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Some Fuzzy Inequalities for Harmonically *s*-Convex Fuzzy Number Valued Functions in the Second Sense Integral

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Abstract: Many fields of mathematics rely on convexity and nonconvexity, especially when studying optimization issues, where it stands out for a variety of practical aspects. Owing to the behavior of its definition, the idea of convexity also contributes significantly to the discussion of inequalities. The concepts of symmetry and convexity are related and we can apply this because of the close link that has grown between the two in recent years. In this study, harmonic convexity, also known as harmonic s-convexity for fuzzy number valued functions (*F-NV-Fs*), is defined in a more thorough manner. In this paper, we extend harmonically convex *F-NV-Fs* and demonstrate Hermite–Hadamard (*H.H*) and Hermite–Hadamard Fejér (*H.H.* Fejér) inequalities. The findings presented here are summaries of a variety of previously published studies.

Keywords: harmonically *s*-convex fuzzy number valued function in the second sense; Hermite–Hadamard inequality; Hermite–Hadamard Fejér inequality

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1. Introduction

A variety of scientific fields, including mathematical analysis, optimization, economics, finance, engineering, management science, and game theory, have greatly benefited from the active, interesting, and appealing field of convexity theory study. Numerous scholars study the idea of convex functions, attempting to broaden and generalize its various manifestations by using cutting-edge concepts and potent methods. Convexity theory offers a comprehensive framework for developing incredibly effective, fascinating, and potent numerical tools to approach and resolve a wide range of issues in both pure and practical sciences. Convexity has been developed, broadened, and extended in several sectors during recent years. Inequality theory has benefited greatly from the introduction of convex functions. Numerous studies have shown strong connections between the theories of inequality and convex functions.

Functional analysis, physics, statistics theory, and optimization theory all benefit from integral inequality. Only a handful of the applications of inequalities in research [1–8] include statistical difficulties, probability, and numerical quadrature equations. Convex analysis and inequalities have developed into an alluring, captivating, and attention-grabbing topic for researchers as a result of various generalizations, variants, extensions, wide-ranging perspectives, and applications; the reader can refer to [9–15]. Kadakal and Iscan recently presented n-polynomial convex functions, which are an extension of convexity [16].

A well-known particular instance of the harmonic mean is the power mean. In areas such as statistics, computer science, trigonometry, geometry, probability, finance, and electric circuit theory, it is frequently employed when average rates are sought. Since it equalizes the weights of each piece of data, the harmonic mean is the ideal statistic for rates and ratios. The harmonic convex set is defined by the harmonic mean. Shi [17] was the first to present the harmonic convex set in 2003. The harmonic and p-harmonic convex functions were introduced and explored for the first time by Anderson et al. [18] and Noor et al. [19], respectively. Awan et al. [20] introduced a new class known as n-polynomial harmonically convex function while maintaining their focus on extensions.

We learned that there is a certain class of function known as the exponential convex function and that there are many people working on this subject currently [21,22]. This information was motivated and encouraged by recent actions and research in the field of convex analysis. Convexity of the exponential type was described by Dragomir [23]. Dragomir's work was continued by Awan et al. [24], who investigated and examined an entirely new family of exponentially convex functions. Kadakal and İşcan presented a fresh idea for exponential type convexity in [25]. The idea of n-polynomial harmonic exponential type convex functions was recently put out by Geo et al. [26]. In statistical learning, information sciences, data mining, stochastic optimization, and sequential prediction [27,28], and the references therein, the benefits and applications of exponential type convexity are used.

Additionally, symmetry and inequality have a direct relationship with convexity. Because of their close association, whatever one we focus on may be applied to the other, demonstrating the important relationship between convexity and symmetry, see [29]. The traditional ideas of convexity have been successfully extended in a number of instances. Weir and Mond, for instance, developed the category of preinvex functions. The concept of h-convexity, which also encompasses several other types of convex functions, was first presented by Varosanec [30]. Additionally, Varosanec has discovered a few h-convex function-related classical inequalities. Noor et al. [31] introduced the class of h-preinvex functions and pointed out that by considering multiple viable options for the real function h, other classes of preinvexity and classical convexity may be retrieved. In a similar work, they also created a number of entirely new Dragomir–Agarwal and Hermite–Hadamard inequalities. Cristescu et al. [32] were the first to introduce the class of (h1, h2)-convex functions, which they also investigated and covered some of its key aspects. By combining ideas from interval analysis and convex analysis, Zhao et al. [33] created a class of interval-valued h-convex functions and created several entirely new iterations of the Hermite–Hadamard inequality.

In order to construct the HH inequalities for harmonic convex functions, Iscan [34] first developed the idea of a harmonic convex set. By defining the Harmonic h-convex functions on the Harmonic convex set and expanding the HH inequalities that Iscan [34] developed, Mihai [35] advanced the concept of harmonic convex functions.

Keep in mind that fuzzy mappings are fuzzy functions with numeric values. On the other hand, Nanda and Kar [36] were the first to introduce the idea of convex *F-NV-Fs*. To this one step further, Khan et al. [37,38] recently proposed h-convex *F-NV-Fs* and (h 1,h 2)-convex *F-NV-Fs* and obtained some to offer new iterations of *HH* and fractional type of inequalities by employing fuzzy Riemann–Liouville fractional integrals and fuzzy Riemannian integrals, respectively. Similarly, using fuzzy order relations and fuzzy Riemann–Liouville fractional integrals, Sana and Khan et al. [39] developed new iterations of fuzzy fractional HH inequalities for harmonically convex *F-NV-Fs*. We direct the readers to [40–73] and the references therein for further information on generalized convex functions, fuzzy intervals, and fuzzy integrals.

We introduced a new generalization of harmonically convex functions known as harmonic s-convex *F-NV-Fs* in the second sense by fuzzy order relation. This work was motivated and inspired by ongoing research. We developed new iterations of the HH inequalities utilizing Riemann–Liouville fractional operators with the help of this class. In addition, we explored the applicability of our study in rare instances.

2. Preliminaries

We begin by recalling the basic notations and definitions. We define interval as

$$[\mathfrak{X}_*,\ \mathfrak{X}^*] = \{\imath \in \mathbb{R} : \mathfrak{X}_* \leq \imath \leq\ \mathfrak{X}^* \text{ and } \mathfrak{X}_*, \mathfrak{X}^* \in \mathbb{R}\ \},\ \text{where } \mathfrak{X}_* \leq\ \mathfrak{X}^*.$$

The collection of all closed and bounded intervals of $\mathbb{R}$ is denoted and defined as $\mathcal{K}_C = \{[\mathfrak{X}_*,\ \mathfrak{X}^*] : \mathfrak{X}_*,\ \mathfrak{X}^* \in \mathbb{R} \text{ and } \mathfrak{X}_* \leq\ \mathfrak{X}^*\}$. If $\mathfrak{X}_* \geq 0$, then $[\mathfrak{X}_*,\ \mathfrak{X}^*]$ is called positive interval. The set of all positive intervals is denoted by ${\mathcal{K}_C}^+$ and defined as ${\mathcal{K}_C}^+ = \{[\mathfrak{X}_*,\ \mathfrak{X}^*] : [\mathfrak{X}_*,\ \mathfrak{X}^*] \in \mathcal{K}_C \text{ and } \mathfrak{X}_* \geq 0\}$.

We now examine some of the properties of intervals using arithmetic operations. Let $[\mathfrak{X}_*,\ \mathfrak{X}^*],\ [\mathfrak{w}_*,\ \mathfrak{w}^*] \in \mathcal{K}_C$ and $\rho \in \mathbb{R}$, then we have

$$[\mathfrak{X}_*,\ \mathfrak{X}^*] + [\mathfrak{w}_*,\ \mathfrak{w}^*] = [\mathfrak{X}_* + \mathfrak{w}_*,\ \mathfrak{X}^* + \mathfrak{w}^*],$$

$$[\mathfrak{X}_*,\ \mathfrak{X}^*] \times [\mathfrak{w}_*,\ \mathfrak{w}^*] = \begin{bmatrix} min\{\mathfrak{X}_*\mathfrak{w}_*,\ \mathfrak{X}^*\mathfrak{w}_*,\ \mathfrak{X}_*\mathfrak{w}^*,\ \mathfrak{X}^*\mathfrak{w}^*\}, \\ max\{\mathfrak{X}_*\mathfrak{w}_*,\ \mathfrak{X}^*\mathfrak{w}_*,\ \mathfrak{X}_*\mathfrak{w}^*,\ \mathfrak{X}^*\mathfrak{w}^*\} \end{bmatrix}$$

$$\rho.[\mathfrak{X}_*,\ \mathfrak{X}^*] = \begin{cases} [\rho\mathfrak{X}_*,\ \rho\mathfrak{X}^*] \text{ if } \rho \geq 0, \\ [\rho\mathfrak{X}^*, \rho\mathfrak{X}_*] \text{ if } \rho < 0. \end{cases}$$

For $[\mathfrak{X}_*,\ \mathfrak{X}^*],\ [\mathfrak{w}_*,\ \mathfrak{w}^*] \in \mathcal{K}_C$, the inclusion " $\subseteq$ " is defined by

$$[\mathfrak{X}_*,\ \mathfrak{X}^*] \subseteq [\mathfrak{w}_*,\ \mathfrak{w}^*],\ \text{if and only if } \mathfrak{w}_* \leq \mathfrak{X}_*,\ \mathfrak{X}^* \leq \mathfrak{w}^*. \tag{1}$$

Remark 1 [40]. *The relation " $\leq_I$ " is defined on $\mathcal{K}_C$ by*

$$[\mathfrak{X}_*,\ \mathfrak{X}^*] \leq_I [\mathfrak{w}_*,\ \mathfrak{w}^*] \text{ if and only if } \mathfrak{X}_* \leq \mathfrak{w}_*,\ \mathfrak{X}^* \leq \mathfrak{w}^*, \tag{2}$$

for all $[\mathfrak{X}_*,\ \mathfrak{X}^*],\ [\mathfrak{w}_*,\ \mathfrak{w}^*] \in \mathcal{K}_C$, it is an order relation. For given $[\mathfrak{X}_*,\ \mathfrak{X}^*],\ [\mathfrak{w}_*,\ \mathfrak{w}^*] \in \mathcal{K}_C$, we say that $[\mathfrak{X}_*,\ \mathfrak{X}^*] \leq_I [\mathfrak{w}_*,\ \mathfrak{w}^*]$ if and only if $\mathfrak{X}_* \leq \mathfrak{w}_*,\ \mathfrak{X}^* \leq \mathfrak{w}^*$ or $\mathfrak{X}_* \leq \mathfrak{w}_*,\ \mathfrak{X}^* < w^*$.

Moore [41] initially proposed the concept of Riemann integral for I-V-F, which is defined as follows:

Theorem 1 [41]. *If $\mathfrak{S} : [\mu, \nu] \subset \mathbb{R} \to \mathcal{K}_C$ is an I-V-F on such that $\mathfrak{S}(\imath) = [\mathfrak{S}_*(\imath),\ \mathfrak{S}^*(\imath)]$. Then $\mathfrak{S}$ is Riemann integrable over $[\mu, \nu]$ if and only if, $\mathfrak{S}_*$ and $\mathfrak{S}^*$ both are Riemann integrable over $[\mu, \nu]$ such that*

$$(IR)\int_\mu^\nu \mathfrak{S}(\imath)d\imath = \left[(R)\int_\mu^\nu \mathfrak{S}_*(\imath)d\imath,\ (R)\int_\mu^\nu \mathfrak{S}^*(\imath)d\imath\right] \tag{3}$$

Let $\mathfrak{X} \in \mathbb{F}_0$ be a real fuzzy number, if and only if φ-cuts $[\mathfrak{X}]^\varphi$ is a nonempty compact convex set of $\mathbb{R}$. This is represented by

$$[\mathfrak{X}]^\varphi = \{\imath \in \mathbb{R} |\ \mathfrak{X}(\imath) \geq \varphi\},$$

From these definitions, we have

$$[\mathfrak{X}]^\varphi = [\mathfrak{X}_*(\varphi),\ \mathfrak{X}^*(\varphi)],$$

where

$$\mathfrak{X}_*(\varphi) = inf\{\imath \in \mathbb{R} |\ \mathfrak{X}(\imath) \geq \varphi\},$$
$$\mathfrak{X}^*(\varphi) = sup\{\imath \in \mathbb{R} |\ \mathfrak{X}(\imath) \geq \varphi\}.$$

Proposition 1 [51]. *Let $\mathfrak{X}, \mathfrak{w} \in \mathbb{F}_0$. Then fuzzy order relation "$\preccurlyeq$" is given on $\mathbb{F}_0$ by*

$$\mathfrak{X} \preccurlyeq \mathfrak{w} \text{ if and only if, } [\mathfrak{X}]^{\varphi} \leq_I [\mathfrak{w}]^{\varphi} \text{ for all } \varphi \in (0,\ 1], \tag{4}$$

It is a partial order relation.

We now use mathematical operations to examine some of the characteristics of fuzzy numbers. Let $\mathfrak{X}, \mathfrak{w} \in \mathbb{F}_0$ and $\rho \in \mathbb{R}$, then we have

$$\left[\mathfrak{X} \widetilde{+} \mathfrak{w}\right]^{\varphi} = [\mathfrak{X}]^{\varphi} + [\mathfrak{w}]^{\varphi}, \tag{5}$$

$$\left[\mathfrak{X} \widetilde{\times} \mathfrak{w}\right]^{\varphi} = [\mathfrak{X}]^{\varphi} \times [\ \mathfrak{w}]^{\varphi}, \tag{6}$$

$$[\rho.\mathfrak{X}]^{\varphi} = \rho.[\mathfrak{X}]^{\varphi} \tag{7}$$

Definition 1 [49]. *A fuzzy number valued map $\widetilde{\mathfrak{S}} : K \subset \mathbb{R} \to \mathbb{F}_0$ is called F-NV-F. For each $\varphi \in (0,\ 1]$ whose φ-cuts define the family of I-V-Fs $\mathfrak{S}_{\varphi} : K \subset \mathbb{R} \to \mathcal{K}_C$ are given by $\mathfrak{S}_{\varphi}(\imath) = [\mathfrak{S}_*(\imath, \varphi),\ \mathfrak{S}^*(\imath, \varphi)]$ for all $\imath \in K$. Here, for each $\varphi \in (0,\ 1]$ the end point real functions $\mathfrak{S}_*(., \varphi),\ \mathfrak{S}^*(., \varphi) : K \to \mathbb{R}$ are called lower and upper functions of $\widetilde{\mathfrak{S}}$.*

Definition 2 [49]. *Let $\widetilde{\mathfrak{S}} : [\mu,\ \nu] \subset \mathbb{R} \to \mathbb{F}_0$ be an F-NV-F. Then fuzzy integral of $\widetilde{\mathfrak{S}}$ over $[\mu,\ \nu]$, denoted by $(FR)\int_{\mu}^{\nu} \widetilde{\mathfrak{S}}(\imath)d\imath$, is given levelwise by*

$$\left[(FR)\int_{\mu}^{\nu} \widetilde{\mathfrak{S}}(\imath)d\imath\right]^{\varphi} = (IR)\int_{\mu}^{\nu} \mathfrak{S}_{\varphi}(\imath)d\imath = \left\{\int_{\mu}^{\nu} \mathfrak{S}(\imath, \varphi)d\imath : \mathfrak{S}(\imath, \varphi) \in \mathcal{R}_{([\mu,\ \nu],\ \varphi)}\right\}, \tag{8}$$

for all $\varphi \in (0,\ 1]$, where $\mathcal{R}_{([\mu,\ \nu],\ \varphi)}$ denotes the collection of Riemannian integrable functions of I-V-Fs. $\widetilde{\mathfrak{S}}$ is FR-integrable over $[\mu,\ \nu]$ if $(FR)\int_{\mu}^{\nu} \widetilde{\mathfrak{S}}(\imath)d\imath \in \mathbb{F}_0$. Note that, if $\mathfrak{S}_*(\imath, \varphi),\ \mathfrak{S}^*(\imath, \varphi)$ are Lebesgue-integrable, then $\mathfrak{S}$ is a fuzzy Aumann-integrable function over $[\mu,\ \nu]$.

Theorem 2 [49]. *Let $\widetilde{\mathfrak{S}} : [\mu,\ \nu] \subset \mathbb{R} \to \mathbb{F}_0$ be a F-NV-F whose φ-cuts define the family of I-V-Fs $\mathfrak{S}_{\varphi} : [\mu,\ \nu] \subset \mathbb{R} \to \mathcal{K}_C$ are given by $\mathfrak{S}_{\varphi}(\imath) = [\mathfrak{S}_*(\imath, \varphi),\ \mathfrak{S}^*(\imath, \varphi)]$ for all $\imath \in [\mu,\ \nu]$ and for all $\varphi \in (0,\ 1]$. Then $\widetilde{\mathfrak{S}}$ is FR-integrable over $[\mu,\ \nu]$ if and only if $\mathfrak{S}_*(\imath, \varphi)$ and $\mathfrak{S}^*(\imath, \varphi)$ both are R-integrable over $[\mu,\ \nu]$. Moreover, if $\widetilde{\mathfrak{S}}$ is FR-integrable over $[\mu,\ \nu]$, then*

$$\left[(FR)\int_{\mu}^{\nu} \widetilde{\mathfrak{S}}(\imath)d\imath\right]^{\varphi} = \left[(R)\int_{\mu}^{\nu} \mathfrak{S}_*(\imath, \varphi)d\imath,\ \ (R)\int_{\mu}^{\nu} \mathfrak{S}^*(\imath, \varphi)d\imath\right] = (IR)\int_{\mu}^{\nu} \mathfrak{S}_{\varphi}(\imath)d\imath \tag{9}$$

for all $\varphi \in (0,\ 1]$. For all $\varphi \in (0,\ 1]$, $\mathcal{FR}_{([\mu,\ \nu],\ \varphi)}$ denotes the collection of all FR-integrable F-NV-Fs over $[\mu,\ \nu]$.

Definition 3 [34]. *A set $K = [\mu,\ \nu] \subset \mathbb{R}^+ = (0, \infty)$ is said to be a convex set, if, for all $\imath,\ \jmath \in K$, $\xi \in [0,\ 1]$, we have*

$$\frac{\imath\jmath}{\xi\imath + (1-\xi)\jmath} \in K. \tag{10}$$

Definition 4 [34]. *The $\mathfrak{S} : [\mu,\ \nu] \to \mathbb{R}^+$ is called a harmonically convex function on $[\mu,\ \nu]$ if*

$$\mathfrak{S}\left(\frac{\imath\jmath}{\xi\imath + (1-\xi)\jmath}\right) \leq (1-\xi)\mathfrak{S}(\imath) + \xi\mathfrak{S}\left(\jmath\right), \tag{11}$$

for all $\imath,\ \jmath \in [\mu,\ \nu]$, $\xi \in [0,\ 1]$, where $\mathfrak{S}(\imath) \geq 0$ for all $\imath \in [\mu,\ \nu]$. If (11) is reversed then, $\mathfrak{S}$ is called a harmonically concave function on $[\mu,\ \nu]$.

Definition 5 [36]. *The positive real-valued function* $\mathfrak{S} : [\mu,\, \upsilon] \to \mathbb{R}^+$ *is called a harmonically s-convex function in the second sense on* $[\mu,\, \upsilon]$ *if*

$$\mathfrak{S}\left(\frac{\imath\jmath}{\xi\imath + (1-\xi)\jmath}\right) \leq (1-\xi)^s \mathfrak{S}(\imath) + \xi^s \mathfrak{S}\left(\jmath\right), \tag{12}$$

for all $\imath,\ \jmath \in [\mu,\, \upsilon]$, $\xi \in [0,\, 1]$, *where* $\mathfrak{S}(\imath) \geq 0$ *for all* $\imath \in [\mu,\, \upsilon]$ *and* $s \in [0,\, 1]$. *If (12) is reversed then,* $\mathfrak{S}$ *is called a harmonically s-concave function in the second sense on* $[\mu,\, \upsilon]$. *The set of all harmonically s-convex (harmonically s-concave) functions is denoted by*

$$HSX\big([\mu,\, \upsilon],\, \mathbb{R}^+, s\big) \qquad \big(HSV\big([\mu,\, \upsilon],\, \mathbb{R}^+,\, s\big)\big).$$

Definition 6 [36]. *The F-NV-F* $\widetilde{\mathfrak{S}} : [\mu,\, \upsilon] \to \mathbb{F}_0$ *is called convex F-NV-F in the second sense on* $[\mu,\, \upsilon]$ *if*

$$\widetilde{\mathfrak{S}}\left((1-\xi)\imath + \xi\jmath\right) \preccurlyeq (1-\xi)\widetilde{\mathfrak{S}}(\imath) \tilde{+} \xi\widetilde{\mathfrak{S}}\left(\jmath\right), \tag{13}$$

for all $\imath,\ \jmath \in [\mu,\, \upsilon]$, $\xi \in [0,\, 1]$, *where* $\mathfrak{S}(\imath) \geq 0$ *for all* $\imath \in [\mu,\, \upsilon]$ *and* $s \in [0,\, 1]$. *If (13) is reversed then,* $\widetilde{\mathfrak{S}}$ *is called concave F-NV-F on* $[\mu,\, \upsilon]$. *The set of all convex (concave) F-NV-Fs is denoted by*

$$FSX([\mu,\, \upsilon],\, \mathbb{F}_0, s) \qquad (FSV([\mu,\, \upsilon],\, \mathbb{F}_0,\, s)).$$

Definition 7 [39]. *The F-NV-F* $\widetilde{\mathfrak{S}} : [\mu,\, \upsilon] \to \mathbb{F}_0$ *is called harmonically convex F-NV-F on* $[\mu,\, \upsilon]$ *if*

$$\widetilde{\mathfrak{S}}\left(\frac{\imath\jmath}{\xi\imath + (1-\xi)\jmath}\right) \preccurlyeq (1-\xi)\widetilde{\mathfrak{S}}(\imath) \tilde{+} \xi\widetilde{\mathfrak{S}}\left(\jmath\right), \tag{14}$$

for all $\imath,\ \jmath \in [\mu,\, \upsilon]$, $\xi \in [0,\, 1]$, *where* $\widetilde{\mathfrak{S}}(\imath) \succcurlyeq \tilde{0}$, *for all* $\jmath \in [\mu,\, \upsilon]$. *If (14) is reversed then,* $\widetilde{\mathfrak{S}}$ *is called harmonically concave F-NV-F on* $[\mu,\, \upsilon]$.

Definition 8. *The F-NV-F* $\widetilde{\mathfrak{S}} : [\mu,\, \upsilon] \to \mathbb{F}_0$ *is called harmonically s-convex F-NV-F in the second sense on* $[\mu,\, \upsilon]$ *if*

$$\widetilde{\mathfrak{S}}\left(\frac{\imath\jmath}{\xi\imath + (1-\xi)\jmath}\right) \preccurlyeq (1-\xi)^s\widetilde{\mathfrak{S}}(\imath) \tilde{+} \xi^s\widetilde{\mathfrak{S}}\left(\jmath\right), \tag{15}$$

for all $\imath,\ \jmath \in [\mu,\, \upsilon]$, $\xi \in [0,\, 1]$, *where* $\widetilde{\mathfrak{S}}(\imath) \succcurlyeq \tilde{0}$, *for all* $\imath \in [\mu,\, \upsilon]$ *and* $s \in [0,\, 1]$. *If (15) is reversed, then* $\widetilde{\mathfrak{S}}$ *is called harmonically s-concave F-NV-F in the second sense on* $[\mu,\, \upsilon]$. *The set of all harmonically s-convex (harmonically s-concave) F-NV-F is denoted by*

$$HFSX([\mu,\, \upsilon],\, \mathbb{F}_0,\, s) \qquad (HFSV([\mu,\, \upsilon],\, \mathbb{F}_0,\, s)).$$

Theorem 3. *Let* $[\mu,\, \upsilon]$ *be a harmonically convex set, and let* $\widetilde{\mathfrak{S}} : [\mu,\, \upsilon] \to \mathbb{F}_C(\mathbb{R})$ *be a F-NV-F, whose* φ*-cuts define the family of I-V-Fs* $\mathfrak{S}_\varphi : [\mu,\, \upsilon] \subset \mathbb{R} \to \mathcal{K}_C^+ \subset \mathcal{K}_C$ *are given by*

$$\mathfrak{S}_\varphi(\imath) = [\mathfrak{S}_*(\imath, \varphi),\, \mathfrak{S}^*(\imath, \varphi)],\ \forall\, \imath \in [\mu,\, \upsilon]. \tag{16}$$

for all $\imath \in [\mu,\, \upsilon]$, $\varphi \in [0,\, 1]$. *Then* $\widetilde{\mathfrak{S}} \in HFSX([\mu,\, \upsilon],\, \mathbb{F}_0,\, s)$, *if and only if, for all* $\in [0,\, 1]$, $\mathfrak{S}_*(\imath,\ \varphi), \mathfrak{S}^*(\imath,\ \varphi) \in HSX([\mu,\, \upsilon],\, \mathbb{R}^+, s)$.

Proof. The proof is similar to the proof of Theorem 2.13, see [39]. □

Example 1. *We consider the F-NV-Fs* $\widetilde{\mathfrak{S}} : [0,\ 2] \to \mathbb{F}_C(\mathbb{R})$ *defined by*

$$\widetilde{\mathfrak{S}}(\imath)(\partial) = \begin{cases} \frac{\partial}{\sqrt{\imath}} & \partial \in \left[0,\ \sqrt{\imath}\right] \\ \frac{2-\partial}{2\sqrt{\imath}} & \partial \in \left(\sqrt{\imath},\ 2\sqrt{\imath}\right] \\ 0 & \text{otherwise,} \end{cases}$$

Then, for each $\varphi \in [0,\ 1]$, *we have* $\mathfrak{S}_\varphi(\imath) = \left[\varphi\sqrt{\imath}, (2-\varphi)\sqrt{\imath}\right]$. *Since* $\mathfrak{S}_*(\imath,\ \varphi), \mathfrak{S}^*(\imath,\ \varphi) \in HSX([\mu,\ \upsilon],\ \mathbb{R}^+, s)$ *with* $s = 1$, *for each* $\varphi \in [0,\ 1]$. *Hence* $\widetilde{\mathfrak{S}} \in HFSX([\mu,\ \upsilon],\ \mathbb{F}_0,\ s)$.

Remark 2. *If* $s = 1$, *then Definition 8 reduces to the Definition 7.*

If $\mathfrak{S}_*(\mu,\ \varphi) = \mathfrak{S}^*(\upsilon,\ \varphi)$ *with* $\varphi = 1$, *then harmonically s-convex F-NV-F in the second sense reduces to the classical harmonically s-convex function in the second sense, see [36].*

If $\mathfrak{S}_*(\mu,\ \varphi) = \mathfrak{S}^*(\upsilon,\ \varphi)$ *with* $\varphi = 1$ *and* $s = 1$, *then harmonically s-convex F-NV-F in the second sense reduces to the classical harmonically convex function, see [34].*

If $\mathfrak{S}_*(\mu,\ \varphi) = \mathfrak{S}^*(\upsilon,\ \varphi)$ *with* $\varphi = 1$ *and* $s = 0$ *then harmonically s-convex F-NV-F in the second sense reduces to the classical harmonical P-convex function, see [36].*

3. Fuzzy Hermite–Hadamard Inequalities

Two different sorts of inequalities are proven in this section. The first is *H.H* and their different forms, and the second is *H.H.* Fejér inequalities for convex *F-NV-Fs* with *F-NV-Fs* as the integrands.

Theorem 4. *Let* $\widetilde{\mathfrak{S}} \in HFSX([\mu,\ \upsilon],\ \mathbb{F}_0,\ s)$, *whose* φ*-cuts define the family of I-V-Fs* $\mathfrak{S}_\varphi : [\mu,\ \upsilon] \subset \mathbb{R} \to \mathcal{K}_C^+$ *are given by* $\mathfrak{S}_\varphi(\imath) = [\mathfrak{S}_*(\imath, \varphi),\ \mathfrak{S}^*(\imath, \varphi)]$ *for all* $\imath \in [\mu,\ \upsilon]$, $\varphi \in [0,\ 1]$. *If* $\widetilde{\mathfrak{S}} \in \mathcal{FR}_{([\mu,\ \upsilon],\ \varphi)}$, *then*

$$2^{s-1}\widetilde{\mathfrak{S}}\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right) \preccurlyeq \frac{\mu\upsilon}{\upsilon-\mu}\int_\mu^\upsilon \frac{\widetilde{\mathfrak{S}}(\imath)}{\imath^2}d\imath \preccurlyeq \frac{\widetilde{\mathfrak{S}}(\mu)\ \widetilde{+}\ \widetilde{\mathfrak{S}}(\upsilon)}{1+s} \tag{17}$$

If $\widetilde{\mathfrak{S}} \in HFSV([\mu,\ \upsilon],\ \mathbb{F}_0,\ s)$, *then*

$$2^{s-1}\widetilde{\mathfrak{S}}\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right) \succcurlyeq \frac{\mu\upsilon}{\upsilon-\mu}\int_\mu^\upsilon \frac{\widetilde{\mathfrak{S}}(\imath)}{\imath^2}d\imath \succcurlyeq \frac{\widetilde{\mathfrak{S}}(\mu)\ \widetilde{+}\ \widetilde{\mathfrak{S}}(\upsilon)}{1+s} \tag{18}$$

Proof. Let $\widetilde{\mathfrak{S}} \in HFSX([\mu,\ \upsilon],\ \mathbb{F}_0,\ s)$. Then, by hypothesis, we have

$$2^s\widetilde{\mathfrak{S}}\left(\frac{2\mu\upsilon}{\mu+\upsilon}\right) \preccurlyeq \widetilde{\mathfrak{S}}\left(\frac{\mu\upsilon}{\xi\mu+(1-\xi)\upsilon}\right)\widetilde{+}\widetilde{\mathfrak{S}}\left(\frac{\mu\upsilon}{(1-\xi)\mu+\xi\upsilon}\right).$$

Therefore, for each $\varphi \in [0,\ 1]$, we have

$$\begin{aligned} 2^s\mathfrak{S}_*\left(\tfrac{2\mu\upsilon}{\mu+\upsilon},\ \varphi\right) &\le \mathfrak{S}_*\left(\tfrac{\mu\upsilon}{\xi\mu+(1-\xi)\upsilon},\ \varphi\right) + \mathfrak{S}_*\left(\tfrac{\mu\upsilon}{(1-\xi)\mu+\xi\upsilon},\ \varphi\right), \\ 2^s\mathfrak{S}^*\left(\tfrac{2\mu\upsilon}{\mu+\upsilon},\ \varphi\right) &\le \mathfrak{S}^*\left(\tfrac{\mu\upsilon}{\xi\mu+(1-\xi)\upsilon},\ \varphi\right) + \mathfrak{S}^*\left(\tfrac{\mu\upsilon}{(1-\xi)\mu+\xi\upsilon}, \varphi\right). \end{aligned}$$

Then

$$\begin{aligned} 2^s\int_0^1 \mathfrak{S}_*\left(\tfrac{2\mu\upsilon}{\mu+\upsilon},\ \varphi\right)d\xi &\le \int_0^1 \mathfrak{S}_*\left(\tfrac{\mu\upsilon}{\xi\mu+(1-\xi)\upsilon}, \varphi\right)d\xi + \int_0^1 \mathfrak{S}_*\left(\tfrac{\mu\upsilon}{(1-\xi)\mu+\xi\upsilon},\ \varphi\right)d\xi, \\ 2^s\int_0^1 \mathfrak{S}^*\left(\tfrac{2\mu\upsilon}{\mu+\upsilon}, \varphi\right)d\xi &\le \int_0^1 \mathfrak{S}^*\left(\tfrac{\mu\upsilon}{\xi\mu+(1-\xi)\upsilon},\ \varphi\right)d\xi + \int_0^1 \mathfrak{S}^*\left(\tfrac{\mu\upsilon}{(1-\xi)\mu+\xi\upsilon}, \varphi\right)d\xi. \end{aligned}$$

It follows that

$$2^{s-1}\mathfrak{S}_*\left(\frac{2\mu v}{\mu+v},\ \varphi\right) \le \frac{\mu v}{v-\mu}\ \int_\mu^v \frac{\mathfrak{S}_*(\imath,\ \varphi)}{\imath^2}d\imath,$$

$$2^{s-1}\mathfrak{S}^*\left(\frac{2\mu v}{\mu+v},\ \varphi\right) \le \frac{\mu v}{v-\mu}\ \int_\mu^v \frac{\mathfrak{S}^*(\imath,\ \varphi)}{\imath^2}d\imath.$$

that is

$$2^{s-1}\left[\mathfrak{S}_*\left(\frac{2\mu v}{\mu+v},\ \varphi\right),\ \mathfrak{S}^*\left(\frac{2\mu v}{\mu+v},\ \varphi\right)\right] \le_I \frac{\mu v}{v-\mu}\left[\int_\mu^v \frac{\mathfrak{S}_*(\imath,\ \varphi)}{\imath^2}d\imath,\ \int_\mu^v \frac{\mathfrak{S}^*(\imath,\ \varphi)}{\imath^2}d\imath\right].$$

Thus,

$$2^{s-1}\,\widetilde{\mathfrak{S}}\left(\frac{2\mu v}{\mu+v}\right) \preccurlyeq \frac{\mu v}{v-\mu}\,(FR)\int_\mu^v \frac{\widetilde{\mathfrak{S}}(\imath)}{\imath^2}d\imath. \tag{19}$$

In a similar way as above, we have

$$\frac{\mu v}{v-\mu}\,(FR)\int_\mu^v \frac{\widetilde{\mathfrak{S}}(\imath)}{\imath^2}d\imath \preccurlyeq \frac{\widetilde{\mathfrak{S}}(\mu)\ \widetilde{+}\ \widetilde{\mathfrak{S}}(v)}{s+1}. \tag{20}$$

Combining (19) and (20), we have

$$2^{s-1}\widetilde{\mathfrak{S}}\left(\frac{2\mu v}{\mu+v}\right) \preccurlyeq \frac{\mu v}{v-\mu}\ \int_\mu^v \frac{\widetilde{\mathfrak{S}}(\imath)}{\imath^2}d\imath \preccurlyeq \frac{\widetilde{\mathfrak{S}}(\mu)\ \widetilde{+}\ \widetilde{\mathfrak{S}}(v)}{s+1}$$

Hence, the required result. □

Remark 3. *If $s = 1$, then from (17) we acquire the following inequality, see [39]:*

$$\widetilde{\mathfrak{S}}\left(\frac{2\mu v}{\mu+v}\right) \preccurlyeq \frac{\mu v}{v-\mu}\,(FR)\int_\mu^v \frac{\widetilde{\mathfrak{S}}(\imath)}{\imath^2}d\imath \preccurlyeq \frac{\widetilde{\mathfrak{S}}(\mu)\ \widetilde{+}\ \widetilde{\mathfrak{S}}(v)}{2}.$$

If $s \equiv 0$, then from (17) we acquire the following inequality, see [36]:

$$\frac{1}{2}\widetilde{\mathfrak{S}}\left(\frac{2\mu v}{\mu+v}\right) \preccurlyeq \frac{\mu v}{v-\mu}\,(FR)\int_\mu^v \frac{\widetilde{\mathfrak{S}}(\imath)}{\imath^2}d\imath \preccurlyeq \widetilde{\mathfrak{S}}(\mu)\ \widetilde{+}\ \widetilde{\mathfrak{S}}(v).$$

If $\mathfrak{S}_*(\imath,\ \varphi) = \mathfrak{S}^*(\imath,\ \varphi)$ with $\varphi = 1$, then from (17) we achieve the following inequality, see [36]:

$$2^{s-1}\,\mathfrak{S}\left(\frac{2\mu v}{\mu+v}\right) \le \frac{\mu v}{v-\mu}\,(R)\int_\mu^v \frac{\mathfrak{S}(\imath)}{\imath^2}d\imath \le \frac{1}{s+1}[\mathfrak{S}(\mu)+\mathfrak{S}(v)].$$

If $\mathfrak{S}_*(\imath,\ \varphi) = \mathfrak{S}^*(\imath,\ \varphi)$ with $\varphi = 1$ and $s = 1$, then from (17) we acquire the following inequality, see [34]:

$$\mathfrak{S}\left(\frac{2\mu v}{\mu+v}\right) \le \frac{\mu v}{v-\mu}\,(R)\int_\mu^v \frac{\mathfrak{S}(\imath)}{\imath^2}d\imath \le \frac{\mathfrak{S}(\mu)+\mathfrak{S}(v)}{2}.$$

If $\mathfrak{S}_*(\imath,\ \varphi) = \mathfrak{S}^*(\imath,\ \varphi)$ with $\varphi = 1$ and $s = 0$, then from (17) we achieve the following inequality, see [36]:

$$\frac{1}{2}\mathfrak{S}\left(\frac{2\mu v}{\mu+v}\right) \le \frac{\mu v}{v-\mu}\,(R)\int_\mu^v \frac{\mathfrak{S}(\imath)}{\imath^2}d\imath \le \mathfrak{S}(\mu)+\mathfrak{S}(v).$$

Theorem 5. *Let* $\widetilde{\mathfrak{S}} \in HFSX([\mu,\, v],\, \mathbb{F}_0,\, s)$*, whose* φ*-cuts define the family of I-V-Fs* $\mathfrak{S}_\varphi : [\mu,\, v] \subset \mathbb{R} \to \mathcal{K}_C^+$ *are given by* $\mathfrak{S}_\varphi(\imath) = [\mathfrak{S}_*(\imath,\varphi),\, \mathfrak{S}^*(\imath,\varphi)]$ *for all* $\imath \in [\mu,\, v]$*,* $\varphi \in [0,\, 1]$*. If* $\widetilde{\mathfrak{S}} \in \mathcal{FR}_{([\mu,\, v],\, \varphi)}$*, then*

$$4^{s-1}\, \widetilde{\mathfrak{S}}\left(\frac{2\mu v}{\mu+v}\right) \preccurlyeq \rhd_2 \preccurlyeq \frac{\mu v}{v-\mu}\, (FR) \int_\mu^v \frac{\widetilde{\mathfrak{S}}(\imath)}{\imath^2} d\imath \preccurlyeq \rhd_1 \preccurlyeq \frac{1}{s+1}\left[\widetilde{\mathfrak{S}}(\mu) \,\widetilde{+}\, \widetilde{\mathfrak{S}}(v)\right]\left[\frac{1}{2}+\frac{1}{2^s}\right], \tag{21}$$

where

$$\rhd_1 = \frac{1}{s+1}\left[\frac{\widetilde{\mathfrak{S}}(\mu) \,\widetilde{+}\, \widetilde{\mathfrak{S}}(v)}{2} \,\widetilde{+}\, \widetilde{\mathfrak{S}}\left(\frac{2\mu v}{\mu+v}\right)\right],$$

$$\rhd_2 = 2^{s-2}\left[\widetilde{\mathfrak{S}}\left(\frac{4\mu v}{\mu+3v}\right) \,\widetilde{+}\, \widetilde{\mathfrak{S}}\left(\frac{4\mu v}{3\mu+v}\right)\right],$$

and

$$\rhd_1 = [\rhd_{1*},\, \rhd_1^{\,*}], \rhd_2 = [\rhd_{2*},\, \rhd_2^{\,*}].$$

If $\widetilde{\mathfrak{S}} \in HFSV([\mu,\, v],\, \mathbb{F}_0,\, s)$*, then inequality (21) is reversed.*

Proof. Take $\left[\mu,\, \frac{2\mu v}{\mu+v}\right]$, we have

$$2^s \widetilde{\mathfrak{S}}\left(\frac{\mu \frac{4\mu v}{\mu+v}}{\xi\mu+(1-\xi)\frac{2\mu v}{\mu+v}} + \frac{\mu \frac{4\mu v}{\mu+v}}{(1-\xi)\mu+\xi\frac{2\mu v}{\mu+v}}\right) \preccurlyeq \widetilde{\mathfrak{S}}\left(\frac{\mu \frac{2\mu v}{\mu+v}}{\xi\mu+(1-\xi)\frac{2\mu v}{\mu+v}}\right) \widetilde{+} \widetilde{\mathfrak{S}}\left(\frac{\mu \frac{2\mu v}{\mu+v}}{(1-\xi)\mu+\xi\frac{2\mu v}{\mu+v}}\right).$$

Therefore, for every $\varphi \in [0,\, 1]$, we have

$$2^s \mathfrak{S}_*\left(\frac{\mu \frac{4\mu v}{\mu+v}}{\xi\mu+(1-\xi)\frac{2\mu v}{\mu+v}} + \frac{\mu \frac{4\mu v}{\mu+v}}{(1-\xi)\mu+\xi\frac{2\mu v}{\mu+v}},\, \varphi\right) \le \mathfrak{S}_*\left(\frac{\mu \frac{2\mu v}{\mu+v}}{\xi\mu+(1-\xi)\frac{2\mu v}{\mu+v}},\, \varphi\right) + \mathfrak{S}_*\left(\frac{\mu \frac{2\mu v}{\mu+v}}{(1-\xi)\mu+\xi\frac{2\mu v}{\mu+v}},\, \varphi\right),$$
$$2^s \mathfrak{S}^*\left(\frac{\mu \frac{4\mu v}{\mu+v}}{\xi\mu+(1-\xi)\frac{2\mu v}{\mu+v}} + \frac{\mu \frac{4\mu v}{\mu+v}}{(1-\xi)\mu+\xi\frac{2\mu v}{\mu+v}},\, \varphi\right) \le \mathfrak{S}^*\left(\frac{\mu \frac{2\mu v}{\mu+v}}{\xi\mu+(1-\xi)\frac{2\mu v}{\mu+v}},\, \varphi\right) + \mathfrak{S}^*\left(\frac{\mu \frac{2\mu v}{\mu+v}}{(1-\xi)\mu+\xi\frac{2\mu v}{\mu+v}},\, \varphi\right).$$

In consequence, we obtain

$$2^{s-2}\mathfrak{S}_*\left(\frac{4\mu v}{\mu+3v},\, \varphi\right) \le \frac{\mu v}{v-\mu} \int_\mu^{\frac{2\mu v}{\mu+v}} \frac{\mathfrak{S}_*(\imath,\varphi)}{\imath^2} d\imath,$$

$$2^{s-2}\mathfrak{S}^*\left(\frac{4\mu v}{\mu+3v},\, \varphi\right) \le \frac{\mu v}{v-\mu} \int_\mu^{\frac{2\mu v}{\mu+v}} \frac{\mathfrak{S}^*(\imath,\varphi)}{\imath^2} d\imath.$$

that is

$$2^{s-2}\left[\mathfrak{S}_*\left(\frac{4\mu v}{\mu+3v},\, \varphi\right),\, \mathfrak{S}^*\left(\frac{4\mu v}{\mu+3v},\, \varphi\right)\right] \le_I \frac{\mu v}{v-\mu}\left[\int_\mu^{\frac{2\mu v}{\mu+v}} \frac{\mathfrak{S}_*(\imath,\, \varphi)}{\imath^2} d\imath,\, \int_\mu^{\frac{2\mu v}{\mu+v}} \frac{\mathfrak{S}^*(\imath,\, \varphi)}{\imath^2} d\imath\right].$$

It follows that

$$2^{s-2}\widetilde{\mathfrak{S}}\left(\frac{4\mu v}{\mu+3v}\right) \preccurlyeq \frac{\mu v}{v-\mu} \int_\mu^{\frac{2\mu v}{\mu+v}} \frac{\widetilde{\mathfrak{S}}(\imath)}{\imath^2} d\imath. \tag{22}$$

In a similar way as above, we have

$$2^{s-2}\widetilde{\mathfrak{S}}\left(\frac{4\mu v}{3\mu+v}\right) \preccurlyeq \frac{\mu v}{v-\mu} \int_{\frac{2\mu v}{\mu+v}}^{v} \frac{\widetilde{\mathfrak{S}}(\imath)}{\imath^2} d\imath. \tag{23}$$

Combining (22) and (23), we have

$$2^{s-2}\left[\widetilde{\mathfrak{S}}\left(\frac{4\mu v}{\mu+3v}\right) \widetilde{+} \widetilde{\mathfrak{S}}\left(\frac{4\mu v}{3\mu+v}\right)\right] \preccurlyeq \frac{\mu v}{v-\mu} \int_\mu^v \frac{\widetilde{\mathfrak{S}}(\imath)}{\imath^2} d\imath. \tag{24}$$

Therefore, for every $\varphi \in [0,\ 1]$, by using Theorem 4, we have

$$
\begin{aligned}
4^{s-1}\mathfrak{S}_*\left(\tfrac{2\mu v}{\mu+v},\ \varphi\right) &\le 4^{s-1}\left[\tfrac{1}{2^s}\mathfrak{S}_*\left(\tfrac{4\mu v}{\mu+3v},\ \varphi\right)+\tfrac{1}{2^s}\mathfrak{S}_*\left(\tfrac{4\mu v}{3\mu+v},\ \varphi\right)\right],\\
4^{s-1}\mathfrak{S}^*\left(\tfrac{2\mu v}{\mu+v},\ \varphi\right) &\le 4^{s-1}\left[\tfrac{1}{2^s}\mathfrak{S}^*\left(\tfrac{4\mu v}{\mu+3v},\ \varphi\right)+\tfrac{1}{2^s}\mathfrak{S}^*\left(\tfrac{4\mu v}{3\mu+v},\ \varphi\right)\right],\\
&= \rhd_{2*},\\
&= \rhd_2{}^*,\\
&\le \tfrac{\mu v}{v-\mu}\int_\mu^v \frac{\mathfrak{S}_*(\imath,\ \varphi)}{\imath^2}d\imath,\\
&\le \tfrac{\mu v}{v-\mu}\int_\mu^v \frac{\mathfrak{S}^*(\imath,\ \varphi)}{\imath^2}d\imath,\\
&\le \tfrac{1}{s+1}\left[\tfrac{\mathfrak{S}_*(\mu,\ \varphi)+\mathfrak{S}_*(v,\ \varphi)}{2}+\mathfrak{S}_*\left(\tfrac{2\mu v}{\mu+v},\ \varphi\right)\right],\\
&\le \tfrac{1}{s+1}\left[\tfrac{\mathfrak{S}^*(\mu,\ \varphi)+\mathfrak{S}^*(v,\ \varphi)}{2}+\mathfrak{S}^*\left(\tfrac{2\mu v}{\mu+v},\ \varphi\right)\right],\\
&= \rhd_{1*},\\
&= \rhd_1{}^*,\\
&\le \tfrac{1}{s+1}\left[\tfrac{\mathfrak{S}_*(\mu,\ \varphi)+\mathfrak{S}_*(v,\ \varphi)}{2}+\tfrac{1}{2^s}(\mathfrak{S}_*(\mu,\ \varphi)+\mathfrak{S}_*(v,\ \varphi))\right],\\
&\le \tfrac{1}{s+1}\left[\tfrac{\mathfrak{S}^*(\mu,\ \varphi)+\mathfrak{S}^*(v,\ \varphi)}{2}+\tfrac{1}{2^s}(\mathfrak{S}^*(\mu,\ \varphi)+\mathfrak{S}^*(v,\ \varphi))\right],\\
&= \tfrac{1}{s+1}[\mathfrak{S}_*(\mu,\ \varphi)+\mathfrak{S}_*(v,\ \varphi)]\left[\tfrac{1}{2}+\tfrac{1}{2^s}\right],\\
&= \tfrac{1}{s+1}[\mathfrak{S}^*(\mu,\ \varphi)+\mathfrak{S}^*(v,\ \varphi)]\left[\tfrac{1}{2}+\tfrac{1}{2^s}\right],
\end{aligned}
$$

that is

$$4^{s-1}\,\widetilde{\mathfrak{S}}\left(\frac{2\mu v}{\mu+v}\right) \preccurlyeq \rhd_2 \preccurlyeq \frac{\mu v}{v-\mu}\ (FR)\int_\mu^v \frac{\widetilde{\mathfrak{S}}(\imath)}{\imath^2}d\imath \preccurlyeq \rhd_1 \preccurlyeq \frac{1}{s+1}\left[\widetilde{\mathfrak{S}}(\mu)\ \widetilde{+}\ \widetilde{\mathfrak{S}}(v)\right]\left[\frac{1}{2}+\frac{1}{2^s}\right]$$

□

Theorem 6. *Let* $\widetilde{\mathfrak{S}} \in HFSX([\mu,\ v],\ \mathbb{F}_0,\ s)$ *and* $\widetilde{\mathcal{Q}} \in HFSX([\mu,\ v],\ \mathbb{F}_0,\ s)$*, whose* φ*-cuts* $\mathfrak{S}_\varphi,\ \mathcal{Q}_\varphi : [\mu,\ v] \subset \mathbb{R} \to \mathcal{K}_C^+$ *are defined by* $\mathfrak{S}_\varphi(\imath) = [\mathfrak{S}_*(\imath,\varphi),\ \mathfrak{S}^*(\imath,\varphi)]$ *and* $\mathcal{Q}_\varphi(\imath) = [\mathcal{Q}_*(\imath,\varphi),\ \mathcal{Q}^*(\imath,\varphi)]$ *for all* $\imath \in [\mu,\ v]$, $\varphi \in [0,\ 1]$*, respectively. If* $\widetilde{\mathfrak{S}}\widetilde{\times}\widetilde{\mathcal{Q}} \in \mathcal{FR}_{([\mu,\ v],\ \varphi)}$*, then*

$$\frac{\mu v}{v-\mu}\ (FR)\int_\mu^v \frac{\widetilde{\mathfrak{S}}(\imath)\widetilde{\times}\widetilde{\mathcal{Q}}(\imath)}{\imath^2}d\imath \preccurlyeq \widetilde{\Lambda}(\mu,v)\int_0^1 \xi^s\cdot\xi^s d\xi \widetilde{+}\ \widetilde{\partial}(\mu,v)\int_0^1 \xi^s(1-\xi)^s d\xi,$$

where $\widetilde{\Lambda}(\mu,v) = \widetilde{\mathfrak{S}}(\mu)\widetilde{\times}\widetilde{\mathcal{Q}}(\mu)\ \widetilde{+}\ \widetilde{\mathfrak{S}}(v)\widetilde{\times}\widetilde{\mathcal{Q}}(v)$, $\widetilde{\partial}(\mu,v) = \widetilde{\mathfrak{S}}(\mu)\widetilde{\times}\widetilde{\mathcal{Q}}(v)\ \widetilde{+}\ \widetilde{\mathfrak{S}}(v)\widetilde{\times}\widetilde{\mathcal{Q}}(\mu)$*, and* $\Lambda_\varphi(\mu,v) = [\Lambda_*((\mu,v),\ \varphi),\ \Lambda^*((\mu,v),\ \varphi)]$ *and* $\partial_\varphi(\mu,v) = [\partial_*((\mu,v),\ \varphi),\ \partial^*((\mu,v),\ \varphi)]$.

Proof. Since $\widetilde{\mathfrak{S}},\ \widetilde{\mathcal{Q}}$ are harmonically s-convex *F-NV-Fs*, then for each, we have

$$
\begin{aligned}
\mathfrak{S}_*\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v},\ \varphi\right) &\le \xi^s\mathfrak{S}_*(\mu,\ \varphi)+(1-\xi)^s\mathfrak{S}_*(v,\ \varphi),\\
\mathfrak{S}^*\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v},\ \varphi\right) &\le \xi^s\mathfrak{S}^*(\mu,\ \varphi)+(1-\xi)^s\mathfrak{S}^*(v,\ \varphi).
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{Q}_*\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v},\ \varphi\right) &\le \xi^s\mathcal{Q}_*(\mu,\ \varphi)+(1-\xi)^s\mathcal{Q}_*(v,\ \varphi),\\
\mathcal{Q}^*\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v},\ \varphi\right) &\le \xi^s\mathcal{Q}^*(\mu,\ \varphi)+(1-\xi)^s\mathcal{Q}^*(v,\ \varphi).
\end{aligned}
$$

From the definition of harmonically s-convexity of *F-NV-Fs*, it follows that $\widetilde{\mathfrak{S}}(\imath) \succcurlyeq \widetilde{0}$ and $\widetilde{\mathcal{Q}}(\imath) \succcurlyeq \widetilde{0}$, so

$$\begin{aligned}
&\mathfrak{S}_*\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v},\, \varphi\right) \times \mathcal{Q}_*\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v},\, \varphi\right) \\
&\quad \le \left(\xi^s \mathfrak{S}_*(\mu,\, \varphi) + (1-\xi)^s \mathfrak{S}_*(v,\, \varphi)\right)\left(\xi^s \mathcal{Q}_*(\mu,\, \varphi) + (1-\xi)^s \mathcal{Q}_*(v,\, \varphi)\right) \\
&\quad = \mathfrak{S}_*(\mu,\, \varphi) \times \mathcal{Q}_*(\mu,\, \varphi)[\xi^s \cdot \xi^s] + \mathfrak{S}_*(v,\, \varphi) \times \mathcal{Q}_*(v,\, \varphi)\left[(1-\xi)^s (1-\xi)^s\right] \\
&\quad + \mathfrak{S}_*(\mu,\, \varphi)\mathcal{Q}_*(v,\, \varphi)\xi^s(1-\xi)^s + \mathfrak{S}_*(v,\, \varphi) \times \mathcal{Q}_*(\mu,\, \varphi)(1-\xi)^s \cdot \xi^s, \\
&\mathfrak{S}^*\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v},\, \varphi\right) \times \mathcal{Q}^*\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v},\, \varphi\right) \\
&\quad \le \left(\xi^s \mathfrak{S}^*(\mu,\, \varphi) + (1-\xi)^s \mathfrak{S}^*(v,\, \varphi)\right)\left(\xi^s \mathcal{Q}^*(\mu,\, \varphi) + (1-\xi)^s \mathcal{Q}^*(v,\, \varphi)\right) \\
&\quad = \mathfrak{S}^*(\mu,\, \varphi) \times \mathcal{Q}^*(\mu,\, \varphi)[\xi^s \cdot \xi^s] + \mathfrak{S}^*(v,\, \varphi) \times \mathcal{Q}^*(v,\, \varphi)\left[(1-\xi)^s (1-\xi)^s\right] \\
&\quad + \mathfrak{S}^*(\mu,\, \varphi) \times \mathcal{Q}^*(v,\, \varphi)\xi^s(1-\xi)^s + \mathfrak{S}^*(v,\, \varphi) \times \mathcal{Q}^*(\mu,\, \varphi)(1-\xi)^s \xi^s.
\end{aligned}$$

Integrating both sides of the above inequality over [0, 1], we obtain

$$\begin{aligned}
&\int_0^1 \mathfrak{S}_*\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v},\, \varphi\right) \times \mathcal{Q}_*\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v},\, \varphi\right) = \frac{\mu v}{v-\mu}\int_\mu^v \frac{\mathfrak{S}_*(\imath,\, \varphi) \times \mathcal{Q}_*(\imath,\, \varphi)}{\imath^2} d\imath \\
&\quad \le \left(\mathfrak{S}_*(\mu,\, \varphi) \times \mathcal{Q}_*(\mu,\, \varphi) + \mathfrak{S}_*(v,\, \varphi) \times \mathcal{Q}_*(v,\, \varphi)\right)\int_0^1 \xi^s \cdot \xi^s d\xi \\
&\quad + \left(\mathfrak{S}_*(\mu,\, \varphi) \times \mathcal{Q}_*(v,\, \varphi) + \mathfrak{S}_*(v,\, \varphi) \times \mathcal{Q}_*(\mu,\, \varphi)\right)\int_0^1 \xi^s(1-\xi)^s d\xi,
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 \mathfrak{S}^*\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v},\, \varphi\right) \times \mathcal{Q}^*\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v},\, \varphi\right) = \frac{\mu v}{v-\mu}\int_\mu^v \frac{\mathfrak{S}^*(\imath,\, \varphi) \times \mathcal{Q}^*(\imath,\, \varphi)}{\imath^2} d\imath \\
&\quad \le \left(\mathfrak{S}^*(\mu,\, \varphi) \times \mathcal{Q}^*(\mu,\, \varphi) + \mathfrak{S}_*(v,\, \varphi) \times \mathcal{Q}^*(v,\, \varphi)\right)\int_0^1 \xi^s \cdot \xi^s d\xi \\
&\quad + \left(\mathfrak{S}_*(\mu,\, \varphi) \times \mathcal{Q}^*(v,\, \varphi) + \mathfrak{S}_*(v,\, \varphi) \times \mathcal{Q}^*(\mu,\, \varphi)\right)\int_0^1 \xi^s(1-\xi)^s d\xi.
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{\mu v}{v-\mu}\int_\mu^v \mathfrak{S}_*(\imath,\, \varphi) \times \mathcal{Q}_*(\imath,\, \varphi) d\imath &\le \Lambda_*((\mu, v),\, \varphi)\int_0^1 \xi^s \cdot \xi^s d\xi \\
&+ \partial_*((\mu, v),\, \varphi)\int_0^1 \xi^s(1-\xi)^s d\xi,
\end{aligned}$$

$$\begin{aligned}
\frac{\mu v}{v-\mu}\int_\mu^v \mathfrak{S}^*(\imath,\, \varphi) \times \mathcal{Q}^*(\imath,\, \varphi) d\imath &\le \Lambda^*((\mu, v),\, \varphi)\int_0^1 \xi^s \cdot \xi^s d\xi \\
&+ \partial_*((\mu, v),\, \varphi)\int_0^1 \xi^s(1-\xi)^s d\xi,
\end{aligned}$$

that is

$$\begin{aligned}
&\frac{\mu v}{v-\mu}\left[\int_\mu^v \mathfrak{S}_*(\imath,\, \varphi) \times \mathcal{Q}_*(\imath,\, \varphi) d\imath,\, \int_\mu^v \mathfrak{S}^*(\imath,\, \varphi) \times \mathcal{Q}^*(\imath,\, \varphi) d\imath\right] \\
&\quad \le_I [\Lambda_*((\mu, v),\, \varphi),\, \Lambda^*((\mu, v),\, \varphi)]\int_0^1 \xi^s \cdot \xi^s d\xi \\
&\quad + [\partial_*((\mu, v),\, \varphi),\, \partial^*((\mu, v),\, \varphi)]\int_0^1 \xi^s(1-\xi)^s d\xi.
\end{aligned}$$

Thus,

$$\frac{\mu v}{v-\mu}\,(FR)\int_\mu^v \frac{\widetilde{\mathfrak{S}}(\imath)\,\widetilde{\times}\,\widetilde{\mathcal{Q}}(\imath)}{\imath^2} d\imath \preccurlyeq \widetilde{\Lambda}(\mu, v)\int_0^1 \xi^s \cdot \xi^s d\xi \,\widetilde{+}\, \widetilde{\partial}(\mu, v)\int_0^1 \xi^s(1-\xi)^s d\xi.$$

□

Theorem 7. *Let* $\widetilde{\mathfrak{S}} \in HFSX([\mu,\, v],\, \mathbb{F}_0,\, s)$, $\widetilde{\mathcal{Q}} \in HFSX([\mu,\, v],\, \mathbb{F}_0,\, s)$, *whose* φ*-cuts* $\mathfrak{S}_\varphi,\, \mathcal{Q}_\varphi : [\mu,\, v] \subset \mathbb{R} \to \mathcal{K}_C^+$ *are defined by* $\mathfrak{S}_\varphi(\imath) = [\mathfrak{S}_*(\imath, \varphi),\, \mathfrak{S}^*(\imath, \varphi)]$ *and* $\mathcal{Q}_\varphi(\imath) = [\mathcal{Q}_*(\imath, \varphi),\, \mathcal{Q}^*(\imath, \varphi)]$ *for all* $\imath \in [\mu,\, v]$, $\varphi \in [0,\, 1]$, *respectively. If* $\widetilde{\mathfrak{S}}\,\widetilde{\times}\,\widetilde{\mathcal{Q}} \in \mathcal{FR}_{([\mu,\, v],\, \varphi)}$, *then*

$$2^{2s-1}\,\widetilde{\mathfrak{S}}\left(\frac{2\mu v}{\mu+v}\right)\widetilde{\times}\,\widetilde{\mathcal{Q}}\left(\frac{2\mu v}{\mu+v}\right)$$

$$\preccurlyeq \frac{\mu v}{v-\mu}\,(FR)\int_{\mu}^{v}\frac{\widetilde{\mathfrak{S}}(\imath)\widetilde{\times}\widetilde{\mathcal{Q}}(\imath)}{\imath^2}d\imath+\widetilde{\Lambda}(\mu,v)\int_0^1\xi^s(1-\xi)^s d\xi\widetilde{+}\,\widetilde{\partial}(\mu,v)\int_0^1\xi^s\cdot\xi^s d\xi,$$

where $\widetilde{\Lambda}(\mu,v)=\widetilde{\mathfrak{S}}(\mu)\widetilde{\times}\widetilde{\mathcal{Q}}(\mu)\ \widetilde{+}\ \widetilde{\mathfrak{S}}(v)\widetilde{\times}\widetilde{\mathcal{Q}}(v)$, $\widetilde{\partial}(\mu,v)=\widetilde{\mathfrak{S}}(\mu)\widetilde{\times}\widetilde{\mathcal{Q}}(v)\ \widetilde{+}\ \widetilde{\mathfrak{S}}(v)\widetilde{\times}\widetilde{\mathcal{Q}}(\mu)$, *and* $\Lambda_{\varphi}(\mu,v)=[\Lambda_*((\mu,v),\ \varphi),\ \Lambda^*((\mu,v),\ \varphi)]$ *and* $\partial_{\varphi}(\mu,v)=[\partial_*((\mu,v),\ \varphi),\ \partial^*((\mu,v),\ \varphi)]$.

Proof. By hypothesis, for each $\varphi\in[0,\ 1]$, we have

$$\begin{gathered}\mathfrak{S}_*\left(\tfrac{2\mu v}{\mu+v},\varphi\right)\times\mathcal{Q}_*\left(\tfrac{2\mu v}{\mu+v},\varphi\right)\\ \mathfrak{S}^*\left(\tfrac{2\mu v}{\mu+v},\varphi\right)\times\mathcal{Q}^*\left(\tfrac{2\mu v}{\mu+v},\varphi\right)\end{gathered}$$

$$\le 2^{2s}\begin{bmatrix}\mathfrak{S}_*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\times\mathcal{Q}_*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\\ +\mathfrak{S}_*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\times\mathcal{Q}_*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\end{bmatrix}$$
$$+2^{2s}\begin{bmatrix}\mathfrak{S}_*\left(\frac{\mu v}{(1-\xi)\mu+\xi v},\ \varphi\right)\times\mathcal{Q}_*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\\ +\mathfrak{S}_*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\times\mathcal{Q}_*\left(\frac{\mu v}{(1-\xi)\mu+\xi v},\ \varphi\right)\end{bmatrix},$$
$$\le 2^{2s}\begin{bmatrix}\mathfrak{S}^*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\times\mathcal{Q}^*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\\ +\mathfrak{S}^*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\times\mathcal{Q}^*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\end{bmatrix}$$
$$+2^{2s}\begin{bmatrix}\mathfrak{S}^*\left(\frac{\mu v}{(1-\xi)\mu+\xi v},\ \varphi\right)\times\mathcal{Q}^*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\\ +\mathfrak{S}^*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\times\mathcal{Q}^*\left(\frac{\mu v}{(1-\xi)\mu+\xi v},\ \varphi\right)\end{bmatrix},$$
$$\le 2^{2s}\begin{bmatrix}\mathfrak{S}_*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\times\mathcal{Q}_*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\\ +\mathfrak{S}_*\left(\frac{\mu v}{(1-\xi)\mu+\xi v},\ \varphi\right)\times\mathcal{Q}_*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\end{bmatrix}$$
$$+2^{2s}\begin{bmatrix}\left(\xi^s\mathfrak{S}_*(\mu,\ \varphi)+(1-\xi)^s\mathfrak{S}_*(v,\ \varphi)\right)\\ \times\left((1-\xi)^s\mathcal{Q}_*(\mu,\ \varphi)+\xi^s\mathcal{Q}_*(v,\ \varphi)\right)\\ +\left((1-\xi)^s\mathfrak{S}_*(\mu,\ \varphi)+\xi^s\mathfrak{S}_*(v,\ \varphi)\right)\\ \times\left(\xi^s\mathcal{Q}_*(\mu,\ \varphi)+(1-\xi)^s\mathcal{Q}_*(v,\ \varphi)\right)\end{bmatrix},$$
$$\le 2^{2s}\begin{bmatrix}\mathfrak{S}^*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\times\mathcal{Q}^*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\\ +\mathfrak{S}^*\left(\frac{\mu v}{\xi\mu+(1-\xi)v}\ \varphi\right)\times\mathcal{Q}^*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\end{bmatrix}$$
$$+2^{2s}\begin{bmatrix}\left(\xi^s\mathfrak{S}^*(\mu,\ \varphi)+(1-\xi)^s\mathfrak{S}^*(v,\ \varphi)\right)\\ \times\left((1-\xi)^s\mathcal{Q}^*(\mu,\ \varphi)+\xi^s\mathcal{Q}^*(v,\ \varphi)\right)\\ +\left((1-\xi)^s\mathfrak{S}^*(\mu,\ \varphi)+\xi^s\mathfrak{S}^*(v,\ \varphi)\right)\\ \times\left(\xi^s\mathcal{Q}^*(\mu,\ \varphi)+(1-\xi)^s\mathcal{Q}^*(v,\ \varphi)\right)\end{bmatrix},$$
$$= 2^{2s}\begin{bmatrix}\mathfrak{S}_*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\times\mathcal{Q}_*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\\ +\mathfrak{S}_*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\times\mathcal{Q}_*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\end{bmatrix}$$
$$+2^{2s}\begin{bmatrix}\left\{\xi^s\cdot\xi^s+(1-\xi)^s(1-\xi)^s\right\}\partial_*((\mu,v),\ \varphi)\\ +\left\{\xi^s(1-\xi)^s+(1-\xi)^s\xi^s\right\}\Lambda_*((\mu,v),\ \varphi)\end{bmatrix},$$
$$= 2^{2s}\begin{bmatrix}\mathfrak{S}^*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\times\mathcal{Q}^*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\\ +\mathfrak{S}^*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\times\mathcal{Q}^*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\end{bmatrix}$$
$$+2^{2s}\begin{bmatrix}\left\{\xi^s\cdot\xi^s+(1-\xi)^s(1-\xi)^s\right\}\partial^*((\mu,v),\ \varphi)\\ +\left\{\xi^s(1-\xi)^s+(1-\xi)^s\xi^s\right\}\Lambda^*((\mu,v),\ \varphi)\end{bmatrix},$$

Integrating over $[0,\ 1]$, we have

$$\begin{aligned}
2^{2s-1}\ \mathfrak{S}_*\left(\tfrac{2\mu v}{\mu+v},\varphi\right)\times\mathcal{Q}_*\left(\tfrac{2\mu v}{\mu+v},\varphi\right) &\le \tfrac{1}{v-\mu}\ (R)\int_\mu^v \mathfrak{S}_*(\imath,\varphi)\times\mathcal{Q}_*(\imath,\varphi)d\imath\\
&+\Lambda_*((\mu,v),\ \varphi)\int_0^1 \xi^s(1-\xi)^s d\xi\\
&+\partial_*((\mu,v),\ \varphi)\int_0^1 \xi^s\cdot\xi^s d\xi,\\
2^{2s-1}\ \mathfrak{S}^*\left(\tfrac{2\mu v}{\mu+v},\varphi\right)\times\mathcal{Q}^*\left(\tfrac{2\mu v}{\mu+v},\varphi\right) &\le \tfrac{1}{v-\mu}\ (R)\int_\mu^v \mathfrak{S}^*(\imath,\varphi)\times\mathcal{Q}^*(\imath,\varphi)d\imath\\
&+\Lambda^*((\mu,v),\ \varphi)\int_0^1 \xi^s(1-\xi)^s d\xi\\
&+\partial^*((\mu,v),\ \varphi)\int_0^1 \xi^s\cdot\xi^s d\xi,
\end{aligned}$$

that is

$$\begin{aligned}
&2^{2s-1}\ \widetilde{\mathfrak{S}}\left(\tfrac{2\mu v}{\mu+v}\right)\widetilde{\times}\widetilde{\mathcal{Q}}\left(\tfrac{2\mu v}{\mu+v}\right)\\
&\preccurlyeq \tfrac{\mu v}{v-\mu}\ (FR)\int_\mu^v \frac{\widetilde{\mathfrak{S}}(\imath)\widetilde{\times}\widetilde{\mathcal{Q}}(\imath)}{\imath^2}d\imath+\widetilde{\Lambda}(\mu,v)\int_0^1 \xi^s(1-\xi)^s d\xi\widetilde{+}\ \widetilde{\partial}(\mu,v)\int_0^1 \xi^s\cdot\xi^s d\xi.
\end{aligned}$$

The theorem has been proved.

The right fuzzy *H.H.* Fejér inequality, which is connected to the right part of the classical *H.H.* Fejér inequality for harmonically s-convex *F-NV-Fs* via fuzzy order relations, is the first inequality we will develop. □

Theorem 8. *(Second fuzzy H.H. Fejér inequality) Let* $\widetilde{\mathfrak{S}}\in HFSX([\mu,\ v],\ \mathbb{F}_0,\ s)$*, whose* φ*-cuts define the family of I-V-Fs* $\mathfrak{S}_\varphi:[\mu,\ v]\subset\mathbb{R}\to\mathcal{K}_C^+$ *are given by* $\mathfrak{S}_\varphi(\imath)=[\mathfrak{S}_*(\imath,\varphi),\ \mathfrak{S}^*(\imath,\varphi)]$ *for all* $\imath\in[\mu,\ v]$, $\varphi\in[0,\ 1]$. *If* $\widetilde{\mathfrak{S}}\in\mathcal{FR}_{([\mu,\ v],\ \varphi)}$ *and* $\nabla:[\mu,\ v]\to\mathbb{R}$, $\nabla\left(\frac{1}{\frac{1}{\mu}+\frac{1}{v}-\frac{1}{\imath}}\right)=\nabla(\imath)\ge 0$, *then*

$$\frac{\mu v}{\nu-\mu}\ (FR)\int_\mu^\nu \frac{\widetilde{\mathfrak{S}}(\imath)}{\imath^2}\nabla(\imath)d\imath\preccurlyeq\left[\widetilde{\mathfrak{S}}(\mu)\ \widetilde{+}\ \widetilde{\mathfrak{S}}(\nu)\right]\int_0^1 \xi^s\nabla\left(\frac{\mu v}{\xi\mu+(1-\xi)v}\right)d\xi. \tag{25}$$

If $\widetilde{\mathfrak{S}}\in HFSV([\mu,\ v],\ \mathbb{F}_0,\ s)$*, then inequality (25) is reversed.*

Proof. Let $\mathfrak{S}$ be a s-convex *F-NV-F*. Then, for each $\varphi\in[0,\ 1]$, we have

$$\begin{aligned}
&\mathfrak{S}_*\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v},\varphi\right)\nabla\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v}\right)\\
&\quad\le\left(\xi^s\mathfrak{S}_*(\mu,\varphi)+(1-\xi)^s\mathfrak{S}_*(v,\varphi)\right)\nabla\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v}\right),\\
&\mathfrak{S}^*\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v},\varphi\right)\nabla\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v}\right)\\
&\quad\le\left(\xi^s\mathfrak{S}^*(\mu,\varphi)+(1-\xi)^s\mathfrak{S}^*(v,\varphi)\right)\nabla\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v}\right).
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
&\mathfrak{S}_*\left(\tfrac{\mu v}{\xi\mu+(1-\xi)v},\varphi\right)\nabla\left(\tfrac{\mu v}{\xi\mu+(1-\xi)v}\right)\\
&\quad\le\left((1-\xi)^s\mathfrak{S}_*(\mu,\varphi)+\xi^s\mathfrak{S}_*(v,\varphi)\right)\nabla\left(\tfrac{\mu v}{\xi\mu+(1-\xi)v}\right),\\
&\mathfrak{S}^*\left(\tfrac{\mu v}{\xi\mu+(1-\xi)v},\varphi\right)\nabla\left(\tfrac{\mu v}{\xi\mu+(1-\xi)v}\right)\\
&\quad\le\left((1-\xi)^s\mathfrak{S}^*(\mu,\varphi)+\xi^s\mathfrak{S}^*(v,\varphi)\right)\nabla\left(\tfrac{\mu v}{\xi\mu+(1-\xi)v}\right).
\end{aligned} \tag{27}$$

After adding (26) and (27), and integrating over $[0,\ 1]$, we obtain

$$\begin{aligned}
&\int_0^1 \mathfrak{S}_*\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v},\varphi\right)\nabla\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v}\right)d\xi \\
&\qquad + \int_0^1 \mathfrak{S}_*\left(\tfrac{\mu v}{\xi\mu+(1-\xi)v},\varphi\right)\nabla\left(\tfrac{\mu v}{\xi\mu+(1-\xi)v}\right)d\xi \\
&\qquad \le \int_0^1 \left[\begin{array}{c} \mathfrak{S}_*(\mu,\varphi)\left\{\begin{array}{c}\xi^s\nabla\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v}\right)\\ +(1-\xi)^s\nabla\left(\tfrac{\mu v}{\xi\mu+(1-\xi)v}\right)\end{array}\right\} \\ +\mathfrak{S}_*(v,\varphi)\left\{\begin{array}{c}(1-\xi)^s\nabla\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v}\right)\\ +\xi^s\nabla\left(\tfrac{\mu v}{\xi\mu+(1-\xi)v}\right)\end{array}\right\}\end{array}\right]d\xi, \\
&\int_0^1 \mathfrak{S}^*\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v},\varphi\right)\nabla\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v}\right)d\xi \\
&\qquad + \int_0^1 \mathfrak{S}^*\left(\tfrac{\mu v}{\xi\mu+(1-\xi)v},\varphi\right)\nabla\left(\tfrac{\mu v}{\xi\mu+(1-\xi)v}\right)d\xi \\
&\qquad \le \int_0^1 \left[\begin{array}{c} \mathfrak{S}^*(\mu,\varphi)\left\{\begin{array}{c}\xi^s\nabla\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v}\right)\\ +(1-\xi)^s\nabla\left(\tfrac{\mu v}{\xi\mu+(1-\xi)v}\right)\end{array}\right\} \\ +\mathfrak{S}^*(v,\varphi)\left\{\begin{array}{c}(1-\xi)^s\nabla\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v}\right)\\ +\xi^s\nabla\left(\tfrac{\mu v}{\xi\mu+(1-\xi)v}\right)\end{array}\right\}\end{array}\right]d\xi. \\
&\qquad = 2\mathfrak{S}_*(\mu,\varphi)\int_0^1 \xi^s\nabla\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v}\right)d\xi \\
&\qquad\quad +2\mathfrak{S}_*(v,\varphi)\int_0^1 \xi^s\nabla\left(\tfrac{\mu v}{\xi\mu+(1-\xi)v}\right)d\xi, \\
&\qquad = 2\mathfrak{S}^*(\mu,\varphi)\int_0^1 \xi^s\nabla\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v}\right)d\xi \\
&\qquad\quad +2\mathfrak{S}^*(v,\varphi)\int_0^1 \xi^s\nabla\left(\tfrac{\mu v}{\xi\mu+(1-\xi)v}\right)d\xi.
\end{aligned}$$

Since ∇ is symmetric, then

$$\begin{aligned}
&= 2[\mathfrak{S}_*(\mu,\ \varphi)+\mathfrak{S}_*(\nu,\ \varphi)]\int_0^1 \xi^s\nabla\left(\tfrac{\mu v}{\xi\mu+(1-\xi)v}\right)d\xi, \\
&= 2[\mathfrak{S}^*(\mu,\ \varphi)+\mathfrak{S}^*(\nu,\ \varphi)]\int_0^1 \xi^s\nabla\left(\tfrac{\mu v}{\xi\mu+(1-\xi)v}\right)d\xi.
\end{aligned} \tag{28}$$

since

$$\begin{aligned}
&\int_0^1 \mathfrak{S}_*\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v},\ \varphi\right)\nabla\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v}\right)d\xi \\
&\qquad = \int_0^1 \mathfrak{S}_*\left(\left(\tfrac{\mu v}{\xi\mu+(1-\xi)v}\right),\ \varphi\right)\nabla\left(\tfrac{\mu v}{\xi\mu+(1-\xi)v}\right)d\xi \\
&\qquad = \tfrac{\mu v}{\nu-\mu}\int_\mu^\nu \mathfrak{S}_*(\imath,\varphi)\nabla(\imath)d\imath \\
&\int_0^1 \mathfrak{S}^*\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v},\ \varphi\right)\nabla\left(\tfrac{\mu v}{(1-\xi)\mu+\xi v}\right)d\xi \\
&\qquad = \int_0^1 \mathfrak{S}^*\left(\left(\tfrac{\mu v}{\xi\mu+(1-\xi)v}\right),\ \varphi\right)\nabla\left(\tfrac{\mu v}{\xi\mu+(1-\xi)v}\right)d\xi \\
&\qquad = \tfrac{\mu v}{\nu-\mu}\int_\mu^\nu \mathfrak{S}^*(\imath,\ \varphi)\nabla(\imath)d\imath.
\end{aligned} \tag{29}$$

From (28) and (29), we have

$$\begin{aligned}
&\tfrac{\mu v}{\nu-\mu}\int_\mu^\nu \mathfrak{S}_*(\imath,\varphi)\nabla(\imath)d\imath \\
&\qquad \le [\mathfrak{S}_*(\mu,\ \varphi)+\mathfrak{S}_*(\nu,\ \varphi)]\int_0^1 \xi^s\nabla\left(\tfrac{\mu v}{\xi\mu+(1-\xi)v}\right)d\xi, \\
&\tfrac{\mu v}{\nu-\mu}\int_\mu^\nu \mathfrak{S}^*(\imath,\varphi)\nabla(\imath)d\imath \\
&\qquad \le [\mathfrak{S}^*(\mu,\ \varphi)+\mathfrak{S}^*(\nu,\ \varphi)]\int_0^1 \xi^s\nabla\left(\tfrac{\mu v}{\xi\mu+(1-\xi)v}\right)d\xi,
\end{aligned}$$

that is

$$\left[\frac{\mu v}{\nu-\mu}\ \int_{\mu}^{\nu}\mathfrak{S}_*(\imath,\varphi)\nabla(\imath)d\imath,\ \frac{\mu v}{\nu-\mu}\ \int_{\mu}^{\nu}\mathfrak{S}^*(\imath,\varphi)\nabla(\imath)d\imath\right]$$
$$\leq_I \left[\mathfrak{S}_*(\mu,\ \varphi)+\mathfrak{S}_*(\nu,\ \varphi),\ \mathfrak{S}^*(\mu,\ \varphi)+\mathfrak{S}^*(\nu,\ \varphi)\right]\int_0^1 \xi^s\nabla\left(\frac{\mu v}{\xi\mu+(1-\xi)v}\right)d\xi,$$

hence

$$\frac{\mu v}{\nu-\mu}\ (FR)\int_{\mu}^{\nu}\frac{\widetilde{\mathfrak{S}}(\imath)}{\imath^2}\nabla(\imath)d\imath \preccurlyeq \left[\widetilde{\mathfrak{S}}(\mu)\ \widetilde{+}\ \widetilde{\mathfrak{S}}(\nu)\right]\int_0^1 \xi^s\nabla\left(\frac{\mu v}{\xi\mu+(1-\xi)v}\right)d\xi,$$

This concludes the proof. □

Next, we construct the first *H.H.* Fejér inequality for harmonically s-convex *F-NV-F*, which generalizes the first *H.H.* Fejér inequality for a harmonically convex function.

Theorem 9 (First fuzzy fractional H.H. Fejér inequality). *Let* $\widetilde{\mathfrak{S}} \in HFSX([\mu,\ v],\ \mathbb{F}_0,\ s)$*, whose* φ*-cuts define the family of I-V-Fs* $\mathfrak{S}_\varphi : [\mu,\ v] \subset \mathbb{R} \to \mathcal{K}_C^+$ *are given by* $\mathfrak{S}_\varphi(\imath) = [\mathfrak{S}_*(\imath,\varphi),\ \mathfrak{S}^*(\imath,\varphi)]$ *for all* $\imath \in [\mu,\ v]$, $\varphi \in [0,\ 1]$. *If* $\widetilde{\mathfrak{S}} \in \mathcal{FR}_{([\mu,\ v],\ \varphi)}$ *and* $\nabla : [\mu,\ v] \to \mathbb{R}$, $\nabla\left(\frac{1}{\frac{1}{\mu}+\frac{1}{v}-\frac{1}{\imath}}\right) = \nabla(\imath) \geq 0$, *then*

$$2^{s-1}\ \widetilde{\mathfrak{S}}\left(\frac{2\mu\nu}{\mu+\nu}\right)\int_{\mu}^{\nu}\frac{\widetilde{\mathfrak{S}}(\imath)}{\imath^2}d\imath \preccurlyeq (FR)\int_{\mu}^{\nu}\frac{\widetilde{\mathfrak{S}}(\imath)}{\imath^2}\nabla(\imath)d\imath \tag{30}$$

If $\widetilde{\mathfrak{S}} \in HFSV([\mu,\ v],\ \mathbb{F}_0,\ s)$*, then inequality (30) is reversed.*

Proof. Since $\mathfrak{S}$ is a *s*-convex, then for $\varphi \in [0,\ 1]$, we have

$$\begin{aligned}\mathfrak{S}_*\left(\frac{2\mu\nu}{\mu+\nu},\ \varphi\right) &\leq \frac{1}{2^s}\left(\mathfrak{S}_*\left(\frac{\mu v}{(1-\xi)\mu+\xi v},\ \varphi\right)+\mathfrak{S}_*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\right)\\ \mathfrak{S}^*\left(\frac{2\mu\nu}{\mu+\nu},\ \varphi\right) &\leq \frac{1}{2^s}\left(\mathfrak{S}^*\left(\frac{\mu v}{(1-\xi)\mu+\xi v},\ \varphi\right)+\mathfrak{S}^*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\right),\end{aligned} \tag{31}$$

By multiplying (31) by $\nabla\left(\frac{\mu v}{(1-\xi)\mu+\xi v}\right) = \nabla\left(\frac{\mu v}{\xi\mu+(1-\xi)v}\right)$ and integrate it by ξ over $[0,\ 1]$, we obtain

$$\begin{aligned}&\mathfrak{S}_*\left(\frac{2\mu\nu}{\mu+\nu},\ \varphi\right)\int_0^1\nabla\left(\frac{\mu v}{\xi\mu+(1-\xi)v}\right)d\xi\\ &\quad\leq \frac{1}{2^s}\begin{pmatrix}\int_0^1\mathfrak{S}_*\left(\frac{\mu v}{(1-\xi)\mu+\xi v},\ \varphi\right)\nabla\left(\frac{\mu v}{\xi\mu+(1-\xi)v}\right)d\xi\\ +\int_0^1\mathfrak{S}_*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\nabla\left(\frac{\mu v}{\xi\mu+(1-\xi)v}\right)d\xi\end{pmatrix}\\ &\mathfrak{S}^*\left(\frac{2\mu\nu}{\mu+\nu},\ \varphi\right)\int_0^1\nabla\left(\frac{\mu v}{\xi\mu+(1-\xi)v}\right)d\xi\\ &\quad\leq \frac{1}{2^s}\begin{pmatrix}\int_0^1\mathfrak{S}^*\left(\frac{\mu v}{(1-\xi)\mu+\xi v},\ \varphi\right)\nabla\left(\frac{\mu v}{\xi\mu+(1-\xi)v}\right)d\xi\\ +\int_0^1\mathfrak{S}^*\left(\frac{\mu v}{\xi\mu+(1-\xi)v},\ \varphi\right)\nabla\left(\frac{\mu v}{\xi\mu+(1-\xi)v}\right)d\xi\end{pmatrix}\end{aligned} \tag{32}$$

since

$$\begin{aligned}
&\int_0^1 \mathfrak{S}_*\left(\frac{\mu\upsilon}{(1-\xi)\mu+\xi\upsilon},\ \varphi\right)\nabla\left(\frac{\mu\upsilon}{(1-\xi)\mu+\xi\upsilon}\right)d\xi \\
&\qquad = \int_0^1 \mathfrak{S}_*\left(\frac{\mu\upsilon}{\xi\mu+(1-\xi)\upsilon},\ \varphi\right)\nabla\left(\frac{\mu\upsilon}{\xi\mu+(1-\xi)\upsilon}\right)d\xi, \\
&\qquad = \frac{\mu\upsilon}{\nu-\mu}\int_\mu^\nu \mathfrak{S}_*(\imath,\varphi)\nabla(\imath)d\imath, \\
&\int_0^1 \mathfrak{S}^*\left(\frac{\mu\upsilon}{\xi\mu+(1-\xi)\upsilon},\ \varphi\right)\nabla\left(\frac{\mu\upsilon}{\xi\mu+(1-\xi)\upsilon}\right)d\xi \\
&\qquad = \int_0^1 \mathfrak{S}^*\left(\frac{\mu\upsilon}{(1-\xi)\mu+\xi\upsilon},\ \varphi\right)\nabla\left(\frac{\mu\upsilon}{(1-\xi)\mu+\xi\upsilon}\right)d\xi, \\
&\qquad = \frac{\mu\upsilon}{\nu-\mu}\int_\mu^\nu \mathfrak{S}^*(\imath,\varphi)\nabla(\imath)d\imath,
\end{aligned} \tag{33}$$

From (32) and (33), we have

$$\begin{aligned}
\mathfrak{S}_*\left(\frac{2\mu\nu}{\mu+\nu},\ \varphi\right) &\le \frac{2^{1-s}}{\int_\mu^\nu \nabla(\imath)d\imath}\int_\mu^\nu \mathfrak{S}_*(\imath,\varphi)\nabla(\imath)d\imath, \\
\mathfrak{S}^*\left(\frac{2\mu\nu}{\mu+\nu},\ \varphi\right) &\le \frac{2^{1-s}}{\int_\mu^\nu \nabla(\imath)d\imath}\int_\mu^\nu \mathfrak{S}^*(\imath,\varphi)\nabla(\imath)d\imath.
\end{aligned}$$

From which, we have

$$\begin{aligned}
&\left[\mathfrak{S}_*\left(\frac{2\mu\nu}{\mu+\nu},\ \varphi\right),\ \mathfrak{S}^*\left(\frac{2\mu\nu}{\mu+\nu},\ \varphi\right)\right] \\
&\qquad \le_I \frac{2^{1-s}}{\int_\mu^\nu \nabla(\imath)d\imath}\left[\int_\mu^\nu \mathfrak{S}_*(\imath,\varphi)\nabla(\imath)d\imath,\ \ \int_\mu^\nu \mathfrak{S}^*(\imath,\varphi)\nabla(\imath)d\imath\right],
\end{aligned}$$

that is

$$2^{s-1}\,\widetilde{\mathfrak{S}}\left(\frac{2\mu\nu}{\mu+\nu}\right)\int_\mu^\nu \frac{\widetilde{\mathfrak{S}}(\imath)}{\imath^2}d\imath \preccurlyeq (FR)\int_\mu^\nu \frac{\widetilde{\mathfrak{S}}(\imath)}{\imath^2}\nabla(\imath)d\imath$$

Then we complete the proof. □

Remark 4. *If $\nabla(\imath) = 1$, then from (25) and (30), we acquire the inequality (17).*

If $s = 1$, then from (25) and (30), we acquire the inequality for harmonically convex F-NV-Fs, see [39].

If $\mathfrak{S}_*(\mu,\varphi) = \mathfrak{S}^*(\mu,\ \varphi)$ with $\varphi = 1$ and $s = 1$, from (25) and (30), we acquire the inequality for a classical harmonically convex function.

4. Conclusions

We presented the idea of fuzzy number valued harmonically s-convex functions in this study. Some new fuzzy Hermite–Hadamard type integral inequalities are produced using this new class. A number of exceptional instances are thoroughly deduced. We also provide some instances to demonstrate the effectiveness and validity of our findings. These findings are novel in the literature, as far as we know. The class of fuzzy number valued harmonically s-convex functions has many uses in mathematics, including convex analysis, fuzzy theory, special functions, related optimization theory, and mathematical inequalities. These applications may encourage further study in a variety of fields of the pure and applied sciences.

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