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A Generalization of Szász–Mirakyan Operators Based on α Non-Negative Parameter

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Abstract: The main purpose of this paper is to define a new family of Szász–Mirakyan operators that depends on a non-negative parameter, say α . This new family of Szász–Mirakyan operators is crucial in that it includes both the existing Szász–Mirakyan operator and allows the construction of new operators for different values of α . Then, the convergence properties of the new operators with the aid of the Popoviciu–Bohman–Korovkin theorem-type property are presented. The Voronovskaja-type theorem and rate of convergence are provided in a detailed proof. Furthermore, with the help of the classical modulus of continuity, we deduce an upper bound for the error of the new operator. In addition to these, in order to show that the convex or monotonic functions produced convex or monotonic operators, we obtain shape-preserving properties of the new family of Szász–Mirakyan operators. The symmetry of the properties of the classical Szász–Mirakyan operator and of the properties of the new sequence is investigated. Moreover, we compare this operator with its classical correspondence to show that the new one has superior properties. Finally, some numerical illustrative examples are presented to strengthen our theoretical results.



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1. Introduction

The approximation theory is one of the significant research topics of mathematical analysis, which originated and spread in the 19th century, and has been studied by a number of mathematicians around the world from this century to the present. The fact that approximation theory sheds light on several scientific problems not only in mathematics but also in other fields, especially in basic sciences and engineering sciences, has led to the increasing importance of approximation theory, day by day. The main purpose of approximation theory is to obtain a representation of an arbitrary function with the help of other functions that are simpler and have more elementary properties, such as differentiability, integrability, etc. The basis of the theory of approximation to functions of real variables was established by the Russian mathematician P. L. Chebyshev and the German mathematician K. Weierstrass proving two significant theorems. Chebyshev's research is based on finding the m -th order polynomial that gives the best approximation to an arbitrary continuous function given in a closed interval. On the other hand, German mathematician K. Weierstrass (1815–1897) made great progress in mathematics when he proved the Weierstrass approximation theory, which bears his name, in 1885 [1]. Since the proof of the Weierstrass theorem is quite long and complex, a number of famous mathematicians have worked on this theorem in order to provide a simpler and more understandable proof. In 1912, Bernstein gave the simplest and most effective proof of Weierstrass's theorem, using a polynomial sequence and based on concepts and ideas in probability theory [2]. There are a number of reasons why Bernstein polynomials are

so popular today, such as being easily differentiable and integrable, being functional in handling a number of problems, having a clear and simple representation of the polynomial, and having various shape-preserving properties.

One of the fields that has an important place in approximation theory is the investigation of the approximation properties of linear positive operators. The concept of linear positive operators gained great importance in the 1950s with the proving of the theorem that gives the uniform convergence of operator sequences to a continuous function on a finite closed interval. Although this theorem is known as the Bohman–Korovkin theorem [3,4] in the literature, it should not be forgotten that T. Popoviciu [5] has worked on this subject before. This theorem contains very simple yet also very effective criteria that show the approximation of the linear positive operators to the unit operator. Along with this theorem, the Bernstein operator is also a set of linear positive operators, satisfying the conditions of the Popoviciu–Bohman–Korovkin theorem on $[0, 1]$. Using this theorem, a number of positive linear operators defined on $[0, 1]$ and the approximation properties of these have been studied [6–10].

As the studies on linear positive operators in the approximation theory deepened, the operators defined in the unlimited intervals alongside the operators defined in the compact interval emerged, and the study of examining their approximation properties accelerated. In 1941 [11], G.M. Mirakyan, extending the Bernstein operators from finite interval to infinite interval, defined linear positive operators sequence in the form

$$M_n(f; x) = \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad x \in [0, \infty), \quad n \in \mathbb{N},$$

for the function $f \in C[0, \infty)$. The $(M_n)_{n \geq 1}$ linear positive operators were also studied separately by J. Favard [12], and O. Szász [13]. This operator is known in the literature as the Szász–Mirakyan operators. After the $(M_n)_{n \geq 1}$ operators were defined, studies on this linear positive operator sequence have become widespread. Some generalizations of this operator can be found in [14–29].

Additionally, in 2017, Chen et al. [30] introduced a new class of generalized Bernstein operators named the α -Bernstein operator, which depends upon a non-negative parameter, that is to say $\alpha \in [0, 1]$. In this study, inspired by Chen's work, we introduced a new family of generalized Szász–Mirakyan operators which are called α -Szász–Mirakyan operators, and their certain fundamental properties, which play a significant role in the theory of the uniform approximation of functions.

The complete structure of the manuscript constitutes eight sections, including that one. The rest of this study is structured as follows. In Section 2, the fundamental facts of the approximation theory are summarized for use in the main results. In Section 3, the construction of the new family of α -Szász–Mirakyan operators and some of its main features are given, such as the linearity, positivity and their moment values, which follow a symmetric pattern. In Section 4, the error bounds of the α -Szász–Mirakyan operators are reviewed, while some shape-preserving properties of these new operators, such as convexity and monotonicity, are given in Section 5. In Section 6, we compare the α -Szász–Mirakyan operators with the standard operator in terms of convergence rate. Finally, in Section 7, some numerical illustrative examples are provided, and in Section 8, some conclusions and further directions of research are provided.

2. Fundamental Facts

In this section, some fundamental concepts and notations related to positive linear operators are discussed. In addition, some basic definitions that will be used in the following sections are given. Since the definitions and notations to be summarized here are well known by those studying in this field, they will be briefly mentioned.

Throughout this and the next sections, we shall denote by $C[0, \infty)$ the space of all continuous real valued functions on $[0, \infty)$. On the other hand, we symbolize by $C_B[0, \infty)$ the space of all bounded functions in the space of $C[0, \infty)$ endowed with the norm,

$$\|f\|_{[0, \infty)} = \sup_{x \in [0, \infty)} |f(x)|.$$

Furthermore, let us define a function by $\tau = 1 + x^2$ and a constant L_f . We denote,

$$\begin{aligned} B_2[0, \infty) &= \{f : [0, \infty) \rightarrow \mathbb{R}, f(x) \leq L_f \tau \text{ whenever } x \geq 0, \text{ for some constant } L_f > 0\}, \\ C_2[0, \infty) &= B_2[0, \infty) \cap C[0, \infty), \\ C_2^*[0, \infty) &= \{f : f \in C_2[0, \infty) \text{ and } \lim_{x \rightarrow \infty} |f(x)|\tau = \text{const.}\}. \end{aligned}$$

Moreover, we shall use the notation e_j for the power functions $e_j(t) = t^j$ for $t \geq 0$ and $j \in \mathbb{N}$ throughout the paper. Additionally, we denote by $(k)_r$ the falling factorials for all integers $r \geq 1$, that is to say

$$(k)_r = \begin{cases} \prod_{i=0}^{r-1} (k-i) = \frac{k!}{(k-r)!} & \text{for all integers } k \geq r \\ 0 & \text{for all integers } 0 \leq k \leq r-1 \end{cases}$$

On the other hand, we set $(k)_r = 1$ for $r = 0$ by convention. In addition to these, we review the forward difference. A forward difference with the increment of $1/n$ is an expression of the form,

$$\Delta f\left(\frac{k}{n}\right) = f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right),$$

and the second order operator,

$$\Delta^2 f\left(\frac{k}{n}\right) = f\left(\frac{k+2}{n}\right) - 2f\left(\frac{k+1}{n}\right) + f\left(\frac{k}{n}\right).$$

As a result, we will use all these definitions briefly mentioned above in the next sections.

3. The New α -Szász–Mirakyan Operators

Szász–Mirakyan operators are one of the most powerful structures in the approximation of the given continuous functions on infinite intervals. Now, we introduce the new class of Szász–Mirakyan operators, which are called α -Szász–Mirakyan operators as follows.

Definition 1. For a given function $f \in C[0, \infty)$, any fixed real α and each positive integers r and n , the α -Szász–Mirakyan operators define as

$$S_{n,\alpha}^r(f; x) = \begin{cases} \sum_{k=0}^{\infty} p_{n,k,r}^{(\alpha)}(x) f\left(\frac{k}{n}\right), & \text{if } x \geq \frac{r}{n} \\ f(x), & \text{if } 0 \leq x < \frac{r}{n}, \end{cases}$$

where

$$p_{n,k,r}^{(\alpha)}(x) = \left[(1-\alpha)e^r (nx-r)^{k-r} (k)_r + \alpha (nx)^k \right] \frac{e^{-nx}}{k!}, \tag{1}$$

and $(k)_r$ is a falling factorial.

Lemma 1. *The relationship between the classical Szász–Mirakyan operator and its α equivalent can be given as follows*

$$S_{n,\alpha}^r(f; x) = M_n(f; x) + (1 - \alpha)M_n \circ t_{-\frac{r}{n}} \left(f \circ t_{\frac{r}{n}}; x \right),$$

where we denote the translation $t_b(a) = a + b$ for $x \in [\frac{r}{n}, \infty)$.

Proof. It should be noted here that for $x \in [\frac{r}{n}, \infty)$, one has

$$\begin{aligned} S_{n,\alpha}^r(f; x) &= \alpha M_n(f; x) + (1 - \alpha)e^{r-nx} \sum_{k=0}^{\infty} \frac{(nx - r)^{k-r}}{k!} (k)_r f\left(\frac{k}{n}\right) \\ &= \alpha M_n(f; x) + (1 - \alpha)e^{r-nx} \sum_{k=r}^{\infty} \frac{(nx - r)^{k-r}}{k!} (k)_r f\left(\frac{k}{n}\right) \\ &= \alpha M_n(f; x) + (1 - \alpha)e^{r-nx} \sum_{h=0}^{\infty} \frac{(nx - r)^h}{h!} (k)_r f\left(\frac{h}{n} + \frac{r}{n}\right) \\ &= \alpha M_n(f; x) + (1 - \alpha)M_n \left(f \circ t_{\frac{r}{n}}; x - \frac{r}{n} \right) \\ &= \alpha M_n(f; x) + (1 - \alpha)M_n \circ t_{-\frac{r}{n}} \left(f \circ t_{\frac{r}{n}}; x \right), \end{aligned}$$

which completes the proof. \square

It is quite apparent that, in the case of $\alpha = 1$, the α -Szász–Mirakyan operators reduce to classical Szász–Mirakyan operators. In other words, we have

$$S_{n,1}^r(f; x) = \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) = M_n(f; x),$$

On the other hand, similarly, in the case of $r = 0$, we obtain

$$S_{n,\alpha}^0(f; x) = \alpha M_n(f; x) + (1 - \alpha)M_n \circ t_0(f \circ t_0; x) = M_n(f; x),$$

which means that the α -Szász–Mirakyan operators comprise their classical correspondence. As a consequence, the α -Szász–Mirakyan operators map a function f , defined on $[0, \infty)$, to $S_{n,\alpha}^r(f)$, where the function $S_{n,\alpha}^r(f)$ evaluated at x is denoted by $S_{n,\alpha}^r(f; x)$.

Now we present and prove here some properties and results of the α -Szász–Mirakyan operators.

Lemma 2. *The α -Szász–Mirakyan operator is linear, that is*

$$S_{n,\alpha}^r(af + bg; x) = aS_{n,\alpha}^r(f; x) + bS_{n,\alpha}^r(g; x),$$

for all functions $f(x)$ and $g(x)$ defined on $C[0, \mathcal{A}]$, where $\mathcal{A} \in \mathbb{R}^+$, and all real a and b .

Proof. The proof is routine and left to the reader. \square

Lemma 3. *Let us assume that the conditions of α -Szász–Mirakyan operators hold. In the circumstances, the moments of the newly defined operators $S_{n,\alpha}^r$ are as follows*

- (i) $S_{n,\alpha}^r(e_0; x) = 1,$
- (ii) $S_{n,\alpha}^r(e_1; x) = x.$

Proof.

(i) Let us denote $s_p(t) = \sum_{k=0}^{\infty} k^p \frac{t^k}{k!}$, where $p \in \mathbb{N}, t \in \mathbb{C}$. Then $s_0(t) = e^t$ and $s_p(t) = ts'_{p-1}(t)$ for $p \geq 1$. It follows that $s_p(t) = tQ_p(t)e^t$, for everywhere $p \in \mathbb{N} \setminus \{0\}$, where the polynomials satisfy the recurrence relation

$$Q_p(t) = tQ'_{p-1}(t) + (t + 1)Q_{p-1}(t).$$

In particular, $Q_1(t) = 1, Q_2(t) = t + 1, Q_3(t) = t^2 + 3t + 1, Q_4(t) = t^3 + 6t^2 + 7t + 1$. Clearly,

$$\begin{aligned} S_{n,\alpha}^r(e_0; x) &= \alpha e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} + (1 - \alpha)e^{r-nx} \sum_{h=0}^{\infty} \frac{(nx - r)^h}{h!} \\ &= \alpha e^{-nx} e^{nx} + (1 - \alpha)e^{r-nx} e^{nx-r} \\ &= 1. \end{aligned}$$

which means that the α -Szász–Mirakyan operators preserve constant functions 1 directly.

(ii) Similarly

$$\begin{aligned} S_{n,\alpha}^r(e_1; x) &= \alpha e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \frac{k}{n} + (1 - \alpha)e^{r-nx} \sum_{h=0}^{\infty} \frac{(nx - r)^h}{h!} \left(\frac{h}{n} + \frac{r}{n} \right) \\ &= \alpha e^{-nx} e^{nx} + (1 - \alpha)e^{r-nx} e^{nx-r}, \end{aligned}$$

hence

$$\begin{aligned} nS_{n,\alpha}^r(e_1; x) &= \alpha e^{-nx} s_1(nx) + (1 - \alpha)e^{r-nx} (s_1(nx - r) + re^{nx-r}) \\ &= \alpha nx Q_1(nx) + (1 - \alpha)((nx - r)Q_1(nx - r) + r) \\ &= \alpha nx + (1 - \alpha)((nx - r) + r) \\ &= nx, \end{aligned}$$

then the unit function t is preserved by the α -Szász–Mirakyan operators.

□

Lemma 4. In general, one can express the moment values of α -Szász–Mirakyan operators as follows for $p \geq 1$:

$$n^p S_{n,\alpha}^r(e_p; x) = \alpha nx Q_p(nx) + (1 - \alpha)(nx - r) \sum_{j=0}^p C_p^j r^{p-j} Q_j(nx - r),$$

where C_p^j is the binomial distribution coefficient and $Q_p(t)$ is given as above.

Proof. Utilizing the definition of α -Szász–Mirakyan operators, we have

$$\begin{aligned} S_{n,\alpha}^r(e_p; x) &= \alpha e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \left(\frac{k}{n} \right)^p + (1 - \alpha)e^{r-nx} \sum_{h=0}^{\infty} \frac{(nx - r)^h}{h!} \left(\frac{h}{n} + \frac{r}{n} \right)^p \\ &= \frac{1}{n^p} \alpha e^{-nx} s_p(nx) + (1 - \alpha)e^{r-nx} \sum_{h=0}^{\infty} \frac{(nx - r)^h}{h!} \sum_{j=0}^p C_p^j \left(\frac{h}{n} \right)^p \left(\frac{r}{n} \right)^{p-j}, \end{aligned}$$

then we have

$$\begin{aligned}
 n^p S_{n,\alpha}^r(e_p; x) &= \alpha e^{-nx} nx Q_p(nx) e^{nx} + (1 - \alpha) e^{r-nx} \sum_{j=0}^p C_p^j S_j(nx - r) r^{p-j} \\
 &= \alpha e^{-nx} nx Q_p(nx) e^{nx} + (1 - \alpha) e^{r-nx} \sum_{j=0}^p C_p^j (nx - r) Q_j(nx - r) e^{nx-r} r^{p-j} \\
 &= \alpha e^{-nx} nx Q_p(nx) e^{nx} + (1 - \alpha) (nx - r) \sum_{j=0}^p C_p^j r^{p-j} Q_j(nx - r),
 \end{aligned}$$

which completes the proof. □

Now we can provide the other moments of α -Szász–Mirakyan operators, say e_2, e_3 and e_4 , which follow a symmetric pattern, as follows.

Lemma 5. *The following equalities hold for the newly defined operators $S_{n,\alpha}^r$*

- (i) $S_{n,\alpha}^r(e_2; x) = x^2 + \frac{x}{n} + \frac{r}{n^2}(\alpha - 1),$
- (ii) $S_{n,\alpha}^r(e_3; x) = x^3 + \frac{3x^2}{n} + \frac{x}{n^2}(3\alpha r - 3r + 1) + \frac{r}{n^3}(\alpha - 1),$
- (iii) $S_{n,\alpha}^r(e_4; x) = x^4 + \frac{6x^3}{n} + \frac{x^2}{n^2}(6\alpha r - 6r + 7) + \frac{x}{n^3}(10\alpha r - 10r + 1) + \frac{r}{n^4}(\alpha - 1)(1 - 3r).$

Proof. By utilizing the result of Lemma 4, the desired results are obtained readily. □

Now we investigate the preservation linear functions, positivity and monotonicity properties of the newly constructed α -Szász–Mirakyan operators, $S_{n,\alpha}^r(f; x)$.

Lemma 6. *For the α -Szász–Mirakyan operators, the following is valid.*

- (i) *The α -Szász–Mirakyan operators leave invariant linear functions, that is*

$$S_{n,\alpha}^r(ax + b; x) = ax + b,$$

for all functions $f(x)$ and $g(x)$ defined on $[0, \infty)$, and all real a and b .

- (ii) *If $S_{n,\alpha}^r(f; x) \geq S_{n,\alpha}^r(g; x)$ holds for the functions which satisfy the condition $f(x) \geq g(x)$ for $x \in [0, \infty)$, that is to say, the α -Szász–Mirakyan operators are said to be a monotone operator for $\alpha \in [0, 1]$.*
- (iii) *The α -Szász–Mirakyan operators are said to be non-negative operators for every $\alpha \in [0, 1]$ since $S_{n,\alpha}^r(f; x) \geq 0$ when $f(x) \geq 0$.*

Proof.

- (i) The first proof of the Lemma is straightforward. That is to say, from Lemma 2, Lemma 3-i and ii, the desired result is obtained readily.
- (ii) For $x \in [0, \infty)$, $f(x)$ and $g(x)$ are two functions which satisfy $f(x) - g(x) \geq 0$. From Lemma 2, we deduce that $S_{n,\alpha}^r(f; x) \geq S_{n,\alpha}^r(g; x)$, which complete the proof of the second part of the lemma.
- (iii) For the last part of the lemma, we know that

$$\text{if } c \leq f(x) \leq C, \text{ for } x \in [0, \infty), \text{ then } c \leq S_{n,\alpha}^r(f; x) \leq C, \text{ for } x \in [0, \infty),$$

from Lemmas 3-i and 6-ii. Particularly, if we choose $c = 0$, we deduce that

$$\text{if } f(x) \geq 0, \text{ for } x \in [0, \infty), \text{ then } S_{n,\alpha}^r(f; x) \geq 0, \text{ for } x \in [0, \infty).$$

□

As a consequence of all of the above, one can conveniently say that the α -Szász–Mirakyan operators are members of the positive linear operator family for $\alpha \in [0, 1]$.

Lemma 7. *The α -Szász–Mirakyan operators are degree preserving on polynomials; in other words, if P is a polynomial of degree d , then $S_{n,\alpha}^r(P; x)$ is a polynomial of degree d .*

Proof. To begin with, let us express the function $p_{n,k,r}^{(\alpha)}$ as the sum of two different functions, as follows:

$$p_{n,k,r}^{(\alpha)} = p_{n,k,r}^{(\alpha),\dagger} + p_{n,k,r}^{(\alpha),\ddagger},$$

where

$$p_{n,k,r}^{(\alpha),\dagger} = (1 - \alpha)e^{r-nx} \frac{(nx - r)^{k-r}}{(k - r)!} \quad \text{and} \quad p_{n,k,r}^{(\alpha),\ddagger} = \alpha e^{-nx} \frac{(nx)^k}{(k)!}.$$

In this circumstance, we have

$$S_{n,\alpha}^r(f; x) = \sum_{k=0}^{\infty} p_{n,k,r}^{(\alpha),\dagger} f\left(\frac{k}{n}\right) + \sum_{k=0}^{\infty} p_{n,k,r}^{(\alpha),\ddagger} f\left(\frac{k}{n}\right) =: S_{n,\alpha}^{r,\dagger}(f; x) + S_{n,\alpha}^{r,\ddagger}(f; x).$$

Now, firstly, we show the degree-preserving properties of $S_{n,\alpha}^{r,\dagger}(f; x)$. For this purpose, let us introduce the (Newton) polynomials of degree d denoted by $\varphi_{n,r}^d$ that is

$$\varphi_{n,r}^d(x) = \prod_{j=0}^{d-1} \left(x - \frac{j+r}{n}\right).$$

The set of Newton polynomials $\{\varphi_{n,r}^d : 1 \leq d \leq D\}$ is the algebraic basis in the space of polynomials of degree, at most, D . For more detail information, please see [31].

So we have,

$$S_{n,\alpha}^{r,\dagger}(\varphi_{n,r}^d; x) = (1 - \alpha) \left(x - \frac{r}{n}\right)^d,$$

which means that $S_{n,\alpha}^{r,\dagger}(f; x)$ is degree preserving for all $d \geq 0$. Similarly, for $S_{n,\alpha}^{r,\ddagger}(f; x)$, let us define another (Newton) polynomial, as follows:

$$\phi_{n,r}^d(x) = \prod_{j=0}^{d-1} \left(x - \frac{j}{n}\right).$$

In this case, we have

$$S_{n,\alpha}^{r,\ddagger}(\phi_{n,r}^d; x) = \alpha \mu_{n,d} m_d(x),$$

where

$$\mu_{n,d} = \prod_{j=0}^{d-1} \left(1 + \frac{j}{n}\right) = \frac{(n)^d}{n^d},$$

where $(n)^d$ is rising factorial and $m_d(x)$ is a monomial of degree d such that,

$$m_d(x) = x^d.$$

So, similarly, $S_{n,\alpha}^{r,\ddagger}(f; x)$ is also degree preserving for all $d \geq 0$. As a consequence, one can easily argue that α -Szász–Mirakyan operators are degree preserving on polynomials. □

Remark 1. Although the α -Szász–Mirakyan operator has a degree-preserving property for all $d \geq 0$ according to Lemma 7, this operator does not reproduce any polynomial of degree greater than one again according to the same lemma.

4. Approximation by the α -Szász–Mirakyan Operators

In this section, we examine some of the approximation properties of the new operator we defined in the previous section. However, before all of this, let us recall the Popoviciu–Bohman–Korovkin theorem, which is one of the most significant and powerful criteria demonstrating that a positive linear operator converges to an identity operator.

Lemma 8 ([3–5]). Let $P_n : C[a, b] \rightarrow C[a, b]$, $n \geq 1$ be linear positive operators. Assume that for every $k \in \{0, 1, 2\}$, the sequence $(P_n(e_k))_{n \geq 1}$ converges to e_k uniformly on $[a, b]$. Then, $(P_n(f))_{n \geq 1}$ converges to f uniformly on $[a, b]$ for every $f \in [a, b]$.

Together with the result expressed above, great progress is made in determining whether linear positive operators converge uniformly to a given function for a given interval. In light of this theorem, it is immediately understood that the α -Szász–Mirakyan operator converges uniformly to the target function. Indeed, considering the results in Lemma 3-i, ii and 5-i, it is possible to conclude that the newly defined operator has a uniform convergence property for the defined interval. Now let us express this result with the theorem given below.

Theorem 1. The α -Szász–Mirakyan operators converge uniformly to the given function $f(x)$ on the interval $C[0, A]$ for each $0 \leq \alpha \leq 1$, if the function $f(x)$ is continuous on $C[0, A]$, where $A \in \mathbb{R}^+$.

Proof. This result can be easily obtained when considering the moment values of the α -Szász–Mirakyan operators. \square

Now, we will focus on two different theorems used to estimate the error value $S_{n,\alpha}^r(f; x) - f(x)$. The first one is related to the asymptotic error term for the α -Szász–Mirakyan operators for twice-differentiable functions, and the other is related to the modulus of continuity. However, first of all, we have to provide the following lemmas for the first theorem.

Lemma 9. Let consider

$$T_s(x) = \sum_{k=0}^{\infty} p_{n,k,r}^{(\alpha)}(x)(k - nx)^s,$$

for $s = 0, 1, 2, 3, 4$. Then the following hold:

- (i) $T_0(x) = 1,$
- (ii) $T_1(x) = 0,$
- (iii) $T_2(x) = nx + r(\alpha - 1),$
- (iv) $T_3(x) = nx + r(\alpha - 1),$
- (v) $T_4(x) = 3n^2x^2 + nx + (6nxr - 3r^2 + r)(\alpha - 1).$

Proof. From Lemmas 3 and 5, one can write the following:

- (i) $\sum_{k=0}^{\infty} p_{n,k,r}^{(\alpha)}(x) = 1,$
- (ii) $\sum_{k=0}^{\infty} p_{n,k,r}^{(\alpha)}(x)k = nx,$

- (iii) $\sum_{k=0}^{\infty} p_{n,k,r}^{(\alpha)}(x)k^2 = n^2x^2 + nx + r(\alpha - 1),$
- (iv) $\sum_{k=0}^{\infty} p_{n,k,r}^{(\alpha)}(x)k^3 = n^3x^3 + 3n^2x^2 + nx(3\alpha r - 3r + 1) + r(\alpha - 1),$
- (v) $\sum_{k=0}^{\infty} p_{n,k,r}^{(\alpha)}(x)k^4 = n^4x^4 + 6n^3x^3 + n^2x^2(6\alpha r - 6r + 7) + nx(10\alpha r - 10r + 1) + r(\alpha - 1)(1 - 3r).$

Then with the help of the well-known binomial expansion of $(k - nx)^s$ for $s = 0, 1, 2, 3, 4$, one can easily deduce the desired results by using the above findings, thus the proof is completed. \square

Lemma 10. *Let δ be a sufficiently small positive number. For every $x \in [0, \infty)$, we have the following inequality, that is to say*

$$\sum_{|\frac{k}{n} - x \geq \delta|} p_{n,k,r}^{(\alpha)}(x) \leq Nn^{-2}x^2\delta^{-2},$$

where N is constant, independent of x and n .

Proof. From the Lemma 9-v, one can easily verify that the sum of $T_4(x)$ is a polynomial with two variables, n and x of degree two. So for arbitrary constant N , we deduce that $|T_4(x)| \leq Nn^2x^2$ for each $x \in [0, \infty)$. In addition to this, we know that $|\frac{k}{n} - x| \geq \delta$ leads to $(k - nx)^4n^{-4}\delta^{-4} \geq 1$ for every chosen sufficiently small δ such that $\delta > 0$. So we deduce that,

$$\sum_{|\frac{k}{n} - x \geq \delta|} p_{n,k,r}^{(\alpha)}(x) \leq n^{-4}\delta^{-4} \sum_{k=0}^{\infty} (k - nx)^4 p_{n,k,r}^{(\alpha)}(x) = n^{-4}\delta^{-4}T_4(x) \leq Nn^{-2}x^2\delta^{-2},$$

which completes the proof. \square

Now we can state and prove the Voronovskaja-type theorem for the α -Szász–Mirakyan operators as follows.

Theorem 2. *Let $f \in C_2^*[0, \infty)$. If f is twice differentiable at $x \in [0, \infty)$, then*

$$\lim_{n \rightarrow \infty} n[S_{n,\alpha}'(f; x) - f(x)] = \frac{1}{2}xf''(x).$$

Proof. By using Taylor’s expansion, we have the following identity:

$$f(t) - f(x) = (t - x)f'(x) + \frac{1}{2}(t - x)^2f''(x) + \rho(t - x)(t - x)^2,$$

where $\rho(z)$ converges to zero with z . Then by choosing $t = \frac{k}{n}$, we deduce that

$$f\left(\frac{k}{n}\right) - f(x) = \left(\frac{k}{n} - x\right)f'(x) + \frac{1}{2}\left(\frac{k}{n} - x\right)^2f''(x) + \rho\left(\frac{k}{n} - x\right)\left(\frac{k}{n} - x\right)^2.$$

Now, let us apply the α -Szász–Mirakyan operator to both sides of this identity and multiply it by n , that is to say,

$$\begin{aligned}
 n \sum_{k=0}^{\infty} p_{n,k,r}^{(\alpha)}(x) \left[f\left(\frac{k}{n}\right) - f(x) \right] &= n \sum_{k=0}^{\infty} p_{n,k,r}^{(\alpha)}(x) \left[\left(\frac{k}{n} - x\right) f'(x) \right] \\
 &+ \frac{1}{2} n \sum_{k=0}^{\infty} p_{n,k,r}^{(\alpha)}(x) \left[\left(\frac{k}{n} - x\right)^2 f''(x) \right] \\
 &+ n \sum_{k=0}^{\infty} p_{n,k,r}^{(\alpha)}(x) \left[\rho\left(\frac{k}{n} - x\right) \left(\frac{k}{n} - x\right)^2 \right],
 \end{aligned}$$

which yields,

$$n [S_{n,\alpha}^r(f; x) - f(x)] = T_1(x) f'(x) + \frac{1}{2n} T_2(x) f''(x) + n \sum_{k=0}^{\infty} p_{n,k,r}^{(\alpha)}(x) \rho\left(\frac{k}{n} - x\right) \left(\frac{k}{n} - x\right)^2.$$

Utilizing Lemma 9-ii and iii, we obtain

$$n [S_{n,\alpha}^r(f; x) - f(x)] = \left(\frac{x}{2} + \frac{r(\alpha - 1)}{2n}\right) f''(x) + n \Lambda_n(x),$$

where

$$\Lambda_n(x) = \sum_{k=0}^{\infty} p_{n,k,r}^{(\alpha)}(x) \rho\left(\frac{k}{n} - x\right) \left(\frac{k}{n} - x\right)^2.$$

In order to finalize the proof of the theorem, we have to show that,

$$\lim_{n \rightarrow \infty} n \Lambda_n(x) = 0.$$

Then, we consider now $\Lambda_n(x)$. For any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|\rho(z)| < \varepsilon$ for $|z| \leq \delta$, and we choose δ sufficiently small that $\delta \leq x$. In this circumstance, we can divide the sum $\Lambda_n(x)$ into two sections for $\alpha \in [0, 1]$ as follows:

$$\begin{aligned}
 \Lambda_n(x) &= \sum_{|\frac{k}{n} - x| < \delta} p_{n,k,r}^{(\alpha)}(x) \rho\left(\frac{k}{n} - x\right) \left(\frac{k}{n} - x\right)^2 + \sum_{|\frac{k}{n} - x| \geq \delta} p_{n,k,r}^{(\alpha)}(x) \rho\left(\frac{k}{n} - x\right) \left(\frac{k}{n} - x\right)^2, \\
 &=: \Lambda_{n,1}(x) + \Lambda_{n,2}(x),
 \end{aligned}$$

say. Let start with the $\Lambda_n^1(x)$. According to Lemma 9-iii again, the following inequality immediately is obtained

$$|\Lambda_{n,1}(x)| \leq \frac{\varepsilon}{n^2} T_2(x) = \frac{\varepsilon}{n^2} (nx + r(1 - \alpha)),$$

which yields

$$\lim_{n \rightarrow \infty} n \Lambda_{n,1}(x) = 0,$$

because ε is arbitrary. Now, let us return the $\Lambda_{n,2}(x)$. In the circumstance, using the Lemma 10, we obtain

$$|\Lambda_{n,2}(x)| \leq \frac{1}{n^2 \delta^2} K N x^2,$$

where $K = \sup_{t \in [0, \infty)} \rho(t - x)(t - x)^2$, which similarly yields,

$$\lim_{n \rightarrow \infty} n \Lambda_{n,2}(x) = 0,$$

and thus, the proof is completed. \square

Another significant problem we encounter when studying with linear positive operators in approximation theory is to determine the approximation speed of the considered operator to a continuous function. One of the most important methods used to determine this is the modulus of continuity which was introduced by H. Lebesgue in 1910 and was also studied by D. Jackson in 1911 in his PhD dissertation. Now, let us review the modulus of continuity as follows.

Definition 2. Let f be a continuous and real valued function defined on $[a, b]$. For arbitrary $\delta > 0$, the modulus of continuity of $f(x)$ on $[a, b]$, denoted by $\omega_f(\delta)$, defined by

$$\omega_f(\delta) := \omega(f; [a, b]; \delta) = \sup\{|f(x) - f(y)| : x, y \in [a, b], |x - y| \leq \delta\}.$$

It is noted that here, the function f is a function of the number δ , ref. [32]. Moreover, in order to present the theorem-related modulus of continuity, we need to provide some fundamental properties of the modulus of continuity.

Lemma 11 ([33]). $\omega_f(\delta)$ satisfies the following conditions:

- (i) For $\delta > 0$, $\omega_f(\delta)$ is a non-negative function.
- (ii) $\omega_f(\delta)$ is increasing function, that is to say, if $\delta_1 \leq \delta_2$, then $\omega_f(\delta_1) \leq \omega_f(\delta_2)$.
- (iii) If $\lambda > 0$, then $\omega_f(\lambda\delta) \leq (1 + \lambda)\omega_f(\delta)$.
- (iv) $\lim_{\delta \rightarrow 0} \omega_f(\delta) = 0$ for the function f , which is uniformly continuous on $[a, b]$.
- (v) The inequality $|f(t) - f(x)| \leq \left(1 + \frac{|t - x|}{\delta}\right)\omega_f(\delta)$ holds.

Now we can present the theorem which provides an upper bound or the approximation error of the α -Szász–Mirakyan operators.

Theorem 3. Let $f \in C_2[0, \infty)$ be a function and $\omega_f(\delta)$ be a classical modulus of continuity. Then for $\alpha \in [0, 1]$, we have

$$\|S_{n,\alpha}^r(f; x) - f(x)\| \leq 2\omega_f(\delta_{n,r}^{(\alpha)}),$$

where $\delta_{n,r}^{(\alpha)} = \frac{\sqrt{nx + r(\alpha - 1)}}{n}$.

Proof. For the α -Szász–Mirakyan operators for $\alpha \in [0, 1]$, we have

$$\begin{aligned} |S_{n,\alpha}^r(f; x) - f(x)| &= \left| \sum_{k=0}^{\infty} p_{n,k,r}^{(\alpha)}(x) f\left(\frac{k}{n}\right) - \sum_{k=0}^{\infty} p_{n,k,r}^{(\alpha)}(x) f(x) \right| \\ &\leq \sum_{k=0}^{\infty} p_{n,k,r}^{(\alpha)}(x) \left| f\left(\frac{k}{n}\right) - f(x) \right|, \end{aligned}$$

where $p_{n,k,r}^{(\alpha)}(x)$ defined in (1). Then, utilizing from Lemma 11-v, we readily deduce that

$$\begin{aligned} |S_{n,\alpha}^r(f; x) - f(x)| &\leq \sum_{k=0}^{\infty} p_{n,k,r}^{(\alpha)}(x) \left(1 + \frac{|k/n - x|}{\delta}\right) \omega_f(\delta) \\ &\leq \left[1 + \frac{1}{\delta} \sum_{k=0}^{\infty} p_{n,k,r}^{(\alpha)}(x) \left|\frac{k}{n} - x\right|\right] \omega_f(\delta). \end{aligned}$$

Now if we apply the Cauchy–Schwarz inequality to the sum in the last inequality above, we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} p_{n,k,r}^{(\alpha)}(x) \left| \frac{k}{n} - x \right| &\leq \left[\sum_{k=0}^{\infty} p_{n,k,r}^{(\alpha)}(x) \left(\frac{k}{n} - x \right)^2 \right]^{1/2} \left[\sum_{k=0}^{\infty} p_{n,k,r}^{(\alpha)}(x) \right]^{1/2} \\ &= \left[\sum_{k=0}^{\infty} p_{n,k,r}^{(\alpha)}(x) \left(\frac{k}{n} - x \right)^2 \right]^{1/2} \\ &\leq \frac{1}{n} \sqrt{T_2(x)} \\ &= \frac{1}{n} \sqrt{nx + r(\alpha - 1)}, \end{aligned}$$

where $T_2(x)$ is given in Lemma 9-iii. Then we deduce that

$$|S_{n,\alpha}^r(f; x) - f(x)| \leq \left[1 + \frac{1}{\delta} \frac{1}{n} \sqrt{nx + r(\alpha - 1)} \right] \omega_f(\delta).$$

If we choose $\delta = \delta_{n,r}^{(\alpha)} = \frac{\sqrt{nx + r(\alpha - 1)}}{n}$, we deduce

$$|S_{n,\alpha}^r(f; x) - f(x)| \leq 2\omega_f(\delta_{n,r}^{(\alpha)}),$$

which completes the proof. \square

Remark 2. The above theorem provides us an upper bound for the error of $S_{n,\alpha}^r(f; x) - f(x)$ in terms of the modulus of continuity. On the other hand, this theorem can be claimed as another proof of Theorem 1. In other words, if f is uniformly continuous on $C_2[0, \infty)$, we have the following identity

$$\lim_{n \rightarrow \infty} \omega \left(\frac{\sqrt{nx + r(\alpha - 1)}}{n} \right) = \lim_{\delta_{n,r}^{(\alpha)} \rightarrow 0} \omega(\delta_{n,r}^{(\alpha)}) = 0,$$

thanks to the Lemma 11-iv. Therefore, this equation is sufficient to prove Theorem 1.

On the other hand, in [34], the author proved the following weighted Korovkin-type theorems. We consider $(L_n)_{n \geq 1}$, a sequence of positive linear operators acting from $C_2[0, \infty)$ to $B_2[0, \infty)$.

Lemma 12 ([34]). The positive linear operators (L_n) , $n \geq 1$, act from $C_2[0, \infty)$ to $B_2[0, \infty)$ if and only if inequality

$$|L_n(\tau; x)| \leq H_n \tau(x),$$

holds, where H_n is a positive constant depending on n .

Theorem 4 ([34]). Let the sequence of linear positive operators (L_n) , $n \geq 1$, acting from $C_2[0, \infty)$ to $B_2[0, \infty)$ satisfy the three conditions

$$\lim_{n \rightarrow \infty} \|L_n t^v - x^v\|_{\tau} = 0, \quad v = 0, 1, 2.$$

Then for any function $f \in C_2[0, \infty)$,

$$\lim_{n \rightarrow \infty} \|L_n f - f\|_{\tau} = 0.$$

Therefore, we can present the following result.

Theorem 5. Let $(S_{n,\alpha}^r)$ be the sequence of linear positive operators defined in Definition 1. Then for each function $f \in C_2^*[0, \infty)$,

$$\lim_{n \rightarrow \infty} \|S_{n,\alpha}^r(f) - f\|_\tau = 0.$$

Proof. Clearly, $\|S_{n,\alpha}^r(e_0) - e_0\|_\tau \rightarrow 0$ and $\|S_{n,\alpha}^r(e_1; x) - e_1\|_\tau \rightarrow 0$ as $n \rightarrow \infty$ on $[0, \infty)$ thanks to the results of Lemma 3. From the equalities in Lemma 5-i, we obtain

$$\sup_{x \in [0, \infty)} \frac{|S_{n,\alpha}^r(e_2; x) - e_2|}{1 + x^2} \leq \frac{1}{n} \sup_{x \in [0, \infty)} \frac{x}{1 + x^2} + \frac{r}{n^2}(\alpha - 1) \sup_{x \in [0, \infty)} \frac{1}{1 + x^2}.$$

Thus, we have

$$\lim_{n \rightarrow \infty} \|S_{n,\alpha}^r(e_2; x) - e_2\|_\tau = 0.$$

Therefore, the desired result follows from Theorem 4. \square

5. Shape-Preserving Properties of $S_{n,\alpha}^r$

In this section, the properties of the α -Szász–Mirakyan operators are examined in case the real-valued function f defined in the interval $C[0, \infty)$ is increasing (decreasing) or convex. In other words, the shape-preserving properties of the α -Szász–Mirakyan operators are presented by proving that $S_{n,\alpha}^r$ preserves the convexity and monotonicity. In line with this objective, firstly, we need to provide the first- and second-order derivatives of $S_{n,\alpha}^r$. The point that should not be forgotten here is that the interval partition in the definition of the α -Szász–Mirakyan operators is also valid here. Namely, since $S_{n,\alpha}^r(f; x) = f(x)$ when $x \in [0, \frac{r}{n})$, we will only deal with the rest of the interval and consider it valid for the whole interval.

Lemma 13. Let $f \in C[0, \infty)$ and $x \in [0, \infty)$. Then, the first- and second-order derivatives of the α -Szász–Mirakyan operators are as follows:

$$(i) \quad (S_{n,\alpha}^r)'(f; x) = \sum_{k=0}^{\infty} n p_{n,k,r}^{(\alpha)}(x) \Delta f \left(\frac{k}{n} \right),$$

$$(ii) \quad (S_{n,\alpha}^r)''(f; x) = \sum_{k=0}^{\infty} n^2 p_{n,k,r}^{(\alpha)}(x) \Delta^2 f \left(\frac{k}{n} \right),$$

where $\Delta^q \left(\frac{k}{n} \right)$ is the q -th order forward difference operator with the increment $\frac{1}{n}$ and $p_{n,k,r}^{(\alpha)}(x)$ defined in (1).

Proof.

(i) First of all, let us calculate the first-order derivative of $S_{n,\alpha}^r$. For this purpose, we use the similar notations in the proof of Lemma 7, that is to say, we have

$$S_{n,\alpha}^r(f; x) = \sum_{k=0}^{\infty} (1 - \alpha) e^{r-nx} \frac{(nx - r)^{k-r}}{(k - r)!} f \left(\frac{k}{n} \right) + \sum_{k=0}^{\infty} \alpha e^{-nx} \frac{(nx)^k}{(k)!} f \left(\frac{k}{n} \right),$$

Then, by taking the first-order derivative of the above identity, we deduce that

$$\begin{aligned}
 (S_{n,\alpha}^r)'(f; x) &= \sum_{k=0}^{\infty} -n(1-\alpha)e^{r-nx} \frac{(nx-r)^{k-r}}{(k-r)!} f\left(\frac{k}{n}\right) \\
 &\quad + \sum_{k=1}^{\infty} n(1-\alpha)e^{r-nx} \frac{(nx-r)^{k-r-1}}{(k-r-1)!} f\left(\frac{k}{n}\right) \\
 &\quad + \sum_{k=0}^{\infty} -n(\alpha)e^{-nx} \frac{(nx)^k}{(k)!} f\left(\frac{k}{n}\right) + \sum_{k=1}^{\infty} n(\alpha)e^{-nx} \frac{(nx)^{k-1}}{(k-1)!} f\left(\frac{k}{n}\right) \\
 &= \sum_{k=0}^{\infty} -n(1-\alpha)e^{r-nx} \frac{(nx-r)^{k-r}}{(k-r)!} f\left(\frac{k}{n}\right) \\
 &\quad + \sum_{k=0}^{\infty} n(1-\alpha)e^{r-nx} \frac{(nx-r)^{k-r}}{(k-r)!} f\left(\frac{k+1}{n}\right) \\
 &\quad + \sum_{k=0}^{\infty} -n(\alpha)e^{-nx} \frac{(nx)^k}{(k)!} f\left(\frac{k}{n}\right) + \sum_{k=0}^{\infty} n(\alpha)e^{-nx} \frac{(nx)^k}{(k)!} f\left(\frac{k+1}{n}\right)
 \end{aligned}$$

which yields

$$(S_{n,\alpha}^r)'(f; x) = \sum_{k=0}^{\infty} np_{n,k,r}^{(\alpha)}(x) \left[f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) \right],$$

then the desired result is obtained.

(ii) Now let us take the second-order derivative of the $S_{n,\alpha}^r$, which is

$$\begin{aligned}
 (S_{n,\alpha}^r)''(f; x) &= \sum_{k=0}^{\infty} n^2(1-\alpha)e^{r-nx} \frac{(nx-r)^{k-r}}{(k-r)!} f\left(\frac{k}{n}\right) \\
 &\quad + \sum_{k=1}^{\infty} -n^2(1-\alpha)e^{r-nx} \frac{(nx-r)^{k-r-1}}{(k-r-1)!} f\left(\frac{k}{n}\right) \\
 &\quad + \sum_{k=0}^{\infty} -n^2(1-\alpha)e^{r-nx} \frac{(nx-r)^{k-r}}{(k-r)!} f\left(\frac{k+1}{n}\right) \\
 &\quad + \sum_{k=1}^{\infty} n^2(1-\alpha)e^{r-nx} \frac{(nx-r)^{k-r-1}}{(k-r-1)!} f\left(\frac{k+1}{n}\right) \\
 &\quad + \sum_{k=0}^{\infty} n^2(\alpha)e^{-nx} \frac{(nx)^k}{(k)!} f\left(\frac{k}{n}\right) + \sum_{k=1}^{\infty} -n^2(\alpha)e^{-nx} \frac{(nx)^{k-1}}{(k-1)!} f\left(\frac{k}{n}\right) \\
 &\quad + \sum_{k=0}^{\infty} -n^2(\alpha)e^{-nx} \frac{(nx)^k}{(k)!} f\left(\frac{k+1}{n}\right) + \sum_{k=1}^{\infty} n^2(\alpha)e^{-nx} \frac{(nx)^{k-1}}{(k-1)!} f\left(\frac{k+1}{n}\right),
 \end{aligned}$$

which yields,

$$\begin{aligned}
 (S_{n,\alpha}^r)''(f; x) &= \sum_{k=0}^{\infty} n^2(1-\alpha)e^{r-nx} \frac{(nx-r)^{k-r}}{(k-r)!} f\left(\frac{k}{n}\right) \\
 &+ \sum_{k=0}^{\infty} -n^2(1-\alpha)e^{r-nx} \frac{(nx-r)^{k-r}}{(k-r)!} f\left(\frac{k+1}{n}\right) \\
 &+ \sum_{k=0}^{\infty} -n^2(1-\alpha)e^{r-nx} \frac{(nx-r)^{k-r}}{(k-r)!} f\left(\frac{k+1}{n}\right) \\
 &+ \sum_{k=0}^{\infty} n^2(1-\alpha)e^{r-nx} \frac{(nx-r)^{k-r}}{(k-r)!} f\left(\frac{k+2}{n}\right) \\
 &+ \sum_{k=0}^{\infty} n^2(\alpha)e^{-nx} \frac{(nx)^k}{(k)!} f\left(\frac{k}{n}\right) + \sum_{k=0}^{\infty} -n^2(\alpha)e^{-nx} \frac{(nx)^k}{(k)!} f\left(\frac{k+1}{n}\right) \\
 &+ \sum_{k=0}^{\infty} -n^2(\alpha)e^{-nx} \frac{(nx)^k}{(k)!} f\left(\frac{k+1}{n}\right) + \sum_{k=0}^{\infty} n^2(\alpha)e^{-nx} \frac{(nx)^k}{(k)!} f\left(\frac{k+2}{n}\right).
 \end{aligned}$$

Similarly, by rearranging the above equation, we obtain

$$(S_{n,\alpha}^r)''(f; x) = \sum_{k=0}^{\infty} n^2 p_{n,k,r}^{(\alpha)}(x) \left[f\left(\frac{k+2}{n}\right) - 2f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) \right],$$

and thus, the proof is completed.

□

Theorem 6. Let $f \in C[0, \infty)$ and $x \in [0, \infty)$. Then the following holds for the α -Szász–Mirakyan operators for $\alpha \in [0, 1]$,

- (i) If f is a monotonically increasing (or decreasing) function, then so are all of its α -Szász–Mirakyan operators.
- (ii) If f is a convex (or concave) function, then so are all of its α -Szász–Mirakyan operators.

Proof.

- (i) With the aid of the results of Lemma 13, one can easily deduce both results in the theorem. That is to say, since $\Delta f\left(\frac{k}{n}\right)$ is non-negative when $f(x)$ is an increasing function, the first derivative of the α -Szász–Mirakyan operators will also be positive. Therefore, this proves our claim.
- (ii) Similarly, if f is a convex function, the second-order divided differences are positive, so the second sum is also positive and the α -Szász–Mirakyan operators and the second derivative are also positive. Thus the proof is completed.

□

With this theorem, the shape-preserving properties of the α -Szász–Mirakyan operators are also presented. Therefore, these results will play a crucial role in determining the application areas of this new operator. Now let us provide some numerical results in the next section in order to observe the approximation properties of the α -Szász–Mirakyan operators.

6. Comparison with Classical Szász–Mirakyan Operators

In this section, the convergence rates of the α -Szász–Mirakyan operators and the classical Szász–Mirakyan operators are compared, and it is shown that the new operators have at least as good a degree of approximation as the classical Szász–Mirakyan operators for a specific interval. In accordance with this purpose, first of all, let us summarize the

rate of convergence of both the new family of Szász–Mirakyan operators and its classical correspondence. For the function, let $f \in C_2[0, \infty)$ be a function and $\omega_f(\delta)$ be a standard modulus of continuity. Then the rates of convergence of these two operators are

$$\|M_n(f; x) - f(x)\| \leq 2\omega_f(\delta^*) \quad \text{and} \quad \|S_{n,\alpha}^r(f; x) - f(x)\| \leq 2\omega_f(\delta_{n,r}^{(\alpha)}),$$

where $\delta^* = \sqrt{\frac{x}{n}}$ and $\delta_{n,r}^{(\alpha)} = \frac{\sqrt{nx+r(\alpha-1)}}{n}$, respectively. Now, we can give the following theorem to prove that the α -Szász–Mirakyan operators have a better rate of convergence performance than the classical one.

Theorem 7. For the function $C_2[0, \infty)$, the α -Szász–Mirakyan operators have a better convergence rate in comparison with the standard Szász–Mirakyan operators since the inequality

$$\delta^* \geq \delta_{n,r}^{(\alpha)},$$

holds for all $x \in [0, \infty)$.

Proof. Let $C[0, \mathcal{A}]$ be a function. Then, in order to show that the α -Szász–Mirakyan operators have a better convergence property than the classical one, we need to be able to show that the following inequality

$$\sqrt{\frac{x}{n}} \geq \frac{\sqrt{nx+r(\alpha-1)}}{n},$$

holds for all $x \in [0, \infty)$. For this purpose, let us define a function which is

$$\psi_{n,r}^{(\alpha)}(x) = \sqrt{\frac{x}{n}} - \frac{\sqrt{nx+r(\alpha-1)}}{n}.$$

In this case, if we can demonstrate that $\psi_{n,r}^{(\alpha)}(x)$ is positive, our claim will be proven correctly. For all $x \in [0, \infty)$ and $n \in \mathbb{N}$, it is obvious that $\sqrt{\frac{x}{n}}$ is greater than $\frac{\sqrt{nx+r(\alpha-1)}}{n}$ in the case of $r > 0$ and $\alpha \in [0, 1]$. It should not be forgotten that the most crucial reason for obtaining this result is that the α is between 0 and 1. It is also clear that $\psi_{n,r}^{(\alpha)}(x)$ does not change any sign on $[0, \infty)$. As a result, it can be concluded that the function $\psi_{n,r}^{(\alpha)}(x)$ is positive under this circumstance. Thus, the proof is completed. \square

In other words, since $\delta_{n,r}^{(\alpha)}$ is decreasing with respect to $\alpha \in (0, 1]$, the right hand side of the approximation estimate in the classical case $\alpha = 1$ has the greatest value, for each x .

This theoretical result obtained in this section will be supported by numerical results in the next section.

7. Illustrative Examples

Until now, we introduced and presented the approximation properties of the new class of Szász–Mirakyan operators in detail. Now, in this section, we will present some numerical illustrative examples in order to observe the approximation behavior of the α -Szász–Mirakyan operators. In these examples, we provide the results obtained with different n and different α values. Additionally, we provide the comparison of the α -Szász–Mirakyan operators with the classical one. All of the numeric operation are executed on an Intel Core i5 individual laptop by running code performed in MATLAB 8.4.1.1290304 (R2019b) software.

In this experiments, we consider a test function given by

$$f(x) = x^2e^{-4x},$$

which we will approximate by α -Szász–Mirakyan operators for different values of the parameter α , different selections of the knots numbers and different choices of the parameter r .

First of all, in Figure 1a, we compare the α -Szász–Mirakyan operators with the classical Szász–Mirakyan operators for the values $\alpha = 0.05$, $n = 10$ and $r = 3$ on the interval $[0, 3]$. Figure 1a shows that the new family of Szász–Mirakyan operators presents better performance in comparison with the standard Szász–Mirakyan operators for given values. Secondly, we wanted to test how the α -Szász–Mirakyan operators would perform for different α values. These results can be seen in Figure 1b. Thirdly, in Figure 1c, we plot the graphs for different values of n . As it is expected, the introduced operator converges better to the test function as the parameter n increases. Finally, we present the performance of α -Szász–Mirakyan operators for different selections of values of r .

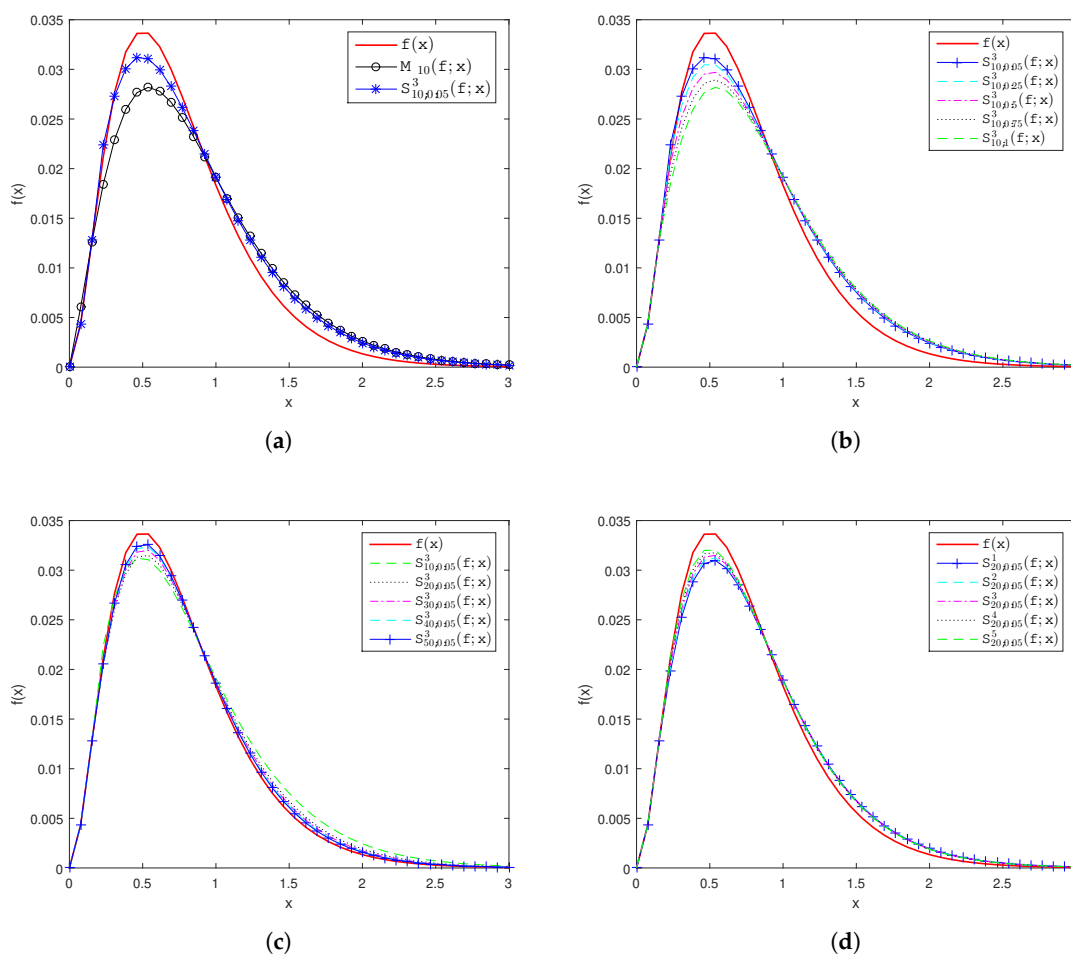


Figure 1. Approximation of f by α -Szász–Mirakyan operators with different parameters on uniform grid in $[0, 3]$. (a) Comparison $S_{n,\alpha}^r(f; x)$ with $M_n(f; x)$. (b) Different choice of α . (c) Different choice of n . (d) Different choice of r .

8. Conclusions

In this paper, the new family of α -Szász–Mirakyan operators are introduced and its approximation properties are presented. In more detail, the shape-preserving, linearity and monotonicity properties of the generalized α -Szász–Mirakyan operators indicated by $S_{n,\alpha}^r$ are examined. Then the asymptotic approximation of operators is investigated by consider-

ing the Voronovskaja-type theorem. At the same time, approximation characteristics of this new operator are discussed with the help of modulus continuity. Finally, Some numerical examples for different choice of the parameters in the definition of the α -Szász–Mirakyan operators are presented to verify the theoretical results. We think that this newly defined operator is applicable to the numerical solutions of integral equations. For future work, we propose to obtain some other variants of the α -Szász–Mirakyan operators, such as α -Szász–Mirakyan–Kantorovich operators $e^{nx} \sum_{k=0}^{m_n} c_{k,n} x^k$.

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