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Similarity Transformations and Linearization for a Family of Dispersionless Integrable PDEs

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Abstract: We apply the theory of Lie point symmetries for the study of a family of partial differential equations which are integrable by the hyperbolic reductions method and are reduced to members of the Painlevé transcendents. The main results of this study are that from the application of the similarity transformations provided by the Lie point symmetries, all the members of the family of the partial differential equations are reduced to second-order differential equations, which are maximal symmetric and can be linearized.

Keywords: Lie symmetries; invariants; similarity transformations; linearization

1. Introduction

In [1], Ferapontov et al. classified the partial differential equations of the form

$$(A(u))_{xx} + (B(u))_{yy} + (C(u))_{yy} + 2\Big((P(u))_{xy} + (Q(u))_{xt} + (P(u))_{yt}\Big) = 0, \tag{1}$$

which are integrable under the method of hydrodynamic reductions [2] and can be reduced into an Painlevé equation [3]. There are five partial differential equations of the form (1) which are integrable by the hydrodynamic reductions method [1]

$$\mathcal{H}_A \equiv u_{xx} + u_{yy} - (\ln(e^u - 1))_{yy} - (\ln(e^u - 1))_{tt} = 0, \tag{2}$$

$$\mathcal{H}_B \equiv u_{xx} + u_{yy} - (e^u)_{tt} = 0, \tag{3}$$

$$\mathcal{H}_C \equiv (e^u - u)_{xx} + 2u_{xy} + (e^u)_{tt} = 0, \tag{4}$$

$$\mathcal{H}_D \equiv u_{xt} - (uu_x)_x - u_{yy} = 0, \tag{5}$$

$$\mathcal{H}_E = \left(u^2\right)_{xx} + u_{yy} + 2u_{xt} = 0. \tag{6}$$

Equations (3) and (5) are the Boyer–Finley [4,5] and dispersionless Kadomtsev–Petviashvili [6] equations, respectively. For these two equations, it is known that they are reduced into Painlevé transcendents by applying the central quadric ansatz. The hydrodynamic reductions method was found to provide more general solutions. Indeed, by studying the dispersionless Kadomtsev–Petviashvili with the hydrodynamic reductions and the central quadric ansatz, it was found that the solutions coming from the later method form a subclass of two-phase solutions provided by the hydrodynamic reductions approach [1].

As far as the reduction of Equations (2)–(6) into a Painlevé equation is concerned, it was found that equation $\mathcal{H}_A$ reduces to the Painlevé P_{VI} equation, and that the Boyer–Finley equation $\mathcal{H}_B$ reduces into the P_V equation reducible to P_{III}. Moreover, $\mathcal{H}_C$ reduces to P_V, the dispersionless Kadomtsev–Petviashvili is related with the P_{II} with a reduction to P_I, while the fifth equation $\mathcal{H}_E$ is reduced to P_{IV} [1]. For extensions of the results of [1]

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Citation: Paliathanasis, A. Similarity Transformations and Linearization for a Family of Dispersionless Integrable PDEs. *Symmetry* **2022**, *14*, 1603. https://doi.org/10.3390/sym14081603

Academic Editor: Dumitru Baleanu

Received: 8 July 2022
Accepted: 2 August 2022
Published: 4 August 2022

and a connection of the hydrodynamic reductions method with the conformal structure of Einstein–Weyl geometry, we refer the reader to [7].

In this work, we apply the Lie symmetry analysis [8–11] in order to investigate the algebraic properties and the similarity transformations for the five partial differential Equations (2)–(6). The method of Lie symmetries of differential equations was established by Sophus Lie at the end of the 19th century, and provides a systematic approach for the study and determination of solutions and conservation laws for nonlinear differential equations.

The novelty of symmetry analysis is that invariant functions can be determined for a given differential equation. From the invariant functions we can define similarity transformations, which are necessary to simplify the differential equation. The similarity transformations are used to reduce the given differential equation into an equivalent equation with less dynamical variables. In the case of partial differential equations, the independent variables are reduced, while in the case of ordinary differential equations, the order of the equation, that is, the dependent variables, are reduced. There is a plethora of applications in the literature on the symmetry analysis of various dynamical systems. The method of symmetry analysis is applied in various systems of fluid dynamics in the studies [12–24]. The Burgers-heat system is investigated by applying the symmetry analysis in the studies [25,26]. A recent application of the Lie symmetry approach on time-fractional systems is presented in [27]. However, Lie symmetries are very useful for the study of ordinary differential equations. Some studies on the symmetry analysis on the geodesic equations in curved spaces are presented in [28–31]. Finally, in [32] a discussion is given on the novelty on the application of the Lie symmetry analysis in gravitational physics and cosmology.

Another important application of the Lie symmetry approach is the classification scheme of differential equations according to the admitted group of symmetries, and to the construction of equivalent transformation which transform a given differential equation into another differential equation of the same order, when the admitted Lie symmetries form the same Lie algebra [33–35]. Recently, in [36] the authors investigated which of the six ordinary differential equations of the Painlevé transcendents admit nontrivial Lie point symmetries. It was found that equations P_{III}, P_V and P_{VI} have nontrivial symmetries for special values of the free parameters. On the other hand, in [37] the method of Jacobi last multiplier is applied in order to determine generalized symmetries for a particular case of the P_{XIV} equation. By using generalized-hidden symmetries, the linearization of the Painleve–Ince equation was proved in [38]. The plan of the paper in as follows.

In Section 2 we present the basic properties and definitions for the Lie symmetry analysis of differential equations. In Sections 3–7 we determine the Lie point symmetries for the five equations of our analysis. We determine the commutators and the Adjoint representation such that to derive, when it is feasible, the one-dimensional optimal system. Equation $\mathcal{H}_A$ admits a finite dimensional Lie algebra of dimension four, in particular the $A_{4,5}$ in the Patera–Winternitz classification scheme [39]. However, the rest of the equations admit infinity Lie point symmetries. We were able to define four-dimensional Lie subalgebras. The main observation of this work is that the application of the Lie invariants define similarity transformation where the partial differential equations reduce to maximal symmetric second-order equations. That is an important result because we are able to investigate the integrability properties of Equations (2)–(6) by using the symmetry analysis. In Section 8 we summarize our results and we draw our conclusions. The main result of this analysis is given in a proposition where we show that Equations (2)–(6) can be linearized with the application of Lie invariants.

2. Preliminaries

Assume the infinitesimal one-parameter point transformation

$$\begin{aligned} t' &= t + \varepsilon\xi^t(t,x,y,u), \\ x' &= x + \varepsilon\xi^x(t,x,y,u), \\ y' &= y + \varepsilon\xi^y(t,x,y,u), \\ u' &= u + \varepsilon\eta(t,x,y,u), \end{aligned}$$

where ε is the infinitesimal parameter, $\varepsilon^2 \to 0$, and infinitesimal generator

$$X = \xi^t(t,x,y,u)\partial_t + \xi^x(t,x,y,u)\partial_x + \xi^y(t,x,y,u)\partial_y + \eta(t,x,y,u)\partial_u.$$

We define the second extension $X^{[2]}$ of X in the jet space $\{t,x,y,u,u_t,u_x,u_y,u_{tt},u_{xx},u_{yy},u_{tx},u_{xy}\}$ as follows

$$X^{[2]} = \xi^\mu\partial_\mu + \eta^A\partial_A + \eta^{A[1]}\partial_{A,\mu} + \eta^{A[2]}\partial_{A,\mu\nu}.$$

in which $\eta^{A[1]}$, $\eta^{A[2]}$ are defined as

$$\eta^{[n]} = D_\mu\eta^{[n-1]} - u_{\mu_1\mu_2,\dots,\mu_{n-1}}D_\mu(\xi^\mu),$$

where $\mu = (t,x,y)$.

By definition a partial differential equation $\mathcal{H} = \mathcal{H}(t,x,y,u,u_t,u_x,u_y,u_{tt},u_{xx},u_{yy},u_{tx},u_{xy})$ is invariant under the action of the one-parameter point transformation with infinitesimal generator the vector field X if and only if there exist a function λ such that [8–11]

$$\mathcal{L}_{X^{[2]}}(\mathcal{H}) = \lambda\mathcal{H},$$

in which $L_{X^{[2]}}$ is the Lie derivative with respect to the vector field $X^{[2]}$.

Lie symmetries are mainly applied for the construction of similarity transformations. The latter are necessary in order to simplify a given differential equation by means of reduction. The exact and analytic solutions which are determined by the application of the Lie symmetries are known as similarity solutions.

In order to perform a complete derivation of all the possible similarity solutions we should find the admitted one-dimensional optimal system. Consider the n-dimensional Lie algebra G_n with elements X_1, X_2, ..., X_n admitted by the differential equation $\mathcal{H}$.

The vector fields [8–11]

$$Z = \sum_{i=1}^{n} a_i X_i \;,\; W = \sum_{i=1}^{n} b_i X_i \;,\;\; a_i,\; b_i \text{ are constants.}$$

are equivalent if and only if $\mathbf{W} = Ad(\exp(\varepsilon_i X_i))\mathbf{Z}$ or $\mathbf{W} = c\mathbf{Z}$ where c is a constant.

Operator $Ad(\exp(\varepsilon X_i))X_j = X_j - \varepsilon[X_i, X_j] + \frac{1}{2}\varepsilon^2[X_i,[X_i,X_j]] + \dots$ is known as the the adjoint representation. They derive all the independent similarity transformations for a given differential equation, so the adjoint representation of the admitted Lie algebra should be determined. This leads to the construction of the one-dimnesional optimal system.

3. Lie Symmetry Analysis for Equation $\mathcal{H}_A$

The first equation of our analysis, equation $\mathcal{H}_A$, admits the Lie point symmetries

$$X_1^A = \partial_t \;,\; X_2^A = \partial_x \;,\; X_3^A = \partial_y \;,\; X_4^A = t\partial_t + x\partial_x + y\partial_y$$

with commutators and Adjoint representation as presented in Tables 1 and 2. The admitted Lie algebra is $A_{4,5}$. The one-dimensional system for the finite Lie algebra consists of the one-dimensional Lie algebras: $\{X_1^A\}$, $\{X_2^A\}$, $\{X_3^A\}$, $\{X_4^A\}$, $\{X_1^A + \alpha X_2^A\}$, $\{X_1^A + \alpha X_3^A\}$, $\{X_2^A + \alpha X_3^A\}$ and $\{X_1^A + \alpha X_2^A + \beta X_3^A\}$.

Table 1. Commutators for the Lie point symmetries of Equation (2).

$[X_I, X_J]$	$X^A{}_1$	$X^A{}_2$	$X^A{}_3$	$X^A{}_4$
$X^A{}_1$	0	0	0	X_1^A
$X^A{}_2$	0	0	0	X_2^A
$X^A{}_3$	0	0	0	X_3^A
$X^A{}_4$	$-X_1^A$	$-X_2^A$	$-X_3^A$	0

Table 2. Adjoint representation for the Lie point symmetries of Equation (2).

$Ad\left(e^{(\varepsilon X_i)}\right)X_j$	$X^A{}_1$	$X^A{}_2$	$X^A{}_3$	$X^A{}_4$
$X^A{}_1$	X_1^A	X_2^A	X_3^A	$X_4^A - \varepsilon X_1^A$
$X^A{}_2$	X_1^A	X_2^A	X_3^A	$X_4^A - \varepsilon X_2^A$
$X^A{}_3$	X_1^A	X_2^A	X_3^A	$X_4^A - \varepsilon X_3^A$
$X^A{}_4$	$e^{\varepsilon} X_1^A$	$e^{\varepsilon} X_2^A$	$e^{\varepsilon} X_3^A$	X_4^A

Application of $\{X_1^A + \alpha X_2^A\}$, $\{X_1^A + \beta X_3^A\}$ provides the reduced equation

$$\left(\beta^2\left(e^U - 1\right)^2 - \alpha^2\left(e^U - 1\right)\left(1 + \beta^2 e^U\right)\right)U_{\sigma\sigma} + \alpha^2\left(\beta^2 + 1\right)e^U (U_\sigma)^2 = 0, \tag{7}$$

with $u = U(\sigma)$ and $\sigma = y - \beta t + \frac{\beta}{\alpha}x$.

Equation (7) is a second-order ordinary differential equation of the form $U_{\sigma\sigma} + L(U)(U_\sigma)^2 = 0$, which means that iy is maximally symmetric. It admits eight Lie point symmetries which form the $sl(3, R)$ algebra. Thus, according to the main theorem of S. Lie theorem, Equation (7) can be linearized [8–11]. Indeed the transformation which linearized the differential equation is of the form $V = \int e^{\int L(U)dU} dU$. We remark that any reduction of Equation (2) with any Lie symmetries provided by the optimal system of the Abelian Lie subalgebra $\{X_1^A, X_2^A, X_3^A\}$ provides a similar result.

On the other hand, reduction with $\{X_4^A\}$ provides the partial differential equation

$$\begin{aligned} 0 \;=\; & \left(e^U - 1\right)\left(U_{\zeta\zeta}\left(\zeta^2 e^U + 1\right) + U_{\omega\omega}\left(e^U\left(\omega^2 - 1\right) + 1\right) + 2\omega\zeta e^U U_{\zeta\omega}\right) \\ & + e^U\left((U_\zeta \zeta + U_\omega \omega)^2 - 2\left(e^U - 1\right)(U_\zeta \zeta + U_\omega \omega) + U_{\zeta\zeta}^2\right), \end{aligned} \tag{8}$$

with $u = U(\zeta, \omega)$, $\zeta = \frac{y}{t}$ and $\omega = \frac{x}{t}$, is the similarity transformation. Equation (8) does not possess any Lie point symmetry; thus, we cannot reduce further the differential equation.

Moreover, reduction with the symmetry vectors $\{X_2^A + \alpha X_3^A\}$ and later with the reduced symmetry of $\{X_4^A\}$, provides the ordinary differential equation

$$\left(\alpha^2\left(e^U - 1\right) - 1 - e^U\rho^2\right)U_{\rho\rho} + \frac{e^U}{(e^U - 1)}\left(\rho^2 + 1\right)(U_\rho)^2 - 2\rho e^U U_\rho = 0, \tag{9}$$

where $u = U(\rho)$ and $\rho = \frac{y - \alpha x}{t}$.

Without loss of generality we assume $\alpha^2 = 1$. Then, Equation (9) is written in the equivalent form

$$\frac{U_{\rho\rho}}{U_\rho} + \frac{e^U(\rho^2 + 1)}{(e^U - 1)(e^U(1 - \rho^2) - 2)}(U_\rho) - \frac{2e^U}{(e^U(1 - \rho^2) - 2)} = 0,$$

that is

$$\frac{U_{\rho\rho}}{U_\rho} + \frac{e^U}{e^U - 1}U_\rho + \frac{U_\rho e^U \rho^2 + 2e^U \rho - U_\rho e^U}{e^U(\rho^2 - 1) + 2} = 0.$$

Hence, we can write easily the later equation as follows

$$\frac{d}{d\rho}\left(\ln(U_\rho) - \ln\left(e^U - 1\right) + \ln\left(e^U\left(\rho^2 - 1\right) + 2\right)\right) = 0.$$

Therefore, the conservation law is

$$\frac{U_\rho(e^U(\rho^2 - 1) + 2)}{e^U - 1} = I_0. \tag{10}$$

We observe that Equation (10) can be written in a linear form after the change of the independent variable $d\kappa = \frac{e^U - 1}{(e^U(\rho^2 - 1) + 2)}d\rho$, and $U = U(\kappa)$. Thus, Equation (10) becomes $U_\kappa = I_0$ which is nothing other than the conservation law for the maximal symmetric second-order ordinary differential equation $U_{\kappa\kappa} = 0$.

Last but not least, we remark that we find similar result if we apply first the reduction of any one-dimensional Lie algebra of the three-dimensional Abelian subalgebra and then we consider the $\{X_4^A\}$.

4. Lie Symmetry Analysis for Equation $\mathcal{H}_B$

In order to proceed with the analysis for equation $\mathcal{H}_B$ we select a new set of independent variables $(z, \bar{z}) = \frac{1}{2}(u + v, i(u - v))$, such that Equation (3) has to be written as follows

$$u_{z\bar{z}} - (e^u)_{tt} = 0. \tag{11}$$

Application of the Lie symmetry conditions indicates that Equation (11) admits the Lie symmetry vectors

$$X_1^B = \partial_t \ , \ X_2^B = t\partial_t + u\partial_u,$$

$$X_3^B = \Phi(z)\partial_z - \Phi(z)_z\partial_u \ , \ X_4^B = \Psi(\bar{z})\partial_z - \Psi(\bar{z})_{\bar{z}}\partial_u.$$

The vector fields X_3^B , X_4^B indicates the infinite number of solutions for the Laplace operator $u_{z\bar{z}}$. The commutators and the Adjoint representation for the admitted symmetry vectors are presented in Tables 3 and 4, respectively, as well as the Lie point symmetries, from the finite Lie algebra $A_{2,1}$ plus the infinite algebra consisted by the vector fields X_3^B and X_4^B. In Tables 3 and 4 we assumed that functions Φ and Ψ are specific and not arbitrary, because in general it holds $\left[X_3^B(\Phi_1), X_3^B(\Phi_2)\right] = X_3^B(\Phi_3)$.

Table 3. Commutator table for the Lie point symmetries of Equation (3).

$[X_I, X_J]$	$X^B{}_1$	$X^B{}_2$	$X^B{}_3$	$X^B{}_4$
$X^B{}_1$	0	X_1^B	0	0
$X^B{}_2$	$-X_1^B$	0	0	0
$X^B{}_3$	0	0	0	0
$X^B{}_4$	0	0	0	0

Table 4. Adjoint representation for the Lie point symmetries of Equation (3).

$Ad\left(e^{(\varepsilon X_i)}\right)X_j$	$X^B{}_1$	$X^B{}_2$	$X^B{}_3$	$X^B{}_4$
$X^B{}_1$	X_1^B	$X_2^B - \varepsilon X_1^B$	X_3^B	X_4^B
$X^B{}_2$	$e^\varepsilon X_1^B$	X_2^B	X_3^B	X_4^B
$X^B{}_3$	X_1^B	X_2^B	X_3^B	X_4^B
$X^B{}_4$	X_1^B	X_2^B	X_3^B	X_4^B

Hence, from Tables 3 and 4 it follows that one-dimensional optimal system is consisted by the following one-dimensional Lie algebras, $\{X_1^B\}$, $\{X_2^B\}$, $\{X_3^B\}$, $\{X_4^B\}$, $\{X_3^B+\alpha X_4^B\}$, $\{X_1^B+\alpha X_3^B\}$, $\{X_1^B+\alpha X_4^B\}$, $\{X_2^B+\alpha X_3^B\}$, $\{X_2^B+\alpha X_4^B\}$, $\{X_1^B+\alpha X_3^B+\beta X_4^B\}$ and $\{X_2^B+\alpha X_3^B+\beta X_4^B\}$. We proceed with the reduction of the equation and the determination of similarity solutions.

Consider now reduction with the use of the symmetry vector $\{X_3^B\}$, then it follows $u=-\ln\Phi(z)+\ln U(t,\bar{z})$, with the reduced equation

$$U_{tt}=0\,,\ U(t,\bar{z})=U_1(\bar{z})t+U_0(z).$$

We remark that the reduced equation is that of the free particle and it is maximally symmetric; thus, it admits eight Lie point symmetries which form the $sl(3,R)$ Lie algebra. A similar result it follows if we assume reduction with respect to the field X_4^B.

Let us assume now reduction with the field $\{X_1^B+X_4^B\}$. The reduced equation is found to be

$$U_{zT}+\left(e^U\right)_{TT}=0 \tag{12}$$

where $T=t-\int\frac{d\bar{z}}{\Psi(\bar{z})}$ and $u=-\ln(\Psi(\bar{z}))+U(T,x)$.

Equation (12) admits the symmetry vectors $\bar{X}_1^B=\partial_T$, $\bar{X}_2=T\partial_T+\partial_U$ and $\bar{X}_3^B=\Phi(z)\partial_z-\Phi(z)_z\partial_U$. Hence, application for the field $\bar{X}_1^B+\bar{X}_3^B$ gives the similarity transformation $U(T,x)=-\ln(\Phi(z))+V(\tau)\,,\tau=T-\int\frac{dz}{\Phi(z)}$, with the reduced equation, maximal symmetric ordinary differential equation

$$V_{\tau\tau}-\frac{e^V}{(1-e^V)}(V_\tau)^2=0.$$

Moreover, application of the vector field $\bar{X}_2+\bar{X}_3^B$ in (12) provides the reduced equation

$$V_{\lambda\lambda}\left(e^V-\lambda\right)+e^V(V_\lambda)^2-V_\lambda=0, \tag{13}$$

where $\lambda=Te^{-S(z)}$, $M(z)=\frac{1}{S_{,z}}$, and $U(T,z)=S(z)+\ln(S(z)_z)+V\left(Te^{-S(z)}\right)$.

Equation (13) is not maximally symmetric, however it can be integrated and it can written in the equivalent form

$$\bar{V}_{\bar{\lambda}}=\frac{e^{\bar{\lambda}}(1+\bar{V})\bar{V}}{e^{\bar{\lambda}-1}}\,,\ \bar{\lambda}=V-\ln\lambda\,,\ \bar{V}=(V_{,\lambda}\lambda-1)^{-1}.$$

Therefore, if we do the change of variables $d\lambda'=I_0\frac{e^{\bar{\lambda}}(1+\bar{V})\bar{V}}{e^{\bar{\lambda}-1}}d\bar{\lambda}$, the latter differential equation becomes $\bar{V}_{\lambda'}=I_0$, which is the conservation law for the maximal symmetric differential equation $\bar{V}_{\lambda'\lambda'}=0$.

5. Lie Symmetry Analysis for Equation $\mathcal{H}_C$

As far as the Lie symmetries of $\mathcal{H}_C$ are concerned, they are calculated

$$X_1^C=\partial_t\,,\ X_2^C=\partial_y\,,\ X_3^C=t\partial_t+x\partial_x+y\partial_y,$$

$$X_4^C(Z(y))=Z(y)\left(\partial_x-2\partial_y\right)+2Z(y)_y\partial_u.$$

Hence, we can infer that equation $\mathcal{H}_C$ admits infinity Lie symmetries. The nonzero commutators are

$$\left[X_2^C,X_3^C\right]=X_2^C\,,\ \left[X_2^C,X_4^C(Z(y))\right]=X_4^C\left(Z(y)_y\right),$$

$$\left[X_3^C,X_4^C(Z(y))\right]=X_4^C\left(yZ(y)_y-Z(y)\right).$$

and

$$\left[X_4^C(Z(y)), X_4^C(W(y))\right] = 2X_4^C(Y(y)) \text{ with } Y(y) = W(y)Z(y)_y - Z(y)W(y)_y.$$

We observe that for $Z(y)_y = 0$, a finite-dimensional Lie algebra exists, the four-dimensional Lie algebra $A_{4,5}$ of equation $\mathcal{H}_A$.

Consider the application of the Lie point symmetries $\{X_1^C + \alpha X_2^C, X_1^C + \alpha\partial_x\}$, then the reduced equation is derived

$$0 = \left(\alpha\left(e^U - 1\right) + 2\beta + \alpha\beta e^U\right)U_{\sigma\sigma} + \left(1+\beta^2\right)\alpha e^U (U_\sigma)^2, \tag{14}$$

where now $u = U(\sigma)$, $\sigma = y - \alpha t + \frac{\alpha}{\beta}x$. We observe that Equation (14) is a maximally symmetric second-order ordinary differential equation.

We proceed with the second reduction approach, where we apply the Lie symmetries $\{X_1^C + \alpha X_2^C, X_1^C + \alpha X_3^C\}$. Hence, equation $\mathcal{H}_C$ is reduced to the second-order ordinary differential equation

$$0 = \left(\left(\alpha^2+\omega^2\right)e^U - \omega(\omega+2)\right)U_{\omega\omega} + \left(\alpha^2+\omega^2\right)e^U(U_\omega)^2 + 2\left(\omega\left(e^U-1\right)-1\right)(U_\omega),$$

where $u = U(\omega)$ and $\omega = \frac{y+\alpha t}{x}$. The latter equation can be integrated as follows

$$U_\omega\left(\left(\alpha^2+\omega^2\right)e^U - \omega(\omega+2)\right) = I_0 \tag{15}$$

which can be written in the equivalent form $U_\varpi = I_0$.

Let us assume reduction with respect to the Lie symmetry vector $\{X_1^C + X_4^C(Z(y))\}$. The similarity transformation is $u = U(T, X)$ with $T = t - \int \frac{d\chi}{Z(y+2x-2\chi)}$ and $X = y + 2x$, while the reduced equation is

$$\left(U_{TT} + (U_T)^2 + 4\left(U_{YY} + (U_Y)^2\right)\right)e^U + 2U_{TY} = 0. \tag{16}$$

The latter equation admits the reduced symmetry vectors $\{\partial_T, \partial_Y, T\partial_T + Y\partial_Y\}$. It follows that reduction with respect to the symmetry vector $\{\partial_T + \beta\partial_Y\}$ gives $U = V(z)$, $z = \frac{1}{\beta}T - Y$ in which $V(z)$ is a solution of the maximal symmetric second-order ordinary differential equation

$$\left(\left(4+\beta^2\right)e^V - 2\beta\right)V_{zz} + \left(\beta^2+4\right)e^V(V_z)^2 = 0.$$

On the other hand, reduction of Equation (16) with respect to the similarity transformation provided by $T\partial_T + Y\partial_Y$ gives

$$\left(\left(4\lambda^2+1\right)e^V - 2\lambda\right)V_{\lambda\lambda} + \left(4\lambda^2+1\right)e^V(V_\lambda)^2 + \left(8\lambda e^V - 2\right)V_\lambda = 0,$$

that is

$$V_\lambda\left(\left(4\lambda^2+1\right)e^V - 2\lambda\right) = I_0,$$

which can be written as a maximal symmetric second-order differential equation.

6. Lie Symmetry Analysis for Equation $\mathcal{H}_D$

We proceed our analysis with the derivation of the Lie symmetry vectors for equation $\mathcal{H}_D$. The application of the Lie symmetry condition shows us that equation $\mathcal{H}_D$ admits infinity Lie symmetries as they are described by the following families of vector fields

$$X_1^D(\Phi(t)) = \Phi(t)\partial_t + \left(\frac{1}{6}\Phi_{tt}y^2 + \frac{1}{3}x\Phi\right)\partial_x + \frac{2}{3}y\Phi_t\partial_y - \frac{1}{3}\left(\Phi_{tt}x + \frac{1}{2}\Phi_{ttt}y^2 - 2\Phi_t u\right)\partial_u ,$$

$$X_2^D(\Psi(t)) = \Psi(t)\partial_x - \Psi(t)_t\partial_u ,$$

$$X_3^D(\Sigma(t)) = \frac{1}{2}\Sigma_t y\partial_x + \Sigma\partial_y - \frac{1}{2}\Sigma_{tt}y\partial_u,$$

and

$$X_4^D = 2x\partial_x + \partial_y + 2u\partial_u.$$

The nonzero commutators of the admitted Lie symmetries are

$$\left[X_1^D(\Phi), X_2^D(\Psi)\right] = X_2^D\left(\Phi\Psi_t - \frac{1}{3}\Phi_t\Psi\right),$$

$$\left[X_1^D(\Phi), X_3^D(\Sigma)\right] = X_3^D\left(\frac{2}{3}\Sigma\Phi_t - 3\Phi\Sigma_t\right),$$

$$\left[X_2^D(\Psi), X_4^D\right] = 2X_2^D(\Psi) , \ \left[X_3^D(\Sigma), X_4^D\right] = X_3^D(\Sigma),$$

$$\left[X_1^D(\Phi(t)), X_1^D(\bar{\Phi}(t))\right] = X_1^D(\Phi\bar{\Phi}_t - \bar{\Phi}\Phi_t),$$

$$\left[X_3^D(\Sigma(t)), X_3^D(\bar{\Sigma}(t))\right] = X_2^D(\bar{\Sigma}\Sigma_t - \Sigma\Sigma_t).$$

For $\Phi(t) = 1$, $\Psi(t) = 1$ and $\Sigma(t) = 1$, we find the four-dimensional subalgebra

$$\bar{X}_1^D = \partial_t , \ \bar{X}_2^D = \partial_x , \ \bar{X}_3^D = \partial_y , \ X_4^D,$$

which form the $A_{3,3} \otimes A_1$ Lie algebra. The commutators and the Adjoint representation of the four dimensional Lie algebra $\{\bar{X}_1^D, \bar{X}_2^D, \bar{X}_3^D, X_4^D\}$ are presented in Tables 5 and 6.

Table 5. Commutators for the elements which form the finite Lie algebra of Equation (5).

$[X_I, X_J]$	$\bar{X}^D{}_1$	$\bar{X}^D{}_2$	$\bar{X}^D{}_3$	$X^D{}_4$
$\bar{X}^D{}_1$	0	0	0	0
$X^D{}_2$	0	0	0	$2\bar{X}_2^D$
$\bar{X}^D{}_3$	0	0	0	$\bar{X}_3^D$
$\mathbf{X}^D{}_4$	0	$-2\bar{X}_2^D$	$-\bar{X}_3^D$	0

Table 6. Adjoint representation for the elements which form the finite Lie algebra (5).

$Ad\left(e^{(\varepsilon X_i)}\right)X_j$	$\bar{X}^D{}_1$	$\bar{X}^D{}_2$	$\bar{X}^D{}_3$	$X^D{}_4$
$\bar{X}^D{}_1$	$\bar{X}_1^D$	$\bar{X}_2^D$	$\bar{X}_3^D$	X_4^D
$\bar{X}^D{}_2$	$\bar{X}_1^D$	$\bar{X}_2^D$	$\bar{X}_3^D$	$X_4^D - 2\varepsilon\bar{X}_2^D$
$\bar{X}^D{}_3$	$\bar{X}_1^D$	$\bar{X}_2^D$	$\bar{X}_3^D$	$X_4^D - \varepsilon\bar{X}_3^D$
$X^D{}_4$	$\bar{X}_1^D$	$e^{2\varepsilon}\bar{X}_2^D$	$e^{\varepsilon}\bar{X}_3^D$	X_4^D

We proceed with the application of the Lie symmetries for the finite Lie algebra $A_{3,3} \otimes A_1$ for the reduction of the partial differential equation $\mathcal{H}_D$. From Table 6 we derive the one-dimensional optimal system, which is consisted by the one-dimensional Lie algebras $\{\bar{X}_1^D\}$, $\{\bar{X}_2^D\}$, $\{\bar{X}_3^D\}$, $\{\bar{X}_4^D\}$, $\{\bar{X}_1^D + \alpha\bar{X}_2^D\}$, $\{\bar{X}_1^D + \alpha\bar{X}_3^D\}$, $\{\bar{X}_1^D + \alpha\bar{X}_4^D\}$, $\{\bar{X}_2^D + \alpha\bar{X}_3^D\}$, $\{\bar{X}_1^D + \alpha\bar{X}_2^D + \beta\bar{X}_3^D\}$.

Therefore, by applying the Lie symmetries $\{\bar{X}_1^D + \alpha\bar{X}_2^D\}$, $\{\bar{X}_1^D + \beta\bar{X}_3^D\}$ for the reduction of Equation (5), we end with the second-order ordinary differential equation

$$\left(\alpha^2 + \beta^2 U + \alpha\beta^2\right)U_{\sigma\sigma} + \beta^2(U_\sigma)^2 = 0, \tag{17}$$

with $u = U(\sigma)$ and $\sigma = y - \beta t + \frac{\beta}{\alpha}x$. Equation (17) is maximally symmetric and can be linearized.

On the other hand, from the Lie symmetry $\{\bar{X}_2^D + \bar{X}_3^D\}$ we find the second-order partial differential equation

$$U_{Yt} + U_{YY} + UU_{YY} + (U_Y)^2 = 0. \tag{18}$$

where $u = U(t, Y)$, $Y = y - x$. Equation (18) admits infinity Lie symmetries consisted by the vector fields $Y_1^D = \partial_t$, $Y_2^D = t\partial_t + (U-1)\partial_U$, $Y_3^D = Y\partial_Y + (U+1)\partial_U$, $Y_4^D = t^2 + tY\partial_t + (Y - t(U+1))\partial_U$ and $Y_5^D = \zeta(t)\partial_Y + \zeta(t)_t\partial_U$. Vector fields $\{Y_1^D, Y_2^D, Y_4^D\}$ form the $sl(2,R)$ Lie algebra.

Hence, from the vector field $\{Y_1^D + \alpha Y_3^D\}$ it follows the similarity transformation $U = -1 + V(\kappa)e^{\alpha t}$, $\kappa = Ye^{-\alpha t}$, with the reduced equation, the differential equation $V_{\kappa\kappa} + \frac{1}{V-\kappa\alpha}(V_\kappa)^2 = 0$, which can be integrated further

$$N_\lambda = \frac{N}{\lambda}(1 + \alpha N)^2 \text{ with } N = (V_\kappa - \alpha)^{-1} \text{ and } \kappa = V - \kappa\alpha. \tag{19}$$

Easily, Equation (19) can be written as $N_{\bar{\lambda}\bar{\lambda}} = 0$. A similar result follows if we perform the reduction of Equation (18) with the rest of the symmetry vectors.

7. Lie Symmetry Analysis for Equation $\mathcal{H}_E$

The fifth equation of our study, namely equation $\mathcal{H}_E$ admits the following Lie symmetries

$$X_1^E(\Phi(t)) = \Phi(t)\partial_t + \frac{1}{2}y\Phi_t\partial_y + \left(\frac{1}{4}\Phi_{tt}y - \frac{1}{2}\Phi_t u\right)\partial_u,$$

$$X_2^E = \partial_x \ , \ X_3^E(\Sigma(t)) = \Sigma(t)\partial_y + \frac{1}{2}\Sigma_t\partial_u,$$

$$X_4^E = 2x\partial_x + y\partial_y + u\partial_u.$$

The Lie symmetries form an infinity Lie algebra, with nonzero commutators

$$\left[X_1^E(\Phi(t)), X_3^E(\Sigma(t))\right] = X_3^D(\Sigma\Phi_t - 2\Phi\Sigma_t)\ ,\ \left[X_2^E, X_4^E\right] = 2X_2^E\ ,$$

$$\left[X_3^E(\Sigma(t)), X_4^E\right] = X_3^E(\Sigma(t)),$$

and

$$\left[X_1^E(\Phi(t)), X_1^E(\bar{\Phi}(t))\right] = X_1^E(\Phi\bar{\Phi}_t - \bar{\Phi}\Phi_t).$$

The four-dimensional finite algebra $A_{3,3} \otimes A_1$ follows for $\Phi(t) = 1$ and $\Sigma(t) = 1$, that is, $A_{3,3} \otimes A_1$ is consisted by the Lie symmetry vectors $\{\bar{X}_1^E, X_2^E, \bar{X}_3^E, X_4^E\}$. We proceed with the application of the Lie symmetry vectors and the determination of the similarity transformations

From the symmetry vectors $\{\bar{X}_1^E + \alpha X_2^E\}$, $\{\bar{X}_1^E + \beta \bar{X}_3^E\}$ we find the similarity transformation $u = U(\sigma)$, $\sigma = y - \beta t + \frac{\beta}{\alpha}x$ with the reduced equation, the maximal symmetric equation

$$\left(2U\beta + \alpha - \beta^2\right)U_{\sigma\sigma} + 2\beta(U_\sigma)^2 = 0.$$

On the other hand, reduction with the symmetry vector $\{X_2^E + \bar{X}_3^E\}$ provides the second-order partial differential equation

$$V_{Yt} - V_{YY} + 2VV_{YY} + 2(V_Y)^2 = 0, \tag{20}$$

where $u = V(t, Y)$, $Y = y - x$. Equation (20) is of the form of Equation (18). Indeed, if we replace in (20) $V = -\frac{1}{2}U$ and $t \to -t$ Equation (20) is written in the form of Equation (18).

8. Conclusions

We applied the Lie symmetry analysis for a family of five partial differential equations of the form (1) which are integrable with the method of hydrodynamic reductions. In particular, we determined the Lie point symmetries and we studied the algebraic properties of the admitted symmetries. Moreover, from the invariant functions provided by the Lie symmetries, we defined similarity transformations which were used to reduce the number of independent variables for the differential equations. With the application of two different similarity transformations we were able to reduce the partial differential equations into a second-order ordinary differential equation. We summarize this result in the following proposition.

Proposition 1. *The five partial differential equations $\mathcal{H}_A$, $\mathcal{H}_B$, $\mathcal{H}_C$, $\mathcal{H}_D$ and $\mathcal{H}_E$ which are integrable with the method of hydrodynamic reductions can be linearized with the use of similaritry transformations given by the Lie point symmetries.*

Equation $\mathcal{H}_A$ admits a finite Lie algebra of four dimensions, while the remaining differential equations, $\mathcal{H}_B$, $\mathcal{H}_C$, $\mathcal{H}_D$ and $\mathcal{H}_E$ admit infinity Lie point symmetries which, however, are constructed by four generic vector fields. The application of the Lie point symmetries for these equations indicates that these five equations posses a common feature: they are reduced to a maximal symmetric ordinary differential equation which can be linearized. We show that this is possible not only when we investigate for "travel-wave" solutions but also for more general reductions.

In a future study we plan to investigate further by applying the theory of Lie symmetries and other differential equations, which are integrable by the method of hydrodynamic reductions.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest: The authors declare no conflict of interest.

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