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# A Class of Fourth-Order Symmetrical Kirchhoff Type Systems

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**Abstract:** This paper deals with the existence and multiplicity of solutions for a perturbed nonlocal fourth-order class of $p(\cdot)\&q(\cdot)$-Kirchhoff elliptic systems under Navier boundary conditions. By using the variational method and Ricceri's critical point theorem, we can find a proper conditions to ensure that the perturbed fourth-order of $(p(x), q(x))$-Kirchhoff systems has at least three weak solutions. We have extended and improved some recent results. The complexity of the combination of variable exponent theory and fourth-order Kirchhoff systems makes the results of this work novel and new contribution. Finally, a very concrete example is given to illustrate the applications of our results.

**Keywords:** multiple solutions; three critical points theory; $(p(x), q(x))$-Kirchhoff system; $p(x)$-biharmonic operator; variational method

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## 1. Introduction

The presence of the variable exponent $p(.)$ provides the fundamental motivation for the study of fourth-order partial differential equations, which makes us open the door to applications for utilizing extremely nonhomogeneous materials that are nowadays becoming increasingly common in industry. One of them has to do with electrorheological fluids, which were found in 1949 by Willis Winslow [1]. These fluids are especially viscous liquids and can significantly change their mechanical properties when they contact an electric field (see Acerbi and Mingione [2], Ružička [3]).Other known applications are related to image restoration (see Chen, Levine and Rao [4]), elastic materials (see Boureanu [5] and Zhikov [6]), mathematical biology (see Fragnelli [7]), dielectric breakdown and electrical resistance (see Bocea and Mihăilescu [8], polycrystal plasticity (see Bocea, Mihăilescu and Popovici [9]) and models of diffusion in sandpiles (see Bocea, Mihăilescu, Perez-Llanos and Rossi [10]).

Problem (3) is a nonlocal problem because of the presence of the term M, which suggests that the equation in (1) is no longer a pointwise identity. Mathematical difficulties arise when attempting to solve problems in calculus. These difficulties provide researchers with an interesting and challenging area of study. Problem (3) is an extension of a model proposed by Kirchhoff [11]. Moreover, Kirchhoff suggested a model that was defined by the form

$$\rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1}$$

which extends D' Alembert's wave equation. One notable feature of model (1) is that it contains a nonlocal term $\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx$. The parameters $L$, $h$, $E$, $\rho$, $\rho_0$ in model (1) represent different physical meanings, which we will not cover here.

It is important to note that if $p(.)$ is a constant, then problem (3) has recently been the subject of extensive research. For example, in [12], Nguyen Thanh Chung studied the existence of non-trivial solutions for the following Kirchhoff-type system involving $p$-biharmonic operator

$$\begin{cases} M\big(\int_\Omega |\Delta u|^p dx\big)\Delta\big(|\Delta u|^{p-2}\Delta u\big) = |u|^{p*-2}u + F_u(x,u,v) & \text{in } \Omega, \\ -M\big(\int_\Omega |\Delta v|^p dx\big)\Delta\big(|\Delta v|^{p-2}\Delta v\big) = |v|^{p^{**}-2}v + F_v(x,u,v) & \text{in } \Omega, \\ u = \frac{\partial u}{\partial v} = 0, \quad v = \frac{\partial v}{\partial v} = 0 & \text{on } \partial\Omega. \end{cases}$$

After that, in [13], the authors proved the existence of infinitely many weak solutions for $(p(x), q(x))$-Laplacian-like system, originating from capillary phenomenon of the following form:

$$\begin{cases} -div\big((1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}})|\nabla u|^{p(x)-2}\nabla u\big) = \lambda F_u(x,u,v) & \text{in } \Omega, \\ -div\big((1+\frac{|\nabla v|^{q(x)}}{\sqrt{1+|\nabla v|^{2q(x)}}})|\nabla v|^{q(x)-2}\nabla v\big) = \lambda F_v(x,u,v) & \text{in } \Omega, \\ u = \frac{\partial u}{\partial v} = 0, \quad v = \frac{\partial v}{\partial v} = 0 & \text{on } \partial\Omega. \end{cases}$$

For more research on such $(p(x), q(x))$-Laplacian problems, the interested reader can refer to the literature [14,15]. In order to consider such problems with variable exponents, we need to use the novel theory of variable exponents Lebesgue and Sobolev spaces. Over the past few decades, these spaces have attracted considerable attention (see Cruz-Uribe and Fiorenza [16], Rădulescu and Repovš [17], Diening, Harjulehto, Hästö and Ružička [18]) and references therein). In [19], Avci et al. investigated the existence and multiplicity of the solutions for the following $p(\cdot)$-Kirchhoff-type problem

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

by means of Krasnoselskii's genus theory. Furthermore, the authors studied a gradient type system with variable exponents and were able to formulate criteria that guaranteed the existence of solutions to the problem in [20] with the help of Ekeland Variational Principle. Our results not only provide a generalization to previous results but also give new contributions in fourth-order elliptic problems. Our aim is to consider the following nonlocal fourth-order systems of $p(\cdot)$ & $q(\cdot)$-Kirchhoff type

$$\begin{cases} -M_1\left(\int_\Omega \frac{|\Delta u|^{p(x)} + |u|^{p(x)}}{p(x)}dx\right)\left(\Delta^2_{p(x)}u - |u|^{p(x)-2}u\right) = \lambda F_u(x,u,v) + \mu G_u(x,u,v) & \text{in } \Omega, \\ -M_2\left(\int_\Omega \frac{|\Delta v|^{q(x)} + |v|^{q(x)}}{q(x)}dx\right)\left(\Delta^2_{q(x)}v - |v|^{q(x)-2}v\right) = \lambda F_v(x,u,v) + \mu G_v(x,u,v) & \text{in } \Omega, \\ u = v = \Delta u = \Delta v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $N \geq 2$, $\Omega \subset \mathbb{R}^N$ is bounded and its boundary $\partial\Omega$ is also smooth, $\Delta^2_{p(x)}u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is a $p(x)$-biharmonic operator and the two functions $p, q$ are continuous on $\overline{\Omega}$.

When $p(x) = p$, the $p(x)$-biharmonic operator $\Delta^2_{p(x)}$ reduces to $p$-biharmonic. The study of fourth-order partial differential equations with constant exponent has intensively developed in recent years. It has a large variety of applications (see Dănet [21], Ferrero and Warnault [22], Myers [23] and references therein). Moreover, many articles [24–28] had investigated $p$-Kirchhoff type elliptic equations in recent years. The authors use a variety of methods to investigate the existence of solutions to problem (3) in the case of $p(x) = p$.

However, there are few authors are interested in the existence and diversity of solutions for situations affecting $p(x)$-biharmonic operators. Inspired by the above literature, we study the existence and multiplicity of solutions for problem (3).

The structure of this paper is as follows: The second part introduces the knowledge of space theory and related lemmas, the third part presents the main results and proofs, and the fourth part considers an important application of the results.

## 2. Preliminaries

In order to study problem (3), we need knowledge of variable exponent Lebesgue Spaces and Sobolev space theory, as well as some properties of $p(x)$-biharmonic operators.

The generalized Lebesgue space is defined as follows:

$$L^{p(x)}(\Omega) = \left\{ u: \ \Omega \longrightarrow \mathbb{R} \text{ is measurable and } \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

where $p \in C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}): \ \ p(x) > 1 \text{ for all } x \in \overline{\Omega}\}$. The norm of $L^{p(x)}(\Omega)$ is that

$$|u|_{p(x)} = \inf \left\{ \mu > 0; \ \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\},$$

and the space $(L^{p(x)}(\Omega), |.|_{p(x)})$ is a Banach.

Denote

$$p^+ = \max_{x \in \overline{\Omega}} p(x), \quad p^- = \min_{x \in \overline{\Omega}} p(x),$$

and for all $x \in \overline{\Omega}$ and $k \geq 1$

$$p^*(x) = \begin{cases} \frac{Np(x)}{N - p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N, \end{cases}$$

and

$$p_k^*(x) = \begin{cases} \frac{Np(x)}{N - kp(x)} & \text{if } kp(x) < N, \\ +\infty & \text{if } kp(x) \geq N. \end{cases}$$

**Proposition 1** (Fan and Zhao [29]). *Let $p, r \in C_+(\overline{\Omega})$, if $r(x) \leq p_k^*(x)$ for any $x \in \overline{\Omega}$, then the embedding*

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$$

*is continuous and if $r(x) < p_k^*(x)$, then the above embedding is compact.*

In the proposition above, the variable exponent Sobolev space $W^{k,p(x)}(\Omega)$ is defined by

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega): \ \ D^\alpha u \in L^{p(x)}(\Omega), \ \ |\alpha| \leq k\},$$

where

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_N^{\alpha_N}}, \alpha = (\alpha_1, \alpha_2, \dots, \alpha_N) \text{ with } |\alpha| = \sum_{i=1}^{i=N} \alpha_i,$$

whose norm is

$$\|u\|_{k,p(x)} = \sum_{|\alpha| \leq k} |D^\alpha u|_{p(x)}.$$

It is well known that the Banach space $W^{k,p(x)}(\Omega)$ is also a separable and reflexive space due to [29–32].

We know that the space $W_0^{k,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$. Obviously, the Banach space $\left(W^{2,p(x)}(\Omega)\cap W_0^{1,p(x)}(\Omega), \|u\|_{p(x)}\right)$ is separable and reflexive, where

$$\|u\|_{p(x)} = \inf\left\{\mu > 0 : \int_\Omega \left|\frac{\Delta u}{\mu}\right|^{p(x)} dx \le 1.\right\}.$$

**Remark 1.** *According to [33], the two norms $\|u\|_{2,p(x)}$ and $\|\Delta u\|_{p(x)}$ are equivalent in $W^{2,p(x)}(\Omega)\cap W_0^{1,p(x)}(\Omega)$. Consequently, the norms $\|.\|_{2,p(x)}, \|.\|_{p(x)}$ and $\|\Delta.\|_{p(x)}$ are also equivalent.*

In order to find weak solutions of problem (3), we define the following work space

$$X = \left(W^{2,p(x)}(\Omega)\cap W_0^{1,p(x)}(\Omega)\right)\times\left(W^{2,q(x)}(\Omega)\cap W_0^{1,q(x)}(\Omega)\right),$$

equipped with the norm $\|(u,v)\| = \max\left\{\|u\|_{p(x)}, \|v\|_{q(x)}\right\}$.

When $p^-, q^- > N$, the spaces $W^{2,p(x)}(\Omega)$ and $W^{2,q(x)}(\Omega)$ are compactly embedded in $C(\overline{\Omega})$, so

$$K = \max\left\{\sup_{u\in W^{2,p(x)}(\Omega)} \frac{\|u\|_\infty}{\|u\|_{p(x)}}; \sup_{v\in W^{2,q(x)}(\Omega)} \frac{\|v\|_\infty}{\|v\|_{q(x)}}\right\}. \tag{4}$$

**Proposition 2.** *Ref. [34] Suppose that*

$$J_{p(x)}(u) = \int_\Omega |\Delta u|^{p(x)} dx,$$

*then for any $u, u_n \in X$ we have that*

*(i)* $\|u\|_{p(x)} < 1 \ (=1; >1) \iff J_{p(x)}(u) < 1 \ (=1; >1)$,

*(ii)* $\|u\|_{p(x)} \le 1 \Longrightarrow \|u\|_{p(x)}^{p^+} \le J(u)_{p(x)} \le \|u\|_{p(x)}^{p^-}$,

*(iii)* $\|u\|_{p(x)} \ge 1 \Longrightarrow \|u\|_{p(x)}^{p^-} \le J(u)_{p(x)} \le \|u\|_{p(x)}^{p^+}$,

*(iv)* $\|u_n\|_{p(x)} \longrightarrow 0 \iff J_{p(x)}(u_n) \longrightarrow 0$,

*(v)* $\|u_n\|_{p(x)} \longrightarrow \infty \iff J_{p(x)}(u_n) \longrightarrow \infty$.

To obtain our main result, we need to make the following assumptions about the functions $M$, $F$ and $G$:

($M_0$) The continuous functions $M_1$ and $M_2 : \mathbb{R}^+ \to \mathbb{R}^+$ are increasing and fulfilling that $M_1(t), M_2(t) \ge m_0 > 0$ for any $t \ge 0$.
$F \in C^1 : \overline{\Omega}\times\mathbb{R}^2 \to \mathbb{R}$ and satisfying the following two conditions:

($F_0$) There exist $\alpha, \beta \in C_+(\overline{\Omega})$ with $1 < \alpha^- \le \alpha^+ \quad 1 < \beta^- \le \beta^+$ and

$$\frac{1+\alpha^+}{p^-} + \frac{1+\beta^+}{q^-} < 1, \tag{5}$$

such that

$$|F_s(x,s,t)| \le C|s|^{\alpha(x)}|t|^{\beta(x)+1}, \quad |F_t(x,s,t)| \le C|s|^{\alpha(x)+1}|t|^{\beta(x)}, \quad \forall (x,s,t) \in \Omega\times\mathbb{R}^2,$$

for some positive constant $C$.

($F_1$) $F(x,s,t) > 0$ for all $x \in \Omega, s > s_0 > 1$ and $t > t_0 > 1$.
$F(x,s,t) \le 0$ for all $x \in \Omega$ and $s,t \in [0,1)$.

$G : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is function such that $G(., s, t)$ is measurable in $\Omega$ for all $(s, t) \in \mathbb{R}^2$ and $G(x, ., .)$ is continuously differentiable in $\mathbb{R}^2$ for example $x \in \Omega, G_u, G_y$ are the partial derivatives of $G$ which satisfy the following condition.

($G_0$) $\sup_{|s|\leq\sigma,|t|\leq\sigma} (|G_u(., s, t) + G_v(., s, t)|) \in L^1(\Omega), \forall \sigma > 0.$

## 3. Main Results and Proofs

Define

$$\Phi(u, v) = \widehat{M}_1\left(\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)} dx\right) + \widehat{M}_2\left(\int_\Omega \frac{|\Delta v|^{q(x)}}{q(x)} dx\right),$$

$$\Psi(u, v) = -\int_\Omega F(x, u, v)dx, \quad J(u, v) = -\int_\Omega G(x, u, v)dx,$$

where $\widehat{M}_i(t) = \int_0^t M_i(s)ds \ (i = 1, 2)$.

**Definition 1.** *$(u, v) \in X$ is a weak solution of problem (3) if only and if*

$$M_1\left(\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)} dx\right) \int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta\varphi dx + M_2\left(\int_\Omega \frac{|\Delta v|^{q(x)}}{q(x)} dx\right) \int_\Omega |\Delta v|^{q(x)-2}\Delta v \Delta\psi dx$$

$$= \lambda \int_\Omega F_u(x, u, v)dx + \lambda \int_\Omega F_v(x, u, v)dx + \mu \int_\Omega G_u(x, u, v)dx + \mu \int_\Omega G_v(x, u, v)dx,$$

*for all $(\varphi, \psi) \in X$.*

According to problem (3), we define the following energy functional

$$I(u, v) = \Phi(u, v) + \lambda\Psi(u, v) + \mu J(u, v) : X \longrightarrow \mathbb{R}.$$

Clearly, $I \in C^1(X, \mathbb{R})$, whose critical points correspond to weak solutions of problem (3). Our main result is as follows:

**Theorem 1.** *If ($M_0$), ($F_0$)–($F_1$) and ($G_0$) hold, then there exist $\rho, \delta > 0$ and an open interval $\Lambda \subseteq [0, +\infty)$ such that for any $\lambda \in \Lambda$ and $\mu \in [0, \delta]$, problem (3) has at least three solutions in $X$ whose norms are less than $\rho$.*

To demonstrate our primary result, we will employ the Ricceri three points theorem [35].

**Theorem 2.** *Ref. [35] Let real Banach space $X$ be separable and reflexive, the continuous Gâteaux differentiable functional $\Phi : X \longrightarrow \mathbb{R}$ be sequentially weakly lower semicontinuous, whose Gâteaux derivative has a continuous inverse on $X^*$ and the Gâteaux derivatives of continuous Gâteaux differentiable functionals $\Psi, J$ be compact. Assume that the following assertions:*

*(1)* $\lim_{\|u\|_X\to\infty} (\Phi(u) + \lambda\Psi(u)) = \pm\infty$ *for all $\lambda > 0$.*

*(2)* *There exist $r \in \mathbb{R}$ and $u_0, u_1 \in X$ such that*

$$\Phi(u_0) < r < \Phi(u_1).$$

*(3)*

$$\inf_{u\in\Phi^{-1}]-\infty,r]} \Psi(u) > \frac{(\Phi(u_1) - r)\Psi(u_0) + (r - \Phi(u_0))\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)}.$$

*Then there exist $\rho, \delta > 0$ and an open interval $\Lambda \subseteq [0, +\infty)$ such that for any $\lambda \in \Lambda$ and $\mu \in [0, \delta]$, the equation*

$$\Phi'(u) + \lambda\Psi'(u) + \mu J'(u) = 0$$

*has at least three solutions in $X$ whose norms are less than $\rho$.*

**Proposition 3.** *Ref. [36] Suppose that $(M_0)$ holds. Then*
*(i) $\Phi$ is bounded on any bounded subset and sequentially weakly lower semicontinuous;*
*(ii) $\Phi' : X \longrightarrow X^*$ is strictly monotone and continuous;*
*(iii) $\Phi' : X \longrightarrow X^*$ is a homeomorphism.*

**Proof of Theorem 1.** Note that $\Psi'$ is compact and Proposition 3 ensures that $\Phi$ is weakly lower semicontinuous and bounded on each bounded subset. We know that $\Phi'$ has a continuous inverse on $X'$. Moreover,

$$\lim_{\|(u,v)\| \to +\infty} \Phi(u,v) + \lambda\Psi(u,v) = +\infty$$

for all $\lambda \in (0, +\infty)$. Indeed,

$$\begin{aligned} \Phi(u,v) &= \widehat{M_1}\left(\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)} dx\right) + \widehat{M_2}\left(\int_\Omega \frac{|\Delta v|^{q(x)}}{q(x)} dx\right) \\ &\geq \frac{m_0}{p^+} J_{p(x)}(u) + \frac{m_0}{q^+} J_{q(x)}(v) \\ &\geq \frac{m_0}{p^+} \min\left(\|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^+}\right) + \frac{m_0}{q^+} \min\left(\|v\|_{q(x)}^{q^-}, \|v\|_{q(x)}^{q^+}\right). \end{aligned} \tag{6}$$

By ($F_0$), we have

$$|F(x,u,v)| \leq C'|u|^{\alpha(x)+1}|v|^{\beta(x)+1}.$$

Therefore

$$\begin{aligned} \Psi(u,v) &= -\int_\Omega F(x,u,v)dx \\ &\geq -C' \int_\Omega |u|^{\alpha(x)+1}|v|^{\beta(x)+1} dx \\ &\geq -C'|\Omega| \max\left(\|u\|_\infty^{1+\alpha^+}, \|u\|_\infty^{1+\alpha^-}\right) \max\left(\|v\|_\infty^{1+\beta^+}, \|v\|_\infty^{1+\beta^-}\right). \end{aligned} \tag{7}$$

Since the embeddings $W^{2,p(x)}(\Omega) \hookrightarrow C(\overline{\Omega})$ and $W^{2,q(x)}(\Omega) \hookrightarrow C(\overline{\Omega})$ are continuous, thus there exists a constant $C_0 > 0$ such that

$$\Psi(u,v) \geq -C'C_0|\Omega| \max\left(\|u\|_{p(x)}^{1+\alpha^+}, \|u\|_{p(x)}^{1+\alpha^-}\right) \max\left(\|v\|_{q(x)}^{1+\beta^+}, \|v\|_{q(x)}^{1+\beta^-}\right). \tag{8}$$

In view (5), there exist $p_1 < p^-$ and $q_1 < q^-$ such that $\frac{1+\alpha^+}{p_1} + \frac{1+\beta^+}{q_1} = 1$. Thus it follows from the Young inequality and (8) that $\Psi(u,v) \geq -C'C_0|\Omega| \left(\frac{1+\alpha^+}{p_1}\|u\|_{p(x)}^{p_1} + \frac{1+\beta^+}{q_1} \max\left(\|v\|_{q(x)}^{q_1}, 1\right)\right)$.

Without loss of generality, we may assume $\|u\|_{p(x)} \geq \|v\|_{q(x)}$. If $\|v\|_{q(x)} > 1$, we get $\Phi(u,v) + \lambda\Psi(u,v) \geq \frac{m_0}{p^+}\|u\|_{p(x)}^{p^+} + \frac{m_0}{q^+}\|v\|_{q(x)}^{q^+} - \lambda C'C_0|\Omega|\left(\frac{1+\alpha^+}{p_1}\|u\|_{p(x)}^{p_1} + \frac{1+\beta^+}{q_1}\|v\|_{q(x)}^{q_1}\right)$. If $\|v\|_{q(x)} < 1$, we get $\Phi(u,v) + \lambda\Psi(u,v) \geq \frac{m_0}{p^+}\|u\|_{p(x)}^{p^+} - \lambda C'C_0|\Omega|\frac{(1+\alpha^+)}{p_1}\|u\|_{p(x)}^{p_1}$. By the assumptions on $q_1$ and $p_1$, we deduce that

$$\lim_{\|(u,v)\| \to \infty} \Phi(u,v) + \lambda\Psi(u,v) = +\infty.$$

Now, from ($F_1$) we can choose $\delta > 1$ such that $F(x,s,t) > 0$ for all $s,t > \delta$, $x \in \Omega$. Then using ($F_2$) we obtain

$$F(x,s,t) > 0 = F(x,0,0) \geq F(x,\tau_1,\tau_2), \quad \text{for all} \quad s,t > \delta, \quad \tau_1,\tau_2 \in [0,1) \quad x \in \Omega. \tag{9}$$

There exist two positive real numbers $a,b$ such that $0 < a < \min(1,K)$, with $K$ is given by (4) and $b > \delta$ with $\min(b^{p^-}, b^{q^-})|\Omega| > 1$. It follows from (9) that

$$\int_\Omega \sup_{0\leq s,t\leq a} F(x,s,t)dx \leq 0 < \int_\Omega F(x,b,b)dx.$$

Set $r := \min\left(\frac{m_0}{p^+}(\frac{a}{K})^{p^+}, \frac{m_0}{q^+}(\frac{a}{K})^{q^+}\right)$

By choosing $(u_0(x),v_0(x)) = (0,0)$, $(u_1(x),v_1(x)) = (b,b)$, we have $\Phi(u_0,v_0) = \Psi(u_0,v_0) = 0$,

$$\begin{aligned}
\Phi(u_1,v_1) &= \widehat{M_1}\left(\int_\Omega \frac{b^{p(x)}}{p(x)}dx\right) + \widehat{M_2}\left(\int_\Omega \frac{b^{q(x)}}{q(x)}dx\right) \\
&\geq m_0 \int_\Omega \left(\frac{b^{p(x)}}{p(x)} + \frac{b^{q(x)}}{q(x)}\right)dx \\
&\geq m_0|\Omega|\left(\frac{b^{p^-}}{p^+} + \frac{b^{q^-}}{q^+}\right) \\
&\geq \frac{m_0}{p^+} + \frac{m_0}{q^+} > r.
\end{aligned}$$

Thus

$$\Phi(u_0,v_0) \leq r \leq \Phi(u_1,v_1).$$

On the other hand

$$-\frac{(\Phi(u_1,v_1)-r)\Psi(u_0,v_0)+(r-\Phi(u_0,v_0))\Psi(u_1,v_1)}{\Phi(u_1,v_1)-\Psi(u_0,v_0)} = r\frac{\int_\Omega F(x,b,b)dx}{\widehat{M_1}\left(\int_\Omega \frac{b^{p(x)}}{p(x)}dx\right)+\widehat{M_2}\left(\int_\Omega \frac{b^{q(x)}}{q(x)}dx\right)} > 0. \tag{10}$$

Let $(u,v) \in X$ such that $\Phi(u,v) \leq r$. Then $\Phi(u,v) \geq m_0\left(\frac{1}{p^+}J_{p(x)}(u) + \frac{1}{q^+}J_{q(x)}(v)\right)$, which implies that

$$J_{p(x)}(u) \leq \frac{rp^+}{m_0} < 1, \quad J_{q(x)}(u) \leq \frac{rq^+}{m_0} < 1. \tag{11}$$

According to Proposition 2, we get $\|u\|_p \leq 1$, $\|v\|_q \leq 1$ and therefore by (11), we have $\|u\|_p \leq \left(\frac{rp^+}{m_0}\right)^{\frac{1}{p^+}}$, $\|u\|_q \leq \left(\frac{rq^+}{m_0}\right)^{\frac{1}{q^+}}$, $|u| \leq K\left(\frac{rp^+}{m_0}\right)^{\frac{1}{p^+}} \leq a$, $|v(x)| \leq K\left(\frac{rp^+}{m_0}\right)^{\frac{1}{q^+}} \leq a$, $\forall x \in \Omega$.

It follows from (10) that

$$\begin{aligned}
-\inf_{(u,v)\in\Phi^{-1}((-\infty,r])} \Psi(u,v) &= \sup_{\Phi(u,v)\leq r} -\Psi(u,v) \leq \sup_{\{(u,v)\in X:|u(x)|,|v(x)|\leq a,\forall x\in\Omega\}} \int_\Omega F(x,u,v)dx \\
&\leq \int_{0\leq s,t\leq a} F(x,s,t)dx \leq 0 \\
&< -\frac{(\Phi(u_1,v_1)-r)\Psi(u_0,v_0)+(r-\Phi(u_0,v_0))\Psi(u_1,v_1)}{\Phi(u_1,v_1)-\Psi(u_0,v_0)}.
\end{aligned}$$

Hence according to Theorem 2, we finish the proof of Theorem 1. □

## 4. Application

Let $\alpha, \beta \in C_+(\Omega)$. Next we will investigate the following specific Kirchhoff problem

$$\begin{cases} -M_1\left(\int_\Omega \frac{|\Delta u|^{p(x)}+|u|^{p(x)}}{p(x)}dx\right)\left(\Delta^2_{p(x)}u-|u|^{p(x)-2}u\right)= \\ \qquad \lambda\left(|u|^{\alpha(x)-1}u|v|^{\alpha(x)+1}-|u|^{\beta(x)-1}u|v|^{\beta(x)+1}\right)+\mu|u|^{\gamma_1(x)-2}u \text{ in } \Omega, \\ -M_2\left(\int_\Omega \frac{|\Delta v|^{q(x)}+|v|^{q(x)}}{q(x)}dx\right)\left(\Delta^2_{q(x)}v-|v|^{q(x)-2}v\right)= \\ \qquad \lambda\left(|u|^{\alpha(x)+1}|v|^{\alpha(x)-1}v-|u|^{\beta(x)+1}|v|^{\beta(x)-1}v\right)+\mu|v|^{\gamma_2(x)-2}v \text{ in } \Omega, \\ u=v=\Delta u=\Delta v=0 \qquad \text{on } \partial\Omega, \end{cases} \tag{12}$$

where

$$(1+\alpha^+)(\frac{1}{p^-}+\frac{1}{q^-})<1, \quad \beta^+<\alpha^- \tag{13}$$

and

$$\gamma_1,\gamma_2\in C_+(\overline{\Omega}), \quad \gamma_1^+<p^-, \quad \gamma_2^+<q^-. \tag{14}$$

**Theorem 3.** *If $(M_0)$ and (13) hold and two functions $\gamma_1,\gamma_2$ satisfy (14), then there exist $\rho,\delta>0$ and an open interval $\Lambda\subseteq[0,+\infty)$ such that for any $\lambda\in\Lambda$ and $\mu\in[0,\delta]$, problem (12) admits at least three solutions whose norms in $X$ are less than $\rho$.*

**Proof of Theorem 3.** Define

$$\Psi(u,v)=-\int_\Omega\left(\frac{1}{\alpha(x)+1}|u|^{\alpha(x)+1}|u|^{\alpha(x)+1}-\frac{1}{\beta(x)+1}|u|^{\beta(x)+1}|u|^{\beta(x)+1}\right)dx$$

and

$$J(u,v)=-\int_\Omega\left(\frac{1}{\gamma_1(x)}|u|^{\gamma_1(x)}+\frac{1}{\gamma_2(x)}|v|^{\gamma_2(x)}\right)dx.$$

Clearly $\Psi'$ and $J'$ are compact. By the Sobolev embedding, there exists $C_1>0$ such that

$$\begin{aligned} \Psi(u,v) &\geq -\int_\Omega\left(\frac{1}{\alpha(x)+1}\|u\|^{\alpha(x)+1}\|v\|^{\alpha(x)+1}\right) \\ &\geq -\frac{|\Omega|}{1+\alpha^-}\max\left(\|u\|_\infty^{1+\alpha^+},\|u\|_\infty^{1+\alpha^-}\right)\max\left(\|v\|_\infty^{1+\alpha^+},\|v\|_\infty^{1+\alpha^-}\right) \\ &\geq -C_1\frac{|\Omega|}{1+\alpha^-}\max\left(\|u\|_{p(x)}^{1+\alpha^+},\|u\|_{p(x)}^{1+\alpha^-}\right)\max\left(\|v\|_{q(x)}^{1+\alpha^+},\|v\|_{q(x)}^{1+\alpha^-}\right). \end{aligned}$$

As in proof of Theorem 1, we may assume that $\|u\|_{p(x)},\|v\|_{q(x)}\longrightarrow\infty$. Then $\Psi(u,v)\geq -C_1\frac{|\Omega|}{1+\alpha^-}\|u\|_{p(x)}^{1+\alpha^+}\|v\|_{q(x)}^{1+\alpha^+}$.

Since $(1+\alpha^+)\left(\frac{1}{p^-}+\frac{1}{q^-}\right)<1$, we can find $p_2\in(1,p^-)$ and $q_2\in(1,q^-)$ such that

$$(1+\alpha^+)\left(\frac{1}{p_2}+\frac{1}{q_2}\right)=1.$$

In view of Young's inequality, we obtain

$$\Psi(u,v)\geq -C_1\frac{|\Omega|}{1+\alpha^-}\left(\frac{1+\alpha^+}{p_2}\|u\|_{p(x)}^{p_2}+\frac{1+\alpha^+}{q_2}\|v\|_{q(x)}^{q_2}\right).$$

This together with (6) imply that $\lim_{\|(u,v)\|\to\infty}\Phi(u,v)+\lambda\Psi(u,v)=+\infty.$

Now we assume that $\|u\|_{p(x)} \longrightarrow \infty$, $\|v\|_{q(x)}$ is bounded. Then $\Psi(u,v) \geq -C_1 \frac{|\Omega|}{1+\alpha^-}$ $\|u\|_{p(x)}^{1+\alpha^+} \max\left(\|v\|_{q(x)}^{1+\alpha^+}, 1\right)$, so,

$$\Psi(u,v) \geq -C_1 \frac{|\Omega|}{1+\alpha^-} \left( \frac{1+\alpha^+}{p_2} \|u\|_{p(x)}^{p_2} + \frac{1+\alpha^+}{q_2} \max\left(\|v\|_{q(x)}^{q_2}, 1\right) \right).$$

Hence, the coercivity of $\Phi + \lambda\Psi$ is achieved. Let

$$H(x,s,t) = \frac{1}{\alpha(x)+1} |s|^{\alpha(x)+1} |t|^{\alpha(x)+1} - \frac{1}{\beta(x)+1} |s|^{\beta(x)+1} |t|^{\beta(x)+1}.$$

Since $\beta^+ < \alpha^-$, we can choose $\delta' > 1$ such that

$$H(x,s,t) > 0 = H(x,0,0) \geq H(x,\tau_1,\tau_2), \quad \forall s,t > \delta', \quad \tau_1,\tau_2 \in [0,1), \quad x \in \Omega.$$

Similar to the proof of Theorem 1, we know that all conditions of Theorem 2 are fulfilled. Hence, the proof of Theorem 3 is complete. □

## 5. Conclusions

The fourth-order Kirchhoff types were extensively investigated in this study. This results supports some previously published research. We believe that researchers working in this domain will be inspired by our work and pave the path for future research in this area.

We have studied the problem (3) under Dirichlet boundary conditions. We used Ricceri's theorem to prove the multiple solutions. The importance of this study is to establish the same result of the following problem under Neuman boundary conditions:

$$\begin{cases} -M_1\left(\int_\Omega \frac{|\Delta u|^{p(x)} + |u|^{p(x)}}{p(x)} dx\right) \left(\Delta^2_{p(x)} u - |u|^{p(x)-2} u\right) = \lambda F_u(x,u,v) + \mu G_u(x,u,v) & \text{in } \Omega, \\ -M_2\left(\int_\Omega \frac{|\Delta v|^{q(x)} + |v|^{q(x)}}{q(x)} dx\right) \left(\Delta^2_{q(x)} v - |v|^{q(x)-2} v\right) = \lambda F_v(x,u,v) + \mu G_v(x,u,v) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial}{\partial \nu}\left(|\Delta u|^{p(x)-2} \Delta u\right) = 0 & \text{on } \partial\Omega. \\ \frac{\partial v}{\partial \nu} = \frac{\partial}{\partial \nu}\left(|\Delta v|^{q(x)-2} \Delta v\right) = 0 & \text{on } \partial\Omega. \end{cases} \tag{15}$$

As we know, the space of study and the embedding properties are not the same, which makes the study more difficult to obtain the desired results.

This problem has been left as an open question for researchers who are interested in the subject.

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