-fixed points of set-valued map $Q : V \to CL(V)$.

Theorem 6. *Let (V, d_V) be a complete metric space and let $Q : V \to CL(V)$ be a set-valued map and $\phi : V \to [0, \infty)$ be another map fulfilling the following inequalities:*

$$\beta_\psi\Big(d_V(l, Ql), \xi_f\big(d_V(k, l)^{\omega_1}, d_V(k, Qk)^{\omega_2}, d_V(l, Ql)^{\omega_3}\big)\Big) \geq 0 \quad (48)$$

for each $k, l \in V \setminus Fix(Q)$ with $l \in Qk$, where $\omega_1, \omega_2 \in [0, 1]$ and $\omega_3 \in (0, 1)$ with $\omega_1 + \omega_2 + \omega_3 = 1$; further, for every $k \in V$, we have

$$\beta_\psi\big(\sup_{l \in Qk} \phi(l), \phi(k)\big) \geq 0. \quad (49)$$

Furthermore, assume that $Graph(Q) = \{(k, l) : k \in V, l \in Qk\}$ is closed. Then at least one ϕ-fixed point of Q exists in V.

3. Application

A suitable application of the work can be seen as an existence theorem for the following type of fractional-order integral equation:

$$k(t) = q(t) + \frac{\mu}{[\Gamma(\alpha)]^2} \int_0^{p(t)} (p(t) - s)^{\alpha - 1} w(s, k(s)) ds, \quad \alpha \in (0, 1), \quad t \in J = [a, b] \quad (50)$$

where $q \colon J \to \mathbb{R}$, $p \colon J \to \mathbb{R}^+ = [0, \infty)$, and $w \colon J \times \mathbb{R} \to \mathbb{R}$ are continuous functions, μ is constant real number, and Γ is the Euler gamma function; that is $\Gamma(\alpha) = \int_0^\infty t^{\alpha - 1} e^{-t} dt$.

Consider $V = (C[a,b], \mathbb{R})$ is the space of all continuous and bounded real-valued functions defined on $J = [a,b]$. Define a metric on V by

$$d_V(k,l) = \|k - l\| = \max_{t \in J} |k(t) - l(t)| \; \forall k,l \in V.$$

Clearly, (V, d_V) is a complete metric space.

Now, we move towards the existence theorem of (50).

Theorem 7. *Consider $V = (C[a,b], \mathbb{R})$ and consider the operator*

$$Q\colon V \to V, \quad Qk(t) = q(t) + \frac{\mu}{[\Gamma(\alpha)]^2} \int_0^{p(t)} (p(t) - s)^{\alpha-1} w(s, k(s)) ds, \quad \alpha \in (0,1), \quad t \in J$$

where $q\colon J \to \mathbb{R}$, $p\colon J \to \mathbb{R}^+ = [0,\infty)$, and $w\colon J \times \mathbb{R} \to \mathbb{R}$ are continuous functions, μ is constant, and Γ is the Euler gamma function; that is $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$. Moreover, consider that there are $\omega_1, \omega_2, \omega_3 \in [0,1]$ with $\omega_1 + \omega_2 + \omega_3 = 1$ satisfying

$$\frac{|w(s,k(s)) - w(s,l(s))|}{\|k - Qk\|^{\omega_2} \|l - Ql\|^{\omega_3}} \leq [\Gamma(\alpha+1)]^2 |k(s) - l(s)|^{\omega_1} \tag{51}$$

for all $s \in J$ and for each $k,l \in V$ with $\min\{\|k-l\|, \|k-Qk\|, \|l-Ql\|\} > 0$, moreover,

$$\sup_{t \in J} \left| \mu(p(t))^\alpha \right| \leq 1.$$

Then, (50) possesses at least one solution.

Proof. For each $k,l \in V$ with $\min\{\|k-l\|, \|k-Qk\|, \|l-Ql\|\} > 0$, we obtain

$$\begin{aligned}
|Qk(t) - Ql(t)| &= \left| \frac{\mu}{[\Gamma(\alpha)]^2} \int_0^{p(t)} (p(t)-s)^{\alpha-1} [w(s,k(s)) - w(s,l(s))] ds \right| \\
&\leq \left| \frac{\mu}{[\Gamma(\alpha)]^2} \int_0^{p(t)} (p(t)-s)^{\alpha-1} ds \right| [\Gamma(\alpha+1)]^2 \|k-l\|^{\omega_1} \|k-Qk\|^{\omega_2} \|l-Ql\|^{\omega_3} \\
&= \left| \frac{\mu}{[\Gamma(\alpha)]^2} \frac{(p(t))^\alpha}{\alpha} \right| [\alpha\Gamma(\alpha)]^2 \|k-l\|^{\omega_1} \|k-Qk\|^{\omega_2} \|l-Ql\|^{\omega_3} \\
&= \alpha \left| \mu(p(t))^\alpha \right| \|k-l\|^{\omega_1} \|k-Qk\|^{\omega_2} \|l-Ql\|^{\omega_3} \; \forall t \in J.
\end{aligned}$$

Thus, we get

$$\|Qk - Ql\| \leq \alpha \|k-l\|^{\omega_1} \|k-Qk\|^{\omega_2} \|l-Ql\|^{\omega_3}$$

for each $k,l \in V \setminus Fix(Q)$ with $k \neq l$. Thus, by Corollary 3, a fixed point of Q occurs; that is, the integral Equation (50) possesses at least one solution. □

Example 3. *Consider $V = \{0, 1, 2 \cdots, 20\}$ and define*

$$d_V(k,l) = \begin{cases} 0, & k = l \\ \max\{k,l\}, & k \neq l. \end{cases}$$

Define $Q : V \to V$ and $\phi : V \to [0,\infty)$ by

$$Q(k) = \begin{cases} 0, & k = 0 \\ k-1, & \textit{otherwise} \end{cases}$$

and

$$\phi(k) = \frac{k}{2}.$$

Then, it is easy to verify that the axioms of Theorem 1 are valid, by taking $\xi_f(a,b,c) = abc$, $\omega_1 = 0.99$, $\omega_2 = 0.005$, $\omega_3 = 0.005$, $L = 0$ *and* $\eta = \frac{99}{100}$. *Thus, there is an element* $k \in V$ *with* $Qk = k$ *and* $\phi(k) = 0$.

Example 4. *Consider* $V = \mathbb{W}$ *the set of all whole numbers and define*

$$d_V(k,l) = \begin{cases} 0, & k = l \\ \max\{k,l\}, & k \neq l. \end{cases}$$

Define $Q : V \to CB(V)$ *and* $\phi : V \to [0,\infty)$ *by*

$$Q(k) = \begin{cases} \{0\}, & k \in \{0,1\} \\ \{0,k-1\}, & k \in \{2,3,\cdots,10\} \\ \{0,k\}, & otherwise \end{cases}$$

and

$$\phi(k) = \begin{cases} k/2, & k \in \{1,2,\cdots,10\} \\ 0, & otherwise. \end{cases}$$

Then, it is easy to check that the axioms of Theorem 6 are valid, by taking $\xi_f(a,b,c) = abc$, $\beta_\psi(k,l) = (49/50)l - k$, $\omega_1 = 0.99$, $\omega_2 = 0.005$, *and* $\omega_3 = 0.005$. *Since*

$$(k-1)^{0.995} \leq (49/50)k^{0.995} \text{ for each } k \in \{1,2,\cdots,10\}.$$

Hence, there is an element $k \in V$ *with* $k \in Qk$ *and* $\phi(k) = 0$.

4. Conclusions

In this article, we have studied the existence of ϕ-fixed points for the mappings satisfying abstract interpolative Reich-Rus-Ćirić-type contractions with a shrink map on a complete metric space. Abstract interpolative Reich-Rus-Ćirić-type contraction with a shrink map has the following characteristics:

- It is an extended form of interpolative Reich-Rus-Ćirić-type contraction.
- It provides an easier proof of the results, ensuring ϕ-fixed points.

Finally, we have studied the existence of a solution for a fractional-order integral equation using our results.

Author Contributions: Both authors contributed equally to this article and approved the final manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: The Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, Saudi Arabia has funded this project, under grant number FP-083-43.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors are grateful to the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, Saudi Arabia for funding this project, under grant number FP-083-43.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Banach, S. Sur les operations dans les ensembles abstraits et leur application aux equations integrales. *Fund. Math.* **1922**, *3*, 133–181. [CrossRef]
2. Kannan, R. Some results on fixed point. *Bull. Cal. Math. Soc.* **1968**, *60*, 71–76.
3. Chatterjea, S.K. Fixed point theorem. *C. R. Acad. Bulg. Sci.***1972**, *25*, 727–730. [CrossRef]
4. Karapınar, E. Revisiting the Kannan type contractions via interpolation. *Adv. Theory Nonlinear Anal. Appl.* **2018**, *2*, 85–87. [CrossRef]
5. Argoubi, H.; Samet, B.; Vetro, C. Nonlinear contractions involving simulation functions in a metric space with a partial order. *J. Nonlinear Sci. Appl.* **2015**, *8*, 1082–1094. [CrossRef]
6. Alqahtani, B.; Alzaid, S.S.; Fulga, A.; Yesilkaya, S.S. Common fixed point theorem on Proinov type mappings via simulation function. *Adv. Diff. Equ.* **2021**, *2021*, 328. [CrossRef]
7. Debnath, P.; de La Sen, M. Fixed-Points of Interpolative Ćirić-Reich-Rus-Type Contractions in b-Metric Spaces. *Symmetry* **2020**, *12*, 12. [CrossRef]
8. Errai, Y.; Marhrani, E.M.; Aamri, M. Fixed Points of g-Interpolative Ćirić-Reich-Rus-Type Contractions in b-Metric Spaces. *Axioms* **2020**, *9*, 132. [CrossRef]
9. Gautam, P.; Sánchez Ruiz, L.M.; Verma, S. Fixed Point of Interpolative Rus-Reich-Ćirić Contraction Mapping on Rectangular Quasi-Partial b-Metric Space. *Symmetry* **2021**, *13*, 32. [CrossRef]
10. Mishra, V.N.; Sánchez Ruiz, L.M.; Gautam, P.; Verma, S. Interpolative Reich-Rus-Ćirić and Hardy–Rogers Contraction on Quasi-Partial b-Metric Space and Related Fixed Point Results. *Mathematics* **2020**, *8*, 1598. [CrossRef]
11. Karapınar, E.; Agarwal, R.P.; Aydi, H. Interpolative Reich-Rus-Ćirić type contractions on partial metric spaces. *Mathematics* **2018**, *6*, 256. [CrossRef]
12. Aydi, H.; Chen, C.M.; Karapınar, E. Interpolative Ćirić-Reich-Rus type contractions via the Branciari distance. *Mathematics* **2019**, *7*, 84. [CrossRef]
13. Debnath, P.; de La Sen, M. Set-valued interpolative Hardy–Rogers and set-valued Reich–Rus–Ćirić-type contractions in b-metric spaces. *Mathematics* **2019**, *7*, 849. [CrossRef]
14. Karapınar, E. Revisiting simulation functions via interpolative contractions. *Appl. Anal. Discrete Math.* **2019**, *13*, 859–870. [CrossRef]
15. Khojasteh, F.; Shukla, S.; Radenovic, S. A new approach to the study of fixed point theorems via simulation functions. *Filomat* **2015**, *29*, 1189–1194. [CrossRef]
16. Gaba, Y.U.; Karapınar, E. A new approach to the interpolative contractions. *Axioms* **2019**, *8*, 110. [CrossRef]
17. Karapınar, E.; Algahtani, O.; Aydi, H. On interpolative Hardy–Rogers type contractions. *Symmetry* **2018**, *11*, 8. [CrossRef]
18. Alansari, M.; Ali, M.U. Unified multivalued interpolative Reich-Rus-Ćirić-type contractions. *Adv. Diff. Equ.* **2021**, *2021*, 311. [CrossRef]
19. Samet, B. Some results on best proximity points. *J. Optim. Theory Appl.* **2013**, *159*, 281–291. [CrossRef]
20. Jleli, M.; Samet, B.; Vetro, C. Fixed point theory in partial metric spaces via ϕ-fixed point's concept in metric spaces. *J. Inequal. Appl.* **2014**, *2014*, 426. [CrossRef]
21. Fulga, A. On interpolative contractions that involve rational forms. *Adv. Diff. Equ.* **2021**, *2021*, 448. [CrossRef]