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On Fractional Newton Inequalities via Coordinated Convex Functions

Pınar Kösem [1], Hasan Kara [1], Hüseyin Budak [1], Muhammad Aamir Ali [2] and Kamsing Nonlaopon [3,*]

[1] Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce 81620, Turkey; pinarksm18@gmail.com (P.K.); hasan64kara@gmail.com (H.K.); hsyn.budak@gmail.com (H.B.)
[2] Jiangsu Key Laboratory of NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China; mahr.muhammad.aamir@gmail.com
[3] Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand
* Correspondence: nkamsi@kku.ac.th; Tel.: +66-8-6642-1582

Abstract: In this paper, firstly, we present an integral identity for functions of two variables via Riemann–Liouville fractional integrals. Then, a Newton-type inequality via partially differentiable coordinated convex mappings is derived by taking the absolute value of the obtained identity. Moreover, several inequalities are obtained with the aid of the Hölder and power mean inequality. In addition, we investigate some Newton-type inequalities utilizing mappings of two variables with bounded variation. Finally, we gave some mathematical examples and their graphical behavior to validate the obtained inequalities.

Keywords: Newton-type inequality; fractional calculus; co-ordinated convex functions; bounded variation functions; Riemann Stieltjes integrals

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1. Introduction

Inequalities are widely recognized as one of the main drivers behind the development of mathematics and various branches of applied mathematics. Fundamental inequalities that have taken their place in the literature over the last decade have greatly contributed to applications in many fields of mathematics. Since inequalities and convex functions play an important role in all areas of mathematics and are an active research area, they have become the focus of attention of researchers, especially in recent years. Among these inequalities, the Simpson and Newton-type inequality has directed much research. The inequality obtained from Simpson's $1/3$ rule, known as Simpson's type inequality in the literature, is as follows.

Suppose that $F:[\sigma,\rho]\to\mathbb{R}$ is a four times continuously differentiable mapping on (σ,ρ) and let $\left\|F^{(4)}\right\|_{\infty}=\sup_{\kappa\in(\sigma,\rho)}\left|F^{(4)}(\kappa)\right|<\infty$. Then, one has the inequality

$$\left|\frac{1}{3}\left[\frac{F(\sigma)+F(\rho)}{2}+2F\left(\frac{\sigma+\rho}{2}\right)\right]-\frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}F(\kappa)d\kappa\right|\leq\frac{1}{2880}\left\|F^{(4)}\right\|_{\infty}(\rho-\sigma)^{4}.$$

The Simpson's $1/3$ type inequality intrigued researchers. For instance, Dragomir et al. [1] proved some new Simpson's type results and their applications to quadrature formulas in numerical integration. Alomari et al. investigated some of Simpson's type inequalities based on the s-convex functions in [2]. Sarikaya et al. gave the variants of Simpson's type inequalities via convexity in [3]. In [4], a Simpson-type inequality via an n-times continuously differentiable function is given. New Simpson-type inequalities are presented based on (s,m)-convexity with the help of the differentiable mappings in [5]. Du et al. introduced the concepts of an m-invex set, generalized (s,m)-preinvex mapping, and explicitly (s,m)-preinvex mapping, provided some properties for the newly introduced mappings, and obtained new Hadamard–Simpson-type integral inequalities via a mapping of which the power of the absolute of the first derivative is generalized (s,m)-preinvex mapping in [6]. Hezenci et al. obtained several fractional Simpson-type inequalities via functions whose second derivatives in modulus are convex in [7]. Sarıkaya et al. obtained Simpson's type inequality via the mapping whose second derivatives modulus is F-convex in [8].

The inequality obtained from Simpson's $3/8$ rule and known as Newtonian inequality in the literature is as follows.

If $F : [\sigma,\rho] \to \mathbb{R}$ is a four times continuously differentiable mapping on (σ,ρ) and let $\left\|F^{(4)}\right\|_\infty = \sup_{\kappa\in(\sigma,\rho)} \left|F^{(4)}(\kappa)\right| < \infty$. Then, one has the inequality

$$\left|\frac{1}{8}\left[F(\sigma) + 3F\left(\frac{2\sigma+\rho}{3}\right) + 3F\left(\frac{\sigma+2\rho}{3}\right) + F(\rho)\right] - \frac{1}{\rho-\sigma}\int_\sigma^\rho F(\kappa)d\kappa\right| \leq \frac{1}{6480}\left\|F^{(4)}\right\|_\infty(\rho-\sigma)^4.$$

There are many studies in the literature on Newton-type inequalities. Gao and Shi derived inequalities of Newton's type with the help of the functions, whose second derivatives moduli are convex in [9]. Erden et al. presented some error estimates of the Newton-type cubature formula with the aid of the bounded variation and Lipschitzian mappings in [10]. Noor et al. gave Newton's type inequalities based on harmonic convex and p-harmonic convex mappings in [11,12], respectively. Iftikhar et al. investigated some new Newton-type integral inequalities on coordinates in [13].

Liouville first introduced the concepts of fractional derivative and fractional integral. The idea of fractional derivative and fractional integral emerged from the question of whether derivatives and integrals exist only for integers. Since the 17th century, it has developed with the pioneering studies of Leibniz, Euler, Lagrange, Abel, Liouville, and many other mathematicians based on the generalization of differential and integration for fractional order. Many researchers have focused on this issue. Sarikaya et al. obtained new inequalities of Hermite–Hadamard type and trapezoid type based on Riemann–Liouville fractional integrals in [14] for the first time. Set proved inequalities of Ostrowski-type inequalities utilizing the Riemann–Liouville fractional integrals via differentiable mappings in [15]. İşcan and Wu obtained inequalities of Hermite–Hadamard type with the aid of the harmonic convexity in [16]. Sarikaya and Yıldırım gave new inequalities of Hermite–Hadamard type and midpoint type inequalities based on Riemann–Liouville fractional integrals in [17]. Chen and Huang established Simpson $1/3$ rule type inequality via s-convex mappings with the aid of the Riemann–Liouville fractional integrals in [18].

Moreover, after Camille Jordan introduced the mappings of bounded variation of a single variable, various studies on mapping this bounded variation were put forward. In p, functions of bounded variations have been the subject of new research in inequality theory. For instance, Dragomir investigated midpoint-type inequalities with the help of the functions of bounded variation in [19]. Then, Dragomir was also obtained for trapezoid-type inequalities in [20]. What is more, Dragomir proved new Simpson's type inequalities based on functions of bounded variations in [21]. Jawarneh and Noorani established results for some inequalities based on mappings of bounded mapping on coordinates in [22]. However, there are minor errors in Lemma 1, which he established here, and Moricz corrected this error in [23]. Then, Budak and Sarıkaya, with the help of the Lemma established by Moricz, obtained the corrections of these results in [22].

Here are the articles that inspire us: Sitthiwirattham et al. [24] gave new Newton's type inequalities with the help of the convex mappings via Riemann–Liouville fractional integrals. The authors gave new fractional Simpson's second formula inequalities via mappings of bounded variation in [24]. Hezenci et al. proved some of Newton's type inequalities with the help of differentiable convex mappings based on the well-known Riemann–Liouville fractional integrals in [25]. For the other paper devoted to Simpson-type inequalities, please refer to [26–28]. With the motivation from these studies, we will establish new Simpson's second rule inequalities via convex mappings on coordinates by using Riemann–Liouville fractional integrals. We will also investigate new fractional Newton formula-type inequalities based on mappings of two variables with bounded variations.

2. Preliminaries

In this section, fundamental definitions of Riemann–Liouville integrals via one and two variables in the literature are given. In addition, Riemann–Liouville fractional Newton-type inequalities are mentioned via a variable. What is more, the two lemmas we will use in the Newton-type inequalities based on bounded variations section will be addressed.

Definition 1 ([29,30])**.** *If $F \in L_1[\sigma,\rho]$ is a mapping with $\alpha > 0$ and $\sigma \geq 0$, then the integrals $\mathcal{J}^{\alpha}_{\sigma+}F(\kappa)$ and $\mathcal{J}^{\alpha}_{\rho-}F(\kappa)$ are described by*

$$\mathcal{J}^{\alpha}_{\sigma+}F(\kappa) = \frac{1}{\Gamma(\sigma)}\int_{\sigma}^{\kappa}(\kappa-\gamma)^{\alpha-1}F(\gamma)d\gamma, \quad \kappa > \sigma$$

and

$$\mathcal{J}^{\alpha}_{\rho-}F(\kappa) = \frac{1}{\Gamma(\alpha)}\int_{\kappa}^{\rho}(\gamma-\kappa)^{\alpha-1}F(\gamma)d\gamma, \quad \kappa < \rho,$$

where $\Gamma(\cdot)$ is the well-known Gamma function. These integrals are called Riemann–Liouville fractional integrals in the literature.

Definition 2 ([31])**.** *If $F \in L_1(\Delta = [\sigma,\rho] \times [\varsigma,d])$ is a mapping and $\alpha,\beta > 0$ and $\sigma,\varsigma \geq 0$, then the impressions $\mathcal{J}^{\alpha,\beta}_{\sigma+,\varsigma+}$, $\mathcal{J}^{\alpha,\beta}_{\sigma+,d-}$, $+\mathcal{J}^{\alpha,\beta}_{\rho-,\varsigma+}$, and $\mathcal{J}^{\alpha,\beta}_{\rho-,d-}$ are defined by*

$$\mathcal{J}^{\alpha,\beta}_{\sigma+,\varsigma+}F(\kappa,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\sigma}^{\kappa}\int_{\varsigma}^{y}(\kappa-\gamma)^{\alpha-1}(y-\delta)^{\beta-1}F(\gamma,\delta)d\delta d\gamma, \quad \kappa > \sigma,\ y > \varsigma,$$

$$\mathcal{J}^{\alpha,\beta}_{\sigma+,d-}F(\kappa,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\sigma}^{\kappa}\int_{y}^{d}(\kappa-\gamma)^{\alpha-1}(\delta-y)^{\beta-1}F(\gamma,\delta)d\delta d\gamma, \quad \kappa > \sigma,\ y > d,$$

$$\mathcal{J}^{\alpha,\beta}_{\rho-,\varsigma+}F(\kappa,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\kappa}^{\rho}\int_{\varsigma}^{y}(\gamma-\kappa)^{\alpha-1}(y-\delta)^{\beta-1}F(\gamma,\delta)d\delta d\gamma, \quad \kappa < \rho,\ y > \varsigma,$$

$$\mathcal{J}^{\alpha,\beta}_{\rho-,d-}F(\kappa,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\kappa}^{\rho}\int_{y}^{d}(\gamma-\kappa)^{\alpha-1}(\delta-y)^{\beta-1}F(\gamma,\delta)d\delta d\gamma,\ quad\kappa < \rho,\ y < d,$$

where Γ is the well-known Gamma function. These impressions are called double Riemann–Liouville fractional integrals in the literature.

For more information and recent results for fractional calculus, one can refer to [32–34].

With the aid of $\mathcal{J}^{\alpha}_{\sigma+}F(\kappa)$ and $\mathcal{J}^{\alpha}_{\rho-}F(\kappa)$, Sitthiwirattham et al. [24] obtained the new Newton-type inequalities as follows:

Theorem 1. *Let $F : [\sigma,\rho] \to \mathbb{R}$ be a differentiable mapping on (σ,ρ) with $F \in L_1[\sigma,\rho]$. If $|F'|$ is convex mapping, then we have the following Newton's type inequality:*

$$\begin{aligned} &\left|\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\sigma+}F\left(\frac{2\sigma+\rho}{3}\right) + \mathcal{J}^{\alpha}_{\frac{2\sigma+\rho}{3}+}F\left(\frac{\sigma+2\rho}{3}\right) + \mathcal{J}^{\alpha}_{\frac{\sigma+2\rho}{3}+}F(\rho)\right]\right. \\ &\quad \left. -\frac{1}{8}\left[F(\sigma) + 3F\left(\frac{2\sigma+\rho}{3}\right) + 3F\left(\frac{\sigma+2\rho}{3}\right) + F(\rho)\right]\right| \\ &\leq \frac{\rho-\sigma}{27}\left[\left|F'(\rho)\right|(3\Omega_2(\alpha) - \Omega_1(\alpha) + 2\Omega_4(\alpha) - \Omega_3(\alpha) + \Omega_6(\alpha) - \Omega_5(\alpha))\right. \\ &\quad \left. + \left|F'(\sigma)\right|(\Omega_1(\alpha) + \Omega_4(\alpha) + \Omega_3(\alpha) + 2\Omega_6(\alpha) + \Omega_5(\alpha))\right], \end{aligned} \tag{1}$$

where

$$\begin{aligned}
\Omega_1(v) &= \int_0^1 \tau\left|\tau^v - \frac{3}{8}\right|d\tau = \frac{v}{v+2}\left(\frac{3}{8}\right)^{\frac{v+2}{v}} + \frac{1}{v+2} - \frac{3}{16},\\
\Omega_2(v) &= \int_0^1 \left|\tau^v - \frac{3}{8}\right|d\tau = \frac{2v}{v+1}\left(\frac{3}{8}\right)^{\frac{v+1}{v}} + \frac{1}{v+1} - \frac{3}{8},\\
\Omega_3(v) &= \int_0^1 \tau\left|\tau^v - \frac{1}{2}\right|d\tau = \frac{v}{v+2}\left(\frac{1}{2}\right)^{\frac{v+2}{v}} + \frac{1}{v+2} - \frac{1}{4},\\
\Omega_4(v) &= \int_0^1 \left|\tau^v - \frac{1}{2}\right|d\tau = \frac{2v}{v+1}\left(\frac{1}{2}\right)^{\frac{v+1}{v}} + \frac{1}{v+1} - \frac{1}{2},\\
\Omega_5(v) &= \int_0^1 \tau\left|\tau^v - \frac{5}{8}\right|d\tau = \frac{v}{v+2}\left(\frac{5}{8}\right)^{\frac{v+2}{v}} + \frac{1}{v+2} - \frac{5}{16},\\
\Omega_6(v) &= \int_0^1 \left|\tau^v - \frac{5}{8}\right|d\tau = \frac{2v}{v+1}\left(\frac{5}{8}\right)^{\frac{v+1}{v}} + \frac{1}{v+1} - \frac{5}{8}.
\end{aligned} \tag{2}$$

The next definition will be used several times in the proofs of the main results:

Definition 3 ([35]). *A function $F : \Delta \to \mathbb{R}$ is called coordinated convex on Δ, for all $(\kappa, u), (y, v) \in \Delta$ and $\gamma, \delta \in [0,1]$, if it satisfies the following inequality:*

$$\begin{aligned}
&f(\gamma\kappa + (1-\gamma)y, \gamma u + (1-\gamma)v)\\
&\leq \gamma\delta F(\kappa, u) + \gamma(1-\delta)F(\kappa, v) + \delta(1-\gamma)F(y, u) + (1-\gamma)(1-\gamma)F(y, v).
\end{aligned} \tag{3}$$

The mapping f is a coordinated concave on Δ if the inequality (3) holds in the reversed direction for all $\gamma, \delta \in [0,1]$ and $(\kappa, u), (y, v) \in \Delta$.

For example, the function $F : [0,1] \times [0,1] \to \mathbb{R}$ defined by $F(\kappa, y) = \kappa^2 e^y$ is coordinated convex on $[0,1] \times [0,1]$.

The following lemmas will be used in the inequalities we will establish based on the mappings of bounded variation.

Lemma 1 ([23]). *If $F(\gamma, \delta)$ is continuous on renctangle Δ and $\alpha(\gamma, \delta)$ is a bounded variation on Δ, then $\alpha(\gamma, \delta)$ is integrable with respect to $F(\gamma, \delta)$ over Δ in the Riemann-Stieltjes sense, and*

$$\begin{aligned}
\int_\sigma^\rho \int_\varsigma^d F(\gamma, \delta) d_\gamma d_\delta \alpha(\gamma, \delta) &= \int_\sigma^\rho \int_\varsigma^d \alpha(\gamma, \delta) d_\gamma d_\delta F(\gamma, \delta)\\
&- \int_\sigma^\rho \alpha(\gamma, d) d_\gamma F(\gamma, d) + \int_\sigma^\rho \alpha(\gamma, \varsigma) d_\gamma F(\gamma, \varsigma)\\
&- \int_\varsigma^d \alpha(\rho, \delta) d_\delta F(\rho, \delta) + \int_\varsigma^d \alpha(\sigma, \delta) d_\delta F(\sigma, \delta)\\
&+ F(\rho, d)\alpha(\rho, d) - F(\rho, \varsigma)\alpha(\rho, \varsigma) - F(\sigma, d)\alpha(\sigma, d) + F(\sigma, \varsigma)\alpha(\sigma, \varsigma).
\end{aligned}$$

Lemma 2 ([22]). *Assume that F is integrable with respect to $g(\gamma, \delta)$ over Δ in the Riemann–Stieltjes sense on Δ and g is of bounded variation on Δ, then*

$$\left|\int_\sigma^\rho \int_\varsigma^d F(\kappa, y) d_\kappa d_y g(\kappa, y)\right| \leq \sup_{(\kappa, y) \in \Delta} |F(\kappa, y)| \bigvee_\sigma^\rho \bigvee_\varsigma^d (g). \tag{4}$$

3. An Identity

In this section, by using $\mathcal{J}^{\alpha}_{\sigma+}F(\kappa)$, $\mathcal{J}^{\alpha}_{\rho-}F(\kappa)$, $\mathcal{J}^{\alpha,\beta}_{\sigma+,\varsigma+}$, $\mathcal{J}^{\alpha,\beta}_{\sigma+,d-}$, $\mathcal{J}^{\alpha,\beta}_{\rho-,\varsigma+}$, and $\mathcal{J}^{\alpha,\beta}_{\rho-,d-}$, we will consider a convex function in differentiable coordinates and obtain a lemma that we will use throughout the article.

Lemma 3. *Let $F : \Delta \to \mathbb{R}$ be a partially differentiable on Δ°; then, the following Riemann–Liouville fractional integrals identity yield:*

$$\Theta^{\alpha,\beta}(\sigma,\rho;\varsigma,d) = \frac{(\rho-\sigma)(d-\varsigma)}{81}\sum_{k=1}^{9}\Phi_k, \tag{5}$$

where

$$\begin{aligned}
&\Theta^{\alpha,\beta}(\sigma,\rho;\varsigma,d)\\
&:= \frac{[F(\sigma,\varsigma)+F(\sigma,d)+F(\rho,\varsigma)+F(\rho,d)]}{64}\\
&+\frac{3}{64}\left[F\left(\sigma,\frac{2\varsigma+d}{3}\right)+F\left(\sigma,\frac{\varsigma+2d}{3}\right)+F\left(\rho,\frac{2\varsigma+d}{3}\right)+F\left(\rho,\frac{\varsigma+2d}{3}\right)\right.\\
&\left.+F\left(\frac{2\sigma+\rho}{3},\varsigma\right)+F\left(\frac{\sigma+2\rho}{3},\varsigma\right)+\left(\frac{2\sigma+\rho}{3},d\right)+F\left(\frac{\sigma+2\rho}{3},d\right)\right]\\
&+\frac{9}{64}\left[F\left(\frac{2\sigma+\rho}{3},\frac{2\varsigma+d}{3}\right)+F\left(\frac{2\sigma+\rho}{3},\frac{\varsigma+2d}{3}\right)+F\left(\frac{\sigma+2\rho}{3},\frac{2\varsigma+d}{3}\right)+F\left(\frac{\sigma+2\rho}{3},\frac{\varsigma+2d}{3}\right)\right]\\
&-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{8(\rho-\sigma)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\rho-}F\left(\frac{\sigma+2\rho}{3},d\right)+\mathcal{J}^{\alpha}_{\rho-}F\left(\frac{\sigma+2\rho}{3},\varsigma\right)\right.\\
&\left.+\mathcal{J}^{\alpha}_{\frac{\sigma+2\rho}{3}-}F\left(\frac{2\sigma+\rho}{3},d\right)+\mathcal{J}^{\alpha}_{\frac{\sigma+2\rho}{3}-}F\left(\frac{2\sigma+\rho}{3},\varsigma\right)+\mathcal{J}^{\alpha}_{\frac{2\sigma+\rho}{3}-}F(\sigma,d)+\mathcal{J}^{\alpha}_{\frac{2\sigma+\rho}{3}-}F(\sigma,\varsigma)\right]\\
&-\frac{3^{\alpha}\Gamma(\alpha+1)}{8(\rho-\sigma)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\rho-}F\left(\frac{\sigma+2\rho}{3},\frac{\varsigma+2d}{3}\right)+\mathcal{J}^{\alpha}_{\rho-}F\left(\frac{\sigma+2\rho}{3},\frac{2\varsigma+d}{3}\right)+\mathcal{J}^{\alpha}_{\frac{2\sigma+\rho}{3}-}F\left(\sigma,\frac{\varsigma+2d}{3}\right)\right.\\
&\left.+\mathcal{J}^{\alpha}_{\frac{2\sigma+\rho}{3}-}F\left(\sigma,\frac{2\varsigma+d}{3}\right)+\mathcal{J}^{\alpha}_{\frac{\sigma+2\rho}{3}-}F\left(\frac{2\sigma+\rho}{3},\frac{\varsigma+2d}{3}\right)+\mathcal{J}^{\alpha}_{\frac{\sigma+2\rho}{3}-}F\left(\frac{2\sigma+\rho}{3},\frac{2\varsigma+d}{3}\right)\right]\\
&-\frac{3^{\beta-1}\Gamma(\beta+1)}{8(d-\varsigma)^{\beta}}\left[\mathcal{J}^{\beta}_{d-}F\left(\rho,\frac{\varsigma+2d}{3}\right)+\mathcal{J}^{\beta}_{\frac{\varsigma+2d}{3}-}F\left(\rho,\frac{2\varsigma+d}{3}\right)+\mathcal{J}^{\beta}_{\frac{2\varsigma+d}{3}-}F(\rho,\varsigma)\right.\\
&\left.+\mathcal{J}^{\beta}_{d-}F\left(\sigma,\frac{\varsigma+2d}{3}\right)+\mathcal{J}^{\beta}_{\frac{\varsigma+2d}{3}-}F\left(\sigma,\frac{2\varsigma+d}{3}\right)+\mathcal{J}^{\beta}_{\frac{2\varsigma+d}{3}-}F(\sigma,\varsigma)\right]\\
&-\frac{3^{\beta}\Gamma(\beta+1)}{8(d-\varsigma)^{\beta}}\left[\mathcal{J}^{\beta}_{d-}F\left(\frac{\sigma+2\rho}{3},\frac{\varsigma+2d}{3}\right)+\mathcal{J}^{\beta}_{\frac{\varsigma+2d}{3}-}F\left(\frac{\sigma+2\rho}{3},\frac{2\varsigma+d}{3}\right)+\mathcal{J}^{\beta}_{\frac{2\varsigma+d}{3}-}F\left(\frac{\sigma+2\rho}{3},\varsigma\right)\right.\\
&\left.+\mathcal{J}^{\beta}_{d-}F\left(\frac{2\sigma+\rho}{3},\frac{\varsigma+2d}{3}\right)+\mathcal{J}^{\beta}_{\frac{\varsigma+2d}{3}-}F\left(\frac{2\sigma+\rho}{3},\frac{2\varsigma+d}{3}\right)+\mathcal{J}^{\beta}_{\frac{2\varsigma+d}{3}-}F\left(\frac{2\sigma+\rho}{3},\varsigma\right)\right]\\
&+\frac{3^{\alpha-1}3^{\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)}{(\rho-\sigma)^{\alpha}(d-\varsigma)^{\beta}}\\
&\times\left[\mathcal{J}^{\alpha,\beta}_{\rho-,d-}F\left(\frac{\sigma+2\rho}{3},\frac{\varsigma+2d}{3}\right)+\mathcal{J}^{\alpha,\beta}_{\rho-,\frac{\varsigma+2d}{3}-}F\left(\frac{\sigma+2\rho}{3},\frac{2\varsigma+d}{3}\right)+\mathcal{J}^{\alpha,\beta}_{\rho-,\frac{2\varsigma+d}{3}-}F\left(\frac{\sigma+2\rho}{3},\varsigma\right)\right.\\
&+\mathcal{J}^{\alpha,\beta}_{\frac{\sigma+2\rho}{3}-,d-}F\left(\frac{2\sigma+\rho}{3},\frac{\varsigma+2d}{3}\right)+\mathcal{J}^{\alpha,\beta}_{\frac{\sigma+2\rho}{3}-,\frac{\varsigma+2d}{3}-}F\left(\frac{2\sigma+\rho}{3},\frac{2\varsigma+d}{3}\right)+\mathcal{J}^{\alpha,\beta}_{\frac{\sigma+2\rho}{3}-,\frac{2\varsigma+d}{3}-}F\left(\frac{2\sigma+\rho}{3},\varsigma\right)\\
&\left.+\mathcal{J}^{\alpha,\beta}_{\frac{2\sigma+\rho}{3}-,d-}F\left(\sigma,\frac{\varsigma+2d}{3}\right)+\mathcal{J}^{\alpha,\beta}_{\frac{2\sigma+\rho}{3}-,\frac{\varsigma+2d}{3}-}F\left(\sigma,\frac{2\varsigma+d}{3}\right)+\mathcal{J}^{\alpha,\beta}_{\frac{2\sigma+\rho}{3}-,\frac{2\varsigma+d}{3}-}F(\sigma,\varsigma)\right]
\end{aligned} \tag{6}$$

and here,

$$\Phi_1=\int_0^1\int_0^1\left(\gamma^{\alpha}-\frac{5}{8}\right)\left(\delta^{\beta}-\frac{5}{8}\right)\frac{\partial^2 F}{\partial\delta\partial\gamma}\left(\left(\frac{1-\gamma}{3}\right)\sigma+\left(\frac{\gamma+2}{3}\right)\rho,\left(\frac{1-\delta}{3}\right)\varsigma+\left(\frac{\delta+2}{3}\right)d\right)d\delta d\gamma,$$
$$\Phi_2=\int_0^1\int_0^1\left(\gamma^{\alpha}-\frac{5}{8}\right)\left(\delta^{\beta}-\frac{1}{2}\right)\frac{\partial^2 F}{\partial\delta\partial\gamma}\left(\left(\frac{1-\gamma}{3}\right)\sigma+\left(\frac{\gamma+2}{3}\right)\rho,\left(\frac{2-\delta}{3}\right)\varsigma+\left(\frac{\delta+1}{3}\right)d\right)d\delta d\gamma,$$
$$\Phi_3=\int_0^1\int_0^1\left(\gamma^{\alpha}-\frac{5}{8}\right)\left(\delta^{\beta}-\frac{3}{8}\right)\frac{\partial^2 F}{\partial\delta\partial\gamma}\left(\left(\frac{1-\gamma}{3}\right)\sigma+\left(\frac{\gamma+2}{3}\right)\rho,\left(\frac{3-\delta}{3}\right)\varsigma+\frac{\delta}{3}d\right)d\delta d\gamma,$$
$$\Phi_4=\int_0^1\int_0^1\left(\gamma^{\alpha}-\frac{1}{2}\right)\left(\delta^{\beta}-\frac{5}{8}\right)\frac{\partial^2 F}{\partial\delta\partial\gamma}\left(\left(\frac{2-\gamma}{3}\right)\sigma+\left(\frac{\gamma+1}{3}\right)\rho,\left(\frac{1-\delta}{3}\right)\varsigma+\left(\frac{\delta+2}{3}\right)d\right)d\delta d\gamma,$$
$$\Phi_5=\int_0^1\int_0^1\left(\gamma^{\alpha}-\frac{1}{2}\right)\left(\delta^{\beta}-\frac{1}{2}\right)\frac{\partial^2 F}{\partial\delta\partial\gamma}\left(\left(\frac{2-\gamma}{3}\right)\sigma+\left(\frac{\gamma+1}{3}\right)\rho,\left(\frac{2-\delta}{3}\right)\varsigma+\left(\frac{\delta+1}{3}\right)d\right)d\delta d\gamma,$$
$$\Phi_6=\int_0^1\int_0^1\left(\gamma^{\alpha}-\frac{1}{2}\right)\left(\delta^{\beta}-\frac{3}{8}\right)\frac{\partial^2 F}{\partial\delta\partial\gamma}\left(\left(\frac{2-\gamma}{3}\right)\sigma+\left(\frac{\gamma+1}{3}\right)\rho,\left(\frac{3-\delta}{3}\right)\varsigma+\frac{\delta}{3}d\right)d\delta d\gamma,$$
$$\Phi_7=\int_0^1\int_0^1\left(\gamma^{\alpha}-\frac{3}{8}\right)\left(\delta^{\beta}-\frac{5}{8}\right)\frac{\partial^2 F}{\partial\delta\partial\gamma}\left(\left(\frac{3-\gamma}{3}\right)\sigma+\frac{\gamma}{3}\rho,\left(\frac{1-\delta}{3}\right)\varsigma+\left(\frac{\delta+2}{3}\right)d\right)d\delta d\gamma,$$
$$\Phi_8=\int_0^1\int_0^1\left(\gamma^{\alpha}-\frac{3}{8}\right)\left(\delta^{\beta}-\frac{1}{2}\right)\frac{\partial^2 F}{\partial\delta\partial\gamma}\left(\left(\frac{3-\gamma}{3}\right)\sigma+\frac{\gamma}{3}\rho,\left(\frac{2-\delta}{3}\right)\varsigma+\left(\frac{\delta+1}{3}\right)d\right)d\delta d\gamma,$$
$$\Phi_9=\int_0^1\int_0^1\left(\gamma^{\alpha}-\frac{3}{8}\right)\left(\delta^{\beta}-\frac{3}{8}\right)\frac{\partial^2 F}{\partial\delta\partial\gamma}\left(\left(\frac{3-\gamma}{3}\right)\sigma+\frac{\gamma}{3}\rho,\left(\frac{3-\delta}{3}\right)\varsigma+\frac{\delta}{3}d\right)d\delta d\gamma.$$

Proof. By utilizing integration by parts and change of variables, we derive

$$\begin{aligned}\Phi_1&=\int_0^1\int_0^1\left(\gamma^{\alpha}-\frac{5}{8}\right)\left(\delta^{\beta}-\frac{5}{8}\right)\frac{\partial^2 F}{\partial\delta\partial\gamma}\left(\left(\frac{1-\gamma}{3}\right)\sigma+\left(\frac{\gamma+2}{3}\right)\rho,\left(\frac{1-\delta}{3}\right)\varsigma+\left(\frac{\delta+2}{3}\right)d\right)d\delta d\gamma\\&=\frac{81}{64(\rho-\sigma)(d-\varsigma)}F(\rho,d)+\frac{135}{64(\rho-\sigma)(d-\varsigma)}F\left(\frac{\sigma+2\rho}{3},d\right)\\&\quad+\frac{135}{64(\rho-\sigma)(d-\varsigma)}F\left(\rho,\frac{\varsigma+2d}{3}\right)+\frac{225}{64(\rho-\sigma)(d-\varsigma)}F\left(\frac{\sigma+2\rho}{3},\frac{\varsigma+2d}{3}\right)\\&\quad-\frac{9}{8(d-\varsigma)}\frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha+1}}\mathcal{J}^{\alpha}_{\rho-}F\left(\frac{\sigma+2\rho}{3},d\right)-\frac{15}{8(d-\varsigma)}\frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha+1}}\mathcal{J}^{\alpha}_{\rho-}F\left(\frac{\sigma+2\rho}{3},\frac{\varsigma+2d}{3}\right)\\&\quad-\frac{9}{8(\rho-\sigma)}\frac{3^{\beta+1}\Gamma(\beta+1)}{(d-\varsigma)^{\beta+1}}\mathcal{J}^{\beta}_{d-}F\left(\rho,\frac{\varsigma+2d}{3}\right)-\frac{15}{8(\rho-\sigma)}\frac{3^{\beta+1}\Gamma(\beta+1)}{(d-\varsigma)^{\beta+1}}\mathcal{J}^{\beta}_{d-}F\left(\frac{\sigma+2\rho}{3},\frac{\varsigma+2d}{3}\right)\\&\quad+\frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha+1}}\frac{3^{\beta+1}\Gamma(\beta+1)}{(d-\varsigma)^{\beta+1}}\mathcal{J}^{\alpha,\beta}_{\rho-,d-}F\left(\frac{\sigma+2\rho}{3},\frac{\varsigma+2d}{3}\right),\end{aligned}\tag{7}$$

$$\begin{aligned}\Phi_2&=\int_0^1\int_0^1\left(\gamma^{\alpha}-\frac{5}{8}\right)\left(\delta^{\beta}-\frac{1}{2}\right)\frac{\partial^2 F}{\partial\delta\partial\gamma}\left(\left(\frac{1-\gamma}{3}\right)\sigma+\left(\frac{\gamma+2}{3}\right)\rho,\left(\frac{2-\delta}{3}\right)\varsigma+\left(\frac{\delta+1}{3}\right)d\right)d\delta d\gamma\\&=\frac{27}{16(\rho-\sigma)(d-\varsigma)}F\left(\rho,\frac{\varsigma+2d}{3}\right)+\frac{45}{16(\rho-\sigma)(d-\varsigma)}F\left(\frac{\sigma+2\rho}{3},\frac{\varsigma+2d}{3}\right)\\&\quad+\frac{27}{16(\rho-\sigma)(d-\varsigma)}F\left(\rho,\frac{2\varsigma+d}{3}\right)+\frac{45}{16(\rho-\sigma)(d-\varsigma)}F\left(\frac{\sigma+2\rho}{3},\frac{2\varsigma+d}{3}\right)\\&\quad-\frac{3}{2(d-\varsigma)}\frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha+1}}\mathcal{J}^{\alpha}_{\rho-}F\left(\frac{\sigma+2\rho}{3},\frac{\varsigma+2d}{3}\right)-\frac{3}{2(d-\varsigma)}\frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha+1}}\mathcal{J}^{\alpha}_{\rho-}F\left(\frac{\sigma+2\rho}{3},\frac{2\varsigma+d}{3}\right)\\&\quad-\frac{9}{8(\rho-\sigma)}\frac{3^{\beta+1}\Gamma(\beta+1)}{(d-\varsigma)^{\beta+1}}\mathcal{J}^{\beta}_{\frac{\varsigma+2d}{3}-}F\left(\rho,\frac{2\varsigma+d}{3}\right)-\frac{15}{8(\rho-\sigma)}\frac{3^{\beta+1}\Gamma(\beta+1)}{(d-\varsigma)^{\beta+1}}\mathcal{J}^{\beta}_{\frac{\varsigma+2d}{3}-}F\left(\frac{\sigma+2\rho}{3},\frac{2\varsigma+d}{3}\right)\\&\quad+\frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha+1}}\frac{3^{\beta+1}\Gamma(\beta+1)}{(d-\varsigma)^{\beta+1}}\mathcal{J}^{\alpha,\beta}_{\rho-,\frac{\varsigma+2d}{3}-}F\left(\frac{\sigma+2\rho}{3},\frac{2\varsigma+d}{3}\right),\end{aligned}\tag{8}$$

$$\begin{aligned}
\Phi_3 &= \int_0^1 \int_0^1 \left(\gamma^\alpha - \frac{5}{8}\right)\left(\delta^\beta - \frac{3}{8}\right)\frac{\partial^2 F}{\partial\delta\partial\gamma}\left(\left(\frac{1-\gamma}{3}\right)\sigma + \left(\frac{\gamma+2}{3}\right)\rho, \left(\frac{3-\delta}{3}\right)\varsigma + \frac{\delta}{3}d\right)d\delta d\gamma \\
&= \frac{135}{64(\rho-\sigma)(d-\varsigma)}F\left(\rho, \frac{2\varsigma+d}{3}\right) + \frac{225}{64(\rho-\sigma)(d-\varsigma)}F\left(\frac{\sigma+2\rho}{3}, \frac{2\varsigma+d}{3}\right) \\
&+ \frac{81}{64(\rho-\sigma)(d-\varsigma)}F(\rho,\varsigma) + \frac{135}{64(\rho-\sigma)(d-\varsigma)}F\left(\frac{\sigma+2\rho}{3}, \varsigma\right) \\
&- \frac{15}{8(d-\varsigma)}\frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha+1}}\mathcal{J}^{\alpha}_{\rho-}F\left(\frac{\sigma+2\rho}{3}, \frac{2\varsigma+d}{3}\right) - \frac{9}{8(d-\varsigma)}\frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha+1}}\mathcal{J}^{\alpha}_{\rho-}F\left(\frac{\sigma+2\rho}{3}, \varsigma\right) \\
&- \frac{9}{8(\rho-\sigma)}\frac{3^{\beta+1}\Gamma(\beta+1)}{(d-\varsigma)^{\beta+1}}\mathcal{J}^{\beta}_{\frac{2\varsigma+d}{3}-}F(\rho,\varsigma) - \frac{15}{8(\rho-\sigma)}\frac{3^{\beta+1}\Gamma(\beta+1)}{(d-\varsigma)^{\beta+1}}\mathcal{J}^{\beta}_{\frac{2\varsigma+d}{3}-}F\left(\frac{\sigma+2\rho}{3}, \varsigma\right) \\
&+ \frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha+1}}\frac{3^{\beta+1}\Gamma(\beta+1)}{(d-\varsigma)^{\beta+1}}\mathcal{J}^{\alpha,\beta}_{\rho-,\frac{2\varsigma+d}{3}-}F\left(\frac{\sigma+2\rho}{3}, \varsigma\right),
\end{aligned} \tag{9}$$

$$\begin{aligned}
\Phi_4 &= \int_0^1 \int_0^1 \left(\gamma^\alpha - \frac{1}{2}\right)\left(\delta^\beta - \frac{5}{8}\right)\frac{\partial^2 F}{\partial\delta\partial\gamma}\left(\left(\frac{2-\gamma}{3}\right)\sigma + \left(\frac{\gamma+1}{3}\right)\rho, \left(\frac{1-\delta}{3}\right)\varsigma + \left(\frac{\delta+2}{3}\right)d\right)d\delta d\gamma \\
&= \frac{27}{16(\rho-\sigma)(d-\varsigma)}F\left(\frac{\sigma+2\rho}{3}, d\right) + \frac{27}{16(\rho-\sigma)(d-\varsigma)}F\left(\frac{2\sigma+\rho}{3}, d\right) \\
&+ \frac{45}{16(\rho-\sigma)(d-\varsigma)}F\left(\frac{\sigma+2\rho}{3}, \frac{\varsigma+2d}{3}\right) + \frac{45}{16(\rho-\sigma)(d-\varsigma)}F\left(\frac{2\sigma+\rho}{3}, \frac{\varsigma+2d}{3}\right) \\
&- \frac{9}{8(d-\varsigma)}\frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha+1}}\mathcal{J}^{\alpha}_{\frac{\sigma+2\rho}{3}-}F\left(\frac{2\sigma+\rho}{3}, d\right) - \frac{15}{8(d-\varsigma)}\frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha+1}}\mathcal{J}^{\alpha}_{\frac{\sigma+2\rho}{3}-}F\left(\frac{2\sigma+\rho}{3}, \frac{\varsigma+2d}{3}\right) \\
&- \frac{3}{2(\rho-\sigma)}\frac{3^{\beta+1}\Gamma(\beta+1)}{(d-\varsigma)^{\beta+1}}\mathcal{J}^{\beta}_{d-}F\left(\frac{\sigma+2\rho}{3}, \frac{\varsigma+2d}{3}\right) - \frac{3}{2(\rho-\sigma)}\frac{3^{\beta+1}\Gamma(\beta+1)}{(d-\varsigma)^{\beta+1}}\mathcal{J}^{\beta}_{d-}F\left(\frac{2\sigma+\rho}{3}, \frac{\varsigma+2d}{3}\right) \\
&+ \frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha+1}}\frac{3^{\beta+1}\Gamma(\beta+1)}{(d-\varsigma)^{\beta+1}}\mathcal{J}^{\alpha,\beta}_{\frac{\sigma+2\rho}{3}-,d-}F\left(\frac{2\sigma+\rho}{3}, \frac{\varsigma+2d}{3}\right),
\end{aligned} \tag{10}$$

$$\begin{aligned}
\Phi_5 &= \int_0^1 \int_0^1 \left(\gamma^\alpha - \frac{1}{2}\right)\left(\delta^\beta - \frac{1}{2}\right)\frac{\partial^2 F}{\partial\delta\partial\gamma}\left(\left(\frac{2-\gamma}{3}\right)\sigma + \left(\frac{\gamma+1}{3}\right)\rho, \left(\frac{2-\delta}{3}\right)\varsigma + \left(\frac{\delta+1}{3}\right)d\right)d\delta d\gamma \\
&= \frac{9}{4(\rho-\sigma)(d-\varsigma)}F\left(\frac{\sigma+2\rho}{3}, \frac{\varsigma+2d}{3}\right) + \frac{9}{4(\rho-\sigma)(d-\varsigma)}F\left(\frac{2\sigma+\rho}{3}, \frac{\varsigma+2d}{3}\right) \\
&+ \frac{9}{4(\rho-\sigma)(d-\varsigma)}F\left(\frac{\sigma+2\rho}{3}, \frac{2\varsigma+d}{3}\right) + \frac{9}{4(\rho-\sigma)(d-\varsigma)}F\left(\frac{2\sigma+\rho}{3}, \frac{2\varsigma+d}{3}\right) \\
&- \frac{3}{2(d-\varsigma)}\frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha+1}}\mathcal{J}^{\alpha}_{\frac{\sigma+2\rho}{3}-}F\left(\frac{2\sigma+\rho}{3}, \frac{\varsigma+2d}{3}\right) - \frac{3}{2(d-\varsigma)}\frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha+1}}\mathcal{J}^{\alpha}_{\frac{\sigma+2\rho}{3}-}F\left(\frac{2\sigma+\rho}{3}, \frac{2\varsigma+d}{3}\right) \\
&- \frac{3}{2(\rho-\sigma)}\frac{3^{\beta+1}\Gamma(\beta+1)}{(d-\varsigma)^{\beta+1}}\mathcal{J}^{\beta}_{\frac{\varsigma+2d}{3}-}F\left(\frac{\sigma+2\rho}{3}, \frac{2\varsigma+d}{3}\right) - \frac{3}{2(\rho-\sigma)}\frac{3^{\beta+1}\Gamma(\beta+1)}{(d-\varsigma)^{\beta+1}}\mathcal{J}^{\beta}_{\frac{\varsigma+2d}{3}-}F\left(\frac{2\sigma+\rho}{3}, \frac{2\varsigma+d}{3}\right) \\
&+ \frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha+1}}\frac{3^{\beta+1}\Gamma(\beta+1)}{(d-\varsigma)^{\beta+1}}\mathcal{J}^{\alpha,\beta}_{\frac{\sigma+2\rho}{3}-,\frac{\varsigma+2d}{3}-}F\left(\frac{2\sigma+\rho}{3}, \frac{2\varsigma+d}{3}\right),
\end{aligned} \tag{11}$$

$$
\begin{aligned}
\Phi_6 = {} & \int_0^1 \int_0^1 \left(\gamma^{\alpha} - \frac{1}{2}\right)\left(\delta^{\beta} - \frac{3}{8}\right) \frac{\partial^2 F}{\partial\delta\partial\gamma}\left(\left(\frac{2-\gamma}{3}\right)\sigma + \left(\frac{\gamma+1}{3}\right)\rho, \left(\frac{3-\delta}{3}\right)\varsigma + \frac{\delta}{3}d\right) d\delta d\gamma \\
= {} & \frac{45}{16(\rho-\sigma)(d-\varsigma)} F\left(\frac{\sigma+2\rho}{3}, \frac{2\varsigma+d}{3}\right) + \frac{15}{16(\rho-\sigma)(d-\varsigma)} F\left(\frac{2\sigma+\rho}{3}, \frac{2\varsigma+d}{3}\right) \\
& + \frac{27}{16(\rho-\sigma)(d-\varsigma)} F\left(\frac{\sigma+2\rho}{3}, \varsigma\right) + \frac{27}{16(\rho-\sigma)(d-\varsigma)} F\left(\frac{2\sigma+\rho}{3}, \varsigma\right) \\
& - \frac{15}{8(d-\varsigma)} \frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha+1}} \mathcal{J}^{\alpha}_{\frac{\sigma+2\rho}{3}-} F\left(\frac{2\sigma+\rho}{3}, \frac{2\varsigma+d}{3}\right) - \frac{9}{8(d-\varsigma)} \frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha+1}} \mathcal{J}^{\alpha}_{\frac{\sigma+2\rho}{3}-} F\left(\frac{2\sigma+\rho}{3}, \varsigma\right) \\
& - \frac{3}{2(\rho-\sigma)} \frac{3^{\beta+1}\Gamma(\beta+1)}{(d-\varsigma)^{\beta+1}} \mathcal{J}^{\beta}_{\frac{2\varsigma+d}{3}-} F\left(\frac{\sigma+2\rho}{3}, \varsigma\right) - \frac{3}{2(\rho-\sigma)} \frac{3^{\beta+1}\Gamma(\beta+1)}{(d-\varsigma)^{\beta+1}} \mathcal{J}^{\beta}_{\frac{\varsigma+2d}{3}+} F\left(\frac{2\sigma+\rho}{3}, \varsigma\right) \\
& + \frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha+1}} \frac{3^{\beta+1}\Gamma(\beta+1)}{(d-\varsigma)^{\beta+1}} \mathcal{J}^{\alpha,\beta}_{\frac{\sigma+2\rho}{3}-, \frac{2\varsigma+d}{3}-} F\left(\frac{2\sigma+\rho}{3}, \varsigma\right),
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
\Phi_7 = {} & \int_0^1 \int_0^1 \left(\gamma^{\alpha} - \frac{3}{8}\right)\left(\delta^{\beta} - \frac{5}{8}\right) \frac{\partial^2 F}{\partial\delta\partial\gamma}\left(\left(\frac{3-\gamma}{3}\right)\sigma + \frac{\gamma}{3}\rho, \left(\frac{1-\delta}{3}\right)\varsigma + \left(\frac{\delta+2}{3}\right)d\right) d\delta d\gamma \\
= {} & \frac{135}{64(\rho-\sigma)(d-\varsigma)} F\left(\frac{2\sigma+\rho}{3}, d\right) + \frac{81}{64(\rho-\sigma)(d-\varsigma)} F\left(\frac{2\sigma+\rho}{3}, \frac{2\varsigma+d}{3}\right) \\
& + \frac{225}{64(\rho-\sigma)(d-\varsigma)} F\left(\frac{2\sigma+\rho}{3}, \frac{\varsigma+2d}{3}\right) + \frac{135}{64(\rho-\sigma)(d-\varsigma)} F\left(\sigma, \frac{\varsigma+2d}{3}\right) \\
& - \frac{9}{8(d-\varsigma)} \frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha+1}} \mathcal{J}^{\alpha}_{\frac{2\sigma+\rho}{3}-} F(\sigma, d) - \frac{15}{8(d-\varsigma)} \frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha+1}} \mathcal{J}^{\alpha}_{\frac{2\sigma+\rho}{3}-} F\left(\sigma, \frac{\varsigma+2d}{3}\right) \\
& - \frac{15}{8(\rho-\sigma)} \frac{3^{\beta+1}\Gamma(\beta+1)}{(d-\varsigma)^{\beta+1}} \mathcal{J}^{\beta}_{d-} F\left(\frac{2\sigma+\rho}{3}, \frac{\varsigma+2d}{3}\right) - \frac{9}{8(\rho-\sigma)} \frac{3^{\beta+1}\Gamma(\beta+1)}{(d-\varsigma)^{\beta+1}} \mathcal{J}^{\beta}_{d-} F\left(\sigma, \frac{\varsigma+2d}{3}\right) \\
& + \frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha+1}} \frac{3^{\beta+1}\Gamma(\beta+1)}{(d-\varsigma)^{\beta+1}} \mathcal{J}^{\alpha,\beta}_{\frac{2\sigma+\rho}{3}-, d-} F\left(\sigma, \frac{\varsigma+2d}{3}\right),
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
\Phi_8 = {} & \int_0^1 \int_0^1 \left(\gamma^{\alpha} - \frac{3}{8}\right)\left(\delta^{\beta} - \frac{1}{2}\right) \frac{\partial^2 F}{\partial\delta\partial\gamma}\left(\left(\frac{3-\gamma}{3}\right)\sigma + \frac{\gamma}{3}\rho, \left(\frac{2-\delta}{3}\right)\varsigma + \left(\frac{\delta+1}{3}\right)d\right) d\delta d\gamma \\
= {} & \frac{45}{16(\rho-\sigma)(d-\varsigma)} F\left(\frac{2\sigma+\rho}{3}, \frac{\varsigma+2d}{3}\right) + \frac{27}{16(\rho-\sigma)(d-\varsigma)} F\left(\sigma, \frac{\varsigma+2d}{3}\right) \\
& - \frac{45}{16(\rho-\sigma)(d-\varsigma)} F\left(\frac{2\sigma+\rho}{3}, \frac{2\varsigma+d}{3}\right) + \frac{27}{16(\rho-\sigma)(d-\varsigma)} F\left(\sigma, \frac{2\varsigma+d}{3}\right) \\
& - \frac{3}{2(d-\varsigma)} \frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha+1}} \mathcal{J}^{\alpha}_{\frac{2\sigma+\rho}{3}-} F\left(\sigma, \frac{\varsigma+2d}{3}\right) - \frac{3}{2(d-\varsigma)} \frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha+1}} \mathcal{J}^{\alpha}_{\frac{2\sigma+\rho}{3}-} F\left(\sigma, \frac{2\varsigma+d}{3}\right) \\
& - \frac{15}{8(\rho-\sigma)} \frac{3^{\beta+1}\Gamma(\beta+1)}{(d-\varsigma)^{\beta+1}} \mathcal{J}^{\beta}_{\frac{\varsigma+2d}{3}-} F\left(\frac{2\sigma+\rho}{3}, \frac{2\varsigma+d}{3}\right) - \frac{9}{8(\rho-\sigma)} \frac{3^{\beta+1}\Gamma(\beta+1)}{(d-\varsigma)^{\beta+1}} \mathcal{J}^{\beta}_{\frac{\varsigma+2d}{3}-} F\left(\sigma, \frac{2\varsigma+d}{3}\right) \\
& + \frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha+1}} \frac{3^{\beta+1}\Gamma(\beta+1)}{(d-\varsigma)^{\beta+1}} \mathcal{J}^{\alpha,\beta}_{\frac{2\sigma+\rho}{3}-, \frac{\varsigma+2d}{3}-} F\left(\sigma, \frac{2\varsigma+d}{3}\right)
\end{aligned} \tag{14}
$$

and

$$\begin{aligned}
\Phi_9 &= \int_0^1 \int_0^1 \left(\gamma^\alpha - \frac{3}{8}\right)\left(\delta^\beta - \frac{3}{8}\right)\frac{\partial^2 F}{\partial\delta\partial\gamma}\left(\left(\frac{3-\gamma}{3}\right)\sigma + \frac{\gamma}{3}\rho, \left(\frac{3-\delta}{3}\right)\varsigma + \frac{\delta}{3}d\right) d\delta d\gamma \\
&= \frac{225}{64(\rho-\sigma)(d-\varsigma)} F\left(\frac{2\sigma+\rho}{3}, \frac{2\varsigma+d}{3}\right) + \frac{135}{64(\rho-\sigma)(d-\varsigma)} F\left(\sigma, \frac{2\varsigma+d}{3}\right) \\
&\quad - \frac{135}{64(\rho-\sigma)(d-\varsigma)} F\left(\frac{2\sigma+\rho}{3}, \varsigma\right) + \frac{81}{64(\rho-\sigma)(d-\varsigma)} F(\sigma,\varsigma) \\
&\quad - \frac{15}{8(d-\varsigma)} \frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha+1}} \mathcal{J}^{\alpha}_{\frac{2\sigma+\rho}{3}-} F\left(\sigma, \frac{2\varsigma+d}{3}\right) - \frac{9}{8(d-\varsigma)} \frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha+1}} \mathcal{J}^{\alpha}_{\frac{2\sigma+\rho}{3}-} F(\sigma,\varsigma) \\
&\quad - \frac{15}{8(\rho-\sigma)} \frac{3^{\beta+1}\Gamma(\beta+1)}{(d-\varsigma)^{\beta+1}} \mathcal{J}^{\beta}_{\frac{2\varsigma+d}{3}-} F\left(\frac{2\sigma+\rho}{3}, \varsigma\right) - \frac{9}{8(\rho-\sigma)} \frac{3^{\beta+1}\Gamma(\beta+1)}{(d-\varsigma)^{\beta+1}} \mathcal{J}^{\beta}_{\frac{2\varsigma+d}{3}-} F(\sigma,\varsigma) \\
&\quad + \frac{3^{\alpha+1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha+1}} \frac{3^{\beta+1}\Gamma(\beta+1)}{(d-\varsigma)^{\beta+1}} \mathcal{J}^{\alpha,\beta}_{\frac{2\sigma+\rho}{3}-, \frac{2\varsigma+d}{3}-} F(\sigma,\varsigma).
\end{aligned} \tag{15}$$

Thus, we obtain the required identity by adding (7)–(15) and multiplying the resultant one by $\frac{(\rho-\sigma)(d-\varsigma)}{81}$. □

4. Fractional Newton-Type Inequalities for Coordinated Convex Functions

In this section, we will present fractional Newton-type inequalities via differentiable coordinated convex mapping.

Theorem 2. *Let the conditions of Lemma 3 be satisfied. If* $\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}\right|$ *is a coordinated convex function, then we obtain the following inequality:*

$$\begin{aligned}
\left|\Theta^{\alpha,\beta}(\sigma,\rho;\varsigma,d)\right| \le \frac{(\rho-\sigma)(d-\varsigma)}{729} &\left[\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right| [\Omega_6(\alpha) - \Omega_5(\alpha) + 2\Omega_4(\alpha) - \Omega_3(\alpha) + 3\Omega_2(\alpha) - \Omega_1(\alpha)]\right. \\
&\times [\Omega_6(\beta) - \Omega_5(\beta) + 2\Omega_4(\beta) - \Omega_3(\beta) + 3\Omega_2(\beta) - \Omega_1(\beta)] \\
&+ \left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right| [\Omega_6(\alpha) - \Omega_5(\alpha) + 2\Omega_4(\alpha) - \Omega_3(\alpha) + 3\Omega_2(\alpha) - \Omega_1(\alpha)] \\
&\times [2\Omega_6(\beta) + \Omega_5(\beta) + \Omega_3(\beta) + \Omega_4(\beta) + \Omega_1(\beta)] \\
&+ \left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right| [2\Omega_6(\alpha) + \Omega_5(\alpha) + \Omega_3(\alpha) + \Omega_4(\alpha) + \Omega_1(\alpha)] \\
&\times [\Omega_6(\beta) - \Omega_5(\beta) + 2\Omega_4(\beta) - \Omega_3(\beta) + 3\Omega_2(\beta) - \Omega_1(\beta)] \\
&+ \left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right| [2\Omega_6(\alpha) + \Omega_5(\alpha) + \Omega_3(\alpha) + \Omega_4(\alpha) + \Omega_1(\alpha)] \\
&\left.\times [2\Omega_6(\beta) + \Omega_5(\beta) + \Omega_3(\beta) + \Omega_4(\beta) + \Omega_1(\beta)]\right],
\end{aligned} \tag{16}$$

where $\Omega_\Phi, \Phi = 1, 2, \ldots, 6$ *are described in* (2) *and* $\Theta^{\alpha,\beta}(\sigma,\rho;\varsigma,d)$ *is defined in* (6).

Proof. Taking the modulus of (5), we obtain

$$\left|\Theta^{\alpha,\beta}(\sigma,\rho;\varsigma,d)\right| \le \frac{(\rho-\sigma)(d-\varsigma)}{81} \sum_{k=1}^{9} |\Phi_k|.$$

With the help of the coordinated convexity of $\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}\right|$, we possess

$$
\begin{aligned}
|\Phi_1| &\le \int_0^1\int_0^1 \left|\gamma^{\alpha}-\frac{5}{8}\right|\left|\delta^{\beta}-\frac{5}{8}\right|\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}\left(\left(\frac{1-\gamma}{3}\right)\sigma+\left(\frac{\gamma+2}{3}\right)\rho,\left(\frac{1-\delta}{3}\right)\varsigma+\left(\frac{\delta+2}{3}\right)d\right)\right|d\delta d\gamma \\
&\le \int_0^1\int_0^1 \left|\gamma^{\alpha}-\frac{5}{8}\right|\left|\delta^{\beta}-\frac{5}{8}\right|\left[\left(\frac{1-\gamma}{3}\right)\left(\frac{1-\delta}{3}\right)\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|+\left(\frac{1-\gamma}{3}\right)\left(\frac{\delta+2}{3}\right)\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|\right. \\
&\quad \left.+\left(\frac{\gamma+2}{3}\right)\left(\frac{1-\delta}{3}\right)\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|+\left(\frac{\gamma+2}{3}\right)\left(\frac{\delta+2}{3}\right)\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|\right]d\delta d\gamma \\
&= \frac{1}{9}\left[\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|(\Omega_6(\alpha)-\Omega_5(\alpha))(\Omega_6(\beta)-\Omega_5(\beta))\right. \\
&\quad +\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|(\Omega_6(\alpha)-\Omega_5(\alpha))(2\Omega_6(\beta)+\Omega_5(\beta)) \\
&\quad +\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|(\Omega_5(\alpha)+2\Omega_6(\alpha))(\Omega_6(\beta)-\Omega_5(\beta)) \\
&\quad \left.+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|(\Omega_5(\alpha)+2\Omega_6(\alpha))(2\Omega_6(\beta)+\Omega_5(\beta))\right].
\end{aligned} \tag{17}
$$

Similarly, we derive

$$
\begin{aligned}
|\Phi_2| \le \frac{1}{9}&\left[\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|(\Omega_6(\alpha)-\Omega_5(\alpha))(2\Omega_4(\beta)-\Omega_3(\beta))\right. \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|(\Omega_6(\alpha)-\Omega_5(\alpha))(\Omega_3(\beta)+\Omega_4(\beta)) \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|(\Omega_5(\alpha)+2\Omega_6(\alpha))(2\Omega_4(\beta)-\Omega_3(\beta)) \\
&\left.+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|(\Omega_5(\alpha)+2\Omega_6(\alpha))(\Omega_3(\beta)-\Omega_4(\beta))\right],
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
|\Phi_3| \le \frac{1}{9}&\left[\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|(\Omega_6(\alpha)-\Omega_5(\alpha))(3\Omega_2(\beta)-\Omega_1(\beta))\right. \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|(\Omega_6(\alpha)-\Omega_5(\alpha))\Omega_1(\beta) \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|(\Omega_5(\alpha)+2\Omega_6(\alpha))(3\Omega_2(\beta)-\Omega_1(\beta)) \\
&\left.+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|(\Omega_5(\alpha)+2\Omega_6(\alpha))\Omega_1(\beta)\right],
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
|\Phi_4| \le \frac{1}{9}&\left[\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|(2\Omega_4(\alpha)-\Omega_3(\alpha))(\Omega_6(\beta)-\Omega_5(\beta))\right. \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|(2\Omega_4(\alpha)-\Omega_3(\alpha))(2\Omega_6(\beta)+\Omega_5(\beta)) \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|(\Omega_4(\alpha)+\Omega_3(\alpha))(\Omega_6(\beta)-\Omega_5(\beta)) \\
&\left.+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|(\Omega_3(\alpha)+\Omega_4(\alpha))(2\Omega_6(\beta)+\Omega_5(\beta))\right],
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
|\Phi_5| \leq \frac{1}{9}\Bigg[&\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|(2\Omega_4(\alpha)-\Omega_3(\alpha))(2\Omega_4(\beta)-\Omega_3(\beta)) \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|(2\Omega_4(\alpha)-\Omega_3(\alpha))(\Omega_3(\beta)+\Omega_4(\beta)) \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|(\Omega_3(\alpha)+\Omega_4(\alpha))(2\Omega_4(\beta)-\Omega_3(\beta)) \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|(\Omega_4(\alpha)+\Omega_3(\alpha))(\Omega_3(\beta)+\Omega_4(\beta))\Bigg],
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
|\Phi_6| \leq \frac{1}{9}\Bigg[&\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|(2\Omega_4(\alpha)-\Omega_3(\alpha))(3\Omega_2(\beta)-\Omega_1(\beta)) \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|(2\Omega_4(\alpha)-\Omega_3(\alpha))\Omega_1(\beta) \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|(\Omega_3(\alpha)+\Omega_4(\alpha))(3\Omega_2(\beta)-\Omega_1(\beta)) \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|(\Omega_3(\alpha)+\Omega_4(\alpha))\Omega_1(\beta)\Bigg],
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
|\Phi_7| \leq \frac{1}{9}\Bigg[&\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|(3\Omega_2(\alpha)-\Omega_1(\alpha))(\Omega_6(\beta)-\Omega_5(\beta)) \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|(3\Omega_2(\alpha)-\Omega_1(\alpha))(2\Omega_6(\beta)+\Omega_5(\beta)) \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|\Omega_1(\alpha)(\Omega_6(\beta)-\Omega_5(\beta)) \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|\Omega_1(\alpha)(2\Omega_6(\beta)+\Omega_5(\beta))\Bigg],
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
|\Phi_8| \leq \frac{1}{9}\Bigg[&\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|(3\Omega_2(\alpha)-\Omega_1(\alpha))(2\Omega_4(\beta)-\Omega_3(\beta)) \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|(3\Omega_2(\alpha)-\Omega_1(\alpha))(\Omega_3(\beta)+\Omega_4(\beta)) \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|\Omega_1(\alpha)(\Omega_3(\beta)+\Omega_4(\beta)) \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|\Omega_1(\alpha)(\Omega_3(\beta)+\Omega_4(\beta))\Bigg]
\end{aligned}
\tag{24}
$$

and

$$
\begin{aligned}
|\Phi_9| \leq \frac{1}{9}\Bigg[&\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|(3\Omega_2(\alpha)-\Omega_1(\alpha))(3\Omega_2(\beta)-\Omega_1(\beta)) \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|(3\Omega_2(\alpha)-\Omega_1(\alpha))\Omega_1(\beta) \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|\Omega_1(\alpha)(3\Omega_2(\beta)-\Omega_1(\beta)) \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|\Omega_1(\alpha)\Omega_1(\beta)\Bigg].
\end{aligned}
\tag{25}
$$

Thus, we establish the desired inequality by adding (17) to (25) and then multiplying by $\frac{(\rho-\sigma)(d-\varsigma)}{81}$. □

Example 1. *Define a mapping* $F : [0,1] \times [0,1] \to \mathbb{R}$ *by* $F(\gamma,\delta) = \gamma^2\delta^2$. *The right-hand side of the inequality* (16) *reduces the following equality*

$$\begin{aligned}
&\frac{(\rho-\sigma)(d-\varsigma)}{729}\Bigg[\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|[\Omega_6(\alpha)-\Omega_5(\alpha)+2\Omega_4(\alpha)-\Omega_3(\alpha)+3\Omega_2(\alpha)-\Omega_1(\alpha)]\\
&\quad\times[\Omega_6(\beta)-\Omega_5(\beta)+2\Omega_4(\beta)-\Omega_3(\beta)+3\Omega_2(\beta)-\Omega_1(\beta)]\\
&\quad\times\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|[\Omega_6(\alpha)-\Omega_5(\alpha)+2\Omega_4(\alpha)-\Omega_3(\alpha)+3\Omega_2(\alpha)-\Omega_1(\alpha)]\\
&\quad\times[2\Omega_6(\beta)+\Omega_5(\beta)+\Omega_3(\beta)+\Omega_4(\beta)+\Omega_1(\beta)]\\
&\quad\times\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|[2\Omega_6(\alpha)+\Omega_5(\alpha)+\Omega_3(\alpha)+\Omega_4(\alpha)+\Omega_1(\alpha)]\\
&\quad\times[\Omega_6(\beta)-\Omega_5(\beta)+2\Omega_4(\beta)-\Omega_3(\beta)+3\Omega_2(\beta)-\Omega_1(\beta)]\\
&\quad\times\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|[2\Omega_6(\alpha)+\Omega_5(\alpha)+\Omega_3(\alpha)+\Omega_4(\alpha)+\Omega_1(\alpha)]\\
&\quad\times[2\Omega_6(\beta)+\Omega_5(\beta)+\Omega_3(\beta)+\Omega_4(\beta)+\Omega_1(\beta)]\Bigg]\\
&=\frac{4}{729}[2\Omega_6(\alpha)+\Omega_5(\alpha)+\Omega_3(\alpha)+\Omega_4(\alpha)+\Omega_1(\alpha)]\\
&\quad\times[2\Omega_6(\beta)+\Omega_5(\beta)+\Omega_3(\beta)+\Omega_4(\beta)+\Omega_1(\beta)]]:=\Psi_1.
\end{aligned}\tag{26}$$

Then, we obtain the following expressions

$$\frac{[F(0,0)+F(0,1)+F(1,0)+F(1,1)]}{64}=\frac{1}{64},\tag{27}$$

$$\begin{aligned}
&\frac{3}{64}\left[F\left(\sigma,\frac{2\varsigma+d}{3}\right)+F\left(\sigma,\frac{\varsigma+2d}{3}\right)+F\left(\rho,\frac{2\varsigma+d}{3}\right)+F\left(\rho,\frac{\varsigma+2d}{3}\right)\right.\\
&\quad\left.+F\left(\frac{2\sigma+\rho}{3},\varsigma\right)+F\left(\frac{\sigma+2\rho}{3},\varsigma\right)+\left(\frac{2\sigma+\rho}{3},d\right)+F\left(\frac{\sigma+2\rho}{3},d\right)\right]\\
&=\frac{3}{64}\cdot\frac{10}{9}=\frac{5}{96}
\end{aligned}\tag{28}$$

and

$$\begin{aligned}
&\frac{9}{64}\left[F\left(\frac{2\sigma+\rho}{3},\frac{2\varsigma+d}{3}\right)+F\left(\frac{2\sigma+\rho}{3},\frac{\varsigma+2d}{3}\right)+F\left(\frac{\sigma+2\rho}{3},\frac{2\varsigma+d}{3}\right)+F\left(\frac{\sigma+2\rho}{3},\frac{\varsigma+2d}{3}\right)\right]\\
&=\frac{25}{576}.
\end{aligned}\tag{29}$$

By utilizing the definition of Riemann–Liouville fractional integrals, we derive

$$\begin{aligned}
&-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{8(\rho-\sigma)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\rho-}F\left(\frac{\sigma+2\rho}{3},d\right)+\mathcal{J}^{\alpha}_{\rho-}F\left(\frac{\sigma+2\rho}{3},\varsigma\right)+\mathcal{J}^{\alpha}_{\frac{\sigma+2\rho}{3}-}F\left(\frac{2\sigma+\rho}{3},d\right)\right.\\
&\quad\left.+\mathcal{J}^{\alpha}_{\frac{\sigma+2\rho}{3}-}F\left(\frac{2\sigma+\rho}{3},\varsigma\right)+\mathcal{J}^{\alpha}_{\frac{2\sigma+\rho}{3}-}F(\sigma,d)+\mathcal{J}^{\alpha}_{\frac{2\sigma+\rho}{3}-}F(\sigma,\varsigma)\right]\\
&=-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{8}\left[\mathcal{J}^{\alpha}_{1-}F\left(\frac{2}{3},1\right)+\mathcal{J}^{\alpha}_{\frac{2}{3}-}F\left(\frac{1}{3},1\right)+\mathcal{J}^{\alpha}_{\frac{1}{3}-}F(0,1)\right]\\
&=-\frac{3^{\alpha-1}\alpha}{8}\left[\int_{\frac{2}{3}}^{1}\left(\gamma-\frac{2}{3}\right)^{\alpha-1}\gamma^2d\gamma+\int_{\frac{1}{3}}^{\frac{2}{3}}\left(\gamma-\frac{1}{3}\right)^{\alpha-1}\gamma^2d\gamma+\int_0^{\frac{1}{3}}\gamma^{\alpha+1}d\gamma\right]\\
&=-\frac{3^{\alpha-1}}{8}\left[\frac{3^{-\alpha-2}(9\alpha^2+21\alpha+8)}{(\alpha+1)(\alpha+2)}+\frac{3^{-\alpha-2}(4\alpha^2+8\alpha+2)}{(\alpha+1)(\alpha+2)}+\frac{3^{-\alpha-2}(\alpha^2+\alpha)}{(\alpha+1)(\alpha+2)}\right]\\
&=-\frac{1}{108}\left(\frac{7\alpha^2+15\alpha+5}{(\alpha+1)(\alpha+2)}\right),
\end{aligned}\tag{30}$$

$$
\begin{aligned}
&-\frac{3^{\alpha}\Gamma(\alpha+1)}{8(\rho-\sigma)^{\alpha}}\Bigg[\mathcal{J}_{\rho-}^{\alpha}F\left(\frac{\sigma+2\rho}{3},\frac{\varsigma+2d}{3}\right)+\mathcal{J}_{\rho-}^{\alpha}F\left(\frac{\sigma+2\rho}{3},\frac{2\varsigma+d}{3}\right)+\mathcal{J}_{\frac{2\sigma+\rho}{3}-}^{\alpha}F\left(\sigma,\frac{\varsigma+2d}{3}\right)\\
&\quad+\mathcal{J}_{\frac{2\sigma+\rho}{3}-}^{\alpha}F\left(\sigma,\frac{2\varsigma+d}{3}\right)+\mathcal{J}_{\frac{\sigma+2\rho}{3}-}^{\alpha}F\left(\frac{2\sigma+\rho}{3},\frac{\varsigma+2d}{3}\right)+\mathcal{J}_{\frac{\sigma+2\rho}{3}-}^{\alpha}F\left(\frac{2\sigma+\rho}{3},\frac{2\varsigma+d}{3}\right)\Bigg]\\
&=-\frac{3^{\alpha}\Gamma(\alpha+1)}{8(\rho-\sigma)^{\alpha}}\Bigg[\mathcal{J}_{1-}^{\alpha}F\left(\frac{2}{3},\frac{2}{3}\right)+\mathcal{J}_{1-}^{\alpha}F\left(\frac{2}{3},\frac{1}{3}\right)+\mathcal{J}_{\frac{1}{3}-}^{\alpha}F\left(0,\frac{2}{3}\right)\\
&\quad+\mathcal{J}_{\frac{1}{3}-}^{\alpha}F\left(0,\frac{1}{3}\right)+\mathcal{J}_{\frac{2}{3}-}^{\alpha}F\left(\frac{1}{3},\frac{2}{3}\right)+\mathcal{J}_{\frac{2}{3}-}^{\alpha}F\left(\frac{1}{3},\frac{1}{3}\right)\Bigg]\\
&=-\frac{3^{\alpha}\alpha}{72}\Bigg[4\int_{\frac{2}{3}}^{1}\left(\gamma-\frac{2}{3}\right)^{\alpha-1}\gamma^{2}d\gamma+\int_{\frac{2}{3}}^{1}\left(\gamma-\frac{2}{3}\right)^{\alpha-1}\gamma^{2}d\gamma+4\int_{0}^{\frac{1}{3}}\gamma^{\alpha+1}d\gamma \qquad (31)\\
&\quad+\int_{0}^{\frac{1}{3}}\gamma^{\alpha+1}d\gamma+4\int_{\frac{1}{3}}^{\frac{2}{3}}\left(\gamma-\frac{1}{3}\right)^{\alpha-1}\gamma^{2}d\gamma+\int_{\frac{1}{3}}^{\frac{2}{3}}\left(\gamma-\frac{1}{3}\right)^{\alpha-1}\gamma^{2}d\gamma\Bigg]\\
&=-\frac{5}{648}\left[\frac{9\alpha^{2}+21\alpha+8}{(\alpha+1)(\alpha+2)}+\frac{\alpha^{2}+\alpha}{(\alpha+1)(\alpha+2)}+\frac{4\alpha^{2}+8\alpha+2}{(\alpha+1)(\alpha+2)}\right]\\
&=-\frac{5}{324}\left[\frac{7\alpha^{2}+15\alpha+5}{(\alpha+1)(\alpha+2)}\right],
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{3^{\beta-1}\Gamma(\beta+1)}{8(d-\varsigma)^{\beta}}\Bigg[\mathcal{J}_{d-}^{\beta}F\left(\rho,\frac{\varsigma+2d}{3}\right)+\mathcal{J}_{\frac{\varsigma+2d}{3}-}^{\beta}F\left(\rho,\frac{2\varsigma+d}{3}\right)+\mathcal{J}_{\frac{2\varsigma+d}{3}-}^{\beta}F(\rho,\varsigma)\\
&\quad+\mathcal{J}_{d-}^{\beta}F\left(\sigma,\frac{\varsigma+2d}{3}\right)+\mathcal{J}_{\frac{\varsigma+2d}{3}-}^{\beta}F\left(\sigma,\frac{2\varsigma+d}{3}\right)+\mathcal{J}_{\frac{2\varsigma+d}{3}-}^{\beta}F(\sigma,\varsigma)\Bigg]\\
&\quad-\frac{3^{\beta-1}\Gamma(\beta+1)}{8(d-\varsigma)^{\beta}}\Bigg[\mathcal{J}_{1-}^{\beta}F\left(1,\frac{2}{3}\right)+\mathcal{J}_{\frac{2}{3}-}^{\beta}F\left(1,\frac{1}{3}\right)+\mathcal{J}_{\frac{1}{3}-}^{\beta}F(1,0)\Bigg] \qquad (32)\\
&=-\frac{1}{108}\left(\frac{\beta^{2}+15\beta+5}{(\beta+1)(\beta+2)}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&-\frac{3^{\beta}\Gamma(\beta+1)}{8(d-\varsigma)^{\beta}}\Bigg[\mathcal{J}_{d-}^{\beta}F\left(\frac{\sigma+2\rho}{3},\frac{\varsigma+2d}{3}\right)+\mathcal{J}_{\frac{\varsigma+2d}{3}-}^{\beta}F\left(\frac{\sigma+2\rho}{3},\frac{2\varsigma+d}{3}\right)+\mathcal{J}_{\frac{2\varsigma+d}{3}-}^{\beta}F\left(\frac{\sigma+2\rho}{3},\varsigma\right)\\
&\quad+\mathcal{J}_{d-}^{\beta}F\left(\frac{2\sigma+\rho}{3},\frac{\varsigma+2d}{3}\right)+\mathcal{J}_{\frac{\varsigma+2d}{3}-}^{\beta}F\left(\frac{2\sigma+\rho}{3},\frac{2\varsigma+d}{3}\right)+\mathcal{J}_{\frac{2\varsigma+d}{3}-}^{\beta}F\left(\frac{2\sigma+\rho}{3},\varsigma\right)\Bigg] \qquad (33)\\
&=-\frac{5}{324}\left[\frac{7\beta^{2}+15\beta+5}{(\beta+1)(\beta+2)}\right].
\end{aligned}
$$

With the help of the definition of double Riemann–Liouville fractional integrals, we possess

$$
\begin{aligned}
&\frac{3^{\alpha-1}3^{\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)}{(\rho-\sigma)^{\alpha}(d-\varsigma)^{\beta}} \\
&\times\left[\mathcal{J}^{\alpha,\beta}_{\rho-,d-}F\left(\frac{\sigma+2\rho}{3},\frac{\varsigma+2d}{3}\right)+\mathcal{J}^{\alpha,\beta}_{\rho-,\frac{\varsigma+2d}{3}-}F\left(\frac{\sigma+2\rho}{3},\frac{2\varsigma+d}{3}\right)+\mathcal{J}^{\alpha,\beta}_{\rho-,\frac{2\varsigma+d}{3}-}F\left(\frac{\sigma+2\rho}{3},\varsigma\right)\right. \\
&+\mathcal{J}^{\alpha,\beta}_{\frac{\sigma+2\rho}{3}-,d-}F\left(\frac{2\sigma+\rho}{3},\frac{\varsigma+2d}{3}\right)+\mathcal{J}^{\alpha,\beta}_{\frac{\sigma+2\rho}{3}-,\frac{\varsigma+2d}{3}-}F\left(\frac{2\sigma+\rho}{3},\frac{2\varsigma+d}{3}\right)+\mathcal{J}^{\alpha,\beta}_{\frac{\sigma+2\rho}{3}-,\frac{2\varsigma+d}{3}-}F\left(\frac{2\sigma+\rho}{3},\varsigma\right) \\
&\left.+\mathcal{J}^{\alpha,\beta}_{\frac{2\sigma+\rho}{3}-,d-}F\left(\sigma,\frac{\varsigma+2d}{3}\right)+\mathcal{J}^{\alpha,\beta}_{\frac{2\sigma+\rho}{3}-,\frac{\varsigma+2d}{3}-}F\left(\sigma,\frac{2\varsigma+d}{3}\right)+\mathcal{J}^{\alpha,\beta}_{\frac{2\sigma+\rho}{3}-,\frac{2\varsigma+d}{3}-}F(\sigma,\varsigma)\right] \\
&=3^{\alpha-1}3^{\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1) \\
&\times\left[\mathcal{J}^{\alpha,\beta}_{1-,1-}F\left(\frac{2}{3},\frac{2}{3}\right)+\mathcal{J}^{\alpha,\beta}_{1-,\frac{2}{3}-}F\left(\frac{2}{3},\frac{1}{3}\right)+\mathcal{J}^{\alpha,\beta}_{1-,\frac{1}{3}-}F\left(\frac{2}{3},0\right)+\mathcal{J}^{\alpha,\beta}_{\frac{2}{3}-,1-}F\left(\frac{1}{3},\frac{2}{3}\right)\right. \\
&\left.+\mathcal{J}^{\alpha,\beta}_{\frac{2}{3}-,\frac{2}{3}-}F\left(\frac{1}{3},\frac{1}{3}\right)+\mathcal{J}^{\alpha,\beta}_{\frac{2}{3}-,\frac{1}{3}-}F\left(\frac{1}{3},0\right)+\mathcal{J}^{\alpha,\beta}_{\frac{1}{3}-,1-}F\left(0,\frac{2}{3}\right)+\mathcal{J}^{\alpha,\beta}_{\frac{1}{3}-,\frac{2}{3}-}F\left(0,\frac{1}{3}\right)+\mathcal{J}^{\alpha,\beta}_{\frac{1}{3}-,\frac{1}{3}-}F(0,0)\right] \\
&=3^{\alpha-1}3^{\beta-1}\alpha\beta\left[\int_{\frac{2}{3}}^{1}\int_{\frac{2}{3}}^{1}\left(\gamma-\frac{2}{3}\right)^{\alpha-1}\left(\delta-\frac{2}{3}\right)^{\beta-1}\gamma^2\delta^2 d\delta d\gamma+\int_{\frac{2}{3}}^{1}\int_{\frac{1}{3}}^{\frac{2}{3}}\left(\gamma-\frac{2}{3}\right)^{\alpha-1}\left(\delta-\frac{1}{3}\right)^{\beta-1}\gamma^2\delta^2 d\delta d\gamma\right. \qquad (34) \\
&+\int_{\frac{2}{3}}^{1}\int_{0}^{\frac{1}{3}}\left(\gamma-\frac{2}{3}\right)^{\alpha-1}(\delta)^{\beta-1}\gamma^2\delta^2 d\delta d\gamma+\int_{\frac{1}{3}}^{\frac{2}{3}}\int_{\frac{2}{3}}^{1}\left(\gamma-\frac{1}{3}\right)^{\alpha-1}\left(\delta-\frac{2}{3}\right)^{\beta-1}\gamma^2\delta^2 d\delta d\gamma \\
&+\int_{\frac{1}{3}}^{\frac{2}{3}}\int_{\frac{1}{3}}^{\frac{2}{3}}\left(\gamma-\frac{1}{3}\right)^{\alpha-1}\left(\delta-\frac{1}{3}\right)^{\beta-1}\gamma^2\delta^2 d\delta d\gamma+\int_{\frac{1}{3}}^{\frac{2}{3}}\int_{0}^{\frac{1}{3}}\left(\gamma-\frac{1}{3}\right)^{\alpha-1}(\delta)^{\beta-1}\gamma^2\delta^2 d\delta d\gamma \\
&\left.+\int_{0}^{\frac{1}{3}}\int_{\frac{2}{3}}^{1}(\gamma)^{\alpha-1}\left(\delta-\frac{2}{3}\right)^{\beta-1}\gamma^2\delta^2 d\delta d\gamma+\int_{0}^{\frac{1}{3}}\int_{\frac{1}{3}}^{\frac{2}{3}}(\gamma)^{\alpha-1}\left(\delta-\frac{1}{3}\right)^{\beta-1}\gamma^2\delta^2 d\delta d\gamma+\int_{0}^{\frac{1}{3}}\int_{0}^{\frac{1}{3}}(\gamma)^{\alpha-1}(\delta)^{\beta-1}\gamma^2\delta^2 d\delta d\gamma\right] \\
&=\frac{4\left(7\alpha^2+15\alpha+5\right)\left(7\beta^2+15\beta+5\right)}{729(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}.
\end{aligned}
$$

If we add the expressions (27)–(34) *and we have the left-hand side of* (16),

$$
\begin{aligned}
\left|\Theta^{\alpha,\beta}(0,1;0,1)\right| &= \left|\frac{1}{9}-\frac{2}{81}\left[\left(\frac{7\alpha^2+15\alpha+5}{(\alpha+1)(\alpha+2)}\right)+\left(\frac{7\beta^2+15\beta+5}{(\beta+1)(\beta+2)}\right)\right]\right. \\
&\left.+\frac{4\left(7\alpha^2+15\alpha+5\right)\left(7\beta^2+15\beta+5\right)}{729(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\right| := \Psi_2. \qquad (35)
\end{aligned}
$$

If we substitute (35) *and* (26) *in* (16), *we derive*

$$
\begin{aligned}
\left|\Theta^{\alpha,\beta}(0,1;0,1)\right| &\leq \frac{1}{729}[2\Omega_6(\alpha)+\Omega_5(\alpha)+\Omega_3(\alpha)+\Omega_4(\alpha)+\Omega_1(\alpha)] \\
&\times [2\Omega_6(\beta)+\Omega_5(\beta)+\Omega_3(\beta)+\Omega_4(\beta)+\Omega_1(\beta)]].
\end{aligned}
$$

If we demonstrate Example 1 on the graph as Figure 1:

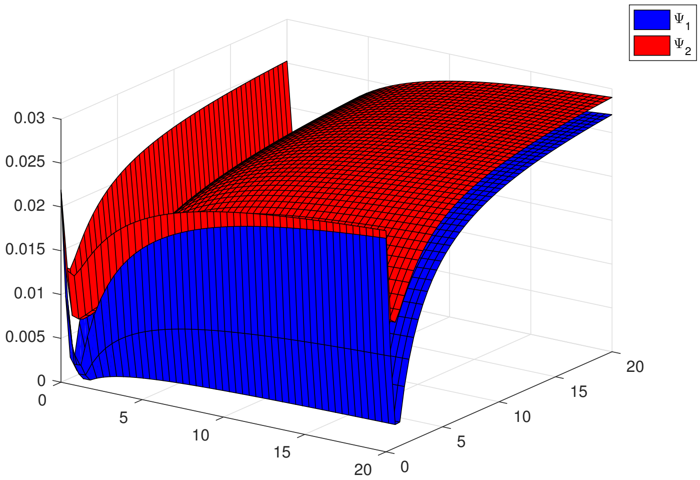

Figure 1. An example to Theorem 2, depending on α and β, computed and plotted with MATLAB.

Remark 1. *If we take $\alpha = \beta = 1$ in Theorem 2, then we obtain*

$$\begin{aligned}
&\left|\Theta^{\alpha,\beta}(\sigma,\rho;\varsigma,d)\right| \\
&\leq \frac{(\rho-\sigma)(d-\varsigma)}{729}\left[\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|[\Omega_6(\alpha)-\Omega_5(\alpha)+2\Omega_4(\alpha)-\Omega_3(\alpha)+3\Omega_2(\alpha)-\Omega_1(\alpha)]\right. \\
&\quad\times[\Omega_6(\beta)-\Omega_5(\beta)+2\Omega_4(\beta)-\Omega_3(\beta)+3\Omega_2(\beta)-\Omega_1(\beta)] \\
&\quad\times\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|[\Omega_6(\alpha)-\Omega_5(\alpha)+2\Omega_4(\alpha)-\Omega_3(\alpha)+3\Omega_2(\alpha)-\Omega_1(\alpha)] \\
&\quad\times[2\Omega_6(\beta)+\Omega_5(\beta)+\Omega_3(\beta)+\Omega_4(\beta)+\Omega_1(\beta)] \\
&\quad\times\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|[2\Omega_6(\alpha)+\Omega_5(\alpha)+\Omega_3(\alpha)+\Omega_4(\alpha)+\Omega_1(\alpha)] \\
&\quad\times[\Omega_6(\beta)-\Omega_5(\beta)+2\Omega_4(\beta)-\Omega_3(\beta)+3\Omega_2(\beta)-\Omega_1(\beta)] \\
&\quad\times\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|[2\Omega_6(\alpha)+\Omega_5(\alpha)+\Omega_3(\alpha)+\Omega_4(\alpha)+\Omega_1(\alpha)] \\
&\quad\left.\times\,[2\Omega_6(\beta)+\Omega_5(\beta)+\Omega_3(\beta)+\Omega_4(\beta)+\Omega_1(\beta)]\right],
\end{aligned} \tag{36}$$

and

$$\begin{aligned}
&|\Upsilon(\sigma,\rho;\varsigma,d)| \\
&\leq(\rho-\sigma)(d-\varsigma)\frac{625}{331776}\left[\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|\right],
\end{aligned}$$

which is given by Iftikhar et al. in [13] and $\Upsilon(\sigma,\rho;\varsigma,d)$ is described by

$$
\begin{aligned}
&\Upsilon(\sigma,\rho;\varsigma,d)\\
&:=\frac{[F(\sigma,\varsigma)+F(\sigma,d)+F(\rho,\varsigma)+F(\rho,d)]}{64}\\
&+\frac{3}{64}\left[F\left(\sigma,\frac{2\varsigma+d}{3}\right)+F\left(\sigma,\frac{\varsigma+2d}{3}\right)+F\left(\rho,\frac{2\varsigma+d}{3}\right)+F\left(\rho,\frac{\varsigma+2d}{3}\right)\right.\\
&\left.+F\left(\frac{2\sigma+\rho}{3},\varsigma\right)+F\left(\frac{\sigma+2\rho}{3},\varsigma\right)+\left(\frac{2\sigma+\rho}{3},d\right)+F\left(\frac{\sigma+2\rho}{3},d\right)\right]\\
&+\frac{9}{64}\left[F\left(\frac{2\sigma+\rho}{3},\frac{2\varsigma+d}{3}\right)+F\left(\frac{2\sigma+\rho}{3},\frac{\varsigma+2d}{3}\right)\right.\\
&\left.+F\left(\frac{2\sigma+\rho}{3},\frac{2\varsigma+d}{3}\right)+F\left(\frac{\sigma+2\rho}{3},\frac{\varsigma+2d}{3}\right)\right] \qquad (37)\\
&-\frac{1}{8(\rho-\sigma)}\left[\int_\sigma^\rho F(\kappa,\varsigma)d\kappa+\int_\sigma^\rho F(\kappa,d)d\kappa+3\int_\sigma^\rho F\left(\kappa,\frac{2\varsigma+d}{3}\right)d\kappa+3\int_\sigma^\rho F\left(\kappa,\frac{\varsigma+2d}{3}\right)d\kappa\right]\\
&-\frac{1}{8(d-\varsigma)}\left[\int_\varsigma^d F(\sigma,y)dy+\int_\varsigma^d F(\rho,y)dy+3\int_\varsigma^d F\left(\frac{2\sigma+\rho}{3},y\right)dy+3\int_\varsigma^d F\left(\frac{\sigma+2\rho}{3},y\right)dy\right]\\
&+\frac{1}{(\rho-\sigma)(d-\varsigma)}\int_\sigma^\rho\int_\varsigma^d F(\kappa,y)dyd\kappa.
\end{aligned}
$$

Theorem 3. *Let the conditions of Lemma 3 be satisfied. If* $\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}\right|^q$, $q>1$ *and* $p^{-1}+q^{-1}=1$ *is a convex function on coordinates, then we establish the following Newton's type inequality:*

$$
\begin{aligned}
&\left|\Theta^{\alpha,\beta}(\sigma,\rho;\varsigma,d)\right|\\
&\le\frac{(\rho-\sigma)(d-\varsigma)}{81}\\
&\times\left[\Omega_9^{\frac{1}{p}}(\alpha,p)\Omega_9^{\frac{1}{p}}(\beta,p)\left(\frac{\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q+25\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q}{36}\right)^{\frac{1}{q}}\right.\\
&+\Omega_9^{\frac{1}{p}}(\alpha,p)\Omega_8^{\frac{1}{p}}(\beta,p)\left(\frac{\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q}{12}\right)^{\frac{1}{q}}\\
&+\Omega_9^{\frac{1}{p}}(\alpha,p)\Omega_7^{\frac{1}{p}}(\beta,p)\left(\frac{5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q+25\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q}{36}\right)^{\frac{1}{q}}\\
&+\Omega_8^{\frac{1}{p}}(\alpha,p)\Omega_9^{\frac{1}{p}}(\beta,p)\left(\frac{\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q}{12}\right)^{\frac{1}{q}}\\
&+\Omega_8^{\frac{1}{p}}(\alpha,p)\Omega_8^{\frac{1}{p}}(\beta,p)\left(\frac{\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q}{4}\right)^{\frac{1}{q}}
\end{aligned}
$$

$$+\Omega_8^{\frac{1}{p}}(\alpha,p)\Omega_7^{\frac{1}{p}}(\beta,p)\left(\frac{\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q}{12}\right)^{\frac{1}{q}}$$

$$+\Omega_7^{\frac{1}{p}}(\alpha,p)\Omega_9^{\frac{1}{p}}(\beta,p)\left(\frac{5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q+25\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q}{36}\right)^{\frac{1}{q}}$$

$$+\Omega_7^{\frac{1}{p}}(\alpha,p)\Omega_8^{\frac{1}{p}}(\beta,p)\left(\frac{5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q}{12}\right)^{\frac{1}{q}}$$

$$+\Omega_7^{\frac{1}{p}}(\alpha,p)\Omega_7^{\frac{1}{p}}(\beta,p)\left(\frac{25\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q}{36}\right)^{\frac{1}{q}},$$

where $q^{-1}+p^{-1}=1$, $\Omega_\Phi, \Phi=1,2,\ldots,6$ *are as in* (2), $\Theta^{\alpha,\beta}(\sigma,\rho;\varsigma,d)$ *is defined by* (6) *and*

$$\begin{aligned}\Omega_7(v,p)&=\int_0^1\left|\tau^v-\frac{3}{8}\right|^p d\tau,\\ \Omega_8(v,p)&=\int_0^1\left|\tau^v-\frac{1}{2}\right|^p d\tau,\\ \Omega_9(v,p)&=\int_0^1\left|\tau^v-\frac{5}{8}\right|^p d\tau.\end{aligned}\tag{38}$$

Proof. With the help of the Hölder inequality and coordinated convexity of $\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\gamma,\delta)\right|^q$, the following inequalities hold:

$$\begin{aligned}|\Phi_1|&\le\int_0^1\int_0^1\left|\gamma^\alpha-\frac{5}{8}\right|\left|\delta^\beta-\frac{5}{8}\right|\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}\left(\left(\frac{1-\gamma}{3}\right)\sigma+\left(\frac{\gamma+2}{3}\right)\rho,\left(\frac{1-\delta}{3}\right)\varsigma+\left(\frac{\delta+2}{3}\right)d\right)\right|d\delta d\gamma\\
&\le\left(\int_0^1\int_0^1\left|\gamma^\alpha-\frac{5}{8}\right|^p\left|\delta^\beta-\frac{5}{8}\right|^p\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial\delta\partial\gamma}F\left(\left(\frac{1-\gamma}{3}\right)\sigma+\left(\frac{\gamma+2}{3}\right)\rho,\left(\frac{1-\delta}{3}\right)\varsigma+\left(\frac{\delta+2}{3}\right)d\right)\right|^q d\delta d\gamma\right)^{\frac{1}{q}}\\
&\le\left(\int_0^1\int_0^1\left|\gamma^\alpha-\frac{5}{8}\right|^p\left|\delta^\beta-\frac{5}{8}\right|^p\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^1\int_0^1\left[\left(\frac{1-\gamma}{3}\right)\left(\frac{1-\delta}{3}\right)\left|\frac{\partial^2}{\partial\delta\partial\gamma}F(\sigma,\varsigma)\right|^q+\left(\frac{1-\gamma}{3}\right)\left(\frac{\delta+2}{3}\right)\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q\right.\right.\\
&\quad\left.\left.+\left(\frac{\gamma+2}{3}\right)\left(\frac{1-\delta}{3}\right)\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q+\left(\frac{\gamma+2}{3}\right)\left(\frac{\delta+2}{3}\right)\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q\right]^q d\delta d\gamma\right)^{\frac{1}{q}}\\
&=\Omega_9^{\frac{1}{p}}(\alpha,p)\Omega_9^{\frac{1}{p}}(\beta,p)\left(\frac{\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q+25\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q}{36}\right)^{\frac{1}{q}}.\end{aligned}$$

Similarly,

$$|\Phi_2| \leq \Omega_9^{\frac{1}{p}}(\alpha,p)\Omega_8^{\frac{1}{p}}(\beta,p)\left(\frac{\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q}{12}\right)^{\frac{1}{q}},$$

$$|\Phi_3| \leq \Omega_9^{\frac{1}{p}}(\alpha,p)\Omega_7^{\frac{1}{p}}(\beta,p)\left(\frac{5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q+25\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q}{36}\right)^{\frac{1}{q}},$$

$$|\Phi_4| \leq \int_0^1 \Omega_8^{\frac{1}{p}}(\alpha,p)\Omega_9^{\frac{1}{p}}(\beta,p)\left(\frac{\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q}{12}\right)^{\frac{1}{q}},$$

$$|\Phi_5| \leq \Omega_8^{\frac{1}{p}}(\alpha,p)\Omega_8^{\frac{1}{p}}(\beta,p)\left(\frac{\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q}{4}\right)^{\frac{1}{q}},$$

$$|\Phi_6| \leq \Omega_8^{\frac{1}{p}}(\alpha,p)\Omega_7^{\frac{1}{p}}(\beta,p)\left(\frac{5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q}{12}\right)^{\frac{1}{q}},$$

$$|\Phi_7| \leq \Omega_7^{\frac{1}{p}}(\alpha,p)\Omega_9^{\frac{1}{p}}(\beta,p)\left(\frac{5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q+25\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q}{36}\right)^{\frac{1}{q}},$$

$$|\Phi_8| \leq \Omega_7^{\frac{1}{p}}(\alpha,p)\Omega_8^{\frac{1}{p}}(\beta,p)\left(\frac{5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q}{12}\right)^{\frac{1}{q}}$$

and

$$|\Phi_9| \leq \Omega_7^{\frac{1}{p}}(\alpha,p)\Omega_7^{\frac{1}{p}}(\beta,p)\left(\frac{25\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q}{36}\right)^{\frac{1}{q}}.$$

This proof is completed. □

Corollary 1. *If we choose $\alpha = \beta = 1$ in Theorem 3, then we derive:*

$$\begin{aligned}
&|\Upsilon(\sigma,\rho;\varsigma,d)| \\
&\leq \frac{(\rho-\sigma)(d-\varsigma)}{81} \\
&\times \Bigg[\left(\frac{5^{p+1}+3^{p+1}}{8^{p+1}(p+1)}\right)^{\frac{2}{p}}\left(\frac{\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q+25\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q}{36}\right)^{\frac{1}{q}} \\
&+\left(\frac{5^{p+1}+3^{p+1}}{8^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{2^p(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q}{12}\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\frac{5^{p+1}+3^{p+1}}{8^{p+1}(p+1)}\right)^{\frac{2}{p}}\left(\frac{5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q+25\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q}{36}\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{2^p(p+1)}\right)^{\frac{2}{p}}\left(\frac{\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q}{12}\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{2^p(p+1)}\right)^{\frac{1}{p}}\left(\frac{5^{p+1}+3^{p+1}}{8^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q}{4}\right)^{\frac{1}{q}}$$

$$+\left(\frac{5^{p+1}+3^{p+1}}{8^{p+1}(p+1)}\right)^{\frac{2}{p}}\left(\frac{\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q}{12}\right)^{\frac{1}{q}}$$

$$+\left(\frac{5^{p+1}+3^{p+1}}{8^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{2^p(p+1)}\right)^{\frac{1}{p}}\left(\frac{5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q}{12}\right)^{\frac{1}{q}}$$

$$+\left(\frac{5^{p+1}+3^{p+1}}{8^{p+1}(p+1)}\right)^{\frac{2}{p}}\left(\frac{25\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q+5\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q}{36}\right)^{\frac{1}{q}},$$

where $\Upsilon(\sigma,\rho;\varsigma,d)$ *is given in* (37).

Theorem 4. *Let the conditions of Lemma 3 be satisfied. If* $\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}\right|^q$ *,*$q\geq 1$*, is a coordinated convex mapping, then we obtain the following Newton-type inequality:*

$$\begin{aligned}
&\left|\Theta^{\alpha,\beta}(\sigma,\rho;\varsigma,d)\right| \\
&\leq \frac{(\rho-\sigma)(d-\varsigma)}{81}\Bigg[\Omega_6^{1-\frac{1}{q}}(\alpha)\Omega_6^{1-\frac{1}{q}}(\beta)\Bigg(\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q\frac{(\Omega_6(\alpha)-\Omega_5(\alpha))(\Omega_6(\beta)-\Omega_5(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q\frac{(\Omega_6(\alpha)-\Omega_5(\alpha))(2\Omega_6(\beta)+\Omega_5(\beta))}{9}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q\frac{(\Omega_5(\alpha)+2\Omega_6(\alpha))(\Omega_6(\beta)-\Omega_5(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q\frac{(\Omega_5(\alpha)+2\Omega_6(\alpha))(2\Omega_6(\beta)+\Omega_5(\beta))}{9}\Bigg)^{\frac{1}{q}} \\
&+\Omega_6^{1-\frac{1}{q}}(\alpha)\Omega_4^{1-\frac{1}{q}}(\beta)\Bigg(\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q\frac{(\Omega_6(\alpha)-\Omega_5(\alpha))(2\Omega_4(\beta)-\Omega_3(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q\frac{(\Omega_6(\alpha)-\Omega_5(\alpha))(\Omega_3(\beta)+\Omega_4(\beta))}{9}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q\frac{(\Omega_5(\alpha)+2\Omega_6(\alpha))(2\Omega_4(\beta)-\Omega_3(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q\frac{(\Omega_5(\alpha)+2\Omega_6(\alpha))(\Omega_3(\beta)+\Omega_4(\beta))}{9}\Bigg)^{\frac{1}{q}} \\
&+\Omega_6^{1-\frac{1}{q}}(\alpha)\Omega_2^{1-\frac{1}{q}}(\beta)\Bigg(\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q\frac{(\Omega_6(\alpha)-\Omega_5(\alpha))(3\Omega_2(\beta)-\Omega_1(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q\frac{(\Omega_6(\alpha)-\Omega_5(\alpha))\Omega_1(\beta)}{9}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q\frac{(\Omega_5(\alpha)+2\Omega_6(\alpha))(3\Omega_2(\beta)-\Omega_1(\beta))}{9}
\end{aligned}$$

$$
\begin{aligned}
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q \frac{(\Omega_5(\alpha)+2\Omega_6(\alpha))\Omega_1(\beta)}{9}\Bigg)^{\frac{1}{q}} \\
&+\Omega_4^{1-\frac{1}{q}}(\alpha)\Omega_6^{1-\frac{1}{q}}(\beta)\Bigg(\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q \frac{(2\Omega_4(\alpha)-\Omega_3(\alpha))(\Omega_6(\beta)-\Omega_5(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q \frac{(2\Omega_4(\alpha)-\Omega_3(\alpha))(2\Omega_6(\beta)+\Omega_5(\beta))}{9}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q \frac{(\Omega_4(\alpha)+\Omega_3(\alpha))(\Omega_6(\beta)-\Omega_5(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q \frac{(\Omega_3(\alpha)+\Omega_4(\alpha))(2\Omega_6(\beta)+\Omega_5(\beta))}{9}\Bigg)^{\frac{1}{q}} \\
&+\Omega_4^{1-\frac{1}{q}}(\alpha)\Omega_4^{1-\frac{1}{q}}(\beta)\Bigg(\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q \frac{(2\Omega_4(\alpha)-\Omega_3(\alpha))(2\Omega_4(\beta)-\Omega_3(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q \frac{(2\Omega_4(\alpha)-\Omega_3(\alpha))(\Omega_3(\beta)+\Omega_4(\beta))}{9}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q \frac{(\Omega_3(\alpha)+\Omega_4(\alpha))(2\Omega_4(\beta)-\Omega_3(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q \frac{(\Omega_4(\alpha)+\Omega_3(\alpha))(\Omega_3(\beta)+\Omega_4(\beta))}{9}\Bigg)^{\frac{1}{q}} \\
&+\Omega_4^{1-\frac{1}{q}}(\alpha).\Omega_2^{1-\frac{1}{q}}(\beta)\Bigg(\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right| \frac{(2\Omega_4(\alpha)-\Omega_3(\alpha))(3\Omega_2(\beta)-\Omega_1(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q \frac{(2\Omega_4(\alpha)-\Omega_3(\alpha))\Omega_1(\beta)}{9}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q \frac{(\Omega_3(\alpha)+\Omega_4(\alpha))(3\Omega_2(\beta)-\Omega_1(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q \frac{(\Omega_3(\alpha)+\Omega_4(\alpha))\Omega_1(\beta)}{9}\Bigg)^{\frac{1}{q}} \\
&+\Omega_2^{1-\frac{1}{q}}(\alpha)\Omega_6^{1-\frac{1}{q}}(\beta)\Bigg(\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q \frac{(3\Omega_2(\alpha)-\Omega_1(\alpha))(\Omega_6(\beta)-\Omega_5(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q \frac{(3\Omega_2(\alpha)-\Omega_1(\alpha))(2\Omega_6(\beta)+\Omega_5(\beta))}{9}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q \frac{\Omega_1(\alpha)(\Omega_6(\beta)-\Omega_5(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q \frac{\Omega_1(\alpha)(2\Omega_6(\beta)+\Omega_5(\beta))}{9}\Bigg)^{\frac{1}{q}} \\
&+\Omega_2^{1-\frac{1}{q}}(\alpha)\Omega_4^{1-\frac{1}{q}}(\beta)\Bigg(\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q \frac{(3\Omega_2(\alpha)-\Omega_1(\alpha))(2\Omega_4(\beta)-\Omega_3(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q \frac{(3\Omega_2(\alpha)-\Omega_1(\alpha))(\Omega_3(\beta)+\Omega_4(\beta))}{9}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q \frac{\Omega_1(\alpha)(\Omega_3(\beta)+\Omega_4(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q \frac{\Omega_1(\alpha)(\Omega_3(\beta)+\Omega_4(\beta))}{9}\Bigg)^{\frac{1}{q}} \\
&+\Omega_2^{1-\frac{1}{q}}(\alpha)\Omega_4^{1-\frac{1}{q}}(\beta)\Bigg(\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q \frac{(3\Omega_2(\alpha)-\Omega_1(\alpha))(3\Omega_2(\beta)-\Omega_1(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q \frac{(3\Omega_2(\alpha)-\Omega_1(\alpha))\Omega_1(\beta)}{9}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q \frac{\Omega_1(\alpha)(3\Omega_2(\beta)-\Omega_1(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q \frac{\Omega_1(\alpha)\Omega_1(\beta)}{9}\Bigg)^{\frac{1}{q}},
\end{aligned}
$$

where Ω_Φ *for* $\Phi = 1, 2, \ldots 6$ *are shown in (2) and* $\Theta^{\alpha,\beta}(\sigma,\rho;\varsigma,d)$ *is as noted in (6).*

Proof. By taking the modulus of Lemma 3 and by using of the power mean inequality, we possess:

$$
\begin{aligned}
|\Phi_1| &\leq \int_0^1\int_0^1 \left|\gamma^\alpha - \frac{5}{8}\right|\left|\delta^\beta - \frac{5}{8}\right|\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}\left(\left(\frac{1-\gamma}{3}\right)\sigma + \left(\frac{\gamma+2}{3}\right)\rho, \left(\frac{1-\delta}{3}\right)\varsigma + \left(\frac{\delta+2}{3}\right)d\right)\right| d\delta d\gamma \\
&\leq \left(\int_0^1\int_0^1 \left|\gamma^\alpha - \frac{5}{8}\right|\left|\delta^\beta - \frac{5}{8}\right|\right)^{1-\frac{1}{q}} \\
&\quad \times \left(\int_0^1\int_0^1 \left|\gamma^\alpha - \frac{5}{8}\right|\left|\delta^\beta - \frac{5}{8}\right|\left|\frac{\partial^2}{\partial\delta\partial\gamma}F\left(\left(\frac{1-\gamma}{3}\right)\sigma + \left(\frac{\gamma+2}{3}\right)\rho, \left(\frac{1-\delta}{3}\right)\varsigma + \left(\frac{\delta+2}{3}\right)d\right)\right|^q d\delta d\gamma\right)^{\frac{1}{q}} \\
&\leq \left(\int_0^1\int_0^1 \left|\gamma^\alpha - \frac{5}{8}\right|\left|\delta^\beta - \frac{5}{8}\right|\right)^{1-\frac{1}{q}} \\
&\quad \times \left(\int_0^1\int_0^1 \left|\gamma^\alpha - \frac{5}{8}\right|\left|\delta^\beta - \frac{5}{8}\right|\left[\left(\frac{1-\gamma}{3}\right)\left(\frac{1-\delta}{3}\right)\left|\frac{\partial^2}{\partial\delta\partial\gamma}F(\sigma,\varsigma)\right|^q + \left(\frac{1-\gamma}{3}\right)\left(\frac{\delta+2}{3}\right)\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q\right.\right. \\
&\quad \left.\left. + \left(\frac{\gamma+2}{3}\right)\left(\frac{1-\delta}{3}\right)\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q + \left(\frac{\gamma+2}{3}\right)\left(\frac{\delta+2}{3}\right)\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q\right] d\delta d\gamma\right)^{\frac{1}{q}} \\
&= \Omega_6^{1-\frac{1}{q}}(\alpha)\Omega_6^{1-\frac{1}{q}}(\beta)\left(\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q \frac{(\Omega_6(\alpha)-\Omega_5(\alpha))(\Omega_6(\beta)-\Omega_5(\beta))}{9}\right. \\
&\quad + \left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q \frac{(\Omega_6(\alpha)-\Omega_5(\alpha))(2\Omega_6(\beta)+\Omega_5(\beta))}{9} \\
&\quad + \left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q \frac{(\Omega_5(\alpha)+2\Omega_6(\alpha))(\Omega_6(\beta)-\Omega_5(\beta))}{9} \\
&\quad \left. + \left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q \frac{(\Omega_5(\alpha)+2\Omega_6(\alpha))(2\Omega_6(\beta)+\Omega_5(\beta))}{9}\right)^{\frac{1}{q}}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|\Phi_2| &\leq \Omega_6^{1-\frac{1}{q}}(\alpha)\Omega_4^{1-\frac{1}{q}}(\beta)\left(\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q \frac{(\Omega_6(\alpha)-\Omega_5(\alpha))(2\Omega_4(\beta)-\Omega_3(\beta))}{9}\right. \\
&\quad + \left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q \frac{(\Omega_6(\alpha)-\Omega_5(\alpha))(\Omega_3(\beta)+\Omega_4(\beta))}{9} \\
&\quad + \left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q \frac{(\Omega_5(\alpha)+2\Omega_6(\alpha))(2\Omega_4(\beta)-\Omega_3(\beta))}{9} \\
&\quad \left. + \left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q \frac{(\Omega_5(\alpha)+2\Omega_6(\alpha))(\Omega_3(\beta)+\Omega_4(\beta))}{9}\right)^{\frac{1}{q}}, \\
|\Phi_3| &\leq \Omega_6^{1-\frac{1}{q}}(\alpha)\Omega_2^{1-\frac{1}{q}}(\beta)\left(\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q \frac{(\Omega_6(\alpha)-\Omega_5(\alpha))(3\Omega_2(\beta)-\Omega_1(\beta))}{9}\right. \\
&\quad + \left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q \frac{(\Omega_6(\alpha)-\Omega_5(\alpha))\Omega_1(\beta)}{9} \\
&\quad + \left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q \frac{(\Omega_5(\alpha)+2\Omega_6(\alpha))(3\Omega_2(\beta)-\Omega_1(\beta))}{9} \\
&\quad \left. + \left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q \frac{(\Omega_5(\alpha)+2\Omega_6(\alpha))\Omega_1(\beta)}{9}\right)^{\frac{1}{q}},
\end{aligned}
$$

$$
\begin{aligned}
|\Phi_4| \le \Omega_4^{1-\frac{1}{q}}(\alpha).\Omega_6^{1-\frac{1}{q}}(\beta)\Bigg(&\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q \frac{(2\Omega_4(\alpha)-\Omega_3(\alpha))(\Omega_6(\beta)-\Omega_5(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q \frac{(2\Omega_4(\alpha)-\Omega_3(\alpha))(2\Omega_6(\beta)+\Omega_5(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q \frac{(\Omega_4(\alpha)+\Omega_3(\alpha))(\Omega_6(\beta)-\Omega_5(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q \frac{(\Omega_3(\alpha)+\Omega_4(\alpha))(2\Omega_6(\beta)+\Omega_5(\beta))}{9}\Bigg)^{\frac{1}{q}}, \\
|\Phi_5| \le \Omega_4^{1-\frac{1}{q}}(\alpha)\Omega_4^{1-\frac{1}{q}}(\beta)\Bigg(&\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q \frac{(2\Omega_4(\alpha)-\Omega_3(\alpha))(2\Omega_4(\beta)-\Omega_3(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q \frac{(2\Omega_4(\alpha)-\Omega_3(\alpha))(\Omega_3(\beta)+\Omega_4(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q \frac{(\Omega_3(\alpha)+\Omega_4(\alpha))(2\Omega_4(\beta)-\Omega_3(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q \frac{(\Omega_4(\alpha)+\Omega_3(\alpha))(\Omega_3(\beta)+\Omega_4(\beta))}{9}\Bigg)^{\frac{1}{q}}, \\
|\Phi_6| \le \Omega_4^{1-\frac{1}{q}}(\alpha).\Omega_2^{1-\frac{1}{q}}(\beta)\Bigg(&\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q \frac{(2\Omega_4(\alpha)-\Omega_3(\alpha))(3\Omega_2(\beta)-\Omega_1(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q \frac{(2\Omega_4(\alpha)-\Omega_3(\alpha))\Omega_1(\beta)}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q \frac{(\Omega_3(\alpha)+\Omega_4(\alpha))(3\Omega_2(\beta)-\Omega_1(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q \frac{(\Omega_3(\alpha)+\Omega_4(\alpha))\Omega_1(\beta)}{9}\Bigg)^{\frac{1}{q}},
\end{aligned}
$$

$$
\begin{aligned}
|\Phi_7| \le \Omega_2^{1-\frac{1}{q}}(\alpha).\Omega_6^{1-\frac{1}{q}}(\beta)\Bigg(&\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q \frac{(3\Omega_2(\alpha)-\Omega_1(\alpha))(\Omega_6(\beta)-\Omega_5(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q \frac{(3\Omega_2(\alpha)-\Omega_1(\alpha))(2\Omega_6(\beta)+\Omega_5(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q \frac{\Omega_1(\alpha)(\Omega_6(\beta)-\Omega_5(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q \frac{\Omega_1(\alpha)(2\Omega_6(\beta)+\Omega_5(\beta))}{9}\Bigg)^{\frac{1}{q}}, \\
|\Phi_8| \le \Omega_2^{1-\frac{1}{q}}(\alpha).\Omega_4^{1-\frac{1}{q}}(\beta)\Bigg(&\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q \frac{(3\Omega_2(\alpha)-\Omega_1(\alpha))(2\Omega_4(\beta)-\Omega_3(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q \frac{(3\Omega_2(\alpha)-\Omega_1(\alpha))(\Omega_3(\beta)+\Omega_4(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q \frac{\Omega_1(\alpha)(\Omega_3(\beta)+\Omega_4(\beta))}{9} \\
&+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q \frac{\Omega_1(\alpha)(\Omega_3(\beta)+\Omega_4(\beta))}{9}\Bigg)^{\frac{1}{q}},
\end{aligned}
$$

and

$$|\Phi_9| \leq \Omega_2^{1-\frac{1}{q}}(\alpha).\Omega_4^{1-\frac{1}{q}}(\beta)\Bigg(\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q \frac{(3\Omega_2(\alpha)-\Omega_1(\alpha))(3\Omega_2(\beta)-\Omega_1(\beta))}{9}$$
$$+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q \frac{(3\Omega_2(\alpha)-\Omega_1(\alpha))\Omega_1(\beta)}{9}$$
$$+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q \frac{\Omega_1(\alpha)(3\Omega_2(\beta)-\Omega_1(\beta))}{9}$$
$$+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q \frac{\Omega_1(\alpha)\Omega_1(\beta)}{9}\Bigg)^{\frac{1}{q}}.$$

Therefore, the proof is completed. □

Corollary 2. *If we consider* $\alpha=\beta=1$ *in Theorem 4, then we obtain*

$$|\Upsilon(\sigma,\rho;\varsigma,d)|$$
$$\leq \frac{(\rho-\sigma)(d-\varsigma)}{6^4}\Bigg[\left(\frac{17^2}{2^8}\right)^{1-\frac{1}{q}}\Bigg(\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q \frac{251^2}{2^{14}\cdot 3^4}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q \frac{7\cdot 139.251}{2^{14}\cdot 3^4}$$
$$+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q \frac{7\cdot 139.251}{2^{14}\cdot 3^4}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q \frac{7^2\cdot 139^2}{2^{14}\cdot 3^4}\Bigg)^{\frac{1}{q}}$$
$$+\left(\frac{17}{2^4}\right)^{1-\frac{1}{q}}\Bigg(\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q \frac{251}{2^8\cdot 3^2}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q \frac{251}{2^8\cdot 3^2}$$
$$+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q \frac{7\cdot 139}{2^8\cdot 3^2}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q \frac{7\cdot 139}{2^8\cdot 3^2}\Bigg)^{\frac{1}{q}}$$
$$+\left(\frac{17^2}{2^8}\right)^{1-\frac{1}{q}}\Bigg(\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q \frac{7\cdot 139.251}{2^{14}\cdot 3^4}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q \frac{251^2}{2^{14}\cdot 3^4}$$
$$+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q \frac{7^2\cdot 139^2}{2^{14}\cdot 3^4}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q \frac{7\cdot 139\cdot 251}{2^{14}\cdot 3^4}\Bigg)^{\frac{1}{q}}$$
$$+\left(\frac{17}{2^4}\right)^{1-\frac{1}{q}}\Bigg(\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q \frac{157}{2^8\cdot 3^2}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q \frac{7\cdot 139}{2^8\cdot 3^2}$$
$$+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q \frac{157}{2^8\cdot 3^2}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q \frac{7\cdot 139}{2^8\cdot 3^2}\Bigg)^{\frac{1}{q}}$$
$$+\Bigg(\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q \frac{1}{4}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q \frac{1}{4}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q \frac{1}{4}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q \frac{1}{4}\Bigg)^{\frac{1}{q}}$$
$$+\left(\frac{17}{2^4}\right)^{1-\frac{1}{q}}\Bigg(\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q \frac{973}{2304}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q \frac{157}{2304}$$
$$+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q \frac{7\cdot 139}{2^8\cdot 3^2}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q \frac{157}{2^8\cdot 3^2}\Bigg)^{\frac{1}{q}}$$
$$+\left(\frac{17^2}{2^8}\right)^{1-\frac{1}{q}}\Bigg(\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q \frac{973.251}{2^{14}\cdot 3^4}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q \frac{157.973}{2^{14}\cdot 3^4}$$

$$+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q \frac{251^2}{2^{14}\cdot 3^4}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q \frac{973.251}{2^{14}\cdot 3^4}\Bigg)^{\frac{1}{q}}$$

$$+\left(\frac{17}{2^4}\right)^{1-\frac{1}{q}}\Bigg(\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q \frac{973}{2304}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q \frac{973}{2304}$$

$$+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q \frac{251}{2304}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q \frac{251}{2304}\Bigg)^{\frac{1}{q}}$$

$$+\left(\frac{17}{2^4}\right)^{1-\frac{1}{q}}\Bigg(\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,\varsigma)\right|^q \frac{973^2}{2^{14}\cdot 3^4}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\sigma,d)\right|^q \frac{251.973}{2^{14}\cdot 3^4}$$

$$+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,\varsigma)\right|^q \frac{973.251}{2^{14}\cdot 3^4}+\left|\frac{\partial^2 F}{\partial\delta\partial\gamma}(\rho,d)\right|^q \frac{251^2}{2^{14}\cdot 3^4}\Bigg)^{\frac{1}{q}},$$

where $\Upsilon(\sigma,\rho;\varsigma,d)$ *is stated as in* (37).

5. Fractional Newton Inequality Based on Functions of Two Variables with Bounded Variation

In this section, with the aid of Riemann–Liouville fractional integrals, we will give Newton-type inequality via the mapping of two variables with bounded variation.

Theorem 5. *If* $F:\Delta\to\mathbb{R}$ *is a mapping of bounded variation on* Δ*, then we obtain the following inequality*

$$\left|\Theta^{\alpha,\beta}(\sigma,\rho;\varsigma,d)\right|\leq\frac{25}{576}\bigvee_{\sigma}^{\rho}\bigvee_{\varsigma}^{d}(F),$$

where $\bigvee_{\sigma}^{\rho}\bigvee_{\varsigma}^{d}(F)$ *denotes the total variation of* F *on interval* $[\sigma,\rho]\times[\varsigma,d]$.

Proof. Describe the impressions $K_\alpha(\kappa)$ and $L_\beta(y)$ by

$$K_\alpha(\kappa)=\begin{cases}(\kappa-\sigma)^\alpha-\dfrac{(\rho-\sigma)^\alpha}{8\cdot 3^{\alpha-1}}, & \text{for } \sigma\leq\kappa\leq\frac{2\sigma+\rho}{3};\\ \left(\kappa-\dfrac{2\sigma+\rho}{3}\right)^\alpha-\dfrac{(\rho-\sigma)^\alpha}{2\cdot 3^\alpha}, & \text{for } \frac{2\sigma+\rho}{3}<\kappa\leq\frac{\sigma+2\rho}{3};\\ \left(\kappa-\dfrac{\sigma+2\rho}{3}\right)^\alpha-\dfrac{5(\rho-\sigma)^\alpha}{8\cdot 3^\alpha} & \text{for } \frac{\sigma+2\rho}{3}<\kappa\leq\rho\end{cases}$$

and

$$L_\beta(y)=\begin{cases}(y-\varsigma)^\beta-\frac{(d-\varsigma)^\alpha}{8\cdot 3^{\alpha-1}}, & \text{for } \varsigma\leq\kappa\leq\frac{2\varsigma+d}{3};\\ \left(y-\frac{2\varsigma+d}{3}\right)^\alpha-\frac{(d-\varsigma)^\alpha}{2\cdot 3^\alpha} & \text{for } \frac{2\varsigma+d}{3}<y\leq\frac{\varsigma+2d}{3};\\ \left(y-\frac{\varsigma+2d}{3}\right)^\alpha-\frac{5(d-\varsigma)^\alpha}{8\cdot 3^\alpha} & \text{for } \frac{\varsigma+2d}{3}<y\leq d,\end{cases}$$

respectively. By Lemma 1, one can easily see that

$$\Theta^{\alpha,\beta}(\sigma,\rho;\varsigma,d)=\frac{3^{\alpha+\beta-2}}{(\rho-\sigma)^\alpha(d-\varsigma)^\beta}\int_{\sigma}^{\rho}\int_{\varsigma}^{d}K_\alpha(\kappa)L_\beta(y)d_\kappa d_y F(\kappa,y), \tag{39}$$

where $\Theta^{\alpha,\beta}(\sigma,\rho;\varsigma,d)$ is stated as in (6). Taking the modulus of (39) and with the aid of Lemma 2, we derive:

$$\begin{aligned}
&\left|\Theta^{\alpha,\beta}(\sigma,\rho;\varsigma,d)\right| \\
&= \frac{3^{\alpha+\beta-2}}{(\rho-\sigma)^{\alpha}(d-\varsigma)^{\beta}}\left|\int_{\sigma}^{\rho}\int_{\varsigma}^{d} K_{\alpha}(\kappa)L_{\beta}(y)d_{\kappa}d_{y}F(\kappa,y)\right| \\
&\leq \frac{3^{\alpha+\beta-2}}{(\rho-\sigma)^{\alpha}(d-\varsigma)^{\beta}} \sup_{(\kappa,y)\in\Delta}\left|K_{\alpha}(\kappa)L_{\beta}(y)\right| \bigvee_{\sigma}^{\rho}\bigvee_{\varsigma}^{d}(F) \\
&= \frac{3^{\alpha+\beta-2}}{(\rho-\sigma)^{\alpha}(d-\varsigma)^{\beta}} \sup_{\kappa\in[\sigma,\rho]}\left|K_{\alpha}(\kappa)\right| \sup_{y\in[\varsigma,d]}\left|L_{\beta}(y)\right| \bigvee_{\sigma}^{\rho}\bigvee_{\varsigma}^{d}(F) \\
&= \frac{3^{\alpha+\beta-2}}{(\rho-\sigma)^{\alpha}(d-\varsigma)^{\beta}} \max\left\{ \sup_{\kappa\in\left[\sigma,\frac{2\sigma+\rho}{3}\right]}\left|(\kappa-\sigma)^{\alpha}-\frac{(\rho-\sigma)^{\alpha}}{8\cdot 3^{\alpha-1}}\right|,\right. \\
&\quad \left. \sup_{\kappa\in\left[\frac{2\sigma+\rho}{3},\frac{\sigma+2\rho}{3}\right]}\left|\left(\kappa-\frac{2\sigma+\rho}{3}\right)^{\alpha}-\frac{(\rho-\sigma)^{\alpha}}{2\cdot 3^{\alpha}}\right|, \sup_{\kappa\in\left[\frac{\sigma+2\rho}{3},\rho\right]}\left|\left(\kappa-\frac{\sigma+2\rho}{3}\right)^{\alpha}-\frac{5(\rho-\sigma)^{\alpha}}{8\cdot 3^{\alpha}}\right|\right\} \\
&\quad \times\max\left\{ \sup_{y\in\left[\varsigma,\frac{2\varsigma+d}{3}\right]}\left|(y-\varsigma)^{\beta}-\frac{(d-\varsigma)^{\alpha}}{8\cdot 3^{\alpha-1}}\right|, \sup_{y\in\left[\frac{2\varsigma+d}{3},\frac{\varsigma+2d}{3}\right]}\left|\left(y-\frac{2\varsigma+d}{3}\right)^{\alpha}-\frac{(d-\varsigma)^{\alpha}}{2\cdot 3^{\alpha}}\right|,\right. \\
&\quad \left. \sup_{y\in\left[\frac{\varsigma+2d}{3},d\right]}\left|\left(y-\frac{\varsigma+2d}{3}\right)^{\alpha}-\frac{5(d-\varsigma)^{\alpha}}{8\cdot 3^{\alpha}}\right|\right\}\bigvee_{\sigma}^{\rho}\bigvee_{\varsigma}^{d}(F) \\
&= \frac{3^{\alpha+\beta-2}}{(\rho-\sigma)^{\alpha}(d-\varsigma)^{\beta}} \max\left\{\frac{5(\rho-\sigma)^{\alpha}}{8\cdot 3^{\alpha}},\frac{(\rho-\sigma)^{\alpha}}{2\cdot 3^{\alpha}}\right\} \\
&\quad \times\max\left\{\frac{5(d-\varsigma)^{\alpha}}{8\cdot 3^{\alpha}},\frac{(d-\varsigma)^{\alpha}}{2\cdot 3^{\alpha}}\right\}\bigvee_{\sigma}^{\rho}\bigvee_{\varsigma}^{d}(F) = \frac{25}{576}\bigvee_{\sigma}^{\rho}\bigvee_{\varsigma}^{d}(F).
\end{aligned}$$

This completes the proof. □

Corollary 3. *If we take $\alpha=1$ and $\rho=1$ in Theorem 5, and we possess*

$$|\Upsilon(\sigma,\rho;\varsigma,d)| \leq \frac{25}{576}\bigvee_{\sigma}^{\rho}\bigvee_{\varsigma}^{d}(F),$$

where $\Upsilon(\sigma,\rho;\varsigma,d)$ is expressed as in (37).

6. Conclusions

In this presented paper, we proved Simpson's second rule formula type inequalities via Riemann–Liouville fractional integrals for differentiable coordinated convex mappings. Moreover, fractional Simpson's 3/8 rule inequalities were obtained via bounded variation functions. The results for symmetric functions can be reached by employing the notions of symmetric convex functions, which will be explored further in future work. Curious readers can investigate new inequalities via inequalities of Newton type utilizing other kinds via fractional integrals. Different types of convexity of these resulting inequalities can be researched in the future.

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