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Analysis of Tempered Fractional Calculus in Hölder and Orlicz Spaces

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Abstract: Here, we propose a general framework covering a wide variety of fractional operators. We consider integral and differential operators and their role in tempered fractional calculus and study their analytic properties. We investigate tempered fractional integral operators acting on subspaces of $L_1[a,b]$, such as Orlicz or Hölder spaces. We prove that in this case, they map Orlicz spaces into (generalized) Hölder spaces. In particular, they map Hölder spaces into the same class of spaces. The obtained results are a generalization of classical results for the Riemann–Liouville fractional operator and constitute the basis for the use of generalized operators in the study of differential and integral equations. However, we will show the non-equivalence differential and integral problems in the spaces under consideration.

Keywords: tempered fractional calculus; Hardy–Littlewood theorem; Hölder space; Orlicz space

MSC: 26A33; 34A08; 26A42; 46G10

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1. Introduction

The integral transform, which is now call the tempered fractional integral, seems to have been first studied in [1], but the associated fractional calculus model is described more explicitly in, e.g., [2,3]. Both of these papers and their references contain a number of applications of tempered fractional calculus to stochastic processes, random walks, Brownian motion, diffusion, turbulence, etc. The recent paper [4] from 2018 also rediscovered tempered fractional calculus by fractionalizing the proportional derivatives defined in [5,6]. In the definitions presented there, it is usually assumed that the domain of fractional-order operators is the set of functions for which the integrals are well defined. To take full advantage of the new possibilities, it is necessary to define the domains and sets of values of such generalized operators.

In this paper, we concentrate on two aspects of this theory. First, let us recall that the classical Riemann–Liouville fractional operator is compact as acting between Lebesgue spaces $L_p[a,b]$) (see [7], (Lemma 3.1)). Since compactness of operators is useful in the study of many fractional problems, we extend this result to the case of generalized fractional operators by further showing that their values lie in some Hölder space.

The second goal is to achieve the optimal exponent (order) of Hölder spaces when acting on it with generalized fractional operators. The classical result by Hardy and Littlewood [8] states that the fractional Riemann–Liouville integral I^α isomorphically maps the space of Hölder-continuous functions of order $\lambda < 1$ on the space of the same type with order $\alpha + \lambda$, provided that $\alpha + \lambda < 1$. This result was then extended both in terms of the integral operators and the spaces on which they act. We follow this idea. Recall that such a class of spaces is useful when studying problems (not only fractional problems) with exponential growth (cf. [9]), or more generally with more than polynomial growth ([10], for instance). From this point of view, this article can also be interesting for studying other equations (for partial integral operators, see [11], for instance).

Our results provide a basis for all applications of the study of (generalized) fractional problems by investigating them by the operator method (e.g., by the fixed-point theorem). However, we emphasize the lack of equivalence for differential and integral problems when looking for solutions in Hölder spaces by presenting relevant examples. This whole paper is, thus, a step towards unifying the fractional-order calculus, due to the symmetry between different fractional-order calculi, by means of formulating problems using the theory of operators and function spaces, and will avoid duplication of papers.

2. Preliminaries

In the study of fractional-order equations, we need to talk about functions in many function spaces of interest. In particular, we are interested in the spaces considered as typical spaces of solutions of fractional-order differential equations, i.e., Hölder spaces. Of course, it is interesting when we study these operators on different space (domains). Let us collect all the auxiliary facts about interesting function spaces, and the operators acting between them, and all necessary definitions, making the paper self-contained.

Function Spaces

By $C[a,b]$ we denote the space of continuous functions defined on a compact interval $[a,b]$. Let $\left(L_p[a,b], \|\cdot\|_p\right)$, $(1 \le p < \infty)$ denote the Banach space of measurable functions for which the p-th power of the absolute value is Lebesgue integrable, where functions which agree almost everywhere are identified, and where the norm is understood as follows:

$$\|f\|_p = \begin{cases} \left(\int_I |f(t)|^p dt\right)^{1/p}, & p \in [1,\infty) \\ \text{ess sup}_{t\in[a,b]}|f(t)| & p = \infty. \end{cases}$$

Let us now recall the concepts related to some special function spaces, namely, Orlicz and Hölder spaces, which will play an important role in this paper. A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a Young function if ψ is increasing, convex, and continuous with $\psi(0) = 0$ and $\lim_{u\to\infty} \psi(u) = \infty$). For any Young function ψ, the function $\widetilde{\psi} : \mathbb{R}^+ \to \mathbb{R}^+$, defined by $\sup_{v\ge 0}\{vu - \psi(v)\}$, is called the Young complement of ψ and it is known that $\widetilde{\psi}$ is also a Young function.

The Orlicz space $L_\psi = L_\psi([a,b],\mathbb{R})$ consists of all (classes of) measurable functions $x : [a,b] \to \mathbb{R}$ for which the norm

$$\|x\|_\psi := \inf\left\{k > 0 : \int_a^b \psi\left(\frac{|x(s)|}{k}\right) ds \le 1\right\} \le 1 + \int_a^b \psi(|x(s)|)\, ds, \tag{1}$$

is finite (see, e.g., [12]). The special choice $\psi(u) = \psi_p(u) := \frac{1}{p}|u|^p$, $p \in [1,\infty)$ leads to the Lebesgue space $L_p = L_p([a,b],\mathbb{R})$. In this case, it can be easily seen that $\tilde{\psi}_p = \psi_{\tilde{p}}$ with $\frac{1}{p} + \frac{1}{\tilde{p}} = 1$ for $p > 1$.

In this regard, it is worth recalling that for any Young function ψ we have $\psi(u - v) \le \psi(u) - \psi(v)$, and $\psi(\rho u) \le \rho\psi(u)$ occurs for any $u, v \in \mathbb{R}$ and $\rho \in [0,1]$. Moreover, for the non-trivial Young function ψ, $L_\infty \subset L_\psi$. For further properties of Young functions and Orlicz spaces generated by such functions, we refer the reader to [12].

In order to apply classical principles of nonlinear analysis, however, we will need to study the compactness property of the tempered fractional integral operator. We recall here the following well-known sufficient condition.

Theorem 1 ([13], (Propostion 6.1.1))**.** *Suppose that ϑ, ψ and ϕ are three Young functions such that $\vartheta(cuv) \le \phi(u)\tilde{\psi}(v)$, $u, v \ge u_0$ for some $c, u_0 > 0$. If $K : [a,b] \times [a,b] \to \mathbb{R}$ is a measurable function such that*

$$\int_a^b \int_a^t \tilde{\vartheta}\left(\frac{|K(t,s)|}{k}\right) ds\, dt < \infty,$$

for all $k > 0$, then the Volterra operator $Vx(t) := \int_a^t K(t,s)x(s)\, ds,\ \ t \in [a,b]$ is compact from L_ϕ into L_ψ.

For the Hammerstein integral operator, we refer to the recent paper [14].

We say that a function f satisfies the Hölder condition of order $\lambda \in [0,1]$ on the interval $[a,b]$ or $f \in \mathbb{H}^\lambda[a,b]$ (or λ-Hölder) if

$$|f(t+h) - f(t)| \le A|h|^\lambda, \quad t, t+h \in [a,b], \tag{2}$$

where A is a constant independent on t and h. For $\lambda \in [0,1)$ we denote by $\mathbb{H}^{0,\lambda}[a,b] := \{f \in \mathbb{H}^\lambda[a,b] : f(a) = 0\}$. The space $\mathbb{H}^\lambda[a,b]$ is called a Hölder space with a fixed order λ and we call the condition (2) a Hölder condition on $[a,b]$. Functions satisfying a Hölder condition are often referred to as Hölder continuous. It is clear that any function satisfying the Hölder condition of order $\lambda \in (0,1]$ is uniformly continuous. In general, we check the fulfillment of the Hölder condition for the case of continuous functions, so for $\lambda = 0$ we are dealing with the space $C[a,b]$. Moreover, it is not difficult to show, for $0 < \lambda_1 < \lambda_2 < 1$, that

$$C^1[a,b] \subset \mathbb{H}^1[a,b] \subset \mathbb{H}^{\lambda_2}[a,b] \subset \mathbb{H}^{\lambda_1}[a,b] \subset \ \mathbb{H}^0[a,b] = C[a,b] \text{ and } \ \mathbb{H}^1[a,b] \subset AC[a,b].$$

Obviously, f is Hölder continuous of order $\lambda \in (0,1]$ if this seminorm is finite:

$$[f]_\lambda := \sup_{t \neq s} \frac{|f(t) - f(s)|}{|t-s|^\lambda} < \infty.$$

It is easy to see that under this definition only the case $0 < \lambda \le 1$ is interesting; however, it is possible to consider the case of $\lambda > 1$. In this case, the definition can be extended using Taylor approximation, but this does not apply to the case we are interested in for the paper. Finally, note that the Hölder space $\left[\mathbb{H}^\lambda[a,b], \|\cdot\|_\lambda\right]$ is a Banach space when equipped with the norm

$$\|f\|_\lambda := \max_{t\in[a,b]} |f(t)| + [f]_\lambda.$$

Clearly, on the space $\mathbb{H}^{0,\lambda}[a,b]$, we can consider only the second term in the above formula as the norm, which may simplify the proofs. Since fractional integral operators are usually vanishing at the initial point of the interval, we will emphasize this subspace of $\mathbb{H}^\lambda[a,b]$. Note that if f is λ-Hölder and g is continuous, then it does not follow that gf is λ-Hölder continuous. It is known that a product of two λ_1-Hölder continuous functions is again λ_1-Holder continuous (it is a Banach algebra), but this property is not true for the product of λ_2-Hölder continuous functions and λ_1-Hölder continuous for $\lambda_2 > \lambda_1$. A very basic example here is the product of $f(t) = 1 + t^{\lambda_1}$ and $g(t) = 1 + t^{\lambda_2}$.

Besides $\mathbb{H}^\lambda[a,b]$, we will have another general class of functions. Let $g \in C^1[a,b]$ be a positive increasing function such that $g'(t) \neq 0$, for all $t \in [a,b]$. If not otherwise stated, we will make these assumptions about the g function throughout the paper. Accordingly, the norm $\|g'\|$ will be always treated as the supremum norm, i.e., $\|g'\| = \sup_{t\in[a,b]} |g'(t)|$. The same agreement applies to the norms of the studied functions x from the space $C[a,b]$ and its subspaces.

For a continuous increasing function $\vartheta : \mathbb{R}^+ \to \mathbb{R}^+$, which has $\vartheta(0) = 0$, we define the (generalized) Hölder space $\mathbb{H}_g^\vartheta[a,b]$ as

$$|x(t) - x(s)| \le L\vartheta(|g(t) - g(s)|), \quad L > 0,\ x \in C[a,b].$$

Equipped with the norm

$$\|x\|_\vartheta := \max_{t\in I} |x(t)| + [x]_\vartheta, \quad \text{where } [x]_\vartheta := \sup_{t\neq s} \frac{|x(t) - x(s)|}{\vartheta(|g(t) - g(s)|)} \quad ,$$

the space $\left[\mathbb{H}_g^{\vartheta}[a,b], \|\cdot\|_{\vartheta}\right]$ becomes a Banach space. The particular choice $g(t) = t$, $\vartheta(t) = t^{\alpha}$, $\alpha \in (0,1]$ leads naturally to the classical Hölder space. In the case of different functions ϑ, we obtain so-called generalized Hölder spaces, which are sometimes studied in the context of fractional differential equations (where this function describes the modulus of continuity), and the use of g leads to the study of generalized fractional integrals, as will be explained later. This Stieltjes-type approach allows us to cover a broad class of spaces and their associated fractional operators.

Example 1. *Let $\alpha \in (0,1]$. Assume that $\widetilde{\psi}$ denotes the Young complement of some Young function ψ. Define $\widetilde{\Psi}_{\alpha} : \mathbb{R}^+ \to \mathbb{R}^+$ by*

$$\widetilde{\Psi}_{\alpha}(\sigma) := \inf\left\{ k > 0 : \frac{k^{\frac{1}{\alpha-1}}}{\|g'\|_{\infty}} \int_0^{\sigma k^{\frac{1}{1-\alpha}}} \widetilde{\psi}(s^{\alpha-1})\, ds \le 1 \right\}, \quad \sigma \ge 0. \tag{3}$$

Arguing similarly to the proof of [15] (Proposition 2), we can show that $\widetilde{\Psi}_{\alpha}$ is increasing and continuous with $\widetilde{\Psi}_{\alpha}(0) = 0$. That is, for any $\alpha \in (0,1]$, the space $\mathbb{H}_g^{\widetilde{\Psi}_{\alpha}}[a,b]$ is a (generalized) Hölder space.

Recall that classical results for Riemann–Liouville fractional operators I^{α} ($\alpha > \frac{1}{p}$) state that when acting on $L_p[0,b]$, it maps this space continuously into $\mathbb{H}^q[a,b]$ ($q = \alpha - \frac{1}{p}$). Thus, due to the compact embedding $\mathbb{H}^p[a,b]$ into $\mathbb{H}^q[a,b]$, we obtain that $I^{\alpha} : \mathbb{H}^p[a,b] \to \mathbb{H}^p[a,b]$ is a compact mapping (by the Arzelà-Ascoli theorem). We will study a more general class of operators, but also a wider class of spaces, namely, generalized Hölder spaces. Note that we follow the idea taken from [10] (Theorem 1), where certain weakly singular Volterra operators are studied as acting between some Orlicz and generalized Hölder spaces. Here, we focus on generalized fractional operators and prove some stronger properties of them.

First, note that $\mathbb{H}_g^{\widetilde{\Psi}_{\alpha}}[a,b]$ is compactly embedded in $C[a,b]$; we are able also to generalize [10] (Theorem 2), but since the later space has a different norm than Hölder spaces, we prefer such a property of Hölder spaces rather than embeddings in $C[a,b]$.

Let us present a compact embedding theorem for generalized Hölder spaces. Denote by H^* the class of non-decreasing functions ϑ from $\mathbb{R}^+$ to $\mathbb{R}^+$, with right limit at zero equals zero and $\vartheta(t)/t \to C > 0$ as $t \to 0^+$.

Lemma 1. *Let $g \in C^1[a,b]$ be a positive increasing function such that $g'(t) \neq 0$, for all $t \in [a,b]$. Suppose that $\phi, \psi \in H^*$.*

$$\liminf_{t \to 0^+} \frac{\psi(t)}{\phi(t)} > 0,$$

then $\mathbb{H}_g^{\phi}[a,b]$ is continuously embedded in $\mathbb{H}_g^{\psi}[a,b]$. If, moreover,

$$\lim_{t \to 0^+} \frac{\psi(t)}{\phi(t)} > 0, \tag{4}$$

then the embedding is also compact.

Proof. Let $(u_n) \subset \mathbb{H}_g^{\psi}[a,b]$ be taken from the unit ball in this space, so, in particular, $\|u_n\|_{\infty} \le 1$ and

$$|u_n(t) - u_n(s)| \le A \cdot \psi(|g(t) - g(s)|)$$

for some $A > 0$ and all $(t,s) \in [a,b]$. By the monotonicity and continuity properties of g and ψ we can apply the Arzelà–Ascoli theorem, and consequently we can subtract a subsequence $(\tilde{u}_n)$ convergent to some $u \in C[a,b]$. However, since

$$|u(t) - u(s)| = \lim_{n \to \infty} |\tilde{u}_n(t) - \tilde{u}_n(s)| \le A \cdot \psi(|g(t) - g(s)|),$$

and again using the properties of ψ and g, we obtain that $u \in \mathbb{H}_g^{\psi}[a,b]$ as well. Now we are in a position to prove that $(u_n - \tilde{u}_n) \subset \mathbb{H}_g^{\psi}[a,b]$ is convergent in $\mathbb{H}_g^{\phi}[a,b]$. Obviously it converges in $C[a,b]$, so it remains to prove its convergence in the seminorm $[x]_{\psi}$. Given $\delta > 0$, by the properties of g, let $t,s \in [a,b]$ be such that $t \neq s$ and $|g(t) - g(s)| < \delta$. Place $v_n = u - \tilde{u}_n$. Since ψ is increasing, we obtain

$$\begin{aligned} [v_n]_{\psi} &= \sup_{t \neq s} \frac{\|v_n(t) - v_n(s)\|}{\psi(|g(t) - g(s)|)} \\ &= \max\left\{ \sup_{t\neq s, 0<|t-s|<\delta} \frac{\|v_n(t) - v_n(s)\|}{\psi(|g(t) - g(s)|)}, \sup_{t\neq s, \delta \leq |t-s| \leq b-a} \frac{\|v_n(t) - v_n(s)\|}{\psi(|g(t) - g(s)|)} \right\} \\ &\leq \max\left\{ \sup_{t\neq s, 0<|t-s|<\delta} \frac{\phi(|g(t) - g(s)|)}{\psi(|g(t) - g(s)|)} \cdot \frac{\|v_n(t) - v_n(s)\|}{\phi(|g(t) - g(s)|)}, \right. \\ &\qquad \left. \sup_{t\neq s, \delta \leq |t-s| \leq b-a} \frac{\|v_n(t) - v_n(s)\|}{\psi(|g(t) - g(s)|)} \right\}. \end{aligned}$$

Consider the case when $\liminf_{\delta \to 0^+} \frac{\psi(\delta)}{\phi(\delta)} = C > 0$. It implies that there exists $\delta_1 > 0$ such that $C \leq \frac{\psi(t)}{\phi(t)}$ for $t \in (0, \delta_1)$ and by monotonicity of these functions $\frac{\psi(\delta_1)}{\phi(b)} \leq \frac{\psi(t)}{\phi(t)}$ for $t \in [\delta_1, b]$. Thus

$$C \cdot \sup_{t\neq s, 0<|t-s|<\delta_1} \frac{\|v_n(t) - v_n(s)\|}{\psi(|g(t) - g(s)|)} \leq \sup_{t\neq s, 0<|t-s|<\delta_1} \frac{\|v_n(t) - v_n(s)\|}{\phi(|g(t) - g(s)|)}$$

and

$$\frac{\psi(\delta_1)}{\phi(b)} \cdot \sup_{t\neq s, 0<|t-s|<\delta_1} \frac{\|v_n(t) - v_n(s)\|}{\psi(|g(t) - g(s)|)} \leq \sup_{t\neq s, 0<|t-s|<\delta_1} \frac{\|v_n(t) - v_n(s)\|}{\phi(|g(t) - g(s)|)}.$$

Hence,

$$[v_n]_{\psi} \leq \frac{[v_n]_{\phi}}{\min\{c, \frac{\phi(\delta_1)}{\phi(b)}\}},$$

(which is true also for any $v \in \mathbb{H}_g^{\phi}[a,b]$ instead of v_n), so the embedding is continuous.

If $\lim_{\delta \to 0^+} \frac{\psi(\delta)}{\phi(\delta)} = K > 0$, then

$$\sup_{t\neq s, 0<|t-s|<\delta} \frac{\phi(|g(t) - g(s)|)}{\psi(|g(t) - g(s)|)} \cdot \frac{\|v_n(t) - v_n(s)\|}{\phi(|g(t) - g(s)|)} \leq K \cdot [v_n]_{\phi} \tag{5}$$

and

$$\sup_{t\neq s, \delta \leq |t-s| \leq b} \frac{\|v_n(t) - v_n(s)\|}{\psi(|g(t) - g(s)|)} \leq \frac{2}{\psi(\delta)} \cdot \|v_n\|_{\infty}. \tag{6}$$

Since both $[v_n]_{\phi}$ and $\|v_n\|_{\infty}$ are convergent to zero as $n \to \infty$, we are finished. □

Remark 1. *Recall that the inclusion of Orlicz spaces are not well-ordered by the Young functions that generate them. We should then give simple examples of functions belonging to the class H^* and satisfying conditions from the above lemma. The most classical example is to take $\phi(t) = t^p$ and then $\psi(t) = t^q$ with $1 \leq q < p$. Then, the condition (4) is satisfied.*

A more interesting case is given in [12], (Lemma 6.3), which in our situation can be interpreted as follows: Take two Young's functions, ϕ and ϑ. Then, the function $\psi(t) = \frac{\phi(t)\cdot\vartheta(t)}{t}$ belongs to the class H^* and satisfies (6) with respect to ϕ. A deeper study of the comparison of Orlicz spaces can be found in [12] (Chapter II, §13).

3. Generalized Fractional Operators

Various modification and generalizations of the classical fractional integration operators are known and are widely used both in theory and in applications. The following definition allows us to unify the different fractional integrals defined for integrable functions, and consequently to solve some initial and/or boundary value problems with different types of fractional integrals and derivatives in a unified way.

Definition 1. *Let $g \in C^1[a,b]$ be a positive increasing function such that $g'(t) \neq 0$, for all $t \in [a,b]$. The generalized g-fractional tempered integral of a given function $x \in L_1[a,b]$ of order $\alpha > 0$ and with parameter $\mu \in \mathbb{R}^+$ is defined by*

$$\Im_{a,g}^{\alpha,\mu} x(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (g(t)-g(s))^{\alpha-1} e^{-\mu(g(t)-g(s))} x(s) g'(s)\, ds, \quad (-\infty \leq a < b \leq \infty). \tag{7}$$

For completeness, we define $\Im_{a,g}^{\alpha,\mu} x(a) := 0$.

Define $\mathbf{d} := \frac{1}{g'} \frac{d}{dt} + \mu$ and note that

$$\mathbf{d}\Im_{a,g}^{1,\mu} x(t) = \begin{cases} \left(\frac{1}{g'(t)}\frac{d}{dt} + \mu\right) \int_a^t e^{-\mu(g(t)-g(s))} x(s) g'(s)\, ds = x, \\ \qquad \forall\, t \in [a,b] \text{ holds for any } x \in C[a,b], & (\maltese) \\ \left(\frac{1}{g'(t)}\frac{d}{dt} + \mu\right) \int_a^t e^{-\mu(g(t)-g(s))} x(s) g'(s)\, ds = x, \\ \qquad a.e.\ t \in [a,b] \text{ holds for any } x \in L_1[a,b]. & (\diamondsuit) \end{cases}$$

Therefore, using the substitution $u = \frac{g(s)-g(a)}{g(t)-g(a)}$, it can be verified that

$$\Im_{a,g}^{\alpha,\mu} \left\{ e^{-\mu g(t)} (g(t)-g(a))^{\beta-1} \right\} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} e^{-\mu g(t)} (g(t)-g(a))^{\alpha+\beta-1}, \ \alpha, \beta > 0, t > a. \tag{8}$$

Additionally (cf. [6]),

$$\Im_{a,g}^{\alpha,\mu} \left\{ (g(t)-g(a))^{\beta-1} \right\} = (g(t)-g(a))^{\alpha+\beta-1} \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\ {}_1\mathbf{F}_1(\alpha, \alpha+\beta, -\mu(g(t)-g(a))), \tag{9}$$

where $\alpha, \beta > 0, t > a,\ {}_1\mathbf{F}_1$ is the confluent hypergeometric function.

Proposition 1 ([2,4] (semi-group property))**.** *For any $\alpha, \beta > 0$, $\mu \in \mathbb{R}^+$ and a positive increasing function $g \in C^1[a,b]$, with $g'(t) \neq 0$, we have*

$$\Im_{a,g}^{\alpha,\mu} \Im_{a,g}^{\beta,\mu} x = \Im_{a,g}^{\beta,\mu} \Im_{a,g}^{\alpha,\mu} x = \Im_{a,g}^{\alpha+\beta,\mu} x$$

holds true for every $x \in L_1[a,b]$.

For completeness, we also include the definition of generalized proportional fractional derivatives.

Definition 2. *Let $g \in C^1[a,b]$ be a positive increasing function such that $g'(t) \neq 0$, for all $t \in [a,b]$. The generalized g-Riemann–Liouville fractional tempered derivative of order $\alpha > 0$ and with parameter $\mu \in \mathbb{R}^+$ applied to the function $x \in C^n[a,b]$ is defined as*

$$\mathfrak{D}_{a,g}^{\alpha,\mu} x := \mathbf{d}^n \Im_{a,g}^{n-\alpha,\mu} x, \tag{10}$$

where the natural number $n \in \mathbb{N}$ is defined by $n = [\alpha] + 1$.

Definition 3. *Let $g \in C^1[a,b]$ be a positive increasing function such that $g'(t) \neq 0$, for all $t \in [a,b]$. The generalized g-Caputo fractional tempered derivative of order $\alpha > 0$ and with parameter $\mu \in \mathbb{R}^+$ applied to the function $x \in C^n[a,b]$ is defined as*

$$\frac{\mathfrak{d}}{\mathfrak{d}\mathfrak{t}}{}_{a,g}^{\alpha,\mu} x := \Im_{a,g}^{n-\alpha,\mu} \mathbf{d}^n x, \tag{11}$$

where the natural number $n \in \mathbb{N}$ is defined by $n = [\alpha] + 1$.

Remark 2. *We note that the generalized fractional operator defined by Definition 1 generalizes several existing fractional vector-valued integral operators (even in the considered context of the norm topology, i.e., with the Bochner integral instead of the weak topology, i.e., for the Pettis one—cf. [15]): Obviously, this new approach allows us to consider as special cases several other classical models of fractional calculus, such as the Hadamard and Erdélyi–Kober fractional operators:*

(1) $\Im_{0,\ln(1+t)}^{\alpha,\mu}$, $t \in [0,1]$, with $\alpha > 0$, $\mu \in \mathbb{R}^+$ is the generalized version of the Hadamard model of fractional calculus. In the particular choice of the function $\mu = 0$, we obtain the standard version the Hadamard fractional integral, discussed by, for example, Cichoń and Salem in [15–17], to investigate solutions to the fractional Cauchy problem.

(2) $\Im_{0,t}^{\alpha,0}$, $t \in [0,1]$ is the classical fractional calculus with the Riemann–Liouville integral.

(3) In the case of $\Im_{a,t}^{\alpha,\mu}$, $t \in [a,b]$ we obtain the tempered fractional calculus [2,3] which has been intensively studied in recent years because of its applications in stochastic and dynamic systems.

(4) In the case of $\Im_{a,t^\rho}^{\alpha,0}$, $t \in [a,b]$, with $a \geq 0, \alpha, \rho > 0, \mu \in \mathbb{R}^+$ we obtain the so-called Katugampola fractional integral calculus [18,19] (e.g., fractional integral operators concerning t^ρ as defined by Erdélyi–Kober in 1964).

(5) When we consider $\rho^{-\alpha} \Im_{a,t}^{\alpha,\frac{1-\rho}{\rho}}$, $t \in [a,b]$ $\rho \in (0,1]$, then we obtain the generalized proportional fractional calculus (cf. [2]).

Before we move on to the next theorem, in what follows we assume that $g(a) = 0$. Define

$$f(x) := (g(t) - g(x))^\gamma - (g(s) - g(x))^\gamma, \quad \gamma \in (0,1), \quad x \in [t,s], \quad t,s \in [a,b].$$

Without loss of generality, suppose that $t \geq s$. Then f is continuous on $[0,s]$, and $f'(x)$ is positive on $(0,s)$. Standard reasoning based on (classical) calculus shows that f is strictly increasing on $[a,s]$, in particular, $f(s) > f(a)$. Thus, by the mean value theorem,

$$[(g(t))^\gamma - (g(s))^\gamma] \leq (g(t) - g(s))^\gamma \leq \|g'\|^\gamma (t-s)^\gamma, \quad \gamma \in (0,1], \quad s \leq t. \tag{12}$$

Additionally, in view of $g(s)/g(t) \leq 1$, $s \leq t$ and $t^\gamma \leq 1 + \gamma(t-1)$, $t \geq 0$), we obtain

$$\begin{aligned}
|(g(t))^\gamma - (g(s))^\gamma| &= (g(t))^\gamma \left| \left(\frac{g(s)}{g(t)}\right)^\gamma - 1 \right| \leq \gamma (g(t))^\gamma \left| \frac{g(s)}{g(t)} - 1 \right| \\
&\leq \gamma (g(t))^\gamma \left| \frac{g(t) - g(s)}{g(t)} \right| \\
&\leq \gamma (g(t))^{\gamma-1} \|g'\| |t-s| \leq \gamma (g(t) - g(t-s))^{\gamma-1} \|g'\| |t-s| \\
&\leq \gamma (s \min_{\xi \in [s,t]} |g'(\xi)|)^{\gamma-1} \|g'\| |t-s| \\
&= \gamma \|g'\| (\min_{\xi \in [a,b]} |g'(\xi)|)^{\gamma-1} s^{\gamma-1} |t-s|.
\end{aligned} \tag{13}$$

Therefore, arguing similarly to [20] (Theorem 4.4) (cf. also [21]), we can investigate the operator $\Im_{a,g}^{\alpha,\mu}$ acting on the Lebesgue spaces, and we can prove the continuity and compactness of our operator.

Theorem 2. *Let $g \in C^1[a,b]$ be a positive increasing function such that $g'(t) \neq 0$, for all $t \in [a,b]$. If $\alpha \in (0,1)$, $\mu > 0$ and $p > \max\{\frac{1}{\alpha}, 1\}$, the map $\Im^{\alpha,\mu}_{a,g} : L_p[a,b] \to \mathbb{H}^{0,\alpha-\frac{1}{p}}[a,b]$ be bounded.*

However, due to the purpose of this paper, we will prove Theorem 2 even in the more general case where the operator acts on Orlicz spaces (i.e., on a wider class of spaces than just Lebesgue spaces).

Theorem 3. *Let $\alpha \in (0,1]$ and let $g \in C^1[a,b]$ be a positive increasing function such that $g'(t) \neq 0$, for all $t \in [a,b]$. For any Young function ψ with its complementary Young function $\widetilde{\psi}$ satisfying*

$$\int_0^t \widetilde{\psi}(s^{\alpha-1})\, ds < \infty, \quad t > 0, \tag{14}$$

the operator $\Im^{\alpha,\mu}_{a,g}$ maps bounded subsets of the Orlicz space $L_\psi([a,b],\mathbb{R})$ into bounded equicontinuous subsets of $C([a,b],\mathbb{R})$. More precisely, $\Im^{\alpha,\mu}_{a,g}$ is bounded from $L_\psi([a,b],\mathbb{R})$ into the (generalized) Hölder space $\mathbb{H}^{\widetilde{\Psi}_\alpha}_g[a,b]$, where $\widetilde{\Psi}_\alpha$ is defined as in (3).

Proof. Let $x \in L_\psi([a,b],\mathbb{R})$ and $t \in [a,b]$. By noting that

$$\left|\Im^{\alpha,\mu}_{a,g} x(t)\right| \leq \frac{1}{\Gamma(\alpha)} \int_a^b |\approxeq(s)||x(s)|\, ds,$$

where

$$\approxeq(s) := \begin{cases} \left(e^{-\mu g(t)}(g(t)-g(s))^{\alpha-1}\right) g'(s) e^{\mu g(s)}, & s \in [a,t], \\ 0 & \text{otherwise} \end{cases}$$

and using the substitution $s \mapsto (g(t)-g(s))k^{\frac{1}{1-\alpha}}, k > 0$, it follows that

$$\begin{aligned}
\int_a^b \widetilde{\psi}\left(\frac{|\approxeq(s)|}{e^{\mu\|g\|}\|g'\|k}\right) ds &= \int_a^t \widetilde{\psi}\left(e^{\mu g(s)} \frac{\left|e^{-\mu g(t)}(g(t)-g(s))^{\alpha-1}\right|}{e^{\mu\|g\|}\|g'\|k} g'(s)\right) ds \\
&\leq \int_a^t \widetilde{\psi}\left(\frac{\left|e^{-\mu g(t)}(g(t)-g(s))^{\alpha-1}\right|}{k}\right) \frac{e^{\mu g(s)} g'(s)}{e^{\mu\|g\|}\|g'\|}\, ds \\
&\leq \int_a^t \widetilde{\psi}\left(\frac{\left|e^{-\mu g(t)}(g(t)-g(s))^{\alpha-1}\right|}{k}\right) \frac{g'(s)}{\|g'\|}\, ds \\
&\leq \frac{1}{\|g'\|} \int_a^t e^{-\mu g(t)} \widetilde{\psi}\left(\frac{(g(t)-g(s))^{\alpha-1}}{k}\right) g'(s)\, ds \\
&\leq \frac{1}{\|g'\|} \int_a^t \widetilde{\psi}\left(\frac{(g(t)-g(s))^{\alpha-1}}{k}\right) g'(s)\, ds \\
&= \frac{k^{\frac{1}{\alpha-1}}}{\|g'\|} \int_0^{g(t)k^{\frac{1}{1-\alpha}}} \widetilde{\psi}(s^{\alpha-1})\, ds,
\end{aligned}$$

where we have used $\widetilde{\psi}(\lambda u) \leq \lambda \widetilde{\psi}(u)$, $\lambda \in (0,1]$. From which, in view of Example 1 together with the definition of the norm in Orlicz spaces, we can deduce that $\approxeq \in L_{\widetilde{\psi}}([a,b])$ with $\|\approxeq\|_{\widetilde{\psi}} \leq e^{\mu\|g\|}\|g'\|K_1$, where

$$K_1 := \inf\left\{k > 0 : \frac{1}{\|g'\|} \int_0^{k^{\frac{1}{1-\alpha}}(g(t))} \widetilde{\psi}(s^{\alpha-1})\, ds \leq k^{\frac{1}{1-\alpha}}\right\} = \widetilde{\Psi}_\alpha(|g(t)|).$$

Hence, by the Hölder inequality in Orlicz spaces we obtain

$$\left|\Im_{a,g}^{\alpha,\mu}x(t)\right| \leq \frac{2e^{\mu\|g\|}\|g'\|\widetilde{\Psi}_{\alpha}(\|g\|)}{\Gamma(\alpha)}\|x\|_{\psi}. \tag{15}$$

Now let $h > 0$, $t, t+h \in [a,b]$. We obtain the following estimate

$$\begin{aligned} & [\Im_{a,g}^{\alpha,\mu}x(t+h) - \Im_{a,g}^{\alpha,\mu}x(t)]\Gamma(\alpha) \\ = & \int_a^t \Big(e^{-\mu g(t+h)}(g(t+h)-g(s))^{\alpha-1} - e^{-\mu g(t)}(g(t)-g(s))^{\alpha-1}\Big)g'(s)e^{\mu g(s)}\,ds \\ + & \int_t^{t+h}(g(t+h)-g(s))^{\alpha-1}e^{-\mu g(t+h)}g'(s)e^{\mu g(s)}\,ds. \end{aligned}$$

and then

$$\left|\Im_{a,g}^{\alpha,\mu}x(t+h) - \Im_{a,g}^{\alpha,\mu}x(t)\right| \leq \frac{1}{\Gamma(\alpha)}\int_a^b |\approxeq_1(s) + \approxeq_2(s)|x(s)|\,ds,$$

where

$$\approxeq_1(s) := \begin{cases} \Big(e^{-\mu g(t+h)}(g(t+h)-g(s))^{\alpha-1} - e^{-\mu g(t)}(g(t)-g(s))^{\alpha-1}\Big)g'(s)e^{\mu g(s)} & , \\ \qquad s \in [a,t], \\ 0 \qquad \text{otherwise} \end{cases}$$

and

$$\approxeq_2(s) := \begin{cases} \Big(g(t+h)-g(s))^{\alpha-1}e^{-\mu g(t+h)}\Big)g'(s)e^{\mu g(s)} & s \in [t,t+h], \\ 0 & \text{otherwise.} \end{cases}$$

We proceed to show that $\approxeq_i \in L_{\widetilde{\psi}}([a,b])$, $(i = 1,2)$. Once we show this, we can conclude, in view of the Hölder inequality, that

$$\left|\Im_{a,g}^{\alpha,\mu}x(t+h) - \Im_{a,g}^{\alpha,\mu}x(t)\right| \leq \frac{2\Big[\|\approxeq_1\|_{\widetilde{\psi}} + \|\approxeq_2\|_{\widetilde{\psi}}\Big]}{\Gamma(\alpha)}\|x\|_{\psi}. \tag{16}$$

In this connection, we fix $k > 0$. After substitutions $s \mapsto (g(t+h)-g(s))k^{\frac{1}{1-\alpha}}$ and $s \mapsto (g(t)-g(s))^{\alpha-1}k^{\frac{1}{1-\alpha}}$, using the properties $\widetilde{\psi}(\lambda u) \leq \lambda\widetilde{\psi}(u)$, $\lambda \in (0,1]$ and $\widetilde{\psi}(u-v) \leq \widetilde{\psi}(u) - \widetilde{\psi}(v), v \leq u$, we obtain the following estimate:

$$\begin{aligned} & \int_a^b \widetilde{\psi}\left(\frac{|\approxeq_1(s)|}{e^{\mu\|g\|}\|g'\|k}\right)ds \\ = & \int_a^t \widetilde{\psi}\left(e^{\mu g(s)}\frac{\left|e^{-\mu g(t+h)}(g(t+h)-g(s))^{\alpha-1} - e^{-\mu g(t)}(g(t)-g(s))^{\alpha-1}\right|}{e^{\mu\|g\|}\|g'\|k}g'(s)\right)ds \\ \leq & \int_a^t e^{-\mu g(t+h)}\widetilde{\psi}\left(\frac{\left|e^{-\mu g(t+h)}(g(t+h)-g(s))^{\alpha-1} - e^{-\mu g(t)}(g(t)-g(s))^{\alpha-1}\right|}{k}\right) \\ & \cdot\frac{e^{\mu g(s)}g'(s)}{e^{\mu\|g\|}\|g'\|}\,ds \\ \leq & \frac{1}{\|g'\|}\int_a^t\left[e^{-\mu g(t)}\widetilde{\psi}\left(\frac{(g(t)-g(s))^{\alpha-1}}{k}\right) - e^{-\mu g(t+h)}\widetilde{\psi}\left(\frac{(g(t+h)-g(s))^{\alpha-1}}{k}\right)\right] \\ & \cdot g'(s)\,ds \\ \leq & \frac{k^{\frac{1}{\alpha-1}}}{\|g'\|}\left[e^{-\mu g(t)}\int_0^{k^{\frac{1}{1-\alpha}}g(t)}\widetilde{\psi}(s^{\alpha-1})\,ds - e^{-\mu g(t+h)}\int_{k^{\frac{1}{1-\alpha}}(g(t+h)-g(t))}^{k^{\frac{1}{1-\alpha}}g(t+h)}\widetilde{\psi}(s^{\alpha-1})\,ds\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{k^{\frac{1}{\alpha-1}}}{\|g'\|}\Bigg[e^{-\mu g(t)}\int_0^{k^{\frac{1}{1-\alpha}}(g(t+h))}\widetilde{\psi}(s^{\alpha-1})\,ds - e^{-\mu g(t+h)}\int_0^{k^{\frac{1}{1-\alpha}}g(t+h)}\widetilde{\psi}(s^{\alpha-1})\,ds \\
&+ e^{-\mu g(t+h)}\int_0^{k^{\frac{1}{1-\alpha}}(g(t+h)-g(t))}\widetilde{\psi}(s^{\alpha-1})\,ds\Bigg] \\
&\leq e^{-\mu g(t+h)}\frac{k^{\frac{1}{\alpha-1}}}{\|g'\|}\int_0^{k^{\frac{1}{1-\alpha}}(g(t+h)-g(t))}\widetilde{\psi}(s^{\alpha-1})\,ds \\
&\leq \frac{k^{\frac{1}{\alpha-1}}}{\|g'\|}\int_0^{k^{\frac{1}{1-\alpha}}(g(t+h)-g(t))}\widetilde{\psi}(s^{\alpha-1})\,ds.
\end{aligned}
$$

From which, it follows $\approxeq_1 \in L_{\widetilde{\psi}}([a,b])$ with $\|\approxeq_1\|_{\widetilde{\psi}} \leq e^{\mu\|g\|}\|g'\|K_2$, where

$$
K_2 := \inf\left\{k>0 : \frac{1}{\|g'\|}\int_0^{k^{\frac{1}{1-\alpha}}(g(t+h)-g(t))}\widetilde{\psi}(s^{\alpha-1})\,ds \leq k^{\frac{1}{1-\alpha}}\right\} = \widetilde{\Psi}_\alpha(|g(t+h)-g(t)|).
$$

Arguing similarly to above, we can show that

$$
\approxeq_2 \in L_{\widetilde{\psi}}([a,b], \quad \text{and } \|\approxeq_2\|_{\widetilde{\psi}} \leq e^{\mu\|g\|}\|g'\|\widetilde{\Psi}_\alpha(|g(t+h)-g(t)|).
$$

Thus, Equation (16) takes the form

$$
\left|\Im_{a,g}^{\alpha,\mu}x(t+h) - \Im_{a,g}^{\alpha,\mu}x(t)\right| \leq \frac{4e^{\mu\|g\|}\|g'\|\widetilde{\Psi}_\alpha(|g(t+h)-g(t)|)}{\Gamma(\alpha)}\|x\|_\psi. \tag{17}
$$

To see that $\Im_{a,g}^{\alpha,\mu} : L_\psi[a,b] \to C[a,b]$ is compact, let $\|x\|_\psi \leq 1$. Given $\epsilon > 0$, choose $\delta > 0$ such that $\widetilde{\Psi}_\alpha(|g(t)-g(\tau)|) < \epsilon$ for $|t-\tau| < \delta$. In view of (17) we conclude that

$$
\left|\Im_{a,g}^{\alpha,\mu}x(t) - \Im_{a,g}^{\alpha,\mu}x(\tau)\right| \leq \frac{4e^{\mu\|g\|}\|g'\|\epsilon}{\Gamma(\alpha)}, \quad |t-\tau| < \delta.
$$

This, together with (15), shows that the set $\{\Im_{a,g}^{\alpha,\mu}x : \|x\|_\psi \leq 1\}$ is uniformly bounded and equicontinuous, and thus the assertion of $\Im_{a,g}^{\alpha,\mu}$ follows from the Arzel'a–Ascoli compactness criterion.

Finally, we note also that (see (15) and (17))

$$
\left[\Im_{a,g}^{\alpha,\mu}x\right]_{\widetilde{\Psi}_\alpha} \leq \frac{4e^{\mu\|g\|}\|g'\|}{\Gamma(\alpha)}\|x\|_\psi, \quad \left|\Im_{a,g}^{\alpha,\mu}x(t)\right| \leq \frac{1}{2}\widetilde{\Psi}_\alpha(\|g\|)\left[\Im_{a,g}^{\alpha,\mu}x\right]_{\widetilde{\Psi}_\alpha}.
$$

Thus,

$$
\left\|\Im_{a,g}^{\alpha,\mu}x\right\|_{\widetilde{\Psi}_\alpha} \leq \frac{2e^{\mu\|g\|}\|g'\|}{\Gamma(\alpha)}\|x\|_\psi(2+\widetilde{\Psi}_\alpha(\|g\|). \tag{18}
$$

So $\Im_{a,g}^{\alpha,\mu} : L_\psi[a,b] \to \mathbb{H}_g^{\widetilde{\Psi}_\alpha}[a,b]$ is bounded. □

We make some comments on Theorem 3:

Proposition 2. *Note that, in Theorem 3, if* $\widetilde{\Psi}_\alpha^* \in H^*$ *and, moreover,*

$$
\lim_{t\to 0^+}\frac{\widetilde{\Psi}_\alpha^*(t)}{\widetilde{\Psi}_\alpha(t)} > 0,
$$

then $\Im_{a,g}^{\alpha,\mu} : L_\psi[a,b] \to \mathbb{H}_g^{\widetilde{\Psi}_\alpha^*}[a,b]$ *is compact.*

Proof. To see this, let us observe that from Theorem 3 we know that the operator $\Im_{a,g}^{\alpha,\mu}$ maps the Orlicz space $L_\psi[a,b]$ into the (generalized) Hölder space $\mathbb{H}_g^{\widetilde{\Psi}_\alpha}[a,b]$ and is bounded. Due to Lemma 1, $\mathbb{H}_g^{\widetilde{\Psi}_\alpha}[a,b]$ is compactly embedded in $\mathbb{H}_g^{\widetilde{\Psi}^*_\alpha}[a,b]$.

Finally, it maps bounded sets in $L_\psi[a,b]$ into compact sets in $\mathbb{H}_g^{\widetilde{\Psi}^*_\alpha}[a,b]$, so the operator $\Im_{a,g}^{\alpha,\mu} : L_\psi[a,b] \to \mathbb{H}_g^{\widetilde{\Psi}^*_\alpha}[a,b]$ is compact. □

It is worth noting that Proposition 2 has a twofold purpose and is an extension of the result known so far only for $I^\alpha : L_p[a,b] \to C[a,b]$ (cf. [21], (Proposition 3.2), for instance) with the use of compact embeddings of Hölder spaces into $C[a,b]$, so our extension applies to both the class of operators ($\Im_{a,g}^{\alpha,\mu}$) and the spaces ($\mathbb{H}_g^{\widetilde{\Psi}^*_\alpha}[a,b]$) under consideration.

Remark 3. *In particular, we covered the following case. Define* $\psi(u) = \psi_p(u) := \frac{1}{p}|u|^p$, $p \in (1,\infty)$. *In this case, we have* $\tilde{\psi}_p = \psi_{\widetilde{p}}$ *with* $1/p + 1/\widetilde{p} = 1$. *It is easy to see that (14) is true if and only if* $p > \frac{1}{\alpha}, \alpha \in (0,1)$. *Additionally,*

$$\widetilde{\Psi}_\alpha(t) = \frac{t^{\alpha-\frac{1}{p}}}{\sqrt[\widetilde{p}]{\|g'\|(\widetilde{p}(\alpha-1)+1)}}.$$

Accordingly, (17) shall read as follows

$$\begin{aligned} \left|\Im_{a,g}^{\alpha,\mu} x(t+h) - \Im_{a,g}^{\alpha,\mu} x(t)\right| &\le \frac{4e^{\mu\|g\|}\|g'\||g(t+h)-g(t)|^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)\sqrt[\widetilde{p}]{\|g'\|(\widetilde{p}(\alpha-1)+1)}}\|x\|_{L_p} \\ &\le \frac{4e^{\mu\|g\|}\|g'\|^\alpha\|x\|_{L_p}}{\Gamma(\alpha)\sqrt[\widetilde{p}]{\widetilde{p}(\alpha-1)+1}} h^{\alpha-\frac{1}{p}}. \end{aligned}$$

From which we conclude that $\Im_{a,g}^{\alpha,\mu}$ *maps the Lebesgue space* $L_p[a,b]$ *into the Hölder space* $\mathbb{H}^{0,\alpha-\frac{1}{p}}[a,b]$.

Theorem 4. *Let* $\alpha \in (0,1]$. *For any Young function* ψ *satisfying* $\psi(\kappa uv) \le \psi(u)\psi(v),\ u,v \ge u_0$ *for some* $u_0, \kappa > 0$ *with its complementary Young function* $\tilde{\psi}$ *satisfying*

$$\int_0^t \tilde{\psi}(s^{\alpha-1})\,ds < \infty, \quad t > 0, \tag{19}$$

the operator $\Im_{a,g}^{\alpha,\mu}$ *is compact from Orlicz space* $L_{\widetilde{\psi}}([a,b],\mathbb{R}) \to L_\psi([a,b],\mathbb{R})$.

Proof. Place $K(t,s) = \begin{cases} \left(e^{-\mu g(t)}(g(t)-g(s))^{\alpha-1}\right)g'(s)e^{\mu g(s)}, & s \in [a,t], \\ 0 & \text{otherwise} \end{cases}$. Arguing similarly to in the proof of Theorem 3 , it can be easily seen that

$$\int_a^b \int_a^t \tilde{\psi}\left(\frac{|K(t,s)|}{k}\right) ds\,dt < \infty,$$

holds true for all $k > 0$. Since ψ satisfies $\psi(\kappa uv) \le \psi(u)\psi(v),\ u,v \ge u_0$, the result follows by Theorem 1. □

We shall now examine further properties of our operator.

Theorem 5. *Let* $g \in C^1[a,b]$ *be a positive increasing function such that* $g'(t) \ne 0$, *for all* $t \in [a,b]$. *For* $\alpha,\ \lambda, 0 < \lambda + \alpha < 1$ *the operator* $\Im_{a,g}^{\alpha,\mu} : \mathbb{H}^{0,\lambda}[a,b] \to \mathbb{H}^{0,\lambda+\alpha}[a,b]$ *is bijective with a continuous inverse* $\mathfrak{D}_{a,g}^{\alpha,\mu}$.

To simplify the proof of Theorem 5 we will divide it up into several stages, by providing some facts. The first one is an extension for the mentioned classical Hardy–Littlewood theorem originally proved for the Riemann–Liouville fractional operator.

Lemma 2. *Let $0 < \lambda + \alpha < 1$. Let $g \in C^1[a,b]$ be a positive increasing function such that $g'(t) \neq 0$, for all $t \in [a,b]$. Then $\Im^{\alpha,\mu}_{a,g}$ maps $\mathbb{H}^{0,\lambda}[a,b]$ into $\mathbb{H}^{0,\lambda+\alpha}[a,b]$.*

Proof. Let $x \in \mathbb{H}^{0,\lambda}[a,b]$. For $h > 0$, $t, t+h \in [a,b]$, after the substitution $s \mapsto t - s$ we obtain in view $x(a) = 0$ and our definition that $\Im^{\alpha,\mu}_{a,g} x(a) = 0$,

$$\begin{aligned}
&\Im^{\alpha,\mu}_{a,g} x(t+h) - \Im^{\alpha,\mu}_{a,g} x(t) \\
= &\ \frac{1}{\Gamma(\alpha)}\Big(\int_{-h}^{t-a} e^{-\mu(g(t+h)-g(t-s))}(g(t+h)-g(t-s))^{\alpha-1}x(t-s)g'(t-s)\,ds \\
- &\int_0^{t-a} e^{-\mu(g(t)-g(t-s))}(g(t)-g(t-s))^{\alpha-1}x(t-s)g'(t-s)\,ds\Big) \\
= &\ \frac{1}{\Gamma(\alpha)}\Big(\int_{-h}^{0} e^{-\mu(g(t+h)-g(t-s))}(g(t+h)-g(t-s))^{\alpha-1}x(t-s)g'(t-s)\,ds \\
+ &\int_0^{t-a}\Big\{e^{-\mu g(t+h)}(g(t+h)-g(t-s))^{\alpha-1} - e^{-\mu g(t)}(g(t)-g(t-s))^{\alpha-1}\Big\} \\
&\qquad\qquad \cdot e^{\mu g(t-s))}x(t-s)g'(t-s)\,ds\Big) \\
= &\ \frac{1}{\Gamma(\alpha)}\Big(\int_{-h}^{t-a} e^{-\mu(g(t+h)-g(t-s))}(g(t+h)-g(t-s))^{\alpha-1}[x(t)-x(a)]g'(t-s)\,ds \\
- &\int_0^{t-a} e^{-\mu(g(t)-g(t-s))}(g(t)-g(t-s))^{\alpha-1}[x(t)-x(a)]g'(t-s)\,ds \\
+ &\int_{-h}^{0} e^{-\mu(g(t+h)-g(t-s))}(g(t+h)-g(t-s))^{\alpha-1}[x(t-s)-x(t)]g'(t-s)\,ds \\
+ &\int_0^{t-a} e^{\mu g(t-s)}\Big\{e^{-\mu g(t+h)}(g(t+h)-g(t-s))^{\alpha-1} - e^{-\mu g(t)}(g(t)-g(t-s))^{\alpha-1}\Big\}\cdot \\
&\qquad\qquad \cdot[x(t-s)-x(t)]g'(t-s)\,ds\Big) \\
&= A + B + C,
\end{aligned}$$

where

$$\begin{aligned}
&|A| := \\
&\frac{|x(t)-x(a)|}{\Gamma(\alpha)}\left|\int_{-h}^{t-a}\frac{e^{-\mu(g(t+h)-g(t-s))}g'(t-s)}{(g(t+h)-g(t-s))^{1-\alpha}}\,ds - \int_0^{t-a}\frac{e^{-\mu(g(t)-g(t-s))}g'(t-s)}{(g(t)-g(t-s))^{1-\alpha}}\,ds\right| \\
\le &\ \frac{[x]_\lambda(t-a)^\lambda}{\Gamma(\alpha)}\left|\int_0^{g(t+h)} e^{-\mu s}s^{\alpha-1}\,ds - \int_0^{g(t)} e^{-\mu s}s^{\alpha-1}\,ds\right| \\
= &\ \frac{[x]_\lambda(t-a)^\lambda}{\Gamma(\alpha)}\left|\int_{g(t)}^{g(t+h)} e^{-\mu s}s^{\alpha-1}\,ds\right| \\
\le &\ \frac{[x]_\lambda(t-a)^\lambda}{\Gamma(\alpha)}\int_{g(t)}^{g(t+h)} s^{\alpha-1}\,ds = \frac{[x]_\lambda(t-a)^\lambda}{\Gamma(1+\alpha)}(g(t+h))^\alpha - (g(t))^\alpha.
\end{aligned}$$

In the above calculations, we used the substitutions $s \mapsto g(t+h) - g(t-s)$ and $s \mapsto g(t) - g(t-s))$.

Now, letting $h \geq t - a$ (in view of (12)), we obtain

$$|A| \le \frac{[x]_\lambda(t-a)^\lambda}{\Gamma(1+\alpha)}\|g'\|^\alpha h^\alpha \le \frac{[x]_\lambda\|g'\|^\alpha}{\Gamma(1+\alpha)}h^{\alpha+\lambda}.$$

Note that the function g' is continuous and positive, so $\min_{s\in[a,b]}|g'(s)|$ is a positive quantity. If $0 < h < t - a$, then we obtain (due to $\lambda + \alpha < 1$, $t^{\alpha-1} \le (t-a)^{\alpha-1}$ and by (13))

$$\begin{aligned}|A| &:= \frac{[x]_\lambda (t-a)^\lambda}{\Gamma(\alpha)} \|g'\| (\min_{\xi\in[t,t+h]} |g'(\xi)|)^{\alpha-1} t^{\alpha-1} h \\ &\leq \frac{[x]_\lambda (t-a)^{\lambda+\alpha-1}}{\Gamma(\alpha)} \|g'\| (\min_{\xi\in[a,b]} |g'(\xi)|)^{\alpha-1} h \\ &\leq \frac{[x]_\lambda h^{\lambda+\alpha}}{\Gamma(\alpha)} \|g'\| (\min_{\xi\in[a,b]} |g'(\xi)|)^{\alpha-1}.\end{aligned}$$

Similarly, we estimate the other components in the sum above:

$$\begin{aligned}|B| &:= \frac{1}{\Gamma(\alpha)} \int_{-h}^{0} \frac{e^{-\mu(g(t+h)-g(t-s))}[x(t-s)-x(t)]g'(t-s)}{(g(t+h)-g(t-s))^{1-\alpha}}\, ds \\ &\leq \frac{[x]_\lambda}{\Gamma(\alpha)} \int_0^{g(t+h)-g(t)} e^{-\mu s} s^{\alpha-1} s^\lambda\, ds \\ &\leq \frac{[x]_\lambda}{\Gamma(\alpha)} \int_0^{g(t+h)-g(t)} s^{\alpha+\lambda-1}\, ds = \frac{[x]_\lambda}{(\alpha+\lambda)\Gamma(\alpha)} (g(t+h)-g(t))^{\alpha+\lambda} \\ &= \frac{[x]_\lambda |h|^{\lambda+\alpha} \|g'\|^{\lambda+\alpha}}{(\alpha+\lambda)\Gamma(\alpha)}.\end{aligned}$$

It remains to estimate

$$\begin{aligned}|C| &:= \frac{1}{\Gamma(\alpha)} \Bigg(\int_0^{t-a} \left| \frac{e^{-\mu(g(t+h)-g(t-s))}}{(g(t+h)-g(t-s))^{1-\alpha}} - \frac{e^{-\mu(g(t)-g(t-s))}}{(g(t)-g(t-s))^{1-\alpha}} \right| \\ &\qquad \cdot |x(t-s)-x(t)| g'(t-s)\, ds \Bigg).\end{aligned}$$

By the mean value theorem, we obtain $g(t+h)-g(t-s) = g'(\xi_1)(h+s)$, $\xi_1 \in (t-s, t+h)$ and $g(t)-g(t-s) = g'(\xi_2)s$, $\xi_2 \in (t-s,t)$. Since $g(t+h)-g(t-s) \to h(t) - g(t-s)$ as $h \to 0$ and simultaneously $g(t+h)-g(t-s) = g'(\xi_1)(h+s)$ (ξ_1 dependent on h), we see that $\xi_1 \to \xi_2$ as $h \to 0$. We note that for $s \in [0, t-a]$, we obtain $\xi_1, \xi_2 \in (a,b)$ After the substitution $s \mapsto s/h$, we obtain

$$\begin{aligned}|C| &\leq \frac{[x]_\lambda}{\Gamma(\alpha)} \left(\int_0^{t-a} \left| \frac{e^{-\mu g'(\xi_1)(h+s)}(g'(\xi_1))^{1-\alpha}}{(s+h)^{1-\alpha}} - \frac{e^{-\mu g'(\xi_2)s}(g'(\xi_2))^{1-\alpha}}{s^{1-\alpha}} \right| s^\lambda \|g'\|\, ds \right) \\ &= \frac{[x]_\lambda \|g'\| h^{\alpha+\lambda}}{\Gamma(\alpha)} \left(\int_0^{\frac{t-a}{h}} \left| \frac{e^{-\mu h g'(\xi_1)(1+s)}(g'(\xi_1))^{1-\alpha}}{(1+s)^{1-\alpha}} - \frac{e^{-\mu h g'(\xi_2)s}(g'(\xi_2))^{1-\alpha}}{s^{1-\alpha}} \right| s^\lambda\, ds \right)\end{aligned}$$

Recall that for any $t \in [a,b]$, we have $0 < c = \min_{s\in[a,b]} g'(s) \leq g'(t) \leq \|g'\|$ and these inequalities will be used hereafter depending on the negative or positive power of this derivative.

If $h \geq t-a$, we obtain (due to $s^{\alpha-1} > (1+s)^{\alpha-1}$ when $s \in (0,1)$)

$$\begin{aligned}|C| &\leq \frac{[x]_\lambda \|g'\| h^{\alpha+\lambda}}{\Gamma(\alpha)} \left(\int_0^1 \left| \frac{e^{-\mu h g'(\xi_1)(1+s)}(g'(\xi_1))^{1-\alpha}}{(1+s)^{1-\alpha}} - \frac{e^{-\mu h g'(\xi_2)s}(g'(\xi_2))^{1-\alpha}}{s^{1-\alpha}} \right| s^\lambda\, ds \right) \\ &\leq \frac{[x]_\lambda \|g'\| h^{\alpha+\lambda}}{\Gamma(\alpha)} \left(\int_0^1 \left| \frac{e^{-\mu h g'(\xi_1)(1+s)} - e^{-\mu h g'(\xi_2)s}}{(\min_{\xi\in[a,b]} |g'(\xi)|)^{\alpha-1} s^{1-\alpha}} \right| s^\lambda\, ds \right) \\ &\leq \frac{[x]_\lambda \|g'\| (\min_{\xi\in[a,b]} |g'(\xi)|)^{1-\alpha} h^{\alpha+\lambda}}{\Gamma(\alpha)} \int_0^1 \frac{2e^{-\mu h g'(\xi_2)s}}{s^{1-\alpha}} s^\lambda\, ds \\ &\leq \frac{[x]_\lambda \|g'\| (\min_{\xi\in[a,b]} |g'(\xi)|)^{\alpha-1} 2h^{\alpha+\lambda}}{\Gamma(\alpha)} \int_0^1 s^{\alpha+\lambda-1}\, ds \\ &\leq \frac{[x]_\lambda \|g'\| (\min_{\xi\in[a,b]} |g'(\xi)|)^{1-\alpha} 2h^{\alpha+\lambda}}{(\alpha+\lambda)\Gamma(\alpha)}.\end{aligned}$$

Since $|C|$ is estimated by integrals with the upper limit depending on h, i.e., it is $\frac{t-a}{h}$ and h can be "small", we extend this integral to the upper limit $+\infty$. Additionally, if $0 < h < t-a$, we obtain

$$\begin{aligned} |C| &\leq \frac{[x]_\lambda \|g'\| h^{\alpha+\lambda}}{\Gamma(\alpha)} \left(\int_0^\infty \left| \frac{e^{-\mu h g'(\xi_1)(1+s)}(g'(\xi_1))^{1-\alpha}}{(1+s)^{1-\alpha}} - \frac{e^{-\mu h g'(\xi_2)s}(g'(\xi_2))^{1-\alpha}}{s^{1-\alpha}} \right| s^\lambda \, ds \right) \\ &\leq \frac{[x]_\lambda \|g'\| (\min_{\xi\in[a,b]} |g'(\xi)|)^{\alpha-1} h^{\alpha+\lambda}}{(\alpha+\lambda)\Gamma(\alpha)} + \frac{[x]_\lambda \|g'\| h^{\alpha+\lambda}}{\Gamma(\alpha)} J_1, \end{aligned}$$

where

$$\begin{aligned} J_1 &:= \int_0^\infty \left| \frac{e^{-\mu h g'(\xi_1)(1+s)}(g'(\xi_1))^{1-\alpha}}{(1+s)^{1-\alpha}} - \frac{e^{-\mu h g'(\xi_2)s}(g'(\xi_2))^{1-\alpha}}{s^{1-\alpha}} \right| s^\lambda \, ds \\ &\leq \int_0^\infty \frac{e^{-\mu h g'(\xi_2)s}}{(g'(\xi_2))^{\alpha-1}} \left| (s+1)^{\alpha-1} - s^{\alpha-1} \right| s^\lambda \, ds \\ &\quad + \int_0^\infty (1+s)^{\alpha-1} \left| \frac{e^{-\mu h g'(\xi_1)} e^{-\mu h g'(\xi_1)s}}{(g'(\xi_1))^{1-\alpha}} - \frac{e^{-\mu h g'(\xi_2)s}}{(g'(\xi_2))^{1-\alpha}} \right| s^\lambda \, ds \\ &\leq \int_0^\infty \frac{e^{-\mu h g'(\xi_2)s}}{(g'(\xi_2))^{\alpha-1}} \left| (s+1)^{\alpha-1} - s^{\alpha-1} \right| s^\lambda \, ds \\ &\quad + \int_0^\infty \left| \frac{e^{-\mu h g'(\xi_1)} e^{-\mu h g'(\xi_1)s}}{(g'(\xi_1))^{1-\alpha}} - \frac{e^{-\mu h g'(\xi_2)s}}{(g'(\xi_2))^{1-\alpha}} \right| s^{\alpha+\lambda-1} \, ds \\ &= \int_0^\infty \frac{e^{-\mu h g'(\xi_2)s}}{(g'(\xi_2))^{\alpha-1}} \left| (s+1)^{\alpha-1} - s^{\alpha-1} \right| s^\lambda \, ds \\ &\quad + \left| e^{-\mu h g'(\xi_1)} \int_0^\infty \frac{e^{-\mu h g'(\xi_1)s}}{(g'(\xi_1))^{1-\alpha}} s^{\alpha+\lambda-1} \, ds - \int_0^\infty \frac{e^{-\mu h g'(\xi_2)s}}{(g'(\xi_2))^{1-\alpha}} s^{\alpha+\lambda-1} \, ds \right| \\ &\leq \int_0^\infty \frac{1}{(g'(\xi_2))^{\alpha-1}} \left| (s+1)^{\alpha-1} - s^{\alpha-1} \right| s^\lambda \, ds \\ &\quad + \left| \frac{e^{-\mu h g'(\xi_1)}}{(g'(\xi_1))^{1+\lambda}} - \frac{1}{(g'(\xi_2))^{1+\lambda}} \right| \frac{\Gamma(\lambda+\alpha)}{(\mu h)^{\lambda+\alpha}}. \end{aligned}$$

Define $\mathbf{h}(t) := t^{\alpha-1}, \ \ t \in [s, s+1], s > 1$. By the mean value theorem, we obtain

$$\left| (s+1)^{\alpha-1} - s^{\alpha-1} \right| s^\lambda \leq s^\lambda \sup_{t\in[s,s+1]} \left| \mathbf{h}'(t) \right| = (1-\alpha) s^{\alpha+\lambda-2}.$$

It follows that

$$\begin{aligned} |C| &\leq \frac{[x]_\lambda \|g'\| h^{\alpha+\lambda}}{\Gamma(\alpha)} \left[\frac{(\min_{\xi\in[a,b]} |g'(\xi)|)^{\alpha-1}}{(\alpha+\lambda)\Gamma(\alpha)} \right. \\ &\quad \left. + \frac{(1-\alpha)}{(1-\alpha-\lambda)(g'(\xi_2))^{\alpha-1}} + \left| \frac{e^{-\mu h g'(\xi_1)}}{(g'(\xi_1))^{1+\lambda}} - \frac{1}{(g'(\xi_2))^{1+\lambda}} \right| \frac{\Gamma(\lambda+\alpha)}{(\mu h)^{\lambda+\alpha}} \right]. \end{aligned} \tag{20}$$

Thus, in view of ($\xi_1 \to \xi_2$ as $h \to 0$), bearing in mind that

$$\lim_{h\to 0} \frac{e^{-\mu h g'(\xi_1)}(g'(\xi_1))^{-1-\lambda} - (g'(\xi_2))^{-1-\lambda}}{(\mu h)^{\lambda+\alpha}} = \lim_{h\to 0} \frac{-\mu(\mu h)^{1-(\lambda+\alpha)}}{(\lambda+\alpha)(g'(\xi_1))^\lambda e^{\mu h g'(\xi_1)}} \to 0,$$

we conclude that

$$\left|\frac{e^{-\mu h g'(\xi_1)}}{(g'(\xi_1))^{1+\lambda}} - \frac{1}{(g'(\xi_2))^{1+\lambda}}\right| \frac{\Gamma(\lambda+\alpha)}{(\mu h)^{\lambda+\alpha}} < \infty, \text{ for any } h > 0.$$

That is, $|C| \leq K[x]_\lambda h^{\alpha+\lambda}$ for some constant $K > 0$. Thus, we have

$$\left|\Im^{\alpha,\mu}_{a,gt} x(t+h) - \Im^{\alpha,\mu}_{a,g} x(t)\right| \leq L[x]_\lambda h^{\alpha+\lambda}, \text{ for any } t \in [a,b], t+h \in [a,b], h > 0.$$

This means that $\Im^{\alpha,\mu}_{a,g}$ maps $\mathbb{H}^{0,\lambda}[a,b]$ into $\mathbb{H}^{0,\lambda+\alpha}[a,b]$ as we expected. This concludes the proof. □

We should note that the order of the space, i.e., the exponent $\alpha+\lambda$ in Theorem 5, is optimal. We will illustrate this with an illuminating example.

Example 2. *Define $x \in \mathbb{H}^{0,\lambda}[0,1]$ by $x(t) := t^\lambda$, $\lambda \in (0,1)$. It is easy to calculate that*

$$\Im^{\alpha,0}_{0,t} x(t) = \frac{\Gamma(1+\lambda)}{\Gamma(1+\alpha+\lambda)} t^{\alpha+\lambda}.$$

Clearly, $\Im^{\alpha,0}_{0,t} x \in \mathbb{H}^{0,\alpha+\lambda}[0,1]$. However, $\Im^{\alpha,0}_{0,t} x \notin \mathbb{H}^{0,\gamma}[0,1]$ for any $\gamma > \alpha+\lambda$. In general, in light of this example, Lemma 2 tells us that it may be the case that $\Im^{\alpha,\mu}_{a,g} : \mathbb{H}^{0,\lambda}[a,b] \to \mathbb{H}^{0,\lambda+\alpha}[a,b] \setminus \mathbb{H}^{0,\gamma}[a,b]$ with $\gamma > \alpha+\lambda$.

Corollary 1. *Let $\alpha \in (0,1)$. Let $g \in C^1[a,b]$ be a positive increasing function such that $g'(t) \neq 0$, for all $t \in [a,b]$. If $f \in \mathbb{H}^{0,\lambda}[a,b]$ with $\lambda+\alpha < 1$, then there exists a unique solution for the equation $x = f + \Im^{\alpha,\mu}_{a,g} x$ in the space $\mathbb{H}^{0,\lambda}[a,b]$.*

Proof. Based on the Banach fixed-point theorem, it is easy to see that the equation $x = f + \Im^{\alpha,\mu}_{a,g} x$ admits a unique continuous solution x.

We proceed by induction to show that $x \in \mathbb{H}^{0,\lambda_k}[a,b]$, with $\lambda_k := \min\{\lambda, k\alpha\}$, $k \in \mathbb{N}$:

$k = 1:$ From Theorem 2 and Lemma 2, we know that $x = f + \Im^{\alpha,\mu}_{a,g} x \in \mathbb{H}^{\lambda_1}[a,b]$. Since $x(a) = f(a) + \Im^{\alpha,\mu}_{a,g} x(a) = 0$, it follows that $x = f + \Im^{\alpha,\mu}_{a,g} x \in \mathbb{H}^{0,\lambda_1}[a,b]$.

$k \to k+1:$ If $x \in \mathbb{H}^{0,\lambda_k}[a,b]$, then it follows from Lemma 2, $\Im^{\alpha,\mu}_{a,g} x \in \mathbb{H}^{0,\alpha+\lambda_k}[a,b]$. Noting that $\alpha + \lambda_k = \min\{\alpha+\lambda, \alpha(k+1)\} \geq \min\{\lambda, \alpha(k+1)\} = \lambda_{k+1}$, results in $\Im^{\alpha,\mu}_{a,g} x \in \mathbb{H}^{0,\lambda_{k+1}}[a,b]$ and $x = f + \Im^{\alpha,\mu}_{a,g} x \in \mathbb{H}^{0,\lambda_{k+1}}[a,b]$ is true for every $k \in \mathbb{N}$.

From this, it follows that there exists a unique continuous solution to $x = f + \Im^{\alpha,\mu}_{a,g} x$ in $\mathbb{H}^{0,\lambda}[a,b]$ as required. □

Our next step, again following the idea of Hardy and Littlewood ([8]), is to prove that

Lemma 3. *Let $0 < \alpha+\lambda < 1$ and let $g \in C^1[a,b]$ be a positive increasing function such that $g'(t) \neq 0$, for all $t \in [a,b]$. For $x \in \mathbb{H}^{0,\lambda+\alpha}[a,b]$ we have $\mathfrak{D}^{\alpha,\mu}_{a,g} x \in \mathbb{H}^{0,\lambda+\alpha}[a,b]$ and it holds*

$$\begin{aligned} &\mathfrak{D}^{\alpha,\mu}_{a,g} x(t) \\ = &\frac{1}{\Gamma(1-\alpha)} \int_a^t \Big[\alpha e^{-\mu(g(t)-g(s))}(g(t)-g(s))^{-1-\alpha}[x(t)-x(s)] \\ &\qquad + \mu x(t) e^{-\mu(g(t)-g(s))}(g(t)-g(s))^{-\alpha}\Big] g'(s)\, ds. \end{aligned}$$

Proof. Let $x \in \mathbb{H}^{0,\lambda}[a,b]$. We have $\mathfrak{D}^{\alpha,\mu}_{a,g} x(a) = 0$. Now for $\epsilon \in (0, b-a]$ and $t \in [a,b]$ we define:

$$\begin{aligned} y_\epsilon(t) &:= \int_a^{t-\epsilon} e^{-\mu(g(t)-g(s))} \frac{x(s) g'(s)\, ds}{|g(t)-g(s)|^\alpha} \\ &= \int_{a+\epsilon}^t e^{-\mu(g(t)-g(s-\epsilon))}(g(t)-g(s-\epsilon))^{-\alpha} x(s-\epsilon) g'(s-\epsilon)\, ds. \end{aligned}$$

Obviously, after the substitution $s \to g(t) - g(s)$,

$$
\begin{aligned}
\left\|\Gamma(1-\alpha)\Im_{a,g}^{1-\alpha,\mu} x - y_\epsilon\right\| &\leq \max_{t\in[a,b]} \int_{t-\epsilon}^{t} e^{-\mu(g(t)-g(s))} \frac{|x(s)|\cdot|g'(s)|\,ds}{|g(t)-g(s)|^{\alpha}} \\
&\leq \max_{t\in[a,b]} \|x\| \int_0^{g(t)-g(t-\epsilon)} e^{-\mu s} s^{-\alpha}\,ds \\
&\leq \max_{t\in[a,b]} \frac{\|x\|}{1-\alpha} (g(t)-g(t-\epsilon))^{1-\alpha} \\
&\leq \frac{\|x\|\|g'\|^{1-\alpha}}{1-\alpha} \epsilon^{1-\alpha} \to 0 \qquad \text{as} \quad \epsilon \to 0,
\end{aligned}
$$

where we used the mean value theorem and the fact that g is increasing. Since y_ϵ approximates our fractional integral, we have to show that its derivative approximates the expected derivative, which can be obtained as a limit when $\epsilon \to 0$.

We have

$$
\begin{aligned}
y'_\epsilon(t) = {} & e^{-\mu(g(t)-g(t-\epsilon))}(g(t)-g(t-\epsilon))^{-\alpha} x(t-\epsilon) g'(t-\epsilon) \\
- {} & \int_{a+\epsilon}^{t} e^{-\mu(g(t)-g(s-\epsilon))} \Big[\alpha(g(t)-g(s-\epsilon))^{-\alpha-1} + \mu(g(t)-g(s-\epsilon))^{-\alpha}\Big] x(s-\epsilon) \\
& \qquad \cdot g'(s-\epsilon) g'(t)\,ds \\
= {} & e^{-\mu(g(t)-g(t-\epsilon))}(g(t)-g(t-\epsilon))^{-\alpha} x(t-\epsilon) g'(t-\epsilon) \\
- {} & \int_{a}^{t-\epsilon} e^{-\mu(g(t)-g(s))} \Big[\alpha(g(t)-g(s))^{-\alpha-1} + \mu(g(t)-g(s))^{-\alpha}\Big] x(s) g'(s) g'(t)\,ds \\
= {} & \Big(e^{-\mu(g(t)-g(t-\epsilon))}(g(t)-g(t-\epsilon))^{-\alpha} x(t-\epsilon) g'(t-\epsilon) \\
- {} & \int_{a}^{t-\epsilon} x(t) \Big[\alpha e^{-\mu(g(t)-g(s))}(g(t)-g(s))^{-\alpha-1}\Big] g'(s) g'(t)\,ds\Big) \\
+ {} & \Big(\int_{a}^{t-\epsilon} x(t) \Big[\alpha e^{-\mu(g(t)-g(s))}(g(t)-g(s))^{-\alpha-1}\Big] g'(s) g'(t)\,ds \\
- {} & \int_{a}^{t-\epsilon} e^{-\mu(g(t)-g(s))} \Big[\alpha(g(t)-g(s))^{-\alpha-1} + \mu(g(t)-g(s))^{-\alpha}\Big] x(s) g'(s) g'(t)\,ds \\
= {} & \Big(e^{-\mu(g(t)-g(t-\epsilon))}(g(t)-g(t-\epsilon))^{-\alpha} x(t-\epsilon) g'(t-\epsilon) \\
- {} & \int_{a}^{t-\epsilon} x(t) \Big[\alpha e^{-\mu(g(t)-g(s))}(g(t)-g(s))^{-\alpha-1}\Big] g'(s) g'(t)\,ds\Big) \\
+ {} & \int_{a}^{t-\epsilon} [x(t)-x(s)] \Big[\alpha e^{-\mu(g(t)-g(s))}(g(t)-g(s))^{-\alpha-1}\Big] g'(s) g'(t)\,ds \\
- {} & \mu \int_{a}^{t-\epsilon} x(s) e^{-\mu(g(t)-g(s))}(g(t)-g(s))^{-\alpha} g'(s) g'(t)\,ds \\
= {} & \Big(e^{-\mu(g(t)-g(t-\epsilon))}(g(t)-g(t-\epsilon))^{-\alpha} x(t-\epsilon) g'(t-\epsilon) \\
- {} & \int_{a}^{t-\epsilon} x(t) \Big[\alpha e^{-\mu(g(t)-g(s))}(g(t)-g(s))^{-\alpha-1}\Big] g'(s) g'(t)\,ds\Big) \\
+ {} & \int_{a}^{t-\epsilon} [x(t)-x(s)] \Big[\alpha e^{-\mu(g(t)-g(s))}(g(t)-g(s))^{-\alpha-1}\Big] g'(s) g'(t)\,ds \\
+ {} & \mu \int_{t-\epsilon}^{t} x(s) e^{-\mu(g(t)-g(s))}(g(t)-g(s))^{-\alpha} g'(s) g'(t)\,ds - \mu g'(t)\Gamma(1-\alpha)\Im_{a,g}^{1-\alpha,\mu} x(t).
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
y'_\epsilon(t) \quad &+ \quad \mu g'(t)\Gamma(1-\alpha)\Im^{1-\alpha,\mu}_{a,t}x(t) \\
&= \Big(e^{-\mu(g(t)-g(t-\epsilon))}(g(t)-g(t-\epsilon))^{-\alpha}x(t-\epsilon)g'(t-\epsilon) \\
&- \int_a^{t-\epsilon} x(t)\Big[\alpha e^{-\mu(g(t)-g(s))}(g(t)-g(s))^{-\alpha-1}\Big]g'(s)g'(t)\,ds\Big) \\
&+ \int_a^{t-\epsilon}[x(t)-x(s)]\Big[\alpha e^{-\mu(g(t)-g(s))}(g(t)-g(s))^{-\alpha-1}\Big]g'(s)g'(t)\,ds \\
&+ \mu\int_{t-\epsilon}^{t} x(s)e^{-\mu(g(t)-g(s))}(g(t)-g(s))^{-\alpha}g'(s)g'(t)\,ds \\
&= \int_a^{t-\epsilon}[x(t)-x(s)]\Big[\alpha e^{-\mu(g(t)-g(s))}(g(t)-g(s))^{-\alpha-1}\Big]g'(s)g'(t)\,ds \\
&+ \mu\int_{t-\epsilon}^{t} x(s)e^{-\mu(g(t)-g(s))}(g(t)-g(s))^{-\alpha}g'(s)g'(t)\,ds + J_2(t),
\end{aligned}
$$

where

$$
\begin{aligned}
J_2(t) \quad &= \quad x(t-\epsilon)g'(t-\epsilon)e^{-\mu(g(t)-g(t-\epsilon))}(g(t)-g(t-\epsilon))^{-\alpha} \\
&+ \int_a^{t-\epsilon} x(t)\Big[\alpha e^{-\mu(g(t)-g(s))}(g(t)-g(s))^{-\alpha}\Big]g'(s)g'(t)\,ds.
\end{aligned}
$$

Define $\Lambda(t) := I(t) + J(t)$, where

$$
I(t) := \mu x(t)\int_a^t e^{-\mu(g(t)-g(s))}(g(t)-g(s))^{-\alpha}g'(s)\,ds.
$$

$$
J(t) := \alpha\int_a^t [x(t)-x(s)]\Big[e^{-\mu(g(t)-g(s))}(g(t)-g(s))^{-\alpha-1}\Big]g'(s)\,ds.
$$

Note that the right-hand side of the formula in the thesis of this Lemma is of the form Λ, and functions I and J describe its parts. Note that $g(t)-g(t-\epsilon) \geq c\epsilon, \quad c := \min_{[a,b]}|g'|$, which implies

$$
\begin{aligned}
&\Big|y'_\epsilon(t) + \mu g'(t)\Gamma(1-\alpha)\Im^{1-\alpha,\mu}_{a,t}x(t) - g'(t)\Lambda(t)\Big| \\
&\leq \quad |x(t-\epsilon)g'(t-\epsilon)|\Big|\Big[e^{-\mu(g(t)-g(t-\epsilon))}(g(t)-g(t-\epsilon))^{-\alpha}\Big| \\
&+ \quad g'(t)\int_{t-\epsilon}^{t}\frac{|x(s)-x(t)|}{e^{\mu(g(t)-g(s))}}\Big[\mu(g(t)-g(s))^{-\alpha} - \alpha(g(t)-g(s))^{-\alpha-1}\Big]g'(s)\,ds \\
&\leq \quad |x(t-\epsilon)g'(t-\epsilon)|c^{-\alpha}\epsilon^{\alpha} \\
&+ \quad \|g'\|^2[x]_{\lambda+\alpha}\int_{t-\epsilon}^{t}(t-s)^{\lambda+\alpha}\Big[\mu c^{-\alpha}(t-s)^{-\alpha} + \alpha(\min_{[a,b]}|g'|)^{-\alpha-1}(t-s)^{-\alpha-1}\Big]\,ds \\
&\leq \quad |x(t-\epsilon)g'(t-\epsilon)|c^{-\alpha}\epsilon^{\lambda} \\
&+ \quad \|g'\|^2[x]_{\lambda+\alpha}\left[\frac{\mu\epsilon^{\lambda}}{(1+\lambda)c^{\alpha}} + \frac{\alpha\epsilon^{\lambda}}{\lambda c^{1-\lambda}}\right].
\end{aligned} \tag{21}
$$

Taking the limit with $\epsilon \to 0$, we conclude that

$$
|x(t-\epsilon)g'(t-\epsilon)e^{-\mu(g(t)-g(t-\epsilon))}(g(t)-g(t-\epsilon))^{\lambda}| \to 0
$$

as $\epsilon \to 0$, and since $1-\lambda > 0$, using the inequality (21), we obtain the desired property

$$
\Big|y'_\epsilon(t) + \mu g'(t)\Gamma(1-\alpha)\Im^{1-\alpha,\mu}_{a,g}x(t) - g'(t)\Lambda(t)\Big| \;\to\; 0 \;\text{ as }\; \epsilon \to 0.
$$

Now, we would like to show that $\Lambda(\cdot) \in \mathbb{H}^{0,\lambda+\alpha}[a,b]$.

Additionally, for $h > 0$, $t, t+h \in [a,b]$, after substitutions $s \mapsto g(t+h) - g(s)$ and $s \mapsto g(t) - g(s)$, we obtain

$$
\begin{aligned}
|I(t+h)-I(t)| &= \left|\mu x(t+h)\int_a^{t+h} e^{-\mu(g(t+h)-g(s))}(g(t+h)-g(s))^{-\alpha}g'(s)\,ds\right.\\
&- \left.\mu x(t)\int_a^{t} e^{-\mu(g(t)-g(s))}(g(t)-g(s))^{-\alpha}g'(s)\,ds\right|\\
&= \left|\mu x(t+h)\int_0^{g(t+h)} e^{-\mu s}s^{-\alpha}\,ds-\mu x(t)\int_0^{g(t)} e^{-\mu s}s^{-\alpha}\,ds\right|\\
&\leq \mu|x(t+h)-x(t)|\int_0^{g(t)} e^{-\mu s}s^{-\alpha}\,ds+\mu x(t+h)\int_{g(t)}^{g(t+h)} e^{-\mu s}s^{-\alpha}\,ds\\
&\leq \mu|x(t+h)-x(t)|\int_0^{g(t)} s^{-\alpha}\,ds+\mu\|x\|\int_{g(t)}^{g(t+h)} s^{-\alpha}\,ds\\
&\leq \mu[x]_{\lambda+\alpha}h^{\lambda+\alpha}\frac{\|g'\|^{1-\alpha}}{1-\alpha}+\mu\|x\|\frac{g(t+h)^{1-\alpha}-g(t)^{1-\alpha}}{\alpha}\\
&\leq \frac{\mu\|g'\|^{\alpha}}{\alpha}\Big[[x]_{\lambda+\alpha}h^{\lambda+\alpha}+\|x\|h^{\lambda}\Big].
\end{aligned}
$$

From which, due to $0<\lambda+\alpha<1$, we know that $\mathbb{H}^{\alpha}[a,b]\subset\mathbb{H}^{\lambda+\alpha}[a,b],\mathbb{H}^{\lambda}[a,b]\subset\mathbb{H}^{\lambda+\alpha}[a,b]$. Thus, we conclude that $I(\cdot)\in\mathbb{H}^{\lambda+\alpha}[a,b]$.

Similarly, we have

$$
\begin{aligned}
|J(t+h)-J(t)| &= \alpha\left|\int_a^{t+h}[x(t+h)-x(s)]\Big[e^{-\mu(g(t+h)-g(s))}(g(t+h)-g(s))^{-\alpha-1}\Big]g'(s)\,ds\right.\\
&\qquad\left.-\int_a^{t}[x(t)-x(s)]\Big[e^{-\mu(g(t)-g(s))}(g(t)-g(s))^{-\alpha-1}\Big]g'(s)\,ds\right|\\
&= \alpha\left|\int_0^{h}[x(t+h)-x(t+h-s)]\Big[e^{-\mu(g(t+h)-g(t+h-s))}(g(t+h)-g(t+h-s))^{-\alpha-1}\Big]\right.\\
&\qquad\cdot g'(t+h-s)\,ds\\
&\quad-\int_h^{t+h}[x(t)-x(t+h-s)]\Big[e^{-\mu(g(t)-g(t+h-s))}(g(t)-g(t+h-s))^{-\alpha-1}\Big]\\
&\qquad\cdot g'(t+h-s)\,ds|\\
&\leq \alpha\left|\int_0^{h}[x(t+h)-x(t+h-s)]\Big[e^{-\mu(g(t+h)-g(t+h-s))}(g(t+h)-g(t+h-s))^{-\alpha-1}\Big]\right.\\
&\qquad\cdot g'(t+h-s)\,ds\\
&\quad+\int_h^{t+h}[x(t+h)-x(t)]\Big[e^{-\mu(g(t+h)-g(t+h-s))}(g(t+h)-g(t+h-s))^{-\alpha-1}\Big]\\
&\qquad\cdot g'(t+h-s)\,ds\\
&\quad+\int_h^{t+h}[x(t)-x(t+h-s)]\Big[e^{-\mu(g(t+h)-g(t+h-s))}(g(t+h)-g(t+h-s))^{-\alpha-1}\Big]\\
&\qquad\cdot g'(t+h-s)\,ds\\
&\quad-\int_h^{t+h}[x(t)-x(t+h-s)]\Big[e^{-\mu(g(t)-g(t+h-s))}(g(t)-g(t+h-s))^{-\alpha-1}\Big]\\
&\qquad\cdot g'(t+h-s)\,ds\\
&\leq |A|+|B|+|C|,
\end{aligned}
$$

where

$$
|A|:=[x]_{\lambda+\alpha}\|g'\|\int_0^{h}s^{\lambda+\alpha}\alpha e^{-\mu cs}(cs)^{\alpha-1}\,ds\leq\frac{\alpha\|g'\|[x]_{\lambda+\alpha}h^{\lambda}}{c^{1+\alpha}},\quad c:=\min_{\xi\in[a,b]}|g'(\xi)|,
$$

$$
|B|:=[x]_{\lambda+\alpha}\|g'\|h^{\lambda+\alpha}\int_h^{t+h}\Big(\alpha e^{-\mu cs}(cs)^{-\alpha-1}\Big)\,ds\leq\frac{\|g'\|[x]_{\lambda+\alpha}h^{\lambda}}{c^{1+\alpha}},
$$

and

$$
\begin{aligned}
|C| \quad := \quad & \|g'\|[x]_{\lambda+\alpha} \int_h^{t+h} |s-h|^{\lambda+\alpha} \alpha \Big| e^{-\mu g'(\zeta_1)(s-h)} (g'(\zeta_1)(s-h)^{-\alpha-1} \\
& -e^{-\mu g'(\zeta_2)s}(g'(\zeta_2)s)^{-\alpha-1} \Big| \, ds,
\end{aligned}
$$

$$
\zeta_1 \in (t+h-s, t+h), \quad \zeta_2 \in (t+h-s, t), \quad s > h, \quad \zeta_1 \to \zeta_2 \text{ as } h \to 0.
$$

Moreover, due to $g(t+h) - g(t) \le g(t+h) - g(t+h-s), \;\; s > h$, we have

$$
\begin{aligned}
g'(\xi_1) - g'(\xi_2) \quad &= \quad \frac{g(t+h) - g(t+h-s)}{s} - \frac{g(t) - g(t+h-s)}{s-h} \\
&= \quad \frac{s[g(t+h) - g(t)] - h[g(t+h) - g(t+h-s)]}{s(s-h)} \ge 0, \\
&\Longrightarrow \quad g'(\xi_1) \ge g'(\xi_2).
\end{aligned}
$$

Thus, after the substitution $s \mapsto s/h$ we obtain

$$
\begin{aligned}
|C| \quad \le \quad & \|g'\|[x]_{\lambda+\alpha} \int_1^{1+\frac{t}{h}} h^{\lambda+\alpha} |s-1|^{\lambda+\alpha} \alpha \Big| h^{\alpha-1} e^{-\mu h g'(\zeta_1)(s-1)} (g'(\zeta_1)(s-1)^{-\alpha-1} \\
& \qquad -h^{\alpha-1} e^{-\mu h g'(\zeta_2)s}(g'(\zeta_2)s)^{-\alpha-1} \Big| h \, ds \\
= \quad & [x]_{\lambda+\alpha} \|g'\|^2 \alpha h^\lambda \Xi,
\end{aligned}
$$

where

$$
\begin{aligned}
\Xi \quad := \quad & \int_1^{1+\frac{t}{h}} |s-1|^{\lambda+\alpha} \Big| e^{-\mu h g'(\zeta_1)(s-1)} (g'(\zeta_1)(s-1))^{-\alpha-1} - e^{-\mu h g'(\zeta_2)s}(g'(\zeta_2)s)^{-\alpha-1} \Big| \, ds \\
\le \quad & \int_1^{1+\frac{t}{h}} s^{\lambda+\alpha} \Bigg| \frac{(s-1)^{-\alpha-1}}{(g'(\zeta_1))^{\alpha+1} e^{\mu h g'(\zeta_1)(s-1)}} - \frac{s^{-\alpha-1}}{(g'(\zeta_1))^{\alpha+1} e^{\mu h g'(\zeta_1)(s-1)}} \\
& + \frac{s^{-\alpha-1}}{(g'(\zeta_1))^{\alpha+1} e^{\mu h g'(\zeta_1)(s-1)}} - \frac{s^{-\alpha-1}}{(g'(\zeta_2))^{\alpha+1} e^{\mu h g'(\zeta_2)s}} \Bigg| \, ds \\
\le \quad & \int_1^{1+\frac{t}{h}} \frac{\big|(s-1)^{-\alpha-1} - s^{-\alpha-1}\big| s^{\lambda+\alpha} \, ds}{(g'(\zeta_1))^{\alpha+1} e^{\mu h g'(\zeta_1)(s-1)}} \\
& + \int_1^{1+\frac{t}{h}} \Bigg| \frac{1}{(g'(\zeta_1))^{\alpha+1} e^{\mu h g'(\zeta_1)(s-1)}} - \frac{1}{(g'(\zeta_2))^{\alpha+1} e^{\mu h g'(\zeta_2)s}} \Bigg| s^{\lambda-1} \, ds \\
\le \quad & \int_1^{1+\frac{t}{h}} \frac{(1-\alpha) s^{\lambda+\alpha-1} \, ds}{(g'(\zeta_1))^{\alpha+1}} \\
& + \int_1^{1+\frac{t}{h}} \Bigg| \frac{e^{\mu h g'(\zeta_1)}}{(g'(\zeta_1))^{\alpha+1} e^{\mu h g'(\zeta_1)s}} - \frac{1}{(g'(\zeta_2))^{\alpha+1} e^{\mu h g'(\zeta_2)s}} \Bigg| s^{\lambda-1} \, ds.
\end{aligned}
$$

Hence,

$$
\Xi \le \frac{(1-\alpha)}{\lambda (g'(\zeta_1))^{1+\alpha}} + \widehat{\Xi},
$$

where

$$
\begin{aligned}
\widehat{\Xi} \quad := \quad & \left| \frac{e^{\mu h g'(\zeta_1)}}{(g'(\zeta_1))^{\alpha+1}} \int_1^{1+\frac{t}{h}} e^{-\mu h g'(\zeta_1)s} s^{\lambda-1} \, ds - \frac{1}{(g'(\zeta_2))^{\alpha+1}} \int_1^{1+\frac{t}{h}} e^{-\mu h g'(\zeta_2)s} s^{\lambda-1} \, ds \right| \\
\le \quad & \left| \frac{e^{\mu h g'(\zeta_1)}}{(g'(\zeta_1))^{\lambda}} - \frac{1}{(g'(\zeta_2))^{\lambda}} \right| \frac{\Gamma(\lambda)}{(\mu h)^{-\lambda}}.
\end{aligned}
$$

Thus, due to $\xi_1 \to \xi_2$ as $h \to 0$ and the continuity of g', we know that

$$\lim_{h\to 0}\frac{e^{\mu h g'(\xi_1)}(g'(\xi_1))^{1-\lambda}-(g'(\xi_2))^{1-\lambda}}{(\mu h)^{-\lambda}}=\lim_{h\to 0}\frac{e^{\mu h g'(\xi_1)}\mu(\mu h)^{1-\lambda}}{-\lambda(g'(\xi_1))^{\lambda}}=0,$$

Accordingly, we conclude that

$$\left|\frac{g'(t+h)e^{\mu h g'(\zeta_1)}}{(g'(\zeta_1))^{-\lambda}}-\frac{g'(t)}{(g'(\zeta_2))^{-\lambda}}\right|\frac{\Gamma(\lambda)}{(\mu h)^{-\lambda}}<\infty,\text{ for any } h>0.$$

That is, $\widehat{\Xi}$ (hence Ξ) is finite; hence $J(\cdot)\in\mathbb{H}^{0,\lambda+\alpha}[a,b]$. Thus, as we already mentioned, this property is also true for $\Lambda(\cdot)=I(\cdot)+J(\cdot)$ and finally $\Lambda\in\mathbb{H}^{0,\lambda+\alpha}[a,b]$.

In summary, for any $t\in[a,b]$ we have

$$y_\epsilon\to\Gamma(1-\alpha)\Im_{a,g}^{1-\alpha,\mu}x\quad\text{and } y'_\epsilon\to g'(t)\Lambda-\mu g'(t)\Gamma(1-\alpha)\Im_{a,g}^{1-\alpha,\mu}x\ \text{ uniformly in } [a,b].$$

Therefore,

$$\frac{1}{g'(t)}\frac{d}{dt}\Im_{a,g}^{1-\alpha,\mu}x=\frac{\Lambda}{\Gamma(1-\alpha)}-\mu\Im_{a,g}^{1-\alpha,\mu}x\ \Rightarrow\ \mathfrak{D}_{a,g}^{\alpha,\mu}x=\frac{\Lambda}{\Gamma(1-\alpha)}\in\mathbb{H}^{0,\lambda+\alpha}[a,b].$$

□

Now we have achieved our goal:

Proof of Theorem 5.

- $\Im_{a,g}^{\alpha,\mu}$ is injective. Indeed, let $x,y\in C[a,b]$ such that $\Im_{a,g}^{\alpha,\mu}x(t)=\Im_{a,g}^{\alpha,\mu}y(t)$, for all $t\in[a,b]$ and define $z:=x-y$. From the semi-group property we obtain

$$\Im_{a,t}^{\alpha,\mu}z(t)=0\Rightarrow 0=\Im_{a,g}^{1-\alpha,\mu}\Im_{a,g}^{\alpha,\mu}z(t)=\Im_{a,g}^{1,\mu}z(t)=\int_a^t e^{-\mu(g(t)-g(s))}g'(s)z(t)\,ds$$

for almost every $t\in[a,b]$. It follows that $z(t)=0$ for almost all $t\in[a,b]$ (even for all $t\in[a,b]$ because of the continuity of z). Thus, $x(t)=y(t)$ for all $t\in[a,b]$.
- $\Im_{a,g}^{\alpha,\mu}$ is surjective with right inverse $\mathfrak{D}_{a,g}^{\alpha,\mu}$. To see this, it suffices to show that for all $x\in\mathbb{H}^{0,\lambda+\alpha}[a,b]$ we have $\Im_{a,g}^{\alpha,\mu}\mathfrak{D}_{a,g}^{\alpha,\mu}x=x$, where $y:=\mathfrak{D}_{a,g}^{\alpha,\mu}x\in\mathbb{H}^{0,\lambda}[a,b]$. From Lemma 3 we know that y is well-defined and $y\in\mathbb{H}^{0,\lambda}[a,b]$. Thus, in light of our definition that $\Im_{a,g}^{2-\alpha,\mu}x(a)=0$), and using integration by parts, we obtain the following:

$$\begin{aligned}\Im_{a,g}^{1-\alpha,\mu}\Im_{a,g}^{\alpha,\mu}y(t) &= \Im_{a,g}^{1,\mu}\mathfrak{D}_{a,g}^{\alpha,\mu}x(t)\\ &= \int_a^t e^{-\mu(g(t)-g(s))}\left[\mu\Im_{a,g}^{1-\alpha,\mu}x(s)+\frac{1}{g'(s)}\left(\Im_{a,g}^{1-\alpha,\mu}x(s)\right)'\right]g'(s)\,ds\\ &= \mu\Im_{a,g}^{2-\alpha,\mu}x(t)+\left[\left(e^{-\mu(g(t)-g(s))}\Im_{a,g}^{1-\alpha,\mu}x(s)\right)_a^t\right.\\ &\qquad\left.-\mu\int_a^t e^{-\mu(g(t)-g(s))}\Im_{a,g}^{1-\alpha,\mu}x(s)g'(s)\,ds\right]\\ &= \mu\Im_{a,g}^{2-\alpha,\mu}x(t)+\Im_{a,g}^{1-\alpha,\mu}x(t)-\mu\Im_{a,g}^{2-\alpha,\mu}x(t)=\Im_{a,g}^{1-\alpha,\mu}x(t).\end{aligned}$$

Consequently, for all $t\in[a,b]$ we obtain

$$\Im_{a,g}^{1-\alpha,\mu}\left[\Im_{a,g}^{\alpha,\mu}y(t)-x(t)\right]=0\implies\Im_{a,g}^{\alpha,\mu}y(t)=x(t)\text{ for every } t\in[a,b].$$

- Since $\Im_{a,g}^{\alpha,\mu}$ is bijective, the right and right inverse of $\Im_{a,g}^{\alpha,\mu}$ are the same (and both are equal to $\left(\Im_{a,g}^{\alpha,\mu}\right)^{-1}$):

$$\mathfrak{D}_{a,g}^{\alpha,\mu}\Im_{a,g}^{\alpha,\mu}=\left(\Im_{a,g}^{\alpha,\mu}\right)^{-1}\Im_{a,g}^{\alpha,\mu}\mathfrak{D}_{a,g}^{\alpha,\mu}\Im_{a,g}^{\alpha,\mu}=\left(\Im_{a,g}^{\alpha,\mu}\right)^{-1}\Im_{a,g}^{\alpha,\mu}=id.$$

Since $\mathbb{H}^{0,\lambda}[a,b]$ and $\mathbb{H}^{0,\lambda+\alpha}[a,b]$ are Banach spaces, the continuity follows from the continuous theorem for operators from $\mathfrak{D}^{\alpha,\mu}_{a,g} = \left(\Im^{\alpha,\mu}_{a,g}\right)^{-1}$ of $\Im^{\alpha,\mu}_{a,g}$.

The following facts are direct consequences of Theorem 5:

Fact 1: There are non-differentiable functions having a Riemann–Liouville fractional tempered derivatives of all orders less than 1. This fact generalizes similar results proved by B. Ross et al. in [22] (see also [23,24]).

Fact 2: Outside of the space of absolutely continuous functions, the equivalence of the fractional integral equations and the corresponding tempered-Caputo differential problem is no longer necessarily true, even in the case of Hölder spaces.

Fact 3: There exists $x \in C[a,b]$ such that $\Im^{\beta,\mu}_{a,g}x$, $\beta \in (0,1)$ is not absolutely continuous on $[a,b]$. This fact generalizes similar results proved by J.L. Webb in [21].

□

It seems like a good place to find that a search of the keywords *Caputo fractional differential problems* will yield a number of specialized manuscripts (e.g., [25–31] in the case of real-valued functions and [32,33] in abstract spaces) on this topic. Unfortunately, by virtue of the assertion of **Fact 2**, most of these manuscripts contain an error in the proof of the equivalence of the fractional-type differential problems and the corresponding integral forms. However, we will modify (slightly) our definition of the g-Caputo tempered fractional differential operators to avoid such an equivalence problem. We also note that according to **Fact 3**, even in the context of generalized fractional operators, we answered the following question posed by Hardy and Littlewood (cf. [8,21]), originally formulated for the case of the Riemann–Liouville fractional operator:
Does there exist a continuous x for which $\Im^{\alpha,\mu}_{a,g}x$ is not absolutely continuous?

1. Proof of **Fact 1**. Let $\alpha \in (0,1)$ and fix $\lambda \in (0, 1-\alpha)$. Since the Hölder spaces of any order contain continuous functions that are nowhere differentiable, there exists a continuous nowhere differentiable function on $[a,b]$ (for example, the well-known Weierstrass function) $y \in \mathbb{H}^{0,\lambda+\alpha}[a,b]$. According to Theorem 5, we know that there exists $x \in \mathbb{H}^{0,\lambda}[a,b]$ such that $\Im^{\alpha,\mu}_{a,g}x = y$. From this, we can deduce that

$$\begin{aligned}\mathfrak{D}^{\alpha,\mu}_{a,g}y &= \left(\mu + \frac{1}{g'(t)}\frac{d}{dt}\right)\Im^{1-\alpha,\mu}_{a,g}y = \left(\mu + \frac{1}{g'(t)}\frac{d}{dt}\right)\Im^{1-\alpha,\mu}_{a,g}\Im^{\alpha,\mu}_{a,g}x \\ &= \left(\mu + \frac{1}{g'(t)}\frac{d}{dt}\right)\Im^{1,\mu}_{a,g}x = x,\end{aligned}$$

is meaningful. This gives rise to the statement that there are functions that do not have a first-order derivative, but have a Riemann–Liouville fractional tempered derivative of all orders less than one.

2. Proof of **Fact 2**. In what follows, we will show that even in the context of Hölder-continuous functions the converse implication from fractional integral equations to the corresponding Caputo fractional tempered differential equations is no longer necessarily true. To see this, let $\mathcal{W}$ be a Hölder-continuous of some order $\lambda < 1$, but nowhere differentiable function on $[a,b]$. According to the assertion of ***Fact 1***, there is a constant $\gamma := \alpha - m + 1 \in (0,1)$, $m = 1, 2, \cdots$ depending only on λ and a continuous function f such that $\Im^{\alpha-m+1,\mu}_{a,g}f = \mathcal{W}$.

   In this connection, let us consider the following (Caputo-type) fractional differential problem:

$$\frac{\mathfrak{d}}{\mathfrak{d}\mathfrak{t}}{}^{\alpha,\mu}_{a,g}x(t) = f(t), \;\; t \in [a,b], \quad \alpha \in (m-1, m], \;\; m = 1, 2, \cdots \tag{22}$$

   combined with appropriate initial or boundary value conditions. Regarding the functions x and g and the constants α and μ, we will make the same assumptions as throughout the article. Since we know that

$$\frac{\partial}{\partial t}{}^{\alpha,\mu}_{a,g} x(t) = e^{-\mu g(t)} \frac{\partial}{\partial t} e^{\mu g(t)} x(t),$$

where $\frac{\partial}{\partial t}$ means the classical Caputo fractional derivative with respect to a function g, then (22) reads as

$$\frac{\partial}{\partial t} e^{\mu g(t)} x(t) = e^{\mu g(t)} f(t), \ \ t \in [a,b], \ \ \alpha \in (m-1,m], \ \ m = 1,2,\cdots$$

from which, for any $f \in C[a,b]$, the integral form of the problem (22) is as follows (cf. [34] (Chapter 3)):

$$x(t) = \sum_{j=1}^{m} c_j e^{-\mu g(t)} (g(t))^{m-j} + \Im^{\alpha,\mu}_{a,g(t)} f(t), \quad t \in [a,b], \tag{23}$$

where $c_j, j = 1,2,\cdots,m$ are arbitrary constants depending only on the boundary or initial conditions.

Accordingly, in view of the following equality

$$\mathbf{d}\Im^{1,\mu}_{a,g} f(t) = \left(\mu + \frac{1}{g(t)}\mathfrak{D}\right) \int_a^t e^{-\mu(g(t)-g(s))} f(s) g'(s)\, ds = f, \text{ holds for any } f \in C[a,b],$$

we conclude that

$$\mathbf{d}^{m-1} x(t) = \left(\mu + \frac{1}{g(t)}\mathfrak{D}\right)^{m-1} x(t) = m! e^{-\mu g(t)} c_1 + \Im^{\alpha-m+1,\mu}_{a,g} f(t), \quad t \in [a,b].$$

Operating both sizes by $\mathbf{d}$, we obtain

$$\mathbf{d}^m x(t) = 0 + \mathbf{d}\Im^{1+(\alpha-m),\mu}_{a,g} f(t) = \mathbf{d}\mathcal{W}.$$

Since $\mathcal{W}$ is nowhere differentiable, then

$$\frac{\partial}{\partial t}{}^{\alpha,\mu}_{a,g} x = \Im^{m-\alpha,\mu}_{a,g} \mathbf{d}^m x = \Im^{m-\alpha,\mu}_{a,g} \mathbf{d}\mathcal{W},$$

is "meaningless". That is, the equivalence between (22) and (23) is not true in this case.

3. Proof of ***Fact* 3.** It is a direct consequence of ***Fact* 2**: Let $\beta = m - \alpha + 1$. We obtain

$$f \in C[a,b] \ \text{ and } \ \Im^{\beta,\mu}_{a,g} f = \mathcal{W} \notin AC[a,b].$$

4. Fractional Differential Equations with g-Caputo Tempered Fractional Derivative

In this section we proceed to avoid the equivalence problem between differential boundary value problems of the g-Caputo tempered-type of fractional orders $\alpha > 1$ and the corresponding integral forms. To do so, we modify (slightly) our definition of the g-Caputo tempered fractional differential operator to a more suitable one:

Definition 4. *The modified g-Caputo tempered fractional derivative of order $n+\alpha$, $n \in \mathbb{N}$, $\alpha \in (0,1)$ with parameter $\mu \in \mathbb{R}^+$ applied to the function x is defined as*

$$\frac{{}^*\partial}{\partial t^{n+}}{}^{n+,}_{a,g} x := \mathbf{d}^n \Im^{1-\alpha,\mu}_{a,g} \mathbf{d} x, \tag{24}$$

whenever the right-hand side is a well-defined function.

Remark 4.

1. *Clearly, Definitions 3 and 4 coincide with those of the g-Caputo-type fractional tempered differential operator when* $n = 0$.
2. *For the existence of (24), we define* $\Im_{a,g}^{n+\alpha,\mu}(L_\psi[a,b]) := \{x \in C[a.b] : x = \Im_{a,g}^{n+\alpha,\mu} y = \Im_{a,g}^{n,\mu}\Im_{a,g}^{\alpha,\mu} y,\ \ y \in L_\psi[a,b]\}$. *Obviously, in view of Theorem 3 we have* $\Im_{a,g}^{n+\alpha,\mu}(L_\psi[a,b]) \subset \mathbb{H}_g^{\widetilde{\Psi}_\alpha}[a,b]$. *However, for any* $x \in \Im_{a,g}^{n+\alpha,\mu}(L_\psi[a,b])$, *we have, for any* $n \geq 1$

$$\frac{{}^*\mathfrak{d}}{\mathfrak{d}\mathfrak{t}}{}^{\alpha,\mu}_{a,g} x = \mathbf{d}^n \Im_{a,g}^{1-\alpha,\mu}\Im_{a,g}^{n-1+\alpha,\mu} y = \mathbf{d}^n \Im_{a,g}^{n,\mu} y = y. \tag{25}$$

Let us complete the goals of the paper with a result on the equivalence of a differential problem and its integral form.

Theorem 6. *Let* ψ *be a Young function such that its complementary function* $\widetilde{\psi}$ *satisfies*

$$\int_0^t \widetilde{\psi}(s^{\alpha-2})\, ds < \infty, \quad t > 0, \tag{26}$$

then for any $f \in L_\psi[a,b]$ *the differential equations of the form*

$$\frac{{}^*\mathfrak{d}}{\mathfrak{d}\mathfrak{t}}{}^{\alpha,\mu}_{a,g} x(t) = f(t),\ \ t \in [a,b],\ \ \alpha \in (m-1,m),\ m \geq 2 \tag{27}$$

combined with appropriate initial or boundary conditions are equivalent to the following integral equation:

$$x(t) = e^{-\mu g(t)} c_0 + \Im_{a,g}^{\alpha,\mu} f(t) + \sum_{j=1}^{m-1} c_j (g(t))^{j+\alpha-m} e^{-\mu g(t)},\ \ c_0 = x(a), \tag{28}$$

where $c_j,\ j = 1, 2, \cdots, m-1$ *are arbitrary constants depending only on the initial or boundary conditions.*

Proof. Let x satisfy the differential form (27). Then, formally, we have

$$\mathbf{d}^{m-1}\Im_{a,g}^{m-\alpha,\mu}\mathbf{d}x(t) = f(t) \ \ \Rightarrow \ \Im_{a,g}^{m-\alpha,\mu}\mathbf{d}x(t) = \Im_{a,g}^{m-1,\mu} f(t) + \sum_{j=1}^{m-1} c_j (g(t))^{j-1} e^{-\mu g(t)}.$$

Operating by $\Im_{a,g}^{1+\alpha-m,\mu}$ and keeping in mind the semi-group property of the g-tempered fractional integral operators yields

$$\Im_{a,g}^{1,\mu}\mathbf{d}x(t) = \Im_{a,g}^{\alpha,\mu} f(t) + \sum_{j=1}^{m-1} c_j (g(t))^{j+\alpha-m} e^{-\mu g(t)}.$$

We can show that (still only formally)

$$x(t) = e^{-\mu g(t)} x(a) + \Im_{a,g}^{\alpha,\mu} f(t) + \sum_{j=1}^{m-1} c_j (g(t))^{j+\alpha-m} e^{-\mu g(t)}.$$

On the other hand, in light of the above discussion, the sufficiency condition is obvious. To see this, let $f \in L_\psi[a,b]$ and consider the integral form. It follows that

$$x(t) = e^{-\mu g(t)}c_0 + \Im^{\alpha,\mu}_{a,g} f(t) + \sum_{j=1}^{m-1} c_j (g(t))^{j+\alpha-m} e^{-\mu g(t)}$$
$$= e^{-\mu g(t)}c_0 + \Im^{1,\mu}_{a,g}\Im^{\alpha-1,\mu}_{a,g} f(t) + \sum_{j=1}^{m-1} c_j (g(t))^{j+\alpha-m} e^{-\mu g(t)},$$

$t \in [a,b]$, $\alpha \in (m-1,m)$. Keeping in mind (26), in view of Theorem 3, it follows that $\Im^{\alpha-1,\mu}_{a,g} f \in C[a,b]$ and so $x \in C^1[a,b]$. In this connection we have

$$\mathbf{d}x(t) = \Im^{\alpha-1,\mu}_{a,g} f(t) + \sum_{j=1}^{m-1} c_j (j+\alpha-m)(g(t))^{j+\alpha-m-1} e^{-\mu g(t)}.$$

Therefore,

$$\Im^{m-\alpha,\mu}_{a,g}\mathbf{d}x(t) = \Im^{m-1,\mu}_{a,g} f(t) + \sum_{j=1}^{m-1} c_j (j+\alpha-m)(g(t))^{j-1} e^{-\mu g(t)},$$

and finally,

$$\frac{{}^{*}\mathfrak{d}}{\mathfrak{d}\mathfrak{t}}{}^{\alpha,\mu}_{a,g} x(t) = \mathbf{d}^{m-1}\Im^{m-\alpha,\mu}_{a,g}\mathbf{d}x(t) = f(t).$$

□

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