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A Continuous Granular Model for Stochastic Reserving with Individual Information

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Abstract: This paper works on the claims data generated by individual policies which are randomly exposed to a period of continuous time. The main aim is to model the occurrence times of individual claims, as well as their developments given the feature information and exposure periods of individual policies, and thus project the outstanding liabilities. In this paper, we also propose a method to compute the moments of outstanding liabilities in an analytic form. It is significant for a general insurance company to more accurately project outstanding liabilities in risk management. It is well-known that the features of individual policies have effects on the occurrence of claims and their developments and thus the projection of outstanding liabilities. Neglecting the information can unquestionably decrease the prediction accuracy of stochastic reserving, where the accuracy is measured by the mean square error of prediction (MSEP), whose analytic form is computed according to the derived moments of outstanding liabilities. The parameters concerned in the proposed model are estimated based on likelihood and quasi-likelihood and the properties of estimated parameters are further studied. The asymptotic behavior of stochastic reserving is also investigated. The asymptotic distribution of parameter estimators is multivariate normal distribution which is a symmetric distribution and the asymptotic distribution of the deviation of the estimated loss reserving from theoretical loss reserve also follows a normal distribution. The confidence intervals for the parameter estimators and the deviation can be easily obtained through the symmetry of the normal distribution. Some simulations are conducted in order to support the main theoretical results.

Keywords: granular model; individual information; stochastic reserving; mean square error of prediction; Monte Carlo

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1. Introduction

The micro/individual data models for stochastic reserving, also called loss reserving, originated from the 1980s, for example, references [1–3]. Most remarkably, references [2,3], by formulating the developments of claims in portfolios as marked Poisson processes, established a framework of loss reserving with individual data. Individual data models have attracted a great deal of interests in the past decade. Ref. [4] proposed semiparametric models for both the occurrence time of the individual claims and the hazard function of delay variables such that they can more flexibly predict loss reserve for incurred but not reported (IBNR) claims. Ref. [5] modeled IBNR loss reserving by applying copula models to investigate the dependence of occurrence times of individual claims with only reporting delays (time lag between occurrence time of claim and reporting time of the claim to insurance company). By parameterizing the probabilistic models of references [2,3], Refs. [6–8] recently analyzed a set of real claim data from an insurance company in Europe and showed that the individual data model provided a more accurate prediction for outstanding liabilities than the traditional reserving methods for aggregate data. Ref. [9] also analyzed the advantage of loss reserving with individual data under a special setting of individual claim data sets, where every claim was paid once at its settlement, and showed that individual loss reserving outperforms aggregate methods such as the classical chain-ladder algorithm and Bornhuetter–Ferguson method and one can refer to the literature [10] for more details about these aggregate methods.

A naive philosophy states that the more information exploited in loss reserving, the greater accuracy of prediction of the outstanding liabilities. Especially, the feature information (individual information) of policies, e.g., the age, gender, driven experience of the driver and model of the cars in automobile insurance, and age, gender and other features characterizing the daily behavior of a policy holder in health insurance, and so on, can increase the accuracy of loss reserving. For example, older people tend to increase the risk of claims in health insurance and experienced drivers are less likely to be involved in traffic accidents. In the spirit of this consideration, a recent work [11] investigated the loss reserving with individual information under a discrete time setting.

Unfortunately, it is hard to incorporate the individual information when modeling loss reserving in the continuous time models formulated by, for example, literature [2,3,6], especially when modeling the occurrence times of claim events. Under a discrete time setting, ref. [11] proposed a reserving model that considered the impacts of individual information and showed that loss reserving neglecting the useful individual information was asymptotically biased. However, they considered a special claim data set where there was only a one-off payment for every claim at its final settlement. In this paper, we consider a granular model for reserving by incorporating policy's features as well as the consideration of more than once payment at the settlement of every claim in a continuous time framework. As we all know, it is a great challenge to derive the analytic expressions of expectation and process variance of outstanding liabilities in continuous time. With the help of discrete time setting or discretization scheme, some literature including some works mentioned above solved this problem, see also literature [12,13]. However, in this paper, we also obtained the moments of outstanding liabilities after some theoretical derivations without discrete time setting or discretization scheme.

Most recently, there is another branch of reserving literature which explores the methods outperforming traditional aggregate reserving models by applying the methods, e.g., neural network in machine learning. Ref. [14] proposed a DeepTriangle model for reserving which jointly modeled the claim amounts and incurred losses by the deep neural network. Ref. [15] illustrated the usage of machine learning techniques in individual loss reserving by an explicit example. Ref. [16] modeled the aggregate run-off triangle of claim amounts using neural network which help to improve prediction accuracy by diminishing the bias of the traditional aggregate method. By adding the information of claim counts, ref. [17] extended the work of literature [16] and the prediction of future claim amounts was improved. It is also a promising direction to study this issue using the reliability approach of studying product failure times in degradation experiments, for example, references [18–21].

In this paper, we propose a method to compute the expectation and variance of outstanding liabilities given historical observations, where the conditional expectation is conventionally called loss reserve–funds prepared by insurers to protect against risks brought by the outstanding liabilities. Specifically, the proposed method is analytic moments computation method for outstanding liabilities (AMPM). AMPM is performed easily in practice because it is a simple analytic approach with a close-form formula. Some Monte Carlo simulations are also performed to verify the analytic expression of AMPM as well as compute the empirical moments.

The rest of this paper is organized as follows. We model the occurrence times of individual claims and their developments in a continuous time framework and parameters are estimated by maximizing likelihood and quasi-likelihood in Section 2. The AMPM method is introduced and the improvement of prediction accuracy by using the proposed model with respect to that without individual information is also investigated in Section 3, where the accuracy was measured by the conditional mean square error of prediction.

Furthermore, the asymptotic behavior of loss reserving is studied. Section 4 reports some simulation results. Section 5 concludes the paper with a few remarks.

2. Modeling Occurrence of Individual Claims and Their Development

This preliminary section consists of three parts: the first concisely describes claim data of a granular individual, the second gives its distributional assumptions and the third is for parameters estimation and its theoretical properties for later use in loss reserving.

2.1. Claims' Occurrence and Developments

Consider a dynamic risk portfolio that evolves in $[0, \infty)$ and has been observed in a time interval conventionally denoted by $[0, \tau]$, where τ is a representative evaluation date when the loss reserving is computed. Assume that the risk portfolio consists of n insurance policies and the ith policy is effective in a period $[\tau_i^s, \tau_i^e]$ (risk exposure period), where τ_i^s is the starting date and τ_i^e is the expiring date of the ith policy in period $[0, \tau]$, where $\tau_i^e = (\tau_i^s + c) \wedge \tau$ with c is a known constant. Figure 1 illustrates the risk exposure of policy i.

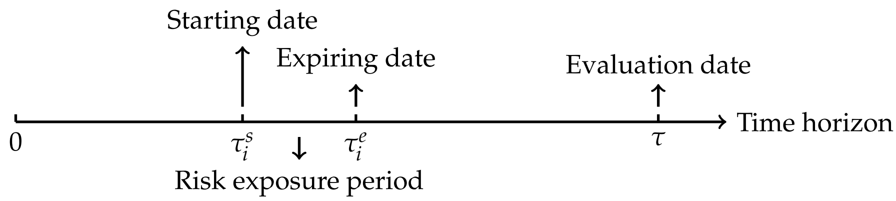

Figure 1. Risk exposure of policy i.

Furthermore, we use a d-dimensional vector of covariates $\boldsymbol{X}_i$ to record the individual information of the ith policy, conventionally with the constant 1 in the first entry. For this policy, the occurrence times of its claims and their developments are recorded by:

(i) A series of occurrence times $T_{ik}, k = 1, 2, \ldots$ taking values between τ_i^s and τ_i^e, and;

(ii) $\{U_{ik}, (V_{ikj}, Y_{ikj})_{j=1}^{\infty}, V_{ik}, (Y_{ik}^s, \delta_{ik})\}$, where U_{ik} is the time lag between the occurrence of kth claim and its report to the insurance company, the sequence $\{(V_{ikj}, Y_{ikj})\}_{j=1}^{\infty}$ records the times when the company pay for this claim and the corresponding payments between its report and final settlement, V_{ik} is the time lag between the report of kth claim to and its final settlement by the insurance company, δ_{ik} indicates whether there is a payment at the settlement: $\delta_{ik} = 1$ if yes in which case $Y_{ik}^s = Y_{ik}$ the corresponding severities, 0 otherwise.

Furthermore, we use N_i to indicate the number of incurred claims from policy i. To summarize, for the ith policy, the random element involved is

$$\boldsymbol{E}_i := (\tau_i^s, \tau_i^e; \boldsymbol{X}_i; \{T_{ik}, U_{ik}, (V_{ikj}, Y_{ikj})_{j=1}^{\infty}, V_{ik}, (Y_{ik}^s, \delta_{ik})\}_{k=1}^{\infty}), \tag{1}$$

where $Y_{ik}^s = Y_{ik}$ a random loss severity if $\delta_{ik} = 1$ and 0 otherwise. All the $\{\boldsymbol{E}_i, i = 1, 2, \cdots, n\}$ are assumed to be iid copies from a representative policy

$$\boldsymbol{E} := (\tau^s, \tau^e; \boldsymbol{X}; \{T_k, U_k, (V_{kj}, Y_{kj})_{j=1}^{\infty}, V_k, (Y_k^s, \delta_k)\}_{k=1}^{\infty}), \tag{2}$$

in which the meanings of the variables are clear. According to the values of its (T_k, U_k, V_k), the claim k is called settled, RBNS or IBNR if (T_k, U_k, V_k) is in $\mathscr{A}^s = \{(t, u, v) : t + u + v \leq \tau\}$ (settled), $\mathscr{A}^{rbns} = \{(t, u, v) : t + u \leq \tau < t + u + v\}$ (RBNS) or $\mathscr{A}^{ibnr} = \{(t, u, v) : t \leq \tau < t + u\}$ (IBNR), respectively.

2.2. Distribution Formulation

This subsection first specifies the joint distribution of the random claims development $\boldsymbol{E}$ in (2).

Assumption 1 (Distributional assumption)**.** *Given a random exposure period* $[\tau^s, \tau^e]$ *and vector* $\boldsymbol{X} = \boldsymbol{x}$ *which are arbitrarily distributed, the generating process of individual claims data is as follows:*

(1) *The occurrence time of individual claims follows a homogeneous Poisson process over* $[\tau^s, \tau^e]$*, with rate* $\lambda = \exp(\boldsymbol{x}'\boldsymbol{\beta})$*, where* $\boldsymbol{\beta} = (\beta_0, \beta_1, \cdots, \beta_{d-1})'$*.*

(2) *For the kth claim, its reporting delays* U_k *follows a distribution* $P_U(u; \alpha)$ *where* $\alpha = \boldsymbol{x}'\boldsymbol{\pi}$ *with* $\boldsymbol{\pi}$ *being* $(\pi_0, \pi_1, \cdots, \pi_{d-1})'$*.*

(3) *After the report of the kth claim, its settlement development is generated by three mutually independent homogeneous Poisson processes: one generates payments without settlement at rate* $h_p := \exp(\boldsymbol{x}'\boldsymbol{\rho}_p)$*, one generates settlement with payment at rate* $h_{sep} := \exp(\boldsymbol{x}'\boldsymbol{\rho}_{sep})$ *and one generates settlement without payment at rate* $h_{se} := \exp(\boldsymbol{x}'\boldsymbol{\rho}_{se})$*, where* $\boldsymbol{\rho}_p, \boldsymbol{\rho}_{sep}$ *and* $\boldsymbol{\rho}_{se}$ *are all d-dimensional vectors of parameters. The development process after claim's reporting will stop when the first settlement event occurs.*

(4) *The payments* Y_{kj}*s,* Y_k *are mutually independent with common mean* μ *and variance* $\phi\mu$ *and independent of the occurrence times of claims, as well as the generating process of payments, where* $\mu = \exp(\boldsymbol{x}'\boldsymbol{\gamma})$ *and* ϕ *is a dispersion parameter.*

Use $\boldsymbol{\theta}$ to denote the vector composed of all the unknown parameters, that is, $\boldsymbol{\theta}' = (\boldsymbol{\beta}', \boldsymbol{\pi}', \boldsymbol{\rho}_p', \boldsymbol{\rho}_{sep}', \boldsymbol{\rho}_{se}', \boldsymbol{\gamma}')$.

2.3. Estimates of the Parameters

In this section, we will separately estimate $\boldsymbol{\theta}_I := (\boldsymbol{\beta}', \boldsymbol{\pi}', \boldsymbol{\rho}_p', \boldsymbol{\rho}_{sep}', \boldsymbol{\rho}_{se}')'$ and $\boldsymbol{\gamma}$. For policy i, one claim can be observed only when its occurrence time and reporting delays belong to $\{(t, u) : t + u \leq \tau\}$. We use N_i^r to represent the number of reported claims that is

$$N_i^r = \sum_{k=1}^{\infty} I_{\{T_{ik}+U_{ik}\leq\tau\}}.$$

Denote N_{ik}^{op} the observed number of payments without settlement for kth reported claim, that is

$$N_{ik}^{op} = \sum_{j=1}^{\infty} I_{\{T_{ik}+U_{ik}+V_{ikj}\leq\tau\}}.$$

The occurrence time of reported claims and their developments excluding payments are denoted by

$$\{(T_{ik}^r, U_{ik}^r, (V_{ikj}^r)_{j=1}^{N_{ik}^{op}}, V_{ik}^r, \delta_{ik}^r) : k = 1, 2, \ldots, N_i^r\}.$$

To derive the overall likelihood of the reported claims, one needs Theorem 2 in Norberg (1993) [2] which shows the thinning properties of the marked Poisson process. In view of the independence among policies, the overall likelihood of the reported claims is

$$\begin{aligned} L(\boldsymbol{\theta}_I) =& \prod_{i=1}^{n} \lambda_i^{N_i^r} \left\{ \prod_{k=1}^{N_i^r} P_U(\tau - T_{ik}^r; \alpha_i) \right\} \exp\left(-\lambda_i \int_{\tau_i^s}^{\tau_i^e} P_U(\tau - t; \alpha_i) dt\right) \\ & \cdot \prod_{i=1}^{n} \prod_{k=1}^{N_i^r} \frac{P_U(dU_{ik}^r; \alpha_i)}{P_U(\tau - T_{ik}^r; \alpha_i)} \\ & \cdot \prod_{i=1}^{n} \left\{ \prod_{k=1}^{N_i^r} h_{i,p}^{N_{ik}^{op}} \left(h_{i,sep}^{\delta_{ik}^r} h_{i,se}^{1-\delta_{ik}^r}\right)^{I_{\{V_{ik}^r \leq \tau - T_{ik}^r - U_{ik}^r\}}} \cdot \exp(-(h_{i,p} + h_{i,sep} + h_{i,se})\tau_{ik}) \right\}, \end{aligned} \quad (3)$$

where $\lambda_i = \exp(\boldsymbol{x}_i'\boldsymbol{\beta})$, $\alpha_i = \boldsymbol{x}_i'\boldsymbol{\pi}$, $h_{i,p} = \exp(\boldsymbol{x}_i'\boldsymbol{\rho}_p)$, $h_{i,sep} = \exp(\boldsymbol{x}_i'\boldsymbol{\rho}_{sep})$, $h_{i,se} = \exp(\boldsymbol{x}_i'\boldsymbol{\rho}_{se})$, which are the policy-specified quantities of λ, α, h_p, h_{sep} and h_{se}, respectively, and

$\tau_{ik} = (\tau - T^r_{ik} - U^r_{ik}) \wedge V^r_{ik}$. One can refer to Theorem 2 in reference [2] for more details in deriving the likelihood above. Then, the log-likelihood is

$$
\begin{aligned}
l(\boldsymbol{\theta}_I) = & \sum_{i=1}^{n} \left\{ N^r_i \log \lambda_i - \lambda_i \int_{\tau^s_i}^{\tau^e_i} P_U(\tau - t; \alpha_i) dt \right\} \\
& + \sum_{i=1}^{n} \sum_{k=1}^{N^r_i} \log P_U(dU^r_{ik}; \alpha_i) \\
& + \sum_{i=1}^{n} [N^{op}_i \log h_{i,p} + N^{sep}_i \log h_{i,sep} + N^{se}_i \log h_{i,se} - (h_{i,p} + h_{i,sep} + h_{i,se})\tau_i],
\end{aligned} \tag{4}
$$

where $N^{op}_i = \sum_{k=1}^{N^r_i} N^{op}_{ik}$, $N^{sep}_i = \sum_{k=1}^{N^r_i} I_{\{V^r_{ik} \le \tau - T^r_{ik} - U^r_{ik}\}} \delta^r_{ik}$, $N^{se}_i = \sum_{k=1}^{N^r_i} I_{\{V^r_{ik} \le \tau - T^r_{ik} - U^r_{ik}\}} (1 - \delta^r_{ik})$, which mean observed number of payments without settlement, settlement with payment and settlement without payment, respectively, of policy i and $\tau_i = \sum_{k=1}^{N^r_i} \tau_{ik}$.

To estimate the parameters about payment severities, arrange all observed payments of the risk portfolio into the set $\{(Y_l, \boldsymbol{x}_l), l = 1, 2, \ldots, N^{tp}\}$, where $\boldsymbol{x}_l$ is covariates associated with payments Y_l and N^{tp} is the total number of payments. Construct quasi-likelihood by independence among policies and the fourth item in Assumption 1,

$$
Q^p(\gamma) = \frac{1}{\phi} \sum_{l=1}^{N^{tp}} (Y_l \log \mu_l - \mu_l), \tag{5}
$$

where $\mu_l = \exp(\boldsymbol{x}'_l \gamma)$. Denote by $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_{N^{tp}})'$ and $\boldsymbol{Y} = (Y_1, \ldots, Y_{N^{tp}})'$. The quasi-score function–partial derivatives of $Q^p(\gamma)$ with respect to the parameters is

$$
\dot{Q}^p(\gamma) := \frac{\partial Q^p(\gamma)}{\partial \gamma} = \frac{1}{\phi} X'(\boldsymbol{Y} - \boldsymbol{\mu}), \tag{6}
$$

where $X = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N^{tp}})'$.

The covariance matrix of $\dot{Q}(\gamma)$, which is also the negative expected value of $\partial \dot{Q}^p(\gamma)/\partial \gamma'$ according to reference [22], is

$$
I^p = \frac{1}{\phi} X' \mathrm{diag}(\boldsymbol{\mu}) X. \tag{7}
$$

The parameters γ are estimated by the iteratively re-weighted least square (IRLS) algorithm, which is as follows,

1. Initialize $\hat{\gamma} = \gamma_0$ such that $\hat{\mu}_l = \exp(\boldsymbol{x}'_l \hat{\gamma})$ and $\hat{\boldsymbol{\mu}} = (\hat{\mu}_1, \hat{\mu}_2, \ldots, \hat{\mu}_{N^{ts}})'$, where γ_0 is usually zero vector.
2. Compute adjusted payments $z_l = \frac{y_l - \hat{\mu}_l}{\hat{\mu}_l} + \boldsymbol{x}'_l \hat{\gamma}$.
3. Update $\hat{\gamma}$ by what follows,

$$
\hat{\gamma} = (X' \mathrm{diag}(\hat{\boldsymbol{\mu}}) X)^{-1} X' \mathrm{diag}(\hat{\boldsymbol{\mu}}) Z,
$$

where $Z = (z_1, z_2, \ldots, z_{N^{tp}})$, and then $\hat{\mu}_l = \exp(\boldsymbol{x}'_l \hat{\gamma})$.

To estimate the dispersion parameter ϕ, we also adopt the conventional method–moment estimation that is,

$$
\hat{\phi} = \frac{1}{N^{tp} - p} \sum_{l=1}^{N^{tp}} \frac{(Y_l - \hat{\mu}_l)^2}{\hat{\mu}_l},
$$

where $\hat{\mu}_l = \exp(\boldsymbol{x}'_l \hat{\gamma})$.

Apparently, it is impossible to acquire analytic expressions of parameter estimators based on the log-likelihood or quasi-likelihood. However, the properties of the estimators can also be studied. Under some regular conditions which can be found in any standard textbook [23] of asymptotic theory, we have the following theorem.

Theorem 1. *The estimators* $\hat{\boldsymbol{\theta}} \xrightarrow{P} \boldsymbol{\theta}$ *and the* $\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \xrightarrow{L} N(0, [I(\boldsymbol{\theta})]^{-1})$ *as n (number of insurance policies) tends to infinity, where* $\xrightarrow{P}$ *and* $\xrightarrow{L}$ *mean converging in probability and distribution, respectively, and*

$$I(\boldsymbol{\theta}) = \begin{bmatrix} I_{\boldsymbol{\beta},\boldsymbol{\pi}} & 0 & 0 & 0 & 0 \\ 0 & I_{\rho_p} & 0 & 0 & 0 \\ 0 & 0 & I_{\rho_{sep}} & 0 & 0 \\ 0 & 0 & 0 & I_{\rho_{se}} & 0 \\ 0 & 0 & 0 & 0 & I_{\gamma} \end{bmatrix},$$

where

$$I_{\boldsymbol{\beta},\boldsymbol{\pi}} = \begin{bmatrix} I_{11} & I_{12} \\ I'_{12} & I_{22} \end{bmatrix},$$

in which

$$I_{11} = \mathbb{E}\left[\lambda \int_{\tau^s}^{\tau^e} P_U(\tau - t; \alpha) dt \boldsymbol{x}\boldsymbol{x}'\right], I_{12} = \mathbb{E}\left[\lambda \frac{\partial}{\partial \alpha} \int_{\tau^s}^{\tau^e} P_U(\tau - t; \alpha) dt \boldsymbol{x}\boldsymbol{x}'\right]$$

$$I_{22} = \mathbb{E}\left[\lambda \frac{\partial^2}{\partial \alpha^2} \int_{\tau^s}^{\tau^e} P_U(\tau - t; \alpha) dt \boldsymbol{x}\boldsymbol{x}'\right] - \mathbb{E}\left[\lambda \int_{\tau^s}^{\tau^e} P_U(\tau - t; \alpha) \mathbb{E}\left[\frac{\partial^2 \log P_U(dU; \alpha)}{\partial \alpha^2} | U \leq \tau - t, \boldsymbol{x}\right] dt \boldsymbol{x}\boldsymbol{x}'\right],$$

$I_{\rho_p} = \mathbb{E}[h_p H_\tau]$, $I_{\rho_{sep}} = \mathbb{E}[h_{sep} H_\tau]$, $I_{\rho_{se}} = \mathbb{E}[h_{se} H_\tau]$, *in which*

$$H_\tau = \lambda \int_{\tau^s}^{\tau^e} P_U(\tau - t; \alpha) dt \mathbb{E}[(\tau - t - U) \wedge V | U \leq \tau - t, \boldsymbol{x}] dt \boldsymbol{x}\boldsymbol{x}',$$

and

$$\begin{aligned} I_\gamma =& \mathbb{E}\left[\lambda\mu \int_{\tau^s}^{\tau^e} \Pr(U \leq \tau - t, U + V \leq \tau - t) dt (h_p \mathbb{E}[V | U \leq \tau - t, U + V \leq \tau - t, \boldsymbol{x}] + \frac{h_{sep}}{h_{sep} + h_{se}}) dt \boldsymbol{x}\boldsymbol{x}'\right] \\ &+ \mathbb{E}\left[\lambda\mu h_p \int_{\tau^s}^{\tau^e} \Pr(U \leq \tau - t, U + V > \tau - t) dt \mathbb{E}[\tau - t - U | U \leq \tau - t, U + V > \tau - t, \boldsymbol{x}] dt \boldsymbol{x}\boldsymbol{x}'\right]. \end{aligned}$$

Proof. The estimators $\hat{\boldsymbol{\theta}}$ is obtained by maximizing $l(\boldsymbol{\theta}) := l(\boldsymbol{\theta}_I; \mathscr{F}_\tau) + Q^p(\gamma)$, which is equivalent to the log-likelihood of observations. Under regular conditions, such as in Sections 5.2 and 5.3 of the book [23]. It is easily justified that the estimators are consistent and weakly converge to multivariate normal distribution. The more details can be find in chapter 5 of the book [23]. According to the law of large number, $I_n(\boldsymbol{\theta})/n \xrightarrow{a.s.} I(\boldsymbol{\theta})$, where $I_n(\boldsymbol{\theta}) := -E[\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'}]$ is the Fisher information matrix. The Hessian matrix $\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'}$ is easy to compute and is a block diagonal matrix. However, the computation of its conditional expectation given the individual information of policies needs to use Corollary to Theorems 1 and 2 in [3]. We only compute $-E[\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \gamma \partial \gamma'}]$ and other parts are easy to obtain. The quasi-likelihood $Q^p(\gamma)$ can be rewrote as

$$\sum_{i=1}^{n} \sum_{k=1}^{N_i^r} \left[\sum_{l=1}^{N_{ik}^{op}} (Y_{ikl} \log \mu_i - \mu_i) + (Y_{ik} \log \mu_i - \mu_i) \delta_{ik} I_{\{V_{ik} \leq \tau - T_{ik}^o - U_{ik}^r\}}\right].$$

Hence, $\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \gamma \partial \gamma'} = \frac{\partial^2 Q^p(\gamma)}{\partial \gamma \partial \gamma'}$, where

$$\frac{\partial^2 Q^p(\gamma)}{\partial \gamma \partial \gamma'} = -\sum_{i=1}^{n} \sum_{k=1}^{N_i^r} (N_{ik}^{op} + \delta_{ik} I_{\{V_{ik} \leq \tau - T_{ik}^r - U_{ik}^r\}}) \mu_i \boldsymbol{x}_i \boldsymbol{x}_i.$$

To compute conditional expectation $-\mathbb{E}\left[\frac{\partial^2 Q^p(\gamma)}{\partial\gamma\partial\gamma'}\right]$, it is enough to compute

$$\mathbb{E}\left[\sum_{k=1}^{N^r}(N_k^{op}+\delta_k I_{\{V_k\le\tau-T_k^o-U_k^o\}})\right]=\mathbb{E}\left[\sum_{k=1}^{N^r}((N_k^{op}+\delta_k)I_{\{V_k\le\tau-T_k^o-U_k^o\}}+N_k^{op}I_{\{V_k>\tau-T_k^o-U_k^o\}})\right].$$

The left part in the equation above can further be computed as

$$\begin{aligned}&\lambda\mu\int_{\tau^s}^{\tau^e}\Pr(U\le\tau-t,U+V\le\tau-t)dt\left(h_p\mathbb{E}[V|U\le\tau-t,U+V\le\tau-t]+\frac{h_{sep}}{h_{sep}+h_{se}}\right)dt\\&+\lambda\mu h_p\int_{\tau^s}^{\tau^e}\Pr(U\le\tau-t,U+V>\tau-t)dt\mathbb{E}[\tau-t-U|U\le\tau-t,U+V>\tau-t]dt.\end{aligned}$$

Then, the proof is complete. □

By the symmetry of the asymptotic distribution, the confidence interval for each parameter can be easily obtained.

3. Loss Reserving

In this section, we detail AMPM for computing the expectation and variance of outstanding liabilities given historical observations. Based on these moments, we will elaborate the loss reserving and assessment of prediction accuracy.

3.1. AMPM Method

At evaluation date τ, the outstanding liabilities to the insureds are caused by both the RBNS and IBNR claims of all policies and hence can be represented as

$$R=R^{rbns}+R^{ibnr}=\sum_{i=1}^{n}R_i^{rbns}+\sum_{i=1}^{n}R_i^{ibnr},\tag{8}$$

where

$$\begin{aligned}R_i^{rbns}&=\sum_{k=1}^{N_i}\left(\sum_{j=1}^{N_{ik}^p}Y_{ikj}I_{\mathscr{A}^{rbns}}(T_{ik},U_{ik},V_{ikj})+Y_{ik}\delta_{ik}\right)I_{\mathscr{A}^{rbns}}(T_{ik},U_{ik},V_{ik}),\text{ and}\\R_i^{ibnr}&=\sum_{k=1}^{N_i}\left(\sum_{j=1}^{N_{ik}^p}Y_{ikj}+Y_{ik}\delta_{ik}\right)I_{\mathscr{A}^{ibnr}}(T_{ik},U_{ik},V_{ik}),\end{aligned}$$

are, respectively, the outstanding liabilities of RBNS and IBNR claims from policy i. Write

$$N_i^{rbns}=\sum_{k=1}^{N_i}I_{\mathscr{A}^{rbns}}(T_{ik},U_{ik},V_{ik})\text{ and }N_{ik}^{rbnsp}=\sum_{j=1}^{N_{ik}^p}I_{\mathscr{A}^{rbns}}(T_{ik},U_{ik},V_{ikj}),$$

which mean the number of RBNS claims of policy i and unobserved number of payments for kth RBNS claim of policy i before its settlement, respectively, and then

$$R_i^{rbns}=\sum_{k=1}^{N_i^{rbns}}\left(\sum_{j=1}^{N_{ik}^{rbnsp}}Y_{ikj}^{rbns}+Y_{ik}^{rbns}\delta_{ik}^{rbns}\right),\tag{9}$$

where Y_{ikj}^{rbns} is amounts of jth payment for kth RBNS claim of policy i, δ_{ik}^{rbns} indicates whether there is payment for kth RBNS claim at its settlement and if so Y_{ik}^{rbns} denotes the payment amounts. Write $N_i^{ibnr} = \sum_{k=1}^{N_i} I_{\mathscr{A}^{ibnr}}(T_{ik}, U_{ik}, V_{ik})$, which means the number of IBNR claims of policy i, and

$$R_i^{ibnr} = \sum_{k=1}^{N_i^{ibnr}} \left(\sum_{j=1}^{N_{ik}^{ibnrp}} Y_{ikj}^{ibnr} + Y_{ik}^{ibnr} \delta_{ik}^{ibnr} \right), \tag{10}$$

where Y_{ikj}^{ibnrp}, $j = 1, 2, \ldots, N_{ik}^{ibnrp}$, are a series of payments for kth IBNR claim before its settlement and Y_{ik}^{ibnr} is the loss severities at the settlement if there is payment, i.e., $\delta_{ik}^{ibnr} = 1$.

According to literature [3], for each policy, the processes which generate settled, RBNS and IBNR claims are independent and still marked Poisson processes. It is easily known that N_i^{ibnr} follows Poisson distribution with mean

$$\Lambda_i := \lambda_i \int_{\tau_i^s}^{\tau_i^e} (1 - P_U(\tau - t; \boldsymbol{x}_i' \boldsymbol{\pi})) dt.$$

Hence, we have the following theorem that gives formulas of the conditional expectation and variance of outstanding liabilities incurred by insurers given the historical observations $\mathscr{F}_\tau$.

Theorem 2. *The conditional expectation and variance of outstanding liabilities are*

$$\mathbb{E}(R|\mathscr{F}_\tau) = \sum_{i=1}^{n} \mathbb{E}\left(R_i^{rbns}|\mathscr{F}_\tau\right) + \sum_{i=1}^{n} \mathbb{E}\left(R_i^{ibnr}|\mathscr{F}_\tau\right), \tag{11}$$

$$\mathrm{Var}(R|\mathscr{F}_\tau) = \sum_{i=1}^{n} \mathrm{Var}\left(R_i^{rbns}|\mathscr{F}_\tau\right) + \sum_{i=1}^{n} \mathrm{Var}\left(R_i^{ibnr}|\mathscr{F}_\tau\right), \tag{12}$$

respectively, where

$$\mathbb{E}\left(R_i^{rbns}|\mathscr{F}_\tau\right) = \frac{N_i^{rbns}(h_{i,p} + h_{i,sep})\mu_i}{h_{i,se} + h_{i,sep}}, \; \mathbb{E}\left(R_i^{ibnr}|\mathscr{F}_\tau\right) = \frac{\Lambda_i \mu_i (h_{i,p} + h_{i,sep})}{h_{i,se} + h_{i,sep}},$$

$$\mathrm{Var}\left(R_i^{rbns}|\mathscr{F}_\tau\right) = N_i^{rbns} \left\{ \frac{\mu_i^2 h_{i,p}(h_{i,p} + h_{i,se} + h_{i,sep})}{(h_{i,se} + h_{i,sep})^2} + \frac{\phi \mu_i (h_{i,p} + h_{i,sep})}{h_{i,se} + h_{i,sep}} \right\},$$

and

$$\mathrm{Var}(R_i^{ibnr}|\mathscr{F}_\tau) = \Lambda_i \left\{ \frac{\mu_i^2 [h_{i,p}(h_{i,p} + h_{i,se} + h_{i,sep}) + (h_{i,p} + h_{i,sep})^2]}{(h_{i,se} + h_{i,sep})^2} + \frac{\phi \mu_i (h_{i,p} + h_{i,sep})}{h_{i,se} + h_{i,sep}} \right\}.$$

Proof. Apparently, the loss reserve is the sum of RBNS and IBNR reserves and the conditional variance of outstanding liabilities is also the sum of two parts thanks to independence between RBNS and IBNR claims that is

$$\mathbb{E}(R|\mathscr{F}_\tau) = \sum_{i=1}^{n} \mathbb{E}\left(R_i^{rbns}|\mathscr{F}_\tau\right) + \sum_{i=1}^{n} \mathbb{E}\left(R_i^{ibnr}|\mathscr{F}_\tau\right),$$

$$\mathrm{Var}(R|\mathscr{F}_\tau) = \sum_{i=1}^{n} \mathrm{Var}\left(R_i^{rbns}|\mathscr{F}_\tau\right) + \sum_{i=1}^{n} \mathrm{Var}\left(R_i^{ibnr}|\mathscr{F}_\tau\right).$$

Firstly, by Assumption 1 and the iteration expectation formula, the RBNS loss reserve of policy i is

$$\begin{aligned}
\mathbb{E}\Big(R_i^{rbns}|\mathscr{F}_\tau\Big) &= \sum_{k=1}^{N_i^{rbns}} \mathbb{E}\left(\sum_{j=1}^{N_{ik}^{rbnsp}} Y_{ikj} + Y_{ik}\delta_{ik}|V_{ik} > \tau - T_{ik} - U_{ik}, \boldsymbol{x}_i\right) \\
&= \sum_{k=1}^{N_i^{rbns}} \left\{\mathbb{E}[(V_{ik} - \tau + T_{ik} + U_{ik})|V_{ik} > \tau - T_{ik} - U_{ik}, \boldsymbol{x}_i]h_{i,p}\mu_i + \frac{h_{i,sep}}{h_{i,se} + h_{i,sep}}\mu_i\right\} \\
&= \frac{N_i^{rbns}(h_{i,p} + h_{i,sep})\mu_i}{h_{i,se} + h_{i,sep}},
\end{aligned}$$

where $N_i^{rbns} = \sum_{k=1}^{N_i} I_{\mathscr{A}^{rbns}}(T_{ik}, U_{ik}, V_{ik})$, $N_{ik}^{rbnsp} = \sum_{j=1}^{N_{ik}} I_{\mathscr{A}^{rbns}}(T_{ik}, U_{ik}, V_{ikj})$, and

$$\begin{aligned}
\mathrm{Var}\Big(R_i^{rbns}|\mathscr{F}_\tau\Big) &= \sum_{k=1}^{N_i^{rbns}} \mathrm{Var}\left(\sum_{j=1}^{N_{ikj}^{rbnsp}} Y_{ikj} + Y_{ik}\delta_{ik}|V_{ik} > \tau - T_{ik} - U_{ik}, \boldsymbol{x}_i\right) \\
&= \sum_{k=1}^{N_i^{rbns}} \big\{\mathrm{Var}((V_{ik} - \tau + T_{ik} + U_{ik})h_{i,p}\mu_i|V_{ik} > \tau - T_{ik} - U_{ik}, \boldsymbol{x}_i) \\
&\quad + \mathbb{E}[(\mu_i^2 + \phi\mu_i)(V_{ik} - \tau + T_{ik} + U_{ik})h_{i,p}|V_{ik} > \tau - T_{ik} - U_{ik}, \boldsymbol{x}_i] + \frac{h_{i,sep}\phi\mu_i}{h_{i,se} + h_{i,sep}}\Big\} \\
&= N_i^{rbns}\left\{\frac{\mu_i^2 h_{i,p}(h_{i,p} + h_{i,se} + h_{i,sep})}{(h_{i,se} + h_{i,sep})^2} + \frac{\phi\mu_i(h_{i,p} + h_{i,sep})}{h_{i,se} + h_{i,sep}}\right\}.
\end{aligned}$$

Second, because the generating process of IBNR claims is independent of the processes which generate RBNS and settled claims, the IBNR loss reserve can be computed by

$$\begin{aligned}
\mathbb{E}\Big(R_i^{ibnr}|\mathscr{F}_\tau\Big) &= E\left[\sum_{k=1}^{N_i^{ibnr}}\left(\sum_{j=1}^{N_{ik}} Y_{ikj} + Y_{ik}\delta_{ik}\right)\Bigg|\boldsymbol{x}_i\right] \\
&= \mathbb{E}[N_i^{ibnr}|\boldsymbol{x}_i]\mathbb{E}\left[\sum_{j=1}^{N_{i1}} Y_{i1j} + Y_{i1}\delta_{i1}\right] \\
&= \frac{\lambda_i\mu_i(h_{i,p} + h_{i,sep})}{h_{i,se} + h_{i,sep}}\int_{\tau_i^s}^{\tau_i^e}(1 - P_U(\tau - t; \boldsymbol{x}_i'\boldsymbol{\pi}))dt,
\end{aligned}$$

in which $N_i^{ibnr} = \sum_{k=1}^{N_i} I_{\mathscr{A}^{ibnr}}(T_{ik}, U_{ik}, V_{ik})$, and

$$\begin{aligned}
\mathrm{Var}(R_i^{ibnr}|\mathscr{F}_\tau) &= \mathrm{Var}\left(\sum_{k=1}^{N_i^{ibnr}}\left(\sum_{j=1}^{N_{ik}} Y_{ikj} + Y_{ik}\delta_{ik}\right)\Bigg|\boldsymbol{x}_i\right) \\
&= \left\{\mathrm{Var}\left(\sum_{j=1}^{N_{ik}} Y_{ikj} + Y_{ik}\delta_{ik}\Bigg|\boldsymbol{x}_i\right) + \left(\mathbb{E}\left[\sum_{j=1}^{N_{ik}} Y_{ikj} + Y_{ik}\delta_{ik}\Bigg|\boldsymbol{x}_i\right]\right)^2\right\}\mathbb{E}[N_i^{ibnr}|\boldsymbol{x}_i] \\
&= \left\{\frac{\mu_i^2[h_{i,p}(h_{i,p} + h_{i,se} + h_{i,sep}) + (h_{i,p} + h_{i,sep})^2]}{(h_{i,se} + h_{i,sep})^2} + \frac{\phi\mu_i(h_{i,p} + h_{i,sep})}{h_{i,se} + h_{i,sep}}\right\} \\
&\quad \cdot \lambda_i\int_{\tau_i^s}^{\tau_i^e}(1 - P_U(\tau - t; \boldsymbol{x}_i'\boldsymbol{\pi}))dt
\end{aligned}$$

The proof is then complete. □

The loss reserve can be expressed as $\mathbb{E}[R|\mathscr{F}_\tau]$, which can be denoted by $R_\tau(\boldsymbol{\theta})$ since it is a function of unknown parameters and related with evaluation date τ. Hence, R_τ needs to be estimated thanks to the unknown parameters. Loss reserving provides a procedure to estimate the loss reserve that is we simply insert the estimated parameters in Section 2.3 into $R_\tau(\boldsymbol{\theta})$ and obtain $\hat{R}_\tau = R_\tau(\hat{\boldsymbol{\theta}})$.

3.2. Accuracy and Asymptotic Behavior of Loss Reserving

It is important to assess the accuracy of loss reserving. A natural idea is to measure the accuracy of loss reserving by the conditional mean square error of prediction (MSEP) that is for any loss reserving $\hat{R}$ which is $\mathscr{F}_\tau$ measurable, its accuracy is measured by

$$\begin{aligned} \mathrm{MSEP}(\hat{R}) &= \mathbb{E}[(R-\hat{R})^2|\mathscr{F}_\tau] \\ &= \mathrm{Var}(R|\mathscr{F}_\tau) + (\mathbb{E}[R|\mathscr{F}_\tau] - \hat{R})^2. \end{aligned} \tag{13}$$

We can see that the conditional MSEP of predictor $\hat{R}$ is the sum of process variance, which remains fixed for any loss reserving method, and the square error of prediction, which depends on specific loss reserving method. Some reserving methods such as the chain-ladder algorithm in the study [24] suffer from information loss in the sense that it aggregates micro-level data such as occurrence times, reporting and settlement dates, payment times and loss severities of individual claims. Although the authors in [6] have shown that the micro-level data model has higher accuracy than aggregate models by empirical analysis and [9] have shown this in theoretical aspects, micro/individual data models also neglect some important information such as the policy's features. It is significant for actuaries to reduce the square error by considering more related information in loss reserving. By the simulations in Section 4, we will show that considering useful individual information in stochastic reserving is helpful to improve the accuracy of loss reserving. Apparently, the conditional MSEP of loss reserving $\hat{R}_\tau$ is $\mathrm{MSEP}(\hat{R}_\tau)$.

In the individual data model formulated by the literature [6,9], where they did not consider the individual information, it is assumed that there is no effects of individual information on the occurrence of claims and their developments that is the coefficients of $x_1, x_2, \ldots, x_{d-1}$ are considered to be zero. For example, $\beta_1 = \beta_2 = \cdots = \beta_{d-1} = 0$ and $\pi_1 = \pi_2 = \cdots = \pi_{d-1} = 0$. Hence, the model parameters in the individual data model only involves $\beta_0, \pi_0, \rho_{p0}, \rho_{se0}, \rho_{sep0}, \gamma_0$, which can be estimated by maximizing loglikelihood (4) and (5) with respect to these parameters. Denote the estimated parameters in the individual data model by $\hat{\beta}_0^{ID}, \hat{\pi}_0^{ID}, \hat{\rho}_{p0}^{ID}, \hat{\rho}_{se0}^{ID}, \hat{\rho}_{sep0}^{ID}, \hat{\gamma}_0^{ID}$ and obtain $\hat{R}_\tau^{ID}$ by inserting these estimated parameters into the formula of $\mathbb{E}[R|\mathscr{F}_\tau]$ in Theorem 2 with other parameters replaced by zero. The conditional MSEP in the individual data model is $\mathrm{MSEP}(\hat{R}_\tau^{ID})$.

We want to explore the improvement of predicting accuracy by using the proposed model with respect to the individual data model. To do so, we just need to compare $\mathrm{MSEP}(\hat{R}_\tau)$ and $\mathrm{MSEP}(\hat{R}_\tau^{ID})$. Then, define the relative predicting accuracy as

$$\begin{aligned} \mathrm{RPA}(\hat{R}_\tau, \hat{R}_\tau^{ID}) &= \frac{\mathrm{MSEP}(\hat{R}_\tau)}{\mathrm{MSEP}(\hat{R}_\tau^{ID})} \\ &= \frac{\mathrm{Var}(R|\mathscr{F}_\tau) + (\mathbb{E}[R|\mathscr{F}_\tau] - \hat{R}_\tau)^2}{\mathrm{Var}(R|\mathscr{F}_\tau) + (\mathbb{E}[R|\mathscr{F}_\tau] - \hat{R}_\tau^{ID})^2}, \end{aligned} \tag{14}$$

which is studied in Section 4 by simulations.

Next, we study the asymptotic behavior of $\hat{R}_\tau - R_\tau$, which is the deviance of loss reserving $\hat{R}_\tau$ from theoretical loss reserve.

Theorem 3. *The asymptotic distribution of the deviation of R_τ from $\hat{R}_\tau$ is*

$$\frac{1}{\sqrt{n}}(\hat{R}_\tau - R_\tau) \xrightarrow{L} N(0, \sigma^2) \text{ with } \sigma^2 = \xi_1' I_{\boldsymbol{\beta},\boldsymbol{\pi}}^{-1} \xi_1 + \xi_2' I_{\boldsymbol{\rho}_p}^{-1} \xi_2 + \xi_3' I_{\boldsymbol{\rho}_{sep}}^{-1} \xi_3 + \xi_4' I_{\boldsymbol{\rho}_{se}}^{-1} \xi_4 + \xi_5' I_{\gamma}^{-1} \xi_5,$$

where $I_{\boldsymbol{\beta},\boldsymbol{\pi}}, I_{\boldsymbol{\rho}_p}, I_{\boldsymbol{\rho}_{sep}}, I_{\boldsymbol{\rho}_{se}}$ and I_γ were defined as in Theorem 1 and

$$
\begin{aligned}
\xi_1 &= \mathbb{E}\left[\frac{\lambda\mu(h_p + h_{sep})}{h_{se} + h_{sep}}\begin{pmatrix} \int_{\tau^s}^{\tau^e}(1 - P_U(\tau - t; \alpha))dt \\ -\frac{\partial}{\partial\alpha}\int_{\tau^s}^{\tau^e} P_U(\tau - t; \alpha)dt \end{pmatrix} \otimes \boldsymbol{x}\right], \\
\xi_2 &= \mathbb{E}\left[\frac{\lambda h_p \mu}{h_{se} + h_{sep}}\int_{\tau^s}^{\tau^e}[\Pr(U \le \tau - t, U + V > \tau - t|\boldsymbol{x}) + 1 - P_U(\tau - t; \alpha)]dt \cdot \boldsymbol{x}\right], \\
\xi_3 &= \mathbb{E}\left[\frac{\lambda(h_{se} - h_p)h_{sep}\mu}{(h_{se} + h_{sep})^2}\int_{\tau^s}^{\tau^e}[\Pr(U \le \tau - t, U + V > \tau - t|\boldsymbol{x}) + 1 - P_U(\tau - t; \alpha)]dt \cdot \boldsymbol{x}\right], \\
\xi_4 &= \mathbb{E}\left[-\frac{\lambda(h_p + h_{sep})h_{se}\mu}{(h_{se} + h_{sep})^2}\int_{\tau^s}^{\tau^e}[\Pr(U \le \tau - t, U + V > \tau - t|\boldsymbol{x}) + 1 - P_U(\tau - t; \alpha)]dt \cdot \boldsymbol{x}\right], \\
\xi_5 &= \mathbb{E}\left[\frac{\lambda(h_p + h_{sep})\mu}{h_{se} + h_{sep}}\int_{\tau^s}^{\tau^e}[\Pr(U \le \tau - t, U + V > \tau - t|\boldsymbol{x}) + 1 - P_U(\tau - t; \alpha)]dt \cdot \boldsymbol{x}\right].
\end{aligned} \tag{15}
$$

So, the asymptotic distribution of the deviation is symmetric.

Proof. This theorem can be proved by Delta method. By Taylor expansion, we have

$$
\frac{1}{\sqrt{n}}(\hat{R}_\tau - R_\tau) = \frac{1}{n}\frac{\partial R_\tau}{\partial\boldsymbol{\theta}}\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) + o_p(||\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}||).
$$

The conclusion holds if $\frac{1}{n}\frac{\partial R_\tau}{\partial\boldsymbol{\theta}}$ converges in probability. We first compute the partial derivatives.

$$
\begin{aligned}
\frac{\partial R_\tau}{\partial\boldsymbol{\beta}} &= \sum_{i=1}^{n}\frac{\lambda_i\mu_i(h_{i,p} + h_{i,sep})}{h_{i,se} + h_{i,sep}}\int_{\tau_i^s}^{\tau_i^e}(1 - P_U(\tau - t; \alpha_i))dt \cdot \boldsymbol{x}_i, \\
\frac{\partial R_\tau}{\partial\boldsymbol{\pi}} &= -\sum_{i=1}^{n}\frac{\lambda_i\mu_i(h_{i,p} + h_{i,sep})}{h_{i,se} + h_{i,sep}}\frac{\partial}{\partial\alpha_i}\int_{\tau_i^s}^{\tau_i^e} P_U(\tau - t; \alpha_i)dt \cdot \boldsymbol{x}_i, \\
\frac{\partial R_\tau}{\partial\boldsymbol{\rho}_p} &= \sum_{i=1}^{n}\frac{h_{i,p}\mu_i(N_i^{rbns} + \Lambda_i)}{h_{i,se} + h_{i,sep}} \cdot \boldsymbol{x}_i, \\
\frac{\partial R_\tau}{\partial\boldsymbol{\rho}_{sep}} &= \sum_{i=1}^{n}\frac{(h_{i,se} - h_{i,p})h_{i,sep}\mu_i(N_i^{rbns} + \Lambda_i)}{(h_{i,se} + h_{i,sep})^2} \cdot \boldsymbol{x}_i, \\
\frac{\partial R_\tau}{\partial\boldsymbol{\rho}_{se}} &= -\sum_{i=1}^{n}\frac{(h_{i,p} + h_{i,sep})h_{i,se}\mu_i(N_i^{rbns} + \Lambda_i)}{(h_{i,se} + h_{i,sep})^2} \cdot \boldsymbol{x}_i, \\
\frac{\partial R_\tau}{\partial\boldsymbol{\gamma}} &= \sum_{i=1}^{n}\frac{(h_{i,p} + h_{i,sep})\mu_i(N_i^{rbns} + \Lambda_i)}{h_{i,se} + h_{i,sep}} \cdot \boldsymbol{x}_i.
\end{aligned}
$$

Given $\boldsymbol{x}_i$ and τ_i^s, N_i^{rbns} follows Poisson distribution with mean $\lambda_i\int_{\tau_i^s}^{\tau_i^e}\Pr(U \le \tau - t, U + V > \tau - t|\boldsymbol{x}_i)dt$. Then, by the law of large number, one can obtain that

$$
\frac{1}{n}\frac{\partial R_\tau}{\partial\boldsymbol{\theta}} \xrightarrow{P} (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)'.
$$

Therefore, combining Theorem 1, the proof is complete. □

4. Simulation Study

Reported in this section are the results from a few small simulations conducted to show the behaviors of estimators of the unknown parameters in the underlying distribution, to demonstrate how prediction accuracy can be affected by neglecting individual information and to justify the results in Theorem 2.

4.1. Simulating Claim Data

In the simulations, we generate loss severities from Poisson distribution, since there is no distribution assumption about the loss severities in our model. Sample n starting times of exposure periods from uniform distribution in $[0, \tau]$ and the constant c is set to be 1. The covariates associated with each policy are independently sampled from multivariate normal distribution with mean 0 and identity covariance matrix with dimension $d-1$. Given the starting dates, expiring dates and evaluation date τ, as well as the associated individual information of policies, the simulated reported claims data for each policy can be generated by the procedure detailed in the following procedure for each value of specified parameters included in Assumption 1:

step 1 For a policy with exposure period $[\tau^s, \tau^e]$ and covariates $\boldsymbol{x}$, the reported claims data is generated by the two-stage procedure detailed in Theorem 1 of [3] . First, generate the N^r reported claims incurred in this period from the Poisson distribution with mean

$$\lambda \int_{\tau^s}^{\tau^e} P_U(\tau - t; \boldsymbol{x}'\boldsymbol{\pi})dt,$$

then, the occurrence times of the N^r reported claims are generated from uniform distribution in $[\tau^s, \tau^e]$ and correspondingly, sample N^r reporting delays from the conditional distribution

$$\Pr(U \le u | U \le \tau - T) = \frac{P_U(u \wedge \tau - T; \alpha)}{P_U(\tau - T; \alpha)},$$

given the simulated occurrence time T of a reported claim.

step 2 Given a reported claim with the simulated occurrence time T and reporting delay U, first sample the settlement delays V from exponential distribution with rate $h_{se} + h_{sep}$, then

- If $V \le \tau - T - U$, generate N^s payments from $Poisson(Vh_p)$ and ordered payment times that are generated by ordering N^s samples sampled from uniform distribution in $(T+U, T+U+V)$, loss severities are generated by assuming that $\frac{Y}{\phi}$ follows $Poisson(\mu)$ and at last generate settlement type δ by Bernoulli distribution with success probability $\frac{h_{sep}}{h_{se}+h_{sep}}$, where $\delta = 1$ represents settlement with payment and $\delta = 0$ for settlement without payment, and if $\delta = 1$, generate the loss severities in the same way as above;
- Otherwise, generate N^{op} payments from $Poisson((\tau - T - U)h_p)$ and ordered payment times that are generated by ordering N^{op} samples sampled from uniform distribution in $(T+U, \tau)$, and corresponding loss severities are generated by assuming that $\frac{Y}{\phi}$ follows $Poisson(\mu)$.

step 3 Order the reported claim data generated according to the above two steps by occurrence time of reported claims.

step 4 For each policy, do Steps 1 to 3 to generate reported claims and their development.

The basic settings of simulations are $\tau = 5$, $d = 3$, $P_U(u; \alpha) = 1 - \exp(-\frac{u}{e^{\alpha}})$ and $\phi = 8$. Furthermore, the constant $c = 1$. It is remarkable that the distribution of reporting delays is arbitrary. Here in this simulation, we used exponential distribution. There are two cases of parameters listed below.

(I) $\boldsymbol{\beta} = (1.3, 0.1, -0.3)'$, $\boldsymbol{\pi} = (-0.5, 0.3, -0.2)'$, $\boldsymbol{\rho}_p = (-2, 1, -0.8)'$, $\boldsymbol{\rho}_{sep} = (-1, 0.2, -0.1)'$, $\boldsymbol{\rho}_{se} = (-0.5, 0.1, -0.1)'$ and $\boldsymbol{\gamma} = (4, 0.6, -0.8)'$.

(II) $\boldsymbol{\beta} = (1.5, 0.3, -0.6)'$, $\boldsymbol{\pi} = (0.3, -0.4, 0.3)'$, $\boldsymbol{\rho}_p = (-0.1, 0.7, -0.5)'$, $\boldsymbol{\rho}_{sep} = (-0.5, 0.8, -0.3)'$, $\boldsymbol{\rho}_{se} = (-0.8, 0.3, -0.3)'$ and $\boldsymbol{\gamma} = (5, 0.6, 0.9)'$.

For each case of parameters set above, simulations are conducted under two portfolio sizes $n_1 = 2000, n_2 = 5000, n_3 = 10000, n_4 = 20000$. Hence, there are eight scenarios that is (I, n_1), (I, n_2),(I, n_3),(I, n_4), (II, n_1), (II, n_2), (II, n_3), (II, n_4).

4.2. Evaluation #1

We duplicated a total of 200 runs of simulations for each scenario. The performance of the estimates of the parameters is summarized in Table 1 in the form of mean ± standard deviation that were computed over the 200 runs. Comparing true values of parameters with Table 1 (their estimates), it is apparent that the estimates look quite good. In real practice, it would usually be the case because a portfolio contains a huge number of policies.

To show the improvement of prediction accuracy of our model with respect to the individual data model, we computed PRA defined in (14) in each run of the simulations and we plotted the simulated values of PRA in Figure 2. It can be easily seen from the figures that adding useful individual information into stochastic loss reserving greatly improve the prediction accuracy since almost all values of PRA are less than 1 and when the sample size becomes larger, most of them are near 0.

Table 1. Simulated means ± standard deviations.

		Parameters					
		β	π	ρ_p	ρ_{sep}	ρ_{se}	γ
I	n_1	1.3 ± 0.0119	-0.5 ± 0.0177	-2 ± 0.0479	-1 ± 0.0262	-0.5 ± 0.0204	4 ± 0.0084
		0.1 ± 0.0144	0.3 ± 0.0164	1 ± 0.0314	0.2 ± 0.0256	0.1 ± 0.0205	0.6 ± 0.0046
		-0.3 ± 0.0146	-0.2 ± 0.0157	-0.8 ± 0.0287	-0.1 ± 0.0242	-0.1 ± 0.0204	-0.8 ± 0.0037
	n_2	1.3 ± 0.0084	-0.5 ± 0.0112	-2 ± 0.0295	-1 ± 0.0157	-0.5 ± 0.0121	4 ± 0.0046
		0.1 ± 0.0089	0.3 ± 0.0103	1 ± 0.0180	0.2 ± 0.0146	0.1 ± 0.0120	0.6 ± 0.0022
		-0.3 ± 0.0083	-0.2 ± 0.0111	-0.8 ± 0.0175	-0.1 ± 0.0171	-0.1 ± 0.0121	-0.8 ± 0.0027
	n_3	1.3 ± 0.0061	-0.5 ± 0.0064	-2 ± 0.0207	-1 ± 0.0126	-0.5 ± 0.0092	4 ± 0.0035
		0.1 ± 0.0062	0.3 ± 0.0075	1 ± 0.0118	0.2 ± 0.0106	0.1 ± 0.0086	0.6 ± 0.0015
		-0.3 ± 0.0060	-0.2 ± 0.0071	-0.8 ± 0.0129	-0.1 ± 0.0119	-0.1 ± 0.0094	-0.8 ± 0.0015
	n_4	1.3 ± 0.0039	-0.5 ± 0.0050	-2 ± 0.0134	-1 ± 0.0078	-0.5 ± 0.0061	4 ± 0.0023
		0.1 ± 0.0040	0.3 ± 0.0048	1 ± 0.0081	0.2 ± 0.0079	0.1 ± 0.0064	0.6 ± 0.0010
		-0.3 ± 0.0038	-0.2 ± 0.0053	-0.8 ± 0.0084	-0.1 ± 0.0082	-0.1 ± 0.0057	-0.8 ± 0.0009
II	n_1	1.5 ± 0.0181	0.3 ± 0.0251	-0.1 ± 0.0183	-0.5 ± 0.0231	-0.8 ± 0.0256	5 ± 0.0037
		0.3 ± 0.0368	-0.4 ± 0.0172	0.7 ± 0.0130	0.8 ± 0.0182	0.3 ± 0.0238	0.6 ± 0.0025
		-0.6 ± 0.0137	0.3 ± 0.0165	-0.5 ± 0.0135	-0.3 ± 0.0207	-0.3 ± 0.0100	0.9 ± 0.0021
	n_2	1.5 ± 0.0106	0.3 ± 0.0166	-0.1 ± 0.0117	-0.5 ± 0.0083	-0.8 ± 0.0171	5 ± 0.0022
		0.3 ± 0.0078	-0.4 ± 0.0109	0.7 ± 0.0083	0.8 ± 0.0106	0.3 ± 0.0139	0.6 ± 0.0016
		-0.6 ± 0.0088	0.3 ± 0.0127	-0.5 ± 0.0083	-0.3 ± 0.0109	-0.3 ± 0.0147	0.9 ± 0.0014
	n_3	1.5 ± 0.0076	0.3 ± 0.0118	-0.1 ± 0.0089	-0.5 ± 0.0104	-0.8 ± 0.0116	5 ± 0.0015
		0.3 ± 0.0057	-0.4 ± 0.0068	0.7 ± 0.0062	0.8 ± 0.0070	0.3 ± 0.0091	0.6 ± 0.0010
		-0.6 ± 0.0060	0.3 ± 0.0078	-0.5 ± 0.0056	-0.3 ± 0.0071	-0.3 ± 0.0100	0.9 ± 0.0009
	n_4	1.5 ± 0.0051	0.3 ± 0.0076	-0.1 ± 0.0058	-0.5 ± 0.0073	-0.8 ± 0.0082	5 ± 0.0011
		0.3 ± 0.0039	-0.4 ± 0.0050	0.7 ± 0.0041	0.8 ± 0.0054	0.3 ± 0.0066	0.6 ± 0.0006
		-0.6 ± 0.0040	0.3 ± 0.0057	-0.5 ± 0.0039	-0.3 ± 0.0056	-0.3 ± 0.0069	0.9 ± 0.0006

4.3. Evaluation #2

With an aim to verify the analytic expression of AMPM, a simulation-based empirical moments calculation algorithm was further developed. We can conduct the following Monte Carlo simulation, given the observations $\mathcal{F}_\tau$ with reported claim data generated by the procedure in the subsection above.

Simulate the IBNR claims and their developments and RBNS claims' developments for each policy exposure in $[0, \tau]$

step 1 For a policy with exposure period $[\tau^s, \tau^e]$ and covariates $\boldsymbol{x}$. First generate the N^{ibnr} claims incurred in this period from the Poisson distribution with mean

$$\Lambda := \lambda \int_{\tau^s}^{\tau^e} (1 - P_U(\tau - t; \boldsymbol{x}'\boldsymbol{\pi}))dt,$$

then, the occurrence times of the N^{ibnr} IBNR claims are generated from uniform distribution in $[\tau^s, \tau^e]$ and correspondingly, sample N^{ibnr} reporting delays from the conditional distribution

$$\Pr(U \leq u | U > \tau - T) = \frac{P_U(u;\alpha) - P_U(\tau - T;\alpha)}{1 - P_U(\tau - T;\alpha)},$$

given the simulated occurrence time T of an IBNR claim.

step 2 Given an IBNR claim with the simulated occurrence time T and reporting delay U, first sample the settlement delays V from exponential distribution with rate $h_{se} + h_{sep}$, then generate N^{ibnrp} payments from $Poisson(Vh_p)$ and ordered payment times that are generated by ordering N^{ibnrp} samples sampled from uniform distribution in $(T + U, T + U + V)$, loss severities are generated by assuming that $\frac{Y}{\phi}$ follows $Poisson(\mu)$ and at last generate settlement type δ by Bernoulli distribution with success probability $\frac{h_{sep}}{h_{se}+h_{sep}}$, where $\delta = 1$ represents settlement with payment and $\delta = 0$ for settlement without payment, and if $\delta = 1$, generate the loss severities in the same way as above.

step 3 Order the IBNR claim data generated according to the above two steps by occurrence time of IBNR claims.

step 4 Given the occurrence time T and reporting delay U of a RBNS claim, first sample the settlement delays $V > \tau - T - U$ from the conditional distribution

$$\Pr(V \leq v | V > \tau - T - U) = 1 - e^{-(h_{se}+h_{sep})(v+T+U-\tau)},$$

then, generate N^{rbnsp} payments from $Poisson((T + U + V - \tau)h_p)$ and ordered payment times that are generated by ordering N^{rbnsp} samples sampled from uniform distribution in $(\tau, T + U + V)$, loss severities and settlement type are generated according to step 2.

step 5 For each policy, do Steps 1 to 4 to generate their IBNR claims and their development and RBNS claims' developments.

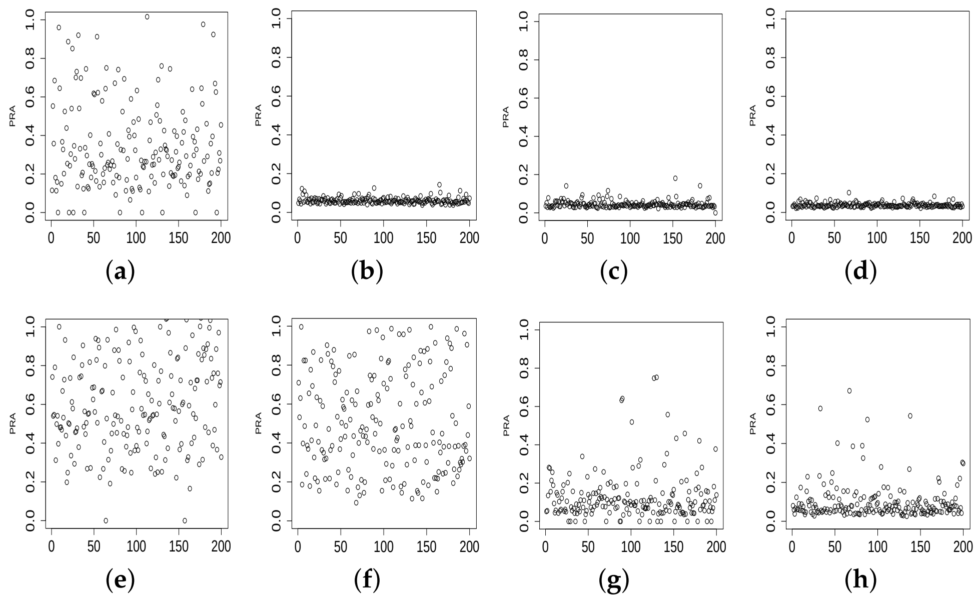

Figure 2. Simulated values of PRA in (14) under eight cases: (**a**) I, n_1; (**b**) I, n_2; (**c**) I, n_3; (**d**) I, n_4; (**e**) II, n_1; (**f**) II, n_2; (**g**) II, n_3; (**h**) II, n_4.

We repeated step 1 to 5 in the procedure above for 10,000 times under both scenario (I, n_3) and (II, n_3), computed outstanding liabilities R by (8) in each run and finally computed the empirical moments of outstanding liabilities by means of sample mean and variance. The results are given in Table 2. Therefore, we obtained 10,000 Rs under both scenarios. To see the distribution of R under each of the two scenarios, we used the 10,000 samples to estimate the distributions of R by kernel density estimation. We plotted the estimated results in Figure 3. One can see that the sample mean is close to the expectation of R under both scenarios. We also computed the sample standard deviations 116,815 for (I, n_3) and 82,498 for (II, n_3), while computed $\sqrt{\mathrm{Var}(R|\mathscr{F}_\tau)}$ were 110,890 and 71,852, respectively. We can see that the simulated values of $\sqrt{\mathrm{Var}(R|\mathscr{F}_\tau)}$ are very close to their true values.

Table 2. The conditional moments of outstanding liabilities and their empirical version.

Scenario		$\mathbb{E}[R|\mathcal{F}_\tau]$	$\mathbf{Var}[R|\mathcal{F}_\tau]$
(I, n_3)	Actual value	2,402,449	109,522
	Empirical value	2,399,602	109,952
(II, n_3)	Actual value	5,189,782	72,441
	Empirical value	5,188,816	75,666

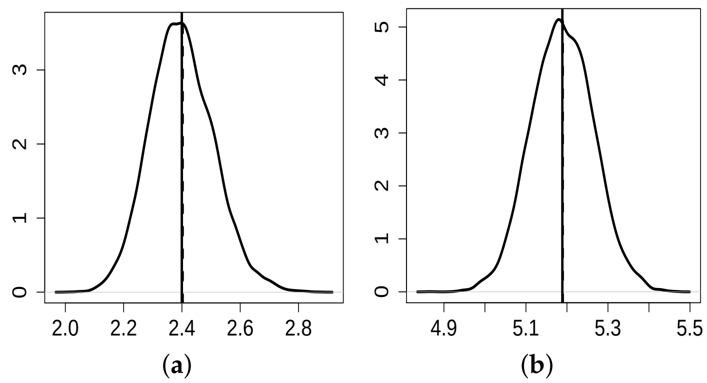

Figure 3. Estimated density of $\frac{R}{10^6}$, where the vertical solid represents sample mean and the vertical dashed represents $\frac{\mathbb{E}[R|\mathscr{F}_\tau]}{10^6}$. (**a**) I, n_3. (**b**) II, n_3.

5. Conclusions

This paper proposed a continuous granular model of loss reserving, which can incorporate the feature information of policies, when modeling loss reserving in the continuous time models. However, for example, studies [2,3,6] are difficult to include in the information on loss reserving. We also show that adding useful individual information into stochastic loss reserving greatly improves the prediction accuracy by numerical simulations.

In our proposed model, we assumed that the occurrence times of claims and their developments of each policy were generated by a Position Independent Marked Poisson Process, which is influenced by the feature information. Furthermore, we considered the situation where there may exist more than one payment for every claim. Based on the model assumption, one method of AMPM was proposed to analytically compute moments of outstanding liabilities. The AMPM method was also verified by Monte Carlo simulation. The parameters concerned in our model were estimated by MLE, as well as maximizing quasi-likelihood, and their asymptotic behaviors were shown. The simulation studies show that the estimates of parameters are quite accurate and stable. Furthermore, asymptotic behavior of the loss reserving was studied. To measure the prediction accuracy of the loss reserving, we computed the mean square error of prediction. In the simulation studies, we showed that neglecting individual information greatly increases the MSEP and hence fails to accurately predict the outstanding liabilities.

The work can be extended to consider the dependence between the processes of settlement and payments. Furthermore, it is more meaningful to further explore how the occurrence times of individual claims and their developments depend on the individual information by, e.g., nonparametric method or neural network.

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