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Preserving Classes of Meromorphic Functions through Integral Operators

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Abstract: We consider three new classes of meromorphic functions defined by an extension of the Wanas operator and two integral operators, in order to study some preservation properties of the classes. The purpose of the paper is to find the conditions such that, when we apply the integral operator $J_{p,\gamma}$ to some function from the new defined classes $\Sigma S^n_{p,q}(\alpha,\delta)$, respectively $\Sigma S^n_{p,q}(\alpha)$, we obtain also a function from the same class. We also define a new integral operator on the class of meromorphic functions, denoted by $J_{p,\gamma,h}$, where h is a normalized analytic function on the unit disc. We study some basic properties of this operator. Then we look for the conditions which allow this operator to preserve a particular subclass of the classes mentioned above.

Keywords: meromorphic functions; Wanas operator; integral operators

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1. Introduction and Preliminaries

Many operators have been used since the beginning of the study of analytic functions. The most interesting of these are the differential and integral operators. Since the beginning of the 20th century, many mathematicians have worked on integral operators applied to classes of analytic functions, but papers on integral operators applied to classes of meromorphic functions are smaller in number. This is happening because there is a need of new integral operators on meromorphic functions.

The first author of the present paper started in 2010 to work on integral operators on meromorphic functions (see [1]). In the same period, new results regarding the same topic were published in papers such as [2–4] etc.

The literature on meromorphic functions is very large, but in the field of geometric theory of meromorphic functions there is still more to say. Recent results on this topic may be found in [5–9].

In this work we introduce a new integral operator on the class of meromorphic functions and we prove that it is well defined. We also introduce new classes of meromorphic functions, with the use of the Wanas operator, and we study some preserving properties of these classes.

Using the integral operator introduced in this paper, beautiful results can be obtained in terms of class conservation.

We consider $U = \{z \in \mathbb{C} : |z| < 1\}$, the unit disc, $\dot{U} = U \setminus \{0\}$ and

$$H(U) = \{f : U \to \mathbb{C} : f \text{ is holomorphic in } U\}.$$

For $p \in \mathbb{N}^*$, we have $\Sigma_p = \left\{g/g(z) = \frac{a_{-p}}{z^p} + a_0 + a_1 z + \cdots, z \in \dot{U}, a_{-p} \neq 0\right\}$, the class of meromorphic functions in U.

We also use:
$\Sigma_p^*(\alpha) = \left\{ g \in \Sigma_p : \text{Re}\left[-\frac{zg'(z)}{g(z)} \right] > \alpha,\ z \in U \right\}$, where $\alpha < p$.
$\Sigma_1^*(\alpha)$ is the class of meromorphic starlike functions of order α, where $0 \le \alpha < 1$.
$\Sigma_p^*(\alpha,\delta) = \left\{ g \in \Sigma_p : \alpha < \text{Re}\left[-\frac{zg'(z)}{g(z)} \right] < \delta,\ z \in U \right\}$, where $\alpha < p < \delta$,
$H[a,n] = \{ f \in H(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots \}$ for $a \in \mathbb{C},\ n \in \mathbb{N}^*$.

Corollary 1 ([1]). *Let $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ and $\alpha < p < \delta \le \text{Re}\,\gamma$. If $g \in \Sigma_p^*(\alpha,\delta)$, then*

$$G = J_{p,\gamma}(g) \in \Sigma_p^*(\alpha,\delta).$$

where $J_{p,\gamma}(g)(z) = \frac{\gamma - p}{z^\gamma} \int_0^z g(t) t^{\gamma-1} dt.$

Corollary 2 ([1]). *Let $p \in \mathbb{N}^*$, $\beta > 0, \gamma \in \mathbb{C}$ and $\alpha < p < \frac{\text{Re}\,\gamma}{\beta}$.*
If $g \in \Sigma_p^*(\alpha)$, with

$$\beta \frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma - p\beta, p}(z),\ z \in U,$$

then $G = J_{p,\beta,\gamma}(g) \in \Sigma_p^*(\alpha)$.

Corollary 3 ([1]). *Let $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ and $\alpha < p < \text{Re}\,\gamma \le \delta$.*
If $g \in \Sigma_p^*(\alpha,\delta)$, with

$$\frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma - p, p}(z),\ z \in U,$$

then $G = J_{p,\gamma}(g) \in \Sigma_p^*(\alpha,\delta)$.

Lemma 1 ([1]). *Let $n \in \mathbb{N}^*$, $\alpha, \beta \in \mathbb{R}$, $\gamma \in \mathbb{C}$ with $\text{Re}\,[\gamma - \alpha\beta] \ge 0$. If we have $P \in H[P(0),n]$ with $P(0) \in \mathbb{R}$ and $P(0) > \alpha$, then*

$$\text{Re}\left[P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} \right] > \alpha \Rightarrow \text{Re}\,P(z) > \alpha,\ z \in U.$$

Theorem 1 ([1]). *Let $p \in \mathbb{N}^*$, $\Phi, \varphi \in H[1,p]$ with $\Phi(z)\varphi(z) \ne 0$, $z \in U$. Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\beta \ne 0$, $\delta + p\beta = \gamma + p\alpha$ and $\text{Re}\,(\gamma - p\beta) > 0$. Let $g \in \Sigma_p$ and suppose that*

$$\alpha \frac{zg'(z)}{g(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta \prec R_{\delta - p\alpha, p}(z),\ z \in U.$$

If $G = J_{p,\alpha,\beta,\gamma,\delta}^{\Phi,\varphi}(g)$ is

$$G(z) = J_{p,\alpha,\beta,\gamma,\delta}^{\Phi,\varphi}(g)(z) = \left[\frac{\gamma - p\beta}{z^\gamma \Phi(z)} \int_0^z g^\alpha(t) \varphi(t) t^{\delta-1} dt \right]^{\frac{1}{\beta}}, \tag{1}$$

then $G \in \Sigma_p$ with $z^p G(z) \ne 0$, $z \in U$, and

$$\text{Re}\left[\beta \frac{zG'(z)}{G(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma \right] > 0,\ z \in U.$$

All powers in (1) are principal ones.

Theorem 2 ([10], ([11], p. 209)). *Let be $p \in H[a,n]$ with $\text{Re}\,a > 0$. If $\psi \in \Psi_n\{a\}$, then*

$$\text{Re}\,\psi(p(z), zp'(z), z^2 p''(z); z) > 0,\ z \in U \Rightarrow \text{Re}\,p(z) > 0,\ z \in U.$$

For $0 < q < p \leq 1$, the (p,q)-derivative operator for a function f is defined by

$$D_{p,q}f(z) = \frac{f(pz) - f(qz)}{(p-q)z} \qquad (z \in \dot{U} = U \setminus \{0\}), \tag{2}$$

and

$$D_{p,q}f(0) = f'(0),$$

see [12,13].

For an analytic function f we have

$$D_{p,q}f(z) = 1 + \sum_{k=2}^{\infty} [k]_{p,q} b_k z^{k-1}, \tag{3}$$

where the (p,q)-bracket number, or twin-basic $[k]_{p,q}$, is given by

$$[k]_{p,q} = \frac{p^k - q^k}{p-q} = p^{k-1} + p^{k-2}q + p^{k-3}q^2 + \cdots + pq^{k-2} + q^{k-1} \quad (p \neq q), \tag{4}$$

which is a natural generalization of the q-number, and we have

$$\lim_{p \to 1^-} [k]_{p,q} = [k]_q = \frac{1-q^k}{1-q}.$$

For more details on the concepts of (p,q)-calculus, or q calculus (in the case when $p = 1$), see [12,13].

There are many interesting works in which the operator $D_{p,q}$ is used; see [14–25].

Inspired by the Wanas operator for analytic functions (see [26–34]), we build an extension of it on the class of meromorphic functions.

For $p \in \mathbb{N}^*$, $n \in \mathbb{N}$ and $0 < q < 1$ we consider the extension of the Wanas operator for meromorphic functions, denoted by $W_q^n : \Sigma_p \to \Sigma_p$, as

$$W_q^n(g)(z) = \frac{a_{-p}}{z^p} + \sum_{k=0}^{\infty} \left(\frac{1-q^{k+1}}{1-q} \right)^n a_k z^k, \ z \in \dot{U},$$

where $g \in \Sigma_p$ is $g(z) = \dfrac{a_{-p}}{z^p} + \sum_{k=0}^{\infty} a_k z^k$, with $z \in \dot{U}$, $a_{-p} \neq 0$.

We have the properties:

(1) $W_q^0 g(z) = g(z)$;

(2) $W_q^1 g(z) = \dfrac{a_{-p}}{z^p} + \sum_{k=0}^{\infty} [k+1]_q a_k z^k$;

(3) $W_q^n(\alpha g_1 + \beta g_2)(z) = \alpha W_q^n(g_1)(z) + \beta W_q^n(g_2)(z)$, $\alpha, \beta \in \mathbb{C}$, $g_1, g_2 \in \Sigma_p$;

(4) $W_q^n\left(W_q^m(g)(z)\right) = W_q^m\left(W_q^n(g)(z)\right) = W_q^{n+m}(g)(z)$;

(5) $W_q^n(zg'(z)) = z \cdot \left(W_q^n g(z)\right)'$.

2. Main Results

Definition 1. *For $p \in \mathbb{N}^*$, $n \in \mathbb{N}$, $0 < q < 1$ and $\alpha < p < \delta$ let*

$$\Sigma S_{p,q}^n(\alpha,\delta) = \left\{ g \in \Sigma_p / \alpha < \operatorname{Re}\left[-\frac{z\left(W_q^n g(z)\right)'}{W_q^n g(z)} \right] < \delta, \ z \in U. \right\}, \tag{5}$$

$$\Sigma S^n_{p,q}(\alpha) = \left\{ g \in \Sigma_p / \mathrm{Re}\left[-\frac{z\left(W^n_q g(z)\right)'}{W^n_q g(z)} \right] > \alpha,\, z \in U. \right\}, \tag{6}$$

and $\Sigma S^n_{p,q} = \Sigma S^n_{p,q}(0)$.

It is easy to see that, for $n = 0$, the class $\Sigma S^0_{p,q}(\alpha, \delta)$ is the class $\Sigma^*_p(\alpha, \delta)$ and the class $\Sigma S^0_{p,q}(\alpha)$ is the class $\Sigma^*_p(\alpha)$, which were studied in [1].

Next, we give the link between the sets $\Sigma S^n_{p,q}(\alpha, \delta)$ and $\Sigma S^{n-1}_{p,q}(\alpha, \delta)$, respectively $\Sigma S^n_{p,q}(\alpha)$ and $\Sigma S^{n-1}_{p,q}(\alpha)$.

Remark 1. *Let p, $n \in \mathbb{N}$, $p \neq 0$, $0 < q < 1$, $\alpha < p < \delta$ and $g \in \Sigma_p$. Then*

$$g \in \Sigma S^n_{p,q}(\alpha, \delta) \Leftrightarrow W_q(g) \in \Sigma S^{n-1}_{p,q}(\alpha, \delta),$$

respectively

$$g \in \Sigma S^n_{p,q}(\alpha) \Leftrightarrow W_q(g) \in \Sigma S^{n-1}_{p,q}(\alpha).$$

Proof. We have $g \in \Sigma S^n_{p,q}(\alpha, \delta)$ equivalent to $W^n_q g \in \Sigma^*_p(\alpha, \delta)$.
Since $W^n_q g = W^{n-1}_q(W^1_q g)$, we get $W^{n-1}_q(W^1_q g) \in \Sigma^*_p(\alpha, \delta)$, which is equivalent to

$$W_q(g) \in \Sigma S^{n-1}_{p,q}(\alpha, \delta).$$

The second equivalence can be proved in the same way. □

Theorem 3. *Let p, $n \in \mathbb{N}$, with $p \neq 0$, and $0 < q < 1$. We consider also α, $\delta \in \mathbb{R}$ and $\gamma \in \mathbb{C}$ satisfying $\alpha < p < \delta \leq \mathrm{Re}\,\gamma$. If $g \in \Sigma S^n_{p,q}(\alpha, \delta)$, then $J_{p,\gamma}(g) \in \Sigma S^n_{p,q}(\alpha, \delta)$, where $J_{p,\gamma}(g)(z) = \frac{\gamma - p}{z^\gamma} \int_0^z g(t) t^{\gamma-1} dt$.*

Proof. Because $g \in \Sigma S^n_{p,q}(\alpha, \delta)$ we have $W^n_q(g) \in \Sigma^*_p(\alpha, \delta)$, hence, from Corollary 1, we get

$$J_{p,\gamma}(W^n_q(g)) \in \Sigma^*_p(\alpha, \delta).$$

We will prove now that we have

$$J_{p,\gamma}(W^n_q(g)) = W^n_q(J_{p,\gamma}(g)).$$

We have

$$W^n_q(g)(z) = \frac{a_{-p}}{z^p} + \sum_{k=0}^{\infty} \left(\frac{1 - q^{k+1}}{1 - q} \right)^n a_k z^k,\ z \in \dot{U},$$

where $g \in \Sigma_p$ is $g(z) = \frac{a_{-p}}{z^p} + \sum_{k=0}^{\infty} a_k z^k$, $z \in \dot{U}$, $a_{-p} \neq 0$.

It is well known that the operator

$$J_{p,\lambda}(g)(z) = \frac{\lambda - p}{z^\lambda} \int_0^z t^{\lambda-1} g(t) dt$$

can be also written as

$$J_{p,\lambda}(g)(z) = \frac{a_{-p}}{z^p} + \sum_{k=0}^{\infty} \frac{\lambda - p}{k + \lambda} a_k z^k, \text{ where } g(z) = \frac{a_{-p}}{z^p} + \sum_{k=0}^{\infty} a_k z^k.$$

Therefore, we have

$$J_{p,\gamma}(W_q^n(g)(z)) = \frac{a_{-p}}{z^p} + \sum_{k=0}^{\infty} \frac{\lambda - p}{k + \lambda} \left(\frac{1 - q^{k+1}}{1 - q} \right)^n a_k z^k,$$

and

$$W_q^n(J_{p,\gamma}(g)(z)) = \frac{a_{-p}}{z^p} + \sum_{k=0}^{\infty} \left(\frac{1 - q^{k+1}}{1 - q} \right)^n \frac{\lambda - p}{k + \lambda} a_k z^k,$$

this meaning that

$$J_{p,\gamma}(W_q^n(g)) = W_q^n(J_{p,\gamma}(g)).$$

We get that

$$W_q^n(J_{p,\gamma}(g)) \in \Sigma_p^*(\alpha, \delta),$$

therefore $J_{p,\gamma}(g) \in \Sigma S_{p,q}^n(\alpha, \delta)$.
□

If we consider, in the above theorem, the case that $n = 0$ we obtain:

Corollary 4. *Let $p \in \mathbb{N}^*$, $0 < q < 1$, $\gamma \in \mathbb{C}$ and $\alpha < p < \delta \leq \operatorname{Re}\gamma$. Then*

$$g \in \Sigma_p^*(\alpha, \delta) \Rightarrow J_{p,\gamma}(g) \in \Sigma_p^*(\alpha, \delta).$$

Proof. The proof is obvious since we have $\Sigma S_{p,q}^0(\alpha, \delta) = \Sigma_p^*(\alpha, \delta)$. □

The result of Corollary 4 was also found in [1].

Theorem 4. *Let $p, n \in \mathbb{N}$ with $p \neq 0$ and $0 < q < 1$, $\gamma \in \mathbb{C}$ with $\alpha < p < \operatorname{Re}\gamma$. If $g \in \Sigma S_{p,q}^n(\alpha)$, with*

$$\frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma-p,p}(z),\ z \in U,$$

then $J_{p,\gamma}(g) \in \Sigma S_{p,q}^n(\alpha)$.

Proof. We omit the proof since it is similar to the proof of Theorem 3, except that we now use, instead of Corollary 1, Corollary 2 with $\beta = 1$. □

Proposition 1. *Let p, $n \in \mathbb{N}$, with $p \neq 0$, and $0 < q < 1$. We consider also α, $\delta \in \mathbb{R}$ and $\gamma \in \mathbb{C}$ satisfying $\alpha < p < \operatorname{Re}\gamma \leq \delta$. If the function $g \in \Sigma S_{p,q}^n(\alpha, \delta)$ satisfies the condition*

$$\frac{z(W_q^n g)'(z)}{W_q^n g(z)} + \gamma \prec R_{\gamma-p,p}(z),\ z \in U,$$

then $J_{p,\gamma}(g) \in \Sigma S_{p,q}^n(\alpha, \delta)$.

Proof. We have $g \in \Sigma S_{p,q}^n(\alpha, \delta) \Leftrightarrow W_q^n(g) \in \Sigma_p^*(\alpha, \delta)$, hence, from Corollary 3, we obtain

$$J_{p,\gamma}(W_q^n(g)) \in \Sigma_p^*(\alpha, \delta).$$

Since

$$J_{p,\gamma}(W_q^n(g)) = W_q^n(J_{p,\gamma}(g)),$$

we obtain that

$$W_q^n(J_{p,\gamma}(g)) \in \Sigma_p^*(\alpha, \delta),$$

which is equivalent to $J_{p,\gamma}(g) \in \Sigma S_{p,q}^n(\alpha, \delta)$. □

If we consider $\delta \to \infty$ in Proposition 1 we get:

Corollary 5. *Let $n \in \mathbb{N}$, $p \in \mathbb{N}^*$, $0 < q < 1$, $\gamma \in \mathbb{C}$ and $\alpha < p < \operatorname{Re}\gamma$. If $g \in \Sigma S^n_{p,q}(\alpha)$ and satisfies the condition*

$$\frac{z(W^n_q g)'(z)}{W^n_q g(z)} + \gamma \prec R_{\gamma-p,p}(z),\ z \in U,$$

then $J_{p,\gamma}(g) \in \Sigma S^n_{p,q}(\alpha)$.

Next we define the operator $J_{p,\gamma,h}$. Let $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$ and $h \in A$. We define

$$J_{p,\gamma,h} : \Sigma_p \to \Sigma_p,\ J_{p,\gamma,h}(g) = \frac{\gamma - p}{h^\gamma(z)} \int_0^z g(t) h^{\gamma-1}(t) h'(t) dt. \tag{7}$$

It is easy to see that for the $h(z) = z$ we have $J_{p,\gamma,h} = J_{p,\gamma}$, where

$$J_{p,\gamma}(g)(z) = \frac{\gamma - p}{z^\gamma} \int_0^z g(t) t^{\gamma-1} dt,$$

found in [1], was used in different papers.

Theorem 5. *Let $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$ and $h \in A$ with $\frac{h(z)}{z} \cdot h'(z) \neq 0$. Let $g \in \Sigma_p$ with*

$$\frac{zg'(z)}{g(z)} + \frac{zh''(z)}{h'(z)} + (\gamma - 1)\frac{zh'(z)}{h(z)} + 1 \prec R_{\gamma-p,p}(z),\ z \in U.$$

If $G = J_{p,\gamma,h}(g)$ is defined by (7), then $G \in \Sigma_p$ with $z^p G(z) \neq 0$, $z \in U$, and

$$\operatorname{Re}\left[\frac{zG'(z)}{G(z)} + \gamma \frac{zh'(z)}{h(z)}\right] > 0,\ z \in U.$$

All powers in (7) are principal ones.

Proof. We consider Theorem 1 with

$$\alpha = \beta = 1,\ \delta = \gamma,\ \Phi(z) = \left(\frac{h(z)}{z}\right)^\gamma,\ \varphi(z) = \left(\frac{h(z)}{z}\right)^{\gamma-1} \cdot h'(z).$$

Using the above notations we show that the subordination

$$\alpha \frac{zg'(z)}{g(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta \prec R_{\delta - p\alpha, p}(z),\ z \in U,$$

is equivalent to

$$\frac{zg'(z)}{g(z)} + \frac{zh''(z)}{h'(z)} + (\gamma - 1)\frac{zh'(z)}{h(z)} + 1 \prec R_{\gamma-p,p}(z),\ z \in U.$$

From

$$\varphi(z) = \left(\frac{h(z)}{z}\right)^{\gamma-1} \cdot h'(z),$$

by using the logarithmic differential, we get

$$\frac{\varphi'(z)}{\varphi(z)} = (\gamma - 1)\frac{h'(z)}{h(z)} - (\gamma - 1)\frac{1}{z} + \frac{h''(z)}{h'(z)},$$

thus

$$\frac{z\varphi'(z)}{\varphi(z)} = (\gamma - 1)\frac{zh'(z)}{h(z)} - \gamma + 1 + \frac{zh''(z)}{h'(z)}. \tag{8}$$

We have now

$$\alpha\frac{zg'(z)}{g(z)}+\frac{z\varphi'(z)}{\varphi(z)}+\delta=\frac{zg'(z)}{g(z)}+(\gamma-1)\frac{zh'(z)}{h(z)}-\gamma+1+\frac{zh''(z)}{h'(z)}+\gamma$$

$$=\frac{zg'(z)}{g(z)}+\frac{zh''(z)}{h'(z)}+(\gamma-1)\frac{zh'(z)}{h(z)}+1.$$

Therefore, the subordination from the hypothesis of Theorem 1 is satisfied.

Since all the other conditions from the hypothesis of Theorem 1 are met, we get from Theorem 1 that

$$G(z)=J^{\Phi,\varphi}_{p,1,1,\gamma,\gamma}(g)(z)=\frac{\gamma-p}{z^{\gamma}\Phi(z)}\int_0^z g(t)\varphi(t)t^{\gamma-1}dt$$

$$=\frac{\gamma-p}{h^{\gamma}(z)}\int_0^z g(t)h^{\gamma-1}(t)h'(t)dt=J_{p,\gamma,h}(g)(z),$$

belongs to the class Σ_p with $z^pG(z)\neq 0$, $z\in U$, and

$$\text{Re}\left[\frac{zG'(z)}{G(z)}+\frac{z\Phi'(z)}{\Phi(z)}+\gamma\right]>0,\ z\in U.$$

Taking into account the fact that we have $\Phi(z)=\left(\frac{h(z)}{z}\right)^{\gamma}$, by using the logarithmic differential we get

$$\frac{\Phi'(z)}{\Phi(z)}=\gamma\frac{h'(z)}{h(z)}-\gamma\frac{1}{z},$$

so

$$\frac{z\Phi'(z)}{\Phi(z)}=\gamma\frac{zh'(z)}{h(z)}-\gamma,$$

this meaning that the inequality

$$\text{Re}\left[\frac{zG'(z)}{G(z)}+\frac{z\Phi'(z)}{\Phi(z)}+\gamma\right]>0,\ z\in U,$$

is equivalent to the inequality

$$\text{Re}\left[\frac{zG'(z)}{G(z)}+\gamma\frac{zh'(z)}{h(z)}\right]>0,\ z\in U.$$

Therefore, the proof of the theorem is complete. □

If we consider in Theorem 5 that $h(z)=z$, since the requirements on h are satisfied, we get:

Corollary 6. *Let $p\in\mathbb{N}^*$, $\gamma\in\mathbb{C}$ with $\text{Re}\,\gamma>p$. Let $g\in\Sigma_p$ with*

$$\frac{zg'(z)}{g(z)}+\gamma\prec R_{\gamma-p,p}(z),\ z\in U.$$

Then $G=J_{p,\gamma}(g)\in\Sigma_p$ with $z^pG(z)\neq 0$, $z\in U$, and

$$\text{Re}\left[\frac{zG'(z)}{G(z)}+\gamma\right]>0,\ z\in U.$$

The above corollary is a particular case of Corollary 2 from [1] (considering $\beta=1$).

Proposition 2. *Let $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$ and $h \in A$ with $\frac{h(z)}{z} \cdot h'(z) \neq 0$. We denote by H the function $H(z) = \frac{h(z)}{h'(z)}$. Let $g \in \Sigma_p$ and $G = J_{p,\gamma,h}(g)$. Then we have the equality*

$$-\frac{zg'(z)}{g(z)} = \alpha(z)p(z) + \frac{zp'(z)}{R(z)},$$

where

$$p(z) = -\frac{zG'(z)}{G(z)},\ \alpha(z) = \frac{z\gamma + zH'(z) - H(z)(1+p(z))}{z\gamma - H(z)p(z)},\ R(z) = \frac{z\gamma - H(z)p(z)}{H(z)}.$$

Proof. From $G(z) = J_{p,\gamma,h}(g)(z) = \frac{\gamma - p}{h^\gamma(z)} \int_0^z g(t)h^{\gamma-1}(t)h'(t)dt$ we have

$$\gamma h^{\gamma-1}h'G + h^\gamma G' = (\gamma - p)gh^{\gamma-1}h' \Leftrightarrow \gamma G + \frac{h}{h'}G' = (\gamma - p)g,$$

thus

$$\gamma G(z) + H(z)G'(z) = (\gamma - p)g(z),\ z \in U. \tag{9}$$

From (9), after differentiating, we obtain

$$\gamma G'(z) + H'(z)G'(z) + H(z)G''(z) = (\gamma - p)g'(z),\ z \in U. \tag{10}$$

We use the notation $p(z) = -\frac{zG'(z)}{G(z)}$ and we get:

$$zG'(z) = -p(z)G(z),\ z^2G''(z) = \left[p(z) + p^2(z) - zp'(z)\right]G(z). \tag{11}$$

From (9) and (10) we obtain

$$\frac{g'(z)}{g(z)} = \frac{\gamma G'(z) + H'(z)G'(z) + H(z)G''(z)}{\gamma G(z) + H(z)G'(z)},$$

so

$$-\frac{zg'(z)}{g(z)} = -\frac{\gamma zG'(z) + zH'(z)G'(z) + zH(z)G''(z)}{\gamma G(z) + H(z)G'(z)}.$$

In the last equality we replace G' and G'', from (11), and we get:

$$-\frac{zg'(z)}{g(z)} = \frac{\gamma p(z)G(z) + p(z)H'(z)G(z) - \frac{H(z)}{z}\left[p(z) + p^2(z) - zp'(z)\right]G(z)}{\gamma G(z) - \frac{H(z)}{z}p(z)G(z)}$$

$$= \frac{\gamma zp(z) + zp(z)H'(z) - H(z)\left[p(z) + p^2(z) - zp'(z)\right]}{\gamma z - H(z)p(z)}$$

$$= p(z) \cdot \frac{z\gamma + zH'(z) - H(z)(1+p(z))}{z\gamma - H(z)p(z)} + zp'(z) \cdot \frac{H(z)}{z\gamma - H(z)p(z)}.$$

Using now the notations from the hypothesis we obtain that

$$-\frac{zg'(z)}{g(z)} = \alpha(z)p(z) + \frac{zp'(z)}{R(z)},$$

where

$$\alpha(z) = \frac{z\gamma + zH'(z) - H(z)(1+p(z))}{z\gamma - H(z)p(z)},\ R(z) = \frac{z\gamma - H(z)p(z)}{H(z)}.$$

□

For the next results we need the following lemma:

Lemma 2. *Let $n \in \mathbb{N}^*$ and the functions $\alpha : U \to \mathbb{R}$, $R : U \to \mathbb{C}$ with $\operatorname{Re} R(z) > 0$, $z \in U$. If $p \in H[a, n]$ with $\operatorname{Re} a > 0$, then*

$$\operatorname{Re}\left[\alpha(z)p(z) + \frac{zp'(z)}{R(z)}\right] > 0 \Rightarrow \operatorname{Re} p(z) > 0,\, z \in U.$$

Proof. To prove this result we use the class of admissible functions. We consider the function $\psi(r, s, t; z) = \alpha(z)r + \dfrac{s}{R(z)}$ and the set $\Omega = \{w \in \mathbb{C} : \operatorname{Re} w > 0\}$.

We need to show that $\operatorname{Re} \psi(\rho i, \sigma, \mu + i\nu; z) \notin \Omega$, when $\rho, \sigma, \mu, \nu \in \mathbb{R}$, $z \in U$, with

$$\sigma \leq -\frac{n}{2} \cdot \frac{|a - i\rho|^2}{\operatorname{Re} a},\, \sigma + \mu \leq 0,$$

this meaning that we have $\psi \in \Psi_n\{a\}$.

We have

$$\operatorname{Re} \psi(\rho i, \sigma, \mu + i\nu; z) = \operatorname{Re}\left[\alpha(z)\rho i + \frac{\sigma}{R(z)}\right] = \operatorname{Re} \frac{\sigma}{R(z)} < 0,$$

since $\sigma < 0$ and $\operatorname{Re} R(z) > 0$.

From Theorem 2, since $\psi \in \Psi_n\{a\}$ and $\operatorname{Re} \psi(p(z), zp'(z), z^2p''(z); z) > 0$, for $z \in U$, we get $\operatorname{Re} p(z) > 0$. □

Theorem 6. *Let $n \in \mathbb{N}$, $p \in \mathbb{N}^*$, $0 < q < 1$, $\gamma \in \mathbb{C}$, $\operatorname{Re} \gamma > p$ and $h \in A$ with $\dfrac{h(z)}{z} \cdot h'(z) \neq 0$. We denote by H the function $H(z) = \dfrac{h(z)}{h'(z)}$. Let $g \in \Sigma S^n_{p,q}$ with*

$$\frac{zg'(z)}{g(z)} + \frac{zh''(z)}{h'(z)} + (\gamma - 1)\frac{zh'(z)}{h(z)} + 1 \prec R_{\gamma-p,p}(z),\, z \in U; \tag{12}$$

$$\frac{z(W^n_q g)'(z)}{W^n_q g(z)} + \frac{zh''(z)}{h'(z)} + (\gamma - 1)\frac{zh'(z)}{h(z)} + 1 \prec R_{\gamma-p,p}(z),\, z \in U. \tag{13}$$

If $G = J_{p,\gamma,h}(g)$ is defined by (8) and verifies

$$\frac{(zH'(z) - H(z))W^n_q G(z)}{z\left[\gamma W^n_q G(z) + H(z) W^n_q G'(z)\right]} \in \mathbb{R},\, z \in U, \tag{14}$$

$$J_{p,\gamma,h}(W^n_q g) = W^n_q (G) \tag{15}$$

then $G \in \Sigma S^n_{p,q}$ with $z^p G(z) \neq 0$, $z \in U$, $z^p W^n_q G(z) \neq 0$, $z \in U$,

$$\operatorname{Re}\left[\frac{zG'(z)}{G(z)} + \gamma\frac{zh'(z)}{h(z)}\right] > 0, \text{ and } \operatorname{Re}\left[\frac{z(W^n_q G)'(z)}{W^n_q G(z)} + \gamma\frac{zh'(z)}{h(z)}\right] > 0,\, z \in U.$$

Proof. We have $g \in \Sigma S^n_{p,q}$, so $g \in \Sigma_p$ with $W^n_q g \in \Sigma^*_p$. Since all the conditions from the hypothesis of Theorem 5 are met we have $G \in \Sigma_p$ with $z^p G(z) \neq 0$, $z \in U$, and

$$\operatorname{Re}\left[\frac{zG'(z)}{G(z)} + \gamma\frac{zh'(z)}{h(z)}\right] > 0,\, z \in U.$$

Let us denote $W_q^n g = g_1$. Since $W_q^n g \in \Sigma_p^* \subset \Sigma_p$ and satisfies (13) it follows from Theorem 5 that $G_1 = J_{p,\gamma,h}(g_1) \in \Sigma_p$ with

$$z^p G_1(z) \neq 0,\ z \in U,\ \text{and } \mathrm{Re}\left[\frac{zG_1'(z)}{G_1(z)} + \gamma\frac{zh'(z)}{h(z)}\right] > 0,\ z \in U. \tag{16}$$

From (15) we have $G_1 = J_{p,\gamma,h}(g_1) = J_{p,\gamma,h}(W_q^n g) = W_q^n G$, therefore (16) is the same with

$$z^p W_q^n G(z) \neq 0,\ z \in U,\ \text{and } \mathrm{Re}\left[\frac{z(W_q^n G)'(z)}{W_q^n G(z)} + \gamma\frac{zh'(z)}{h(z)}\right] > 0,\ z \in U.$$

We also have $g_1 \in \Sigma_p^*$, this meaning that $\mathrm{Re}\left[-\frac{zg_1'(z)}{g_1(z)}\right] > 0$.

Since $G_1 = J_{p,\gamma,h}(g_1)$ we get from Proposition 2 that

$$-\frac{zg_1'(z)}{g_1(z)} = \alpha(z)p(z) + \frac{zp'(z)}{R(z)},$$

where

$$p(z) = -\frac{zG_1'(z)}{G_1(z)},\ \alpha(z) = \frac{z\gamma + zH'(z) - H(z)(1+p(z))}{z\gamma - H(z)p(z)},\ R(z) = \frac{z\gamma - H(z)p(z)}{H(z)}.$$

We have $\mathrm{Re}\left[-\frac{zg_1'(z)}{g_1(z)}\right] > 0,\ z \in U$, therefore

$$\mathrm{Re}\left[\alpha(z)p(z) + \frac{zp'(z)}{R(z)}\right] > 0,\ z \in U.$$

Next, we prove that $\alpha(z) \in \mathbb{R},\ z \in U$.

We have

$$\alpha(z) = 1 + \frac{zH'(z) - H(z)}{z\gamma - H(z)p(z)} = 1 + \frac{(zH'(z) - H(z))G_1(z)}{z\left[\gamma G_1(z) + H(z)G_1'(z)\right]}$$

$$= 1 + \frac{(zH'(z) - H(z))W_q^n G(z)}{z\left[\gamma W_q^n G(z) + H(z)W_q^n G'(z)\right]} \in \mathbb{R},\ z \in U,$$

because, from (14), $\dfrac{(zH'(z) - H(z))W_q^n G(z)}{z\left[\gamma W_q^n G(z) + H(z)W_q^n G'(z)\right]} \in \mathbb{R},\ z \in U$.

On the other hand, since

$$R(z) = \frac{z\gamma - H(z)p(z)}{H(z)} = \frac{z\gamma}{H(z)} - p(z) = \gamma\frac{zh'(z)}{h(z)} + \frac{zG_1'(z)}{G_1(z)}$$

we obtain, from (16), $\mathrm{Re}\,R(z) > 0$.

We have the functions $\alpha : U \to \mathbb{R}$, $R : U \to \mathbb{C}$ with $\mathrm{Re}\,R(z) > 0, z \in U$ and $p \in H[p,n]$. Therefore, since

$$\mathrm{Re}\left[\alpha(z)p(z) + \frac{zp'(z)}{R(z)}\right] > 0$$

we get from Lemma 2 that $\mathrm{Re}\,p(z) > 0,\ z \in U$.

Thus, $\mathrm{Re}\left[-\frac{zG_1'(z)}{G_1(z)}\right] > 0,\ z \in U$, this meaning that $G_1 = W_q^n G \in \Sigma_p^*$, which is equivalent to $G \in \Sigma S_{p,q}^n$. □

Taking $n = 0$ in Theorem 6, since $\Sigma S^0_{p,q} = \Sigma^*_p$, $W^0_q g = g$, we get the next result:

Corollary 7. *Let $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$ and $h \in A$ with $\frac{h(z)}{z} \cdot h'(z) \neq 0$. We denote by H the function $H(z) = \frac{h(z)}{h'(z)}$. Let $g \in \Sigma^*_p$ with*

$$\frac{zg'(z)}{g(z)} + \frac{zh''(z)}{h'(z)} + (\gamma - 1)\frac{zh'(z)}{h(z)} + 1 \prec R_{\gamma - p,p}(z),\ z \in U.$$

If $G = J_{p,\gamma,h}(g)$ is defined by (8) and verifies $\frac{(zH'(z) - H(z))G(z)}{z[\gamma G(z) + H(z)G'(z)]} \in \mathbb{R}$, $z \in U$, then $G \in \Sigma^*_p$ with $z^p G(z) \neq 0$, $z \in U$, and

$$\operatorname{Re}\left[\frac{zG'(z)}{G(z)} + \gamma\frac{zh'(z)}{h(z)}\right] > 0,\ z \in U.$$

Considering in Theorem 6 $h(z) = z$, we have

$$J_{p,\gamma,h} = J_{p,\gamma},\ J_{p,\gamma}(W^n_q g) = W^n_q(G),\ \frac{zh'(z)}{h(z)} = 1,\ H(z) = z,\ zH'(z) - H(z) = 0$$

and

$$\frac{zh''(z)}{h'(z)} + (\gamma - 1)\frac{zh'(z)}{h(z)} + 1 = \gamma.$$

Thus we get:

Corollary 8. *Let $n \in \mathbb{N}$, $p \in \mathbb{N}^*$, $0 < q < 1$, $\gamma \in \mathbb{C}$, $\operatorname{Re}\gamma > p$. Let $g \in \Sigma S^n_{p,q}$ with*

$$\frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma - p,p}(z),\ z \in U,\quad \frac{z(W^n_q g)'(z)}{W^n_q g(z)} + \gamma \prec R_{\gamma - p,p}(z),\ z \in U.$$

If $G = J_{p,\gamma}(g)$, then $G \in \Sigma S^n_{p,q}$ with $z^p G(z) \neq 0$, $z^p W^n_q G(z) \neq 0$, $z \in U$,

$$\operatorname{Re}\left[\frac{zG'(z)}{G(z)} + \gamma\right] > 0,\ \text{and } \operatorname{Re}\left[\frac{z(W^n_q G)'(z)}{W^n_q G(z)} + \gamma\right] > 0,\ z \in U.$$

Taking $n = 0$ in the above result, since $\Sigma S^0_{p,q} = \Sigma^*_p$, $W^0_q g = g$, we have:

Corollary 9. *Let $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$. Let $g \in \Sigma^*_p$ with*

$$\frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma - p,p}(z),\ z \in U.$$

If $G = J_{p,\gamma}(g)$, then $G \in \Sigma^*_p$ with $z^p G(z) \neq 0$, $z \in U$, and

$$\operatorname{Re}\left[\frac{zG'(z)}{G(z)} + \gamma\right] > 0,\ U.$$

This Corollary was also obtained in [1].

3. Discussion

In this paper we first introduced two new classes of meromorphic functions, denoted by $\Sigma S^n_{p,q}(\alpha, \delta)$, respectively $\Sigma S^n_{p,q}(\alpha)$, that used an extension of the Wanas operator to meromorphic functions. We appealed to the Wanas operator because we noticed that it is a well-known operator in recent papers. It is shown that classes of starlike functions of the order α are obtained for specific values of n. Some interesting preserving problems concerning these classes are discussed in the theorems and corollaries.

We have given the conditions for having the function $J_{p,\gamma}(g)$ (where $J_{p,\gamma}$ is a well-known integral operator) in one of the classes $\Sigma S^n_{p,q}(\alpha, \delta)$, respectively $\Sigma S^n_{p,q}(\alpha)$, when g is a function from the same class. It can be seen that these conditions are relatively simple.

Next, we have introduced a new integral operator on meromorphic functions, denoted by $J_{p,\gamma,h}$, proved that it is well-defined and looked for the conditions which allow this operator to preserve the class $\Sigma S^n_{p,q}$. The preservation of $\Sigma S^n_{p,q}$-like classes, following the application of this operator, can be investigated in future works.

Examples were given as corollaries for particular cases of the function h. The new operator defined in this paper can be used to introduce other subclasses of meromorphic functions. Quantum calculus can be also associated for future studies and symmetry properties can be investigated.

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