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Explicit Properties of q-Cosine and q-Sine Array-Type Polynomials Containing Symmetric Structures

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Abstract: The main aim of this study is to define parametric kinds of λ-Array-type polynomials by using q-trigonometric polynomials and to investigate some of their analytical properties and applications. For this purpose, many formulas and relations for these polynomials, including some implicit summation formulas, differentiation rules, and relations with the earlier polynomials by utilizing some series manipulation method are derived. Additionally, as an application, the zero values of q-Array-type polynomials are presented by the tables and multifarious graphical representations for these zero values are drawn.

Keywords: q-calculus; q-array type polynomials; Stirling numbers of the second kind

MSC: 11B68; 11B73; 05A15; 05A19

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Citation: Alatawi, M.S.; Khan, W.A.; Ryoo, C.S. Explicit Properties of q-Cosine and q-Sine Array-Type Polynomials Containing Symmetric Structures. *Symmetry* **2022**, *14*, 1675. https://doi.org/10.3390/sym14081675

Academic Editor: Manuel Manas

Received: 24 July 2022
Accepted: 9 August 2022
Published: 12 August 2022

1. Introduction

Recently, many authors [1–4] have introduced and constructed generating functions for new families of special polynomials including two parametric kinds of polynomials, such as Bernoulli, Euler, and Genocchi. They have given the fundamental properties of these polynomials, and they have also established more identities and relations among trigonometric functions with two parametric kinds of special polynomials by using generating functions. Special polynomials have important roles in several subjects, such as mathematics, approximation theory, engineering, and theoretical physics. By applying the partial derivative operator to these generating functions, some derivative formulas and finite combinatorial sums involving the aforementioned polynomials and numbers are obtained. In addition, these special polynomials allow the derivation of different useful identities in a fairly straightforward way and help introduce new families of special polynomials. The array-type polynomials can be seen in combinatorial mathematics and play a crucial role in the principle and applications of arithmetic. Hence, a wide variety of idea and combinatorics experts have extensively studied their residences and received a series of exciting results (see [5–9]). By inspiring and motivating the above polynomials, in this study, we propose defining a parametric type of λ-array-type polynomials by introducing the two specific q-exponential generating functions. In addition, we show many formulations and family members for those polynomials, such as a few implicit summation formulas, differentiation policies, and correlations with the earlier polynomials with the aid of utilizing a collection manipulation approach.

The concern with q-calculus started in the 19th century due to its packages in various fields such as mathematics, physics, and engineering. The definitions and notations of q-calculus reviewed here are taken from [10,11].

The q-analogue of the shifted factorial $(\alpha)_\omega$ is given by

$$(\alpha;q)_0 = 1, (\alpha;q)_\omega = \prod_{\gamma=0}^{\omega-1}(1-q^\gamma\alpha) \ \ \omega \in \mathbb{N}.$$

The q-analogue of a complex number α and of the factorial function is given by

$$[\alpha]_q = \frac{1-q^\alpha}{1-q} \ \ q \in \mathbb{C}-\{1\}; \alpha \in \mathbb{C},$$

$$[\omega]_q! = \prod_{\gamma=1}^{\omega}[\gamma]_q = [1]_q[2]_q\cdots[\omega]_q = \frac{(q;q)_\omega}{(1-q)^\omega} \ \ q \neq 1; \omega \in \mathbb{N},$$

$$[0]_q! = 1, q \in \mathbb{C}; 0 < q < 1.$$

The Gauss q-binomial coefficient $\left(\begin{array}{c}\omega\\k\end{array}\right)_q$ is given by

$$\left(\begin{array}{c}\omega\\\gamma\end{array}\right)_q = \frac{[\omega]_q!}{[\gamma]_q![\omega-\gamma]_q!} = \frac{(q;q)_\omega}{(q;q)_\gamma(q;q)_{\omega-\gamma}} \ \ \gamma = 0, 1, \cdots, \omega.$$

The q-analogue of the function $(x+y)_q^\omega$ is given by

$$(x+y)_q^\omega = \sum_{\gamma=0}^{\omega}\left(\begin{array}{c}\omega\\\gamma\end{array}\right)_q q^{\gamma(\gamma-1)/2}x^{\omega-\gamma}y^\gamma \ \ \omega \in \mathbb{N}_0. \tag{1}$$

The q-analogues of exponential functions are given by

$$e_q(x) = \sum_{\omega=0}^{\infty}\frac{x^\omega}{[\omega]_q!} = \frac{1}{((1-q)x;q)_\infty} \ \ 0 <\mid q \mid< 1; \mid x \mid<\mid 1-q \mid^{-1}, \tag{2}$$

$$E_q(x) = \sum_{\omega=0}^{\infty}\frac{q^{\binom{\omega}{2}}}{[\omega]_q!}x^\omega = (-(1-q)x;q)_\infty \ \ 0 <\mid q \mid< 1; x \in \mathbb{C}. \tag{3}$$

These two functions are related by the following equation (see [10–12]):

$$e_q(x)E_q(-x) = 1.$$

Remark 1. *It is not difficult to see that [10]*

$$e_q(x) = \frac{1}{((1-q)x;q)_\infty}, \ \ 0 <\mid q \mid< 1, \ \ \mid x \mid< 1$$

$$E_q(x) = (-(1-q)x;q)_\infty, \ \ 0 <\mid q \mid< 1.$$

Definition 1. *Let x and y be two complex numbers and ω be a nonnegative integer. We define the q-addition in the following way (see [13]):*

$$(x \oplus_q y)^\omega = \sum_{\gamma}^{\gamma}\binom{\omega}{\gamma}_q x^\gamma y^{\omega-\gamma}. \tag{4}$$

The q-derivative operator is defined by

$$D_q f(z) = \frac{f(qz)-f(z)}{qz-z}, 0 <\mid q \mid< 1,$$

where $D_q f(0) = f'(0)$ provided that f is differentiable at $x = 0$.

The q-derivative fulfills the following product and quotient rules:

$$D_{q,z}(f(z)g(z)) = f(z)D_{q,z}g(z) + g(qz)D_{q,z}f(z), \tag{5}$$

$$D_{q,z}\left(\frac{f(z)}{g(z)}\right) = \frac{g(qz)D_{q,z}f(z) - f(qz)D_{q,z}g(z)}{g(z)g(qz)}. \tag{6}$$

Definition 2. *The q-trigonometric functions are*

$$sin_q(x) = \frac{e_q(ix) - e_q(-ix)}{2i}, \quad SIN_q(x) = \frac{E_q(ix) - E_q(-ix)}{2i},$$

and

$$cos_q(x) = \frac{e_q(ix) + e_q(-ix)}{2}, \quad COS_q(x) = \frac{E_q(ix) + E_q(-ix)}{2},$$

where $SIN_q(x) = sin_{q^{-1}}(x)$, $COS_q(x) = cos_{q^{-1}}(x)$.

Lemma 1. *Let $y \in \mathbb{R}$ and $i = \sqrt{-1} \in \mathbb{C}$. Then, we have*
(1)

$$E_q(ity) = COS_q(ty) + iSIN_q(ty)$$

(2)

$$E_q(-ity) = COS_q(ty) - iSIN_q(ty),$$

where $SIN_q(x) = sin_{q^{-1}}(x)$, $COS_q(x) = cos_{q^{-1}}(x)$.

Lemma 2. *Let $y \in \mathbb{R}$ and $i = \sqrt{-1} \in \mathbb{C}$. Then, we have*
(1)

$$e_q(tx)E_q(ity) = e_q(t(x \oplus iy)_q),$$

(2)

$$e_q(tx)E_q(-ity) = e_q(t(x \ominus iy)_q).$$

The Apostol-type q-Bernoulli polynomials $\mathbb{B}^{(\alpha)}_{\omega,q}(x;\lambda)$ of the order α, the Apostol-type q-Euler polynomials $\mathbb{E}^{(\alpha)}_{\omega,q}(x;\lambda)$ of the order α, and the Apostol-type q-Genocchi polynomials $\mathbb{G}^{(\alpha)}_{\omega,q}(x;\lambda)$ of the order α are defined as follows, respectively (see [14,15]):

$$\left(\frac{t}{\lambda e_q(t) - 1}\right)^{\alpha} e^{xt} = \sum_{\omega=0}^{\infty} \mathbb{B}^{(\alpha)}_{\omega,q}(x;\lambda)\frac{t^{\omega}}{[\omega]_q!} \quad (|\ t + \log\lambda\ |) < 2\pi, \tag{7}$$

$$\left(\frac{2}{\lambda e_q(t) + 1}\right)^{\alpha} e^{xt} = \sum_{\omega=0}^{\infty} \mathbb{E}^{(\alpha)}_{\omega,q}(x;\lambda)\frac{t^{\omega}}{[\omega]_q!} \quad (|\ t + \log\lambda\ |) < \pi, \tag{8}$$

$$\left(\frac{2t}{\lambda e_q(t) + 1}\right)^{\alpha} e^{xt} = \sum_{\omega=0}^{\infty} \mathbb{G}^{(\alpha)}_{\omega,q}(x;\lambda)\frac{t^{\omega}}{[\omega]_q!} \quad (|\ t + \log\lambda\ | < \pi), \tag{9}$$

Clearly, we have

$$\mathbb{B}^{(\alpha)}_{\omega,q}(\lambda) = \mathbb{B}^{(\alpha)}_{\omega,q}(0;\lambda), \mathbb{E}^{(\alpha)}_{\omega,q}(\lambda) = \mathbb{E}^{(\alpha)}_{\omega,q}(0;\lambda), \mathbb{G}^{(\alpha)}_{n,q}(\lambda) = \mathbb{G}^{(\alpha)}_{\omega,q}(0;\lambda). \tag{10}$$

Kang and Ryoo [13] introduced the q-Bernoulli and q-Euler polynomials, defined by the following respective equations:

$$\frac{t}{e_q(t) - 1} e_q(xt) COS_q(yt) = \sum_{j=0}^{\infty} \frac{\mathbb{B}_{j,q}((x \oplus iy)_q) + \mathbb{B}_j((x \ominus iy)_q)}{2} \frac{t^j}{[j]_q!} = \sum_{j=0}^{\infty} \mathbb{B}^{(C)}_{j,q}(x,y)\frac{t^j}{[j]_q!}, \tag{11}$$

$$\frac{t}{e_q(t)-1}e_q(xt)SIN_q(yt)=\sum_{j=0}^{\infty}\frac{\mathbb{B}_{j,q}((x\oplus iy)_q)-\mathbb{B}_{j,q}((x\ominus iy)_q)}{2i}\frac{t^j}{[j]_q!}=\sum_{j=0}^{\infty}\mathbb{B}_{j,q}^{(S)}(x,y)\frac{t^j}{[j]_q!}, \tag{12}$$

and

$$\frac{2}{e_q(t)+1}e_q(xt)COS_q(yt)=\sum_{j=0}^{\infty}\frac{\mathbb{E}_{j,q}((x\oplus iy)_q)+\mathbb{E}_{j,q}((x\ominus iy)_q)}{2}\frac{t^j}{[j]_q!}=\sum_{j=0}^{\infty}\mathbb{E}_{j,q}^{(C)}(x,y)\frac{t^j}{[j]_q!}, \tag{13}$$

$$\frac{2}{e_q(t)+1}e_q(xt)SIN_q(yt)=\sum_{j=0}^{\infty}\frac{\mathbb{E}_{j}((x\oplus iy)_q)-\mathbb{E}_{j}((x\ominus iy))_q}{2i}\frac{t^j}{[j]_q!}=\sum_{j=0}^{\infty}\mathbb{E}_{j,q}^{(S)}(x,y)\frac{t^j}{[j]_q!}, \tag{14}$$

Kang and Ryoo proved the following (see [13,16]):

$$e_q(xt)COS_q(yt)=\sum_{r=0}^{\infty}C_{r,q}(x,y)\frac{t^r}{[r]_q!}, \tag{15}$$

and

$$e_q(xt)SIN_q(yt)=\sum_{r=0}^{\infty}S_{r,q}(x,y)\frac{t^r}{[r]_q!}, \tag{16}$$

where

$$C_{r,q}(x,y)=\sum_{j=0}^{[\frac{r}{2}]}(-1)^j\binom{r}{2j}_q(-1)^jq^{2j-1}x^{r-2j}y^{2j}, \tag{17}$$

and

$$S_{r,q}(x,y)=\sum_{j=0}^{[\frac{r-1}{2}]}\binom{r}{2j+1}_q(-1)^jq^{(2j+1)j}x^{r-2j-1}y^{2j+1}. \tag{18}$$

For $\lambda\in\mathbb{C}$, the generalized λ-Stirling numbers of the second kind $S_m^n(\lambda)$ are given by the following (see [17,18]):

$$\frac{(\lambda e^t-1)^m}{m!}=\sum_{\omega=0}^{\infty}S_m^{\omega}(\lambda)\frac{t^{\omega}}{\omega!}\ \ m\in\mathbb{N}_0=\{0,1,2,\cdots,\}. \tag{19}$$

Given that $\lambda=1$, Equation (19) reduces to the Stirling numbers of the second kind as follows:

$$\frac{(e^t-1)^m}{m!}=\sum_{\omega=m}^{\infty}S_2(\omega,m)\frac{t^{\omega}}{\omega!}.$$

The λ-array-type polynomials $S_m^n(x,\lambda)$ are defined by the following (see [6]):

$$\frac{(\lambda e^t-1)^m}{m!}e^{xt}=\sum_{\omega=0}^{\infty}S_m^{\omega}(x,\lambda)\frac{t^{\omega}}{\omega!}. \tag{20}$$

2. λ-Array-Type Polynomials of Complex Variables

In this section, we consider the q-Cosine and q-Sine λ-array-type polynomials of complex variables and deduce some identities of these polynomials. First, we present the following definition:

$$\frac{(\lambda e_q(t)-1)^m}{m!}e_q(xt)E_q(ity)=\sum_{n=0}^{\infty}\mathbb{S}_{m,q}(n,(x+iy)_q,\lambda)\frac{t^n}{[n]_q!}. \tag{21}$$

On the other hand, we suppose that

$$e_q(xt)E_q(ity)=e_q(xt)(COS_q(yt)+iSIN_q(yt)). \tag{22}$$

Thus, by Equations (21) and (22), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbb{S}_{m,q}(n,(x+iy)_q,\lambda)\frac{t^n}{[n]_q!}&=\frac{(\lambda e_q(t)-1)^m}{m!}e_q(xt)E_q(ity)\\&=\frac{(\lambda e_q(t)-1)^m}{m!}e_q(xt)(COS_q(yz)+iSIN_q(yz)),\end{aligned} \tag{23}$$

and

$$\sum_{n=0}^{\infty}\mathbb{S}_{m,q}\big(n,(x-iy)_q,\lambda\big)\frac{t^n}{[n]_q!}=\frac{(\lambda e^t-1)^m}{m!}e_q(xt)E_q(-ity)$$
$$=\frac{(\lambda e^t-1)^m}{m!}e_q(xt)(COS_q(yt)-iSIN_q(yt)). \tag{24}$$

From Equations (23) and (24), we find

$$\frac{(\lambda e_q(t)-1)^m}{m!}e_q(xt)COS_q(yt)=\sum_{n=0}^{\infty}\left(\frac{\mathbb{S}_{m,q}\big(n,(x\oplus iy)_q,\lambda\big)+\mathbb{S}_{m,q}\big(n,(x\ominus iy)_q,\lambda\big)}{2}\right)\frac{t^n}{[n]_q!}, \tag{25}$$

and

$$\frac{(\lambda e^t-1)^m}{m!}e_q(xt)SIN_q(yt)=\sum_{n=0}^{\infty}\left(\frac{\mathbb{S}_{m,q}\big(n,(x\oplus iy)_q,\lambda\big)-\mathbb{S}_{m,q}\big(n,(x\ominus iy)_q,\lambda\big)}{2}\right)\frac{t^n}{[n]_q!}. \tag{26}$$

Definition 3. *Let $n\geq 0$. We define two parametric kinds of q-Cosine λ-array-type polynomials $\mathbb{S}^{(c)}_{m,q}(n,x,y,\lambda)$ and q-Sine λ-array-type polynomials $\mathbb{S}^{(s)}_{m,q}(n,x,y,\lambda)$, which for a nonnegative integer n are defined, respectively, by*

$$\frac{(\lambda e_q(t)-1)^m}{m!}e_q(xt)COS_q(yt)=\sum_{n=0}^{\infty}\mathbb{S}^{(c)}_{m,q}(n,x,y,\lambda)\frac{t^n}{[n]_q!}, \tag{27}$$

and

$$\frac{(\lambda e_q(t)-1)^m}{m!}e_q(xt)SIN_q(yt)=\sum_{n=0}^{\infty}\mathbb{S}^{(s)}_{m,q}(n,x,y,\lambda)\frac{t^n}{[n]_q!}, \tag{28}$$

Note that $\mathbb{S}^{(c)}_{m,q}(n,0,0,\lambda)=\mathbb{S}_{m,q}(n,\lambda),\ \ \mathbb{S}^{(s)}_{m,q}(n,0,0,\lambda)=0\ \ (n\geq 0)$.

From Equations (25)–(28), we have

$$\mathbb{S}^{(c)}_{m,q}(n,x,y,\lambda)=\frac{\mathbb{S}_{m,q}\big(n,(x\oplus iy)_q,\lambda\big)+\mathbb{S}_{m,q}\big(n,(x\ominus iy)_q,\lambda\big)}{2}, \tag{29}$$

$$\mathbb{S}^{(s)}_{m,q}(n,x,y,\lambda)=\frac{\mathbb{S}_{m,q}\big(n,(x\oplus iy)_q,\lambda\big)-\mathbb{S}_{m,q}\big(n,(x\ominus iy)_q,\lambda\big)}{2i}. \tag{30}$$

Remark 2. *For $x=0$ in Equations (27) and (28), we obtain a new type of q-Cosine λ-array-type polynomial $\mathbb{S}^{(c)}_{m,q}(n,y,\lambda)$ and q-Sine λ-array-type polynomial $\mathbb{S}^{(s)}_{m,q}(n,y,\lambda)$, respectively, as*

$$\frac{(\lambda e^t-1)^m}{m!}COS_q(yt)=\sum_{j=0}^{\infty}\mathbb{S}^{(c)}_{m,q}(n,y,\lambda)\frac{t^n}{[n]_q!}, \tag{31}$$

and

$$\frac{(\lambda e_q(t)-1)^m}{m!}SIN_q(yt)=\sum_{n=0}^{\infty}\mathbb{S}^{(s)}_{m,q}(n,y,\lambda)\frac{t^n}{[n]_q!}, \tag{32}$$

It is clear that

$$\mathbb{S}^{(c)}_{m,q}(n,0,\lambda)=\mathbb{S}_{m,q}(n,\lambda),\ \ \mathbb{S}^{(s)}_{m,q}(n,0,\lambda)=0,(n\geq 0).$$

Remark 3. *Letting $q\to 1$ in Definition 3, we find two parametric types of λ-array-type polynomials as follows (see [6]):*

$$\frac{(\lambda e^t-1)^m}{m!}e^{xt}COS(yt)=\sum_{n=0}^{\infty}\mathbb{S}^{(c)}_{m}(n,x,y,\lambda)\frac{t^n}{n!},$$

and

$$\frac{(\lambda e^t-1)^m}{m!}e^{xt}SIN(yt)=\sum_{n=0}^{\infty}\mathbb{S}^{(s)}_{m}(n,x,y,\lambda)\frac{t^n}{n!}$$

Now, we start with some basic properties of these polynomials.

Theorem 1. *If we let $n \geq 0$, then*

$$\mathbb{S}_{m,q}^{(c)}(n,y;u;\lambda)=\sum_{v=0}^{[\frac{n}{2}]}\left(\begin{array}{c} n+v \\ 2v \end{array}\right)_q(-1)^v q^{(2v-1)v}y^{2v}\mathbb{S}_{m,q}(n-2v,\lambda), \tag{33}$$

and

$$\mathbb{S}_{m,q}^{(s)}(n,\lambda)=\sum_{v=0}^{[\frac{n-1}{2}]}\left(\begin{array}{c} n+v \\ 2v+1 \end{array}\right)_q(-1)^v q^{(2v+1)v}y^{2v+1}\mathbb{S}_{m,q}(n-2v-1,\lambda). \tag{34}$$

Proof. By Equations (31) and (32), we can derive the following equations:

$$\sum_{n=0}^{\infty}\mathbb{S}_{m,q}^{(c)}(n,y,\lambda)\frac{t^n}{[n]_q!}=\frac{(\lambda e_q(t)-1)^m}{m!}COS_q(yt)$$

$$=\sum_{n=0}^{\infty}\mathbb{S}_{m,q}(n,\lambda)\frac{t^n}{[n]_q!}\sum_{v=0}^{\infty}(-1)^v q^{(2v-1)v}\eta^{2v}\frac{t^v}{[2v]_q!}.$$

$$=\sum_{n=0}^{\infty}\left(\sum_{v=0}^{[\frac{n}{2}]}\left(\begin{array}{c} n+v \\ 2v \end{array}\right)_q(-1)^v q^{(2v-1)v}\eta^{2v}\mathbb{S}_{m,q}(n-2v,\lambda)\right)\frac{t^n}{[n]_q!}, \tag{35}$$

and

$$\sum_{n=0}^{\infty}\mathbb{S}_{m,q}^{(s)}(n,y,\lambda)\frac{t^n}{[n]_q!}=\frac{(\lambda e^t-1)^m}{m!}SIN_q(yt)$$

$$=\sum_{n=0}^{\infty}\left(\sum_{v=0}^{[\frac{n-1}{2}]}\left(\begin{array}{c} n \\ 2v+1 \end{array}\right)_q(-1)^v q^{(2v+1)v}y^{2v+1}\mathbb{S}_{m,q}(j-2v-1,\lambda)\right)\frac{t^n}{[n]_q!}. \tag{36}$$

Therefore, by Equations (35) and (36), we find Equations (33) and (34). □

Theorem 2. *If we let $n \geq 0$, then*

$$\mathbb{S}_{m,q}\big(n,(x\oplus iy)_q,\lambda\big)=\sum_{k=0}^{n}\binom{n}{k}_q(x\oplus iy)_q^k\mathbb{S}_{m,q}(n-k,\lambda)$$

$$=\sum_{k=0}^{n}\binom{n}{k}_q(iy)^k\mathbb{S}_{m,q}(n-k,x,\lambda), \tag{37}$$

and

$$\mathbb{S}_{m,q}\big(n,(x\ominus iy)_q,\lambda\big)=\sum_{k0}^{n}\binom{n}{k}_q(x\ominus iy)_q^k\mathbb{S}_{m,q}(n-k,\lambda)$$

$$=\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k(iy)^k\mathbb{S}_{m,q}(n-k,x,\lambda). \tag{38}$$

Proof. By using Equations (23) and (24), we obtained Equations (37) and (38), so we omitted the proof. □

Theorem 3. *If we let $n \geq 0$, then*

$$\mathbb{S}_{m,q}^{(c)}(n,x,y,\lambda)=\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)_q\mathbb{S}_{m,q}(k,\lambda)C_{n-k,q}(x,y), \tag{39}$$

and

$$\mathbb{S}_{m,q}^{(s)}(n,x,y,\lambda)=\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)_q\mathbb{S}_{m,q}(k,\lambda)S_{n-k,q}(x,y). \tag{40}$$

Proof. Consider the following:

$$\left(\sum_{n=0}^{\infty}a_n\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}b_k\frac{t^k}{k!}\right)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{j}a_{n-k}b_k\right)\frac{t^n}{n!}.$$

Now, we have

$$\sum_{n=0}^{\infty} \mathbb{S}_{m,q}^{(c)}(n,x,y,\lambda)\frac{t^n}{[n]_q!} = \frac{(\lambda e_q(t)-1)^m}{m!}e_q(xt)COS_q(yt)$$

$$= \left(\sum_{k=0}^{\infty} \mathbb{S}_{m,q}(k,\lambda)\frac{t^k}{[k]_q!}\right)\left(\sum_{n=0}^{\infty} C_{n,q}(x,y)\frac{t^n}{[n]_q!}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)_q \mathbb{H}_{m,q}(k,\lambda)C_{n-k,q}(x,y)\right)\frac{t^n}{[n]_q!},$$

which proves Equation (39). The proof of Equation (40) is similar. □

Theorem 4. *If we let $n \geq 0$, then*

$$\mathbb{S}_{m,q}^{(c)}(n,x+s,y,\lambda) = \sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)_q \mathbb{S}_{m,q}^{(c)}(k,x,y,\lambda)r^{n-k}, \quad (41)$$

and

$$\mathbb{S}_{m,q}^{(s)}(n,x,y,\lambda) = \sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)_q \mathbb{S}_{m,q}^{(s)}(k,x,y,\lambda)r^{n-k}. \quad (42)$$

Proof. By changing x with $x+r$ in Equation (27), we have

$$\sum_{n=0}^{\infty} \mathbb{S}_{m,q}^{(c)}(n,x+s,y,\lambda)\frac{t^n}{[n]_q!} = \frac{(\lambda e_q(t)-1)^m}{m!}e_q(xt)COS_q(yt)$$

$$= \left(\sum_{n=0}^{\infty} \mathbb{S}_{m,q}^{(c)}(n,x,y,\lambda)\frac{t^n}{[n]_q!}\right)\left(\sum_{k=0}^{\infty} r^k\frac{t^k}{[k]_q!}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)_q \mathbb{S}_{m,q}^{(c)}(k,x,y,\lambda)r^{n-k}\right)\frac{t^n}{[n]_q!},$$

which completes the proof for Equation (41). The result for Equation (42) can be proven in a similar manner. □

Theorem 5. *If we let $n \geq 1$, then*

$$\frac{\partial}{\partial x}\mathbb{S}_{m,q}^{(c)}(n,x,y,\lambda) = [n]_q\mathbb{S}_{m,q}^{(c)}(n-1,x,y,\lambda), \quad (43)$$

$$\frac{\partial}{\partial y}\mathbb{S}_{m,q}^{(c)}(n,x,y,\lambda) = -[n]_q\mathbb{S}_{m,q}^{(s)}(n-1,x,qy,\lambda), \quad (44)$$

and

$$\frac{\partial}{\partial x}\mathbb{S}_{m,q}^{(s)}(n,x,y,\lambda) = [n]_q\mathbb{S}_{m,q}^{(s)}(n-1,x,y,\lambda), \quad (45)$$

$$\frac{\partial}{\partial y}\mathbb{S}_{m,q}^{(s)}(n,x,y,\lambda) = [n]_q\mathbb{S}_{m,q}^{(c)}(n-1,x,qy,\lambda). \quad (46)$$

Proof. Equation (27) yields

$$\sum_{n=1}^{\infty}\frac{\partial}{\partial x}\mathbb{S}_{m,q}^{(c)}(n,x,y,\lambda)\frac{t^n}{[n]_q!} = \frac{(\lambda e_q(t)-1)^m}{m!}\frac{\partial}{\partial x}e_q(xt)COS_q(yt) = \sum_{n=0}^{\infty}\mathbb{S}_{m,q}^{(c)}(n,x,y,\lambda)\frac{t^{n+1}}{[n]_q!}$$

$$= \sum_{n=1}^{\infty}\mathbb{S}_{m,q}^{(c)}(n-1,x,y,\lambda)\frac{t^n}{[(n-1)]_q!} = \sum_{n=1}^{\infty}[n]_q\mathbb{S}_{m,q}^{(c)}(n-1,x,y,\lambda)\frac{t^n}{[n]_q!},$$

which proves Equation (43). Equations (44)–(46) can be similarly derived. □

Theorem 6. *If we let $N \in \mathbb{N}^*$, then the following formula holds true:*

$$\mathbb{S}_{m,q}^{(c)}(2x,y,\lambda) = \sum_{k=0}^{n}\binom{n}{k}_q \mathbb{S}_{m,q}^{(c)}(n,x,y,\lambda)x^{n-k}, \quad (47)$$

and

$$\mathbb{S}_{m,q}^{(s)}(2x,y,\lambda)=\sum_{k=0}^{n}\binom{n}{k}_q\mathbb{S}_{m,q}^{(s)}(n,x,y,\lambda)x^{n-k}. \tag{48}$$

Proof. By using Definition 3, we can easily prove Equations (47) and (48). Therefore, we omitted the proof. □

Theorem 7. *The following relation holds true:*

$$\mathbb{S}_{m,q}^{(c)}(n,x+1,y,\lambda)=(m+1)\mathbb{S}_{m,q}^{(c)}(n,x+1,y,\lambda)+\mathbb{S}_{m,q}^{(c)}(n,x,y,\lambda), \tag{49}$$

and

$$\mathbb{S}_{m,q}^{(s)}(n,x+1,y,\lambda)=(m+1)\mathbb{S}_{m,q}^{(s)}(n,x+1,y,\lambda)+\mathbb{S}_{m,q}^{(s)}(n,x,y,\lambda). \tag{50}$$

Proof. By using Definition 3, we can easily obtain

$$\begin{aligned}\lambda\sum_{n=0}^{\infty}\mathbb{S}_{m,q}^{(c)}(n,x+1,y,\lambda)\frac{t^n}{[n]_q!}&=\lambda\frac{(\lambda e_q(t)-1)^m}{m!}e_q((x+1)t)COS_q(yt)\\&=\frac{(\lambda e_q(t)-1)^m}{m!}e_q(xt)COS_q(yt)\left[\lambda e_q(t)-1+1\right]\\&=(m+1)\frac{(\lambda e_q(t)-1)^{m+1}}{(m+1)!}e_q(xt)Cos_q(yt)+\frac{(\lambda e_q(t)-1)^m}{m!}e_q(xt)COS_q(yt)\\&=(m+1)\sum_{n=0}^{\infty}\mathbb{S}_{m,q}^{(c)}(n,x+1,y,\lambda)\frac{t^n}{[n]_q!}+\sum_{n=0}^{\infty}\mathbb{S}_{m,q}^{(c)}(n,x,y,\lambda)\frac{t^n}{[n]_q!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$ on both sides of the last equality leads to the desired identity in Equation (49). The relation in Equation (50) follows easily in a similar way. □

Theorem 8. *The following summation formulas are true:*

$$\sum_{m=0}^{n}\binom{n}{m}_q C_{n-m,q}(x,y)=\sum_{k=0}^{n}\binom{n}{k}_q B_{n-k,q}^{(m)}(\lambda)\mathbb{S}_{m,q}^{(c)}(k,x,y,\lambda), \tag{51}$$

and

$$\sum_{m=0}^{n}\binom{n}{m}_q S_{n-m,q}(x,y)=\sum_{k=0}^{n}\binom{n}{k}_q B_{n-k,q}^{(m)}(\lambda)\mathbb{S}_{m,q}^{(s)}(k,x,y,\lambda). \tag{52}$$

Proof. Consider the following equality:

$$\left(\frac{t}{\lambda e_q(t)-1}\right)^m\frac{(\lambda e_q(t)-1)^m}{m!}e_q(xt)COS_q(yt)=\frac{t^m}{m!}e_q(xt)COS_q(yt).$$

By making use of Equation (7) for $x=0$, through Equations (7) and (27), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}B_{n,q}^{(m)}(\lambda)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}S_{m,q}^{(c)}(k,x,y,\lambda)\frac{t^k}{[k]_q!}&=\frac{t^m}{m!}\sum_{n=0}^{\infty}C_{n,q}(x,y)\frac{t^n}{[n]_q!}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}_q B_{n-k,q}^{(m)}(\lambda)\mathbb{S}_{m,q}^{(c)}(k,x,y,\lambda)\frac{t^n}{[n]_q!}\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}_q C_{n-m,q}(x,y)\frac{t^n}{[n]_q!}.\end{aligned} \tag{53}$$

Now, if we compare the coefficients of t^n on both sides of Equation (53), we reach the formula in Equation (51). The relation in Equation (52) can be derived in a similar manner. □

Theorem 9. *Let x, y, and r be any real numbers. Then, we have*
(1)

$$\mathbb{S}_{m,q}^{(c)}(n,(x\oplus r)_q,y,\lambda)+\mathbb{S}_{m,q}^{(s)}(n,(x\ominus r)_q,y,\lambda)$$

$$= \sum_{k=0}^{n} \binom{n}{l}_q q^{\binom{n-l}{2}} r^{n-l} \Big(\mathbb{S}_{m,q}^{(c)}(n,x,y,\lambda) + (-1)^{n-k} \mathbb{S}_{m,q}^{(s)}(n,x,y,\lambda) \Big), \tag{54}$$

(2)

$$\mathbb{S}_{m,q}^{(s)}(n,(x \oplus r)_q,y,\lambda) + \mathbb{S}_{m,q}^{(c)}(n,(x \ominus r)_q,y,\lambda)$$
$$= \sum_{k=0}^{n} \binom{n}{l}_q q^{\binom{n-l}{2}} r^{n-l} \Big(\mathbb{S}_{m,q}^{(s)}(n,x,y,\lambda) + (-1)^{n-k} \mathbb{S}_{m,q}^{(c)}(n,x,y,\lambda) \Big). \tag{55}$$

Proof. By substituting $(\xi \oplus r)_q$ into ξ in the generating function of q-Cosine array-type polynomials, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \mathbb{S}_{m,q}^{(c)}(n,(x \oplus r)_q,y,\lambda) \frac{t^n}{[n]_q!} &= \frac{(\lambda e_q(t)-1)^m}{m!} e_q(t(x \oplus r)_q) COS_q(yt) \\ &= \frac{(\lambda e_q(t)-1)^m}{m!} e_q(tx) COS_q(yt) E_q(tr) \\ &= \sum_{n=0}^{\infty} \mathbb{S}_{m,q}^{(c)}(n,x,y,\lambda) \frac{t^n}{[n]_q!} \sum_{l=0}^{\infty} q^{\binom{l}{2}} r^l \frac{t^l}{[l]_q!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l}_q \mathbb{S}_{m,q}^{(c)}(n-l,x,y,\lambda) q^{\binom{l}{2}} r^l \right) \frac{t^n}{[n]_q!}. \end{aligned} \tag{56}$$

Through a similar method, we can find the following equation:

$$\begin{aligned} \sum_{n=0}^{\infty} \mathbb{S}_{m,q}^{(s)}(n,(x \ominus r)_q,y,\lambda) \frac{t^n}{[n]_q!} &= \frac{(\lambda e_q(t)-1)^m}{m!} e_q(t(x \ominus r)_q) SIN_q(yt) \\ & \frac{(\lambda e_q(t)-1)^m}{m!} e_q(tx) SIN_q(yt) E_q(-tr) \\ &= \sum_{n=0}^{\infty} \mathbb{S}_{m,q}^{(s)}(n,x,y,\lambda) \frac{t^n}{[n]_q!} \sum_{l=0}^{\infty} q^{\binom{l}{2}} (-1)^l r^l \frac{t^l}{[l]_q!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{j}{l}_q \mathbb{S}_{m,q}^{(s)}(n-l,x,y,\lambda) q^{\binom{l}{2}} (-1)^l r^l \right) \frac{t^n}{[n]_q!}. \end{aligned} \tag{57}$$

By adding Equations (56) with (57), we can derive result (1) of Theorem 9.
For results (2) in Theorem 9, we also can find the following equations:

$$\begin{aligned} \sum_{n=0}^{\infty} \mathbb{S}_{m,q}^{(s)}(n,(x \oplus r)_q,y,\lambda) \frac{t^n}{[n]_q!} &= \frac{(\lambda e_q(t)-1)^m}{m!} e_q(t(x \oplus r)_q) SIN_q(yt) \\ &= \frac{(\lambda e_q(t)-1)^m}{m!} e_q(tx) SIN_q(yt) E_q(tr), \end{aligned} \tag{58}$$

$$\begin{aligned} \sum_{n=0}^{\infty} \mathbb{S}_{m,q}^{(c)}(n,(x \ominus r)_q,y,\lambda) \frac{t^n}{[n]_q!} &= \frac{(\lambda e_q(t)-1)^m}{m!} e_q(t(x \ominus r)_q) COS_q(yt) \\ & \frac{(\lambda e_q(t)-1)^m}{m!} e_q(tx) COS_q(yt) E_q(-tr). \end{aligned} \tag{59}$$

Using Equation (59) appropriately, we can find result (2) in Theorem 9. □

Corollary 1. *If we let $n \geq 0$, then*

$$\mathbb{S}_{m,q}^{(c)}\big(n,(x \oplus r)_q,y,\lambda\big) + \mathbb{S}_{m,q}^{(c)}\big(n,(x \ominus r)_q,y,\lambda\big)$$
$$= \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}} r^k \Big(\mathbb{S}_{m,q}^{(c)}(n-k,x,y,\lambda) + (-1)^k \mathbb{S}_{m,q}^{(c)}(n-k,x,y,\lambda) \Big), \tag{60}$$

and

$$\mathbb{S}_{m,q}^{(s)}\big(n,(x \oplus r)_q,y,\lambda\big) + \mathbb{S}_{m,q}^{(s)}\big(n,(x \ominus r)_q,y,\lambda\big)$$
$$= \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}} r^k \Big(\mathbb{S}_{m,q}^{(s)}(n-k,x,y,\lambda) + (-1)^k \mathbb{S}_{m,q}^{(s)}(n-k,x,y,\lambda) \Big). \tag{61}$$

Corollary 2. *For $r = 1$ in Corollary 1, we have*

$$\begin{aligned}&\mathbb{S}_{m,q}^{(c)}\big(n,(x\oplus 1)_q,y,\lambda\big)+\mathbb{S}_{m,q}^{(s)}\big(n,(x\ominus 1)_q,y,\lambda\big)\\&=\sum_{k=0}^{n}\binom{n}{k}_q q^{\binom{k}{2}}r^k\Big(\mathbb{S}_{m,q}^{(c)}(n-k,x,y,\lambda)+(-1)^k\mathbb{S}_{m,q}^{(s)}(n-k,x,y,\lambda)\Big),\end{aligned} \tag{62}$$

and

$$\begin{aligned}&\mathbb{S}_{m,q}^{(s)}\big(n,(x\oplus 1)_q,y,\lambda\big)+\mathbb{S}_{m,q}^{(c)}\big(n,(x\ominus 1)_q,y,\lambda\big)\\&=\sum_{k=0}^{n}\binom{n}{k}_q q^{\binom{n-k}{2}}r^{n-k}\Big(\mathbb{S}_{m,q}^{(s)}(k,x,y,\lambda)+(-1)^{n-k}\mathbb{S}_{m,q}^{(c)}(k,x,y,\lambda)\Big).\end{aligned} \tag{63}$$

3. Symmetric Structures of the Approximate Roots for q-Cosine λ-Array-Type Polynomials and Their Application

In this section, certain zeros of the q-Cosine λ-array-type polynomials $\mathbb{S}_{m,q}^{(c)}(n,x,y,\lambda)$ and beautiful graphical representations are shown. Let $m = 5$.

A few of examples of these include

$$\mathbb{S}_{5,q}^{(c)}(0,x,y,\lambda)=\frac{1}{120}(-1+\lambda)^5,$$

$$\mathbb{S}_{5,q}^{(c)}(1,x,y,\lambda)=-\frac{x}{120}+\frac{\lambda}{24}+\frac{x\lambda}{24}-\frac{\lambda^2}{6}-\frac{x\lambda^2}{12}+\frac{\lambda^3}{4}+\frac{x\lambda^3}{12}-\frac{\lambda^4}{6}-\frac{x\lambda^4}{24}+\frac{\lambda^5}{24}+\frac{x\lambda^5}{120},$$

$$\begin{aligned}\mathbb{S}_{5,q}^{(c)}(2,x,y,\lambda)=&-\frac{x^2}{120}+\frac{qy^2}{120}+\frac{\lambda}{24}+\frac{x^2\lambda}{24}-\frac{qy^2\lambda}{24}-\frac{\lambda^2}{6}-\frac{x^2\lambda^2}{12}+\frac{qy^2\lambda^2}{12}\\&+\frac{\lambda^3}{4}+\frac{x^2\lambda^3}{12}-\frac{qy^2\lambda^3}{12}-\frac{\lambda^4}{6}-\frac{x^2\lambda^4}{24}+\frac{qy^2\lambda^4}{24}+\frac{\lambda^5}{24}+\frac{x^2\lambda^5}{120}-\frac{qy^2\lambda^5}{120}\\&+\frac{x\lambda[2]_q!}{24}-\frac{\lambda^2[2]_q!}{12}-\frac{x\lambda^2[2]_q!}{6}+\frac{\lambda^3[2]_q!}{4}+\frac{x\lambda^3[2]_q!}{4}-\frac{\lambda^4[2]_q!}{4}\\&-\frac{x\lambda^4[2]_q!}{6}+\frac{\lambda^5[2]_q!}{12}+\frac{x\lambda^5[2]_q!}{24},\end{aligned}$$

$$\begin{aligned}\mathbb{S}_{5,q}^{(c)}(3,x,y,\lambda)=&-\frac{x^3}{120}+\frac{\lambda}{24}+\frac{x^3\lambda}{24}-\frac{\lambda^2}{6}-\frac{x^3\lambda^2}{12}+\frac{\lambda^3}{4}+\frac{x^3\lambda^3}{12}-\frac{\lambda^4}{6}-\frac{x^3\lambda^4}{24}\\&+\frac{\lambda^5}{24}+\frac{x^3\lambda^5}{120}-\frac{x\lambda^2[3]_q!}{12}+\frac{\lambda^3[3]_q!}{12}+\frac{x\lambda^3[3]_q!}{4}-\frac{\lambda^4[3]_q!}{6}\\&-\frac{x\lambda^4[3]_q!}{4}+\frac{\lambda^5[3]_q!}{12}+\frac{x\lambda^5[3]_q!}{12}+\frac{qxy^2[3]_q!}{120[2]_q!}+\frac{x\lambda[3]_q!}{24[2]_q!}+\frac{x^2\lambda[3]_q!}{24[2]_q!}-\frac{qy^2\lambda[3]_q!}{24[2]_q!}\\&-\frac{qxy^2\lambda[3]_q!}{24[2]_q!}-\frac{\lambda^2[3]_q!}{6[2]_q!}-\frac{x\lambda^2[3]_q!}{6[2]_q!}-\frac{x^2\lambda^2[3]_q!}{6[2]_q!}+\frac{qy^2\lambda^2[3]_q!}{6[2]_q!}+\frac{qxy^2\lambda^2[3]_q!}{12[2]_q!}+\frac{\lambda^3[3]_q!}{2[2]_q!}\\&+\frac{x\lambda^3[3]_q!}{4[2]_q!}+\frac{x^2\lambda^3[3]_q!}{4[2]_q!}-\frac{qy^2\lambda^3[3]_q!}{4[2]_q!}-\frac{qxy^2\lambda^3[3]_q!}{12[2]_q!}-\frac{\lambda^4[3]_q!}{2[2]_q!}-\frac{x\lambda^4[3]_q!}{6[2]_q!}-\frac{x^2\lambda^4[3]_q!}{6[2]_q!}\\&+\frac{qy^2\lambda^4[3]_q!}{6[2]_q!}+\frac{qxy^2\lambda^4[3]_q!}{24[2]_q!}+\frac{\lambda^5[3]_q!}{6[2]_q!}+\frac{x\lambda^5[3]_q!}{24[2]_q!}+\frac{x^2\lambda^5[3]_q!}{24[2]_q!}-\frac{qy^2\lambda^5[3]_q!}{24[2]_q!}-\frac{qxy^2\lambda^5[3]_q!}{120[2]_q!}.\end{aligned}$$

We investigated the beautiful zeros of the q-Cosine λ-array-type polynomials $\mathbb{S}_{m,q}^{(c)}(n,x,y,\lambda)$ by using a computer. We plotted the zeros of the q-Cosine λ-array-type polynomials $\mathbb{S}_{m,q}^{(c)}(n,x,y,\lambda)=0$ for $n = 40$ (Figure 1).

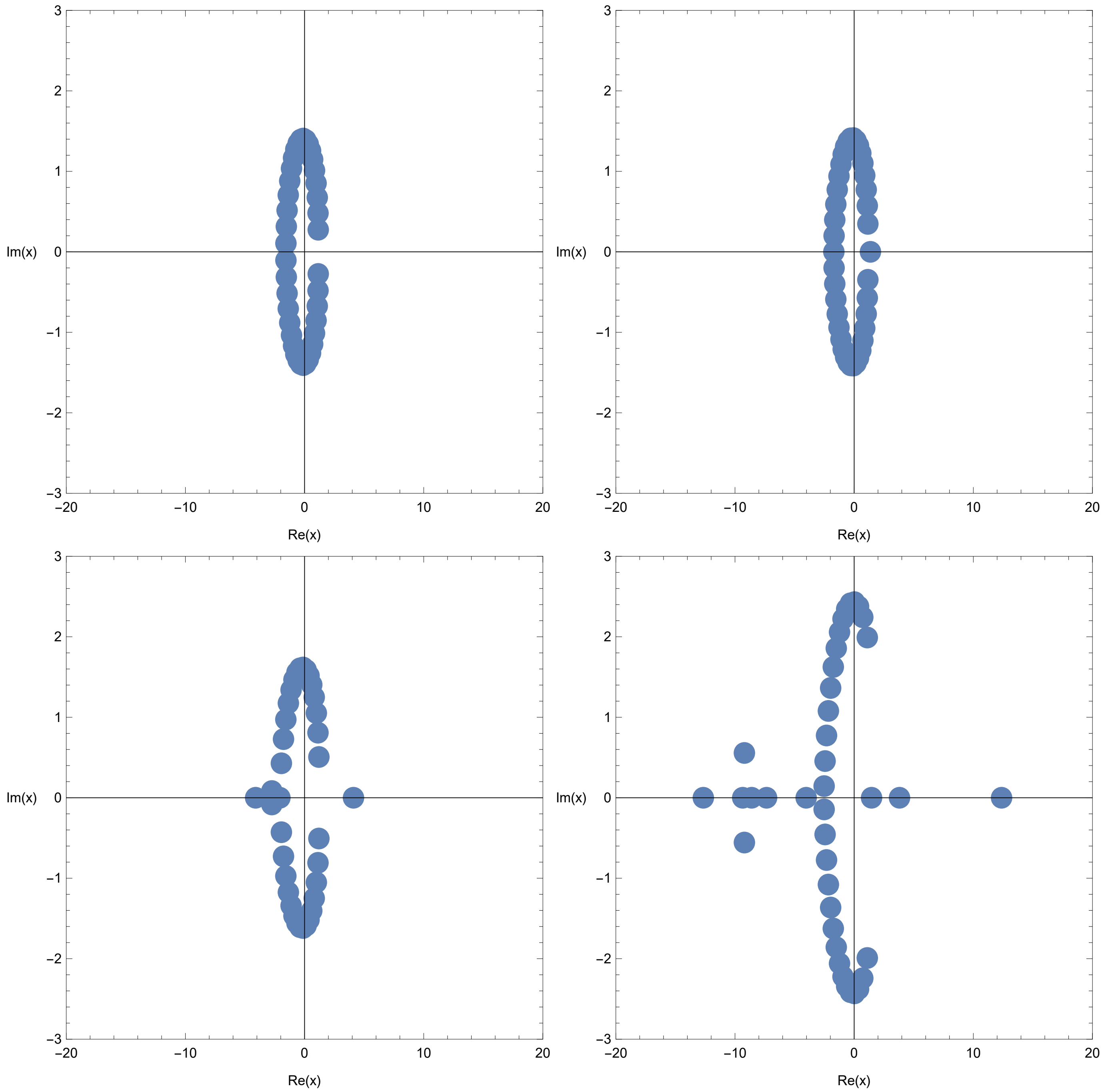

Figure 1. Zeros of $\mathbb{S}_{m,q}^{(c)}(n,x,y,\lambda)$.

In Figure 1 (top left), we chose $\lambda = 3, m = 5, y = 2$, and $q = \frac{1}{10}$. In Figure 1 (top right), we chose $\lambda = 3, m = 5, y = 2$, and $q = \frac{3}{10}$. In Figure 1 (bottom left), we chose $\lambda = 3, m = 5, y = 2$, and $q = \frac{7}{10}$. In Figure 1 (bottom right), we chose $\lambda = 3, m = 5, y = 2$, and $q = \frac{9}{10}$.

The stacks of zeros of the q-Cosine λ-array-type polynomials $\mathbb{S}_{m,q}^{(c)}(n,x,y,\lambda) = 0$ for $1 \leq n \leq 40$, forming a 3D structure, are presented in Figure 2.

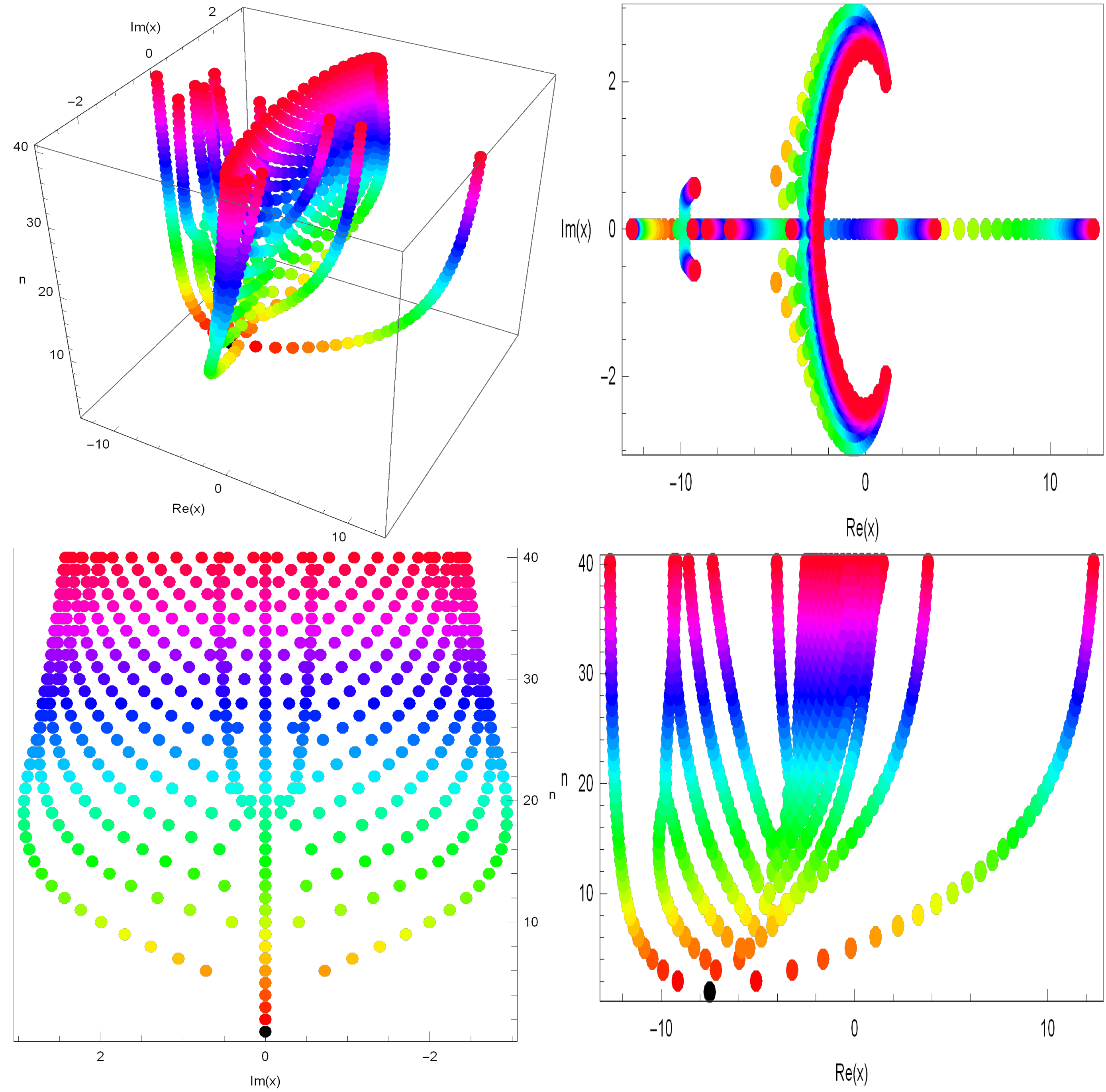

Figure 2. Zeros of $\mathbb{S}^{(c)}_{m,q}(n, x, y, \lambda)$.

In Figure 2 (top left), we plotted the stacks of zeros of the q-Cosine λ-array-type polynomials $\mathbb{S}^{(c)}_{m,q}(n, x, y, \lambda) = 0$ for $1 \leq n \leq 40$, $q = \frac{9}{10}$, $\lambda = 3$, $m = 5$, and $y = 2$. In Figure 2 (top right), we drew the x and y axes but no z axis of the three dimensions. In Figure 2 (bottom left), we drew the y and z axes but no x axis of the three dimensions. In Figure 2 (bottom right), we drew the x and z axes but no y axis of the three dimensions.

We plotted the real zeros of the q-Cosine λ-array-type polynomials $\mathbb{S}^{(c)}_{m,q}(n, x, y, \lambda) = 0$ for $1 \leq n \leq 40$ (Figure 3).

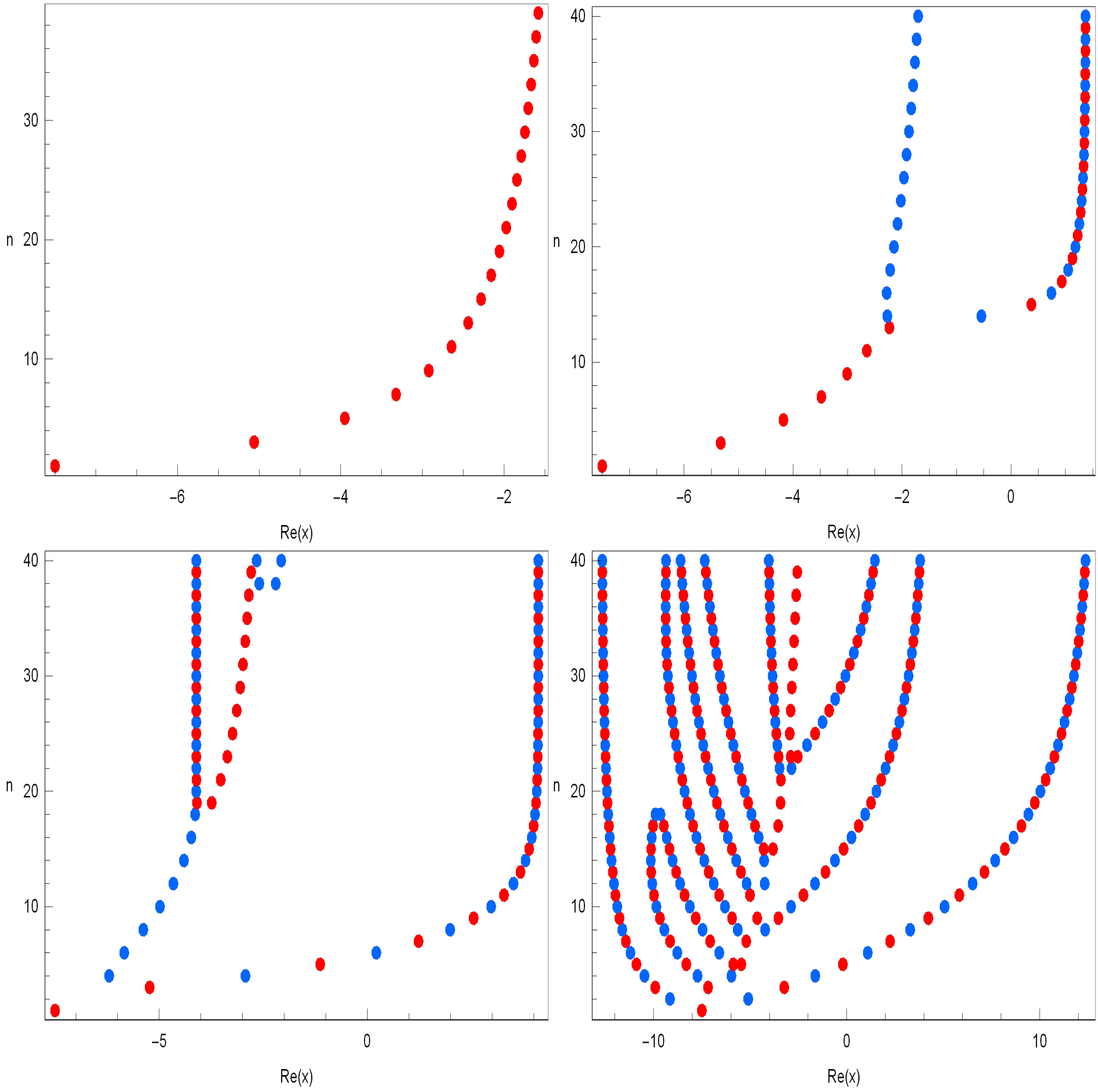

Figure 3. Real zeros of $\mathbb{S}_{m,q}^{(c)}(n,x,y,\lambda)$.

In Figure 3 (top left), we chose $\lambda = 3, m = 5, y = 2$, and $q = \frac{1}{10}$. In Figure 3 (top right), we chose $\lambda = 3, m = 5, y = 2$, and $q = \frac{3}{10}$. In Figure 3 (bottom left), we chose $\lambda = 3, m = 5, y = 2$, and $q = \frac{7}{10}$. In Figure 3 (bottom right), we chose $\lambda = 3, m = 5, y = 2$, and $q = \frac{9}{10}$.

Next, we calculated an approximate solution satisfying the q-Cosine λ-array-type polynomials $\mathbb{S}_{m,q}^{(c)}(n,x,y,\lambda) = 0$ for $q = \frac{9}{10}$. The results are given in Table 1.

Table 1. Approximate solutions of $S^{(c)}_{5,q}(n,x,2,3)=0, x\in\mathbb{R}$.

Degree *n*	***x***
1	−7.5000
2	−9.1537, −5.0963
3	−9.9132, −7.1792, −3.2326
4	−10.473, −7.7237, −5.9691, −1.6272
5	−10.887, −8.3057, −5.8697, −5.4538, −0.19684
6	−11.200, −8.7716, −6.5985 , 1.0889
7	−11.438, −9.1440, −7.0712, −5.2023 , 2.2467
8	−11.622, −9.4417, −7.4646, −5.6413, −4.2220, 3.2895
9	−11.766, −9.6751, −7.8177, −5.9369, −4.6328, −3.5384, 4.2287
10	−11.881, −9.8531, −8.1242, −6.3032, −2.8841, 5.0745

4. Symmetric Structures of the Approximate Roots for *q*-Sine *λ*-Array-Type Polynomials and Their Application

In this section, certain zeros of q-Sine-λ-array-type polynomials $\mathbb{S}^{(s)}_{m,q}(n,x,y,\lambda)$ and beautiful graphical representations are shown. Let $m=5$.

A few of these include the following:

$$\mathbb{S}^{(s)}_{5,q}(0,x,y,\lambda)=0,$$
$$\mathbb{S}^{(s)}_{5,q}(1,x,y,\lambda)=-\frac{y}{120}+\frac{y\lambda}{24}-\frac{y\lambda^2}{12}+\frac{y\lambda^3}{12}-\frac{y\lambda^4}{24}+\frac{y\lambda^5}{120},$$

$$\begin{aligned}\mathbb{S}^{(s)}_{5,q}(2,x,y,\lambda)=&-\frac{xy[2]_q!}{120}+\frac{y\lambda[2]_q!}{24}+\frac{xy\lambda[2]_q!}{24}-\frac{y\lambda^2[2]_q!}{6}-\frac{xy\lambda^2[2]_q!}{12}\\&+\frac{y\lambda^3[2]_q!}{4}+\frac{xy\lambda^3[2]_q!}{12}-\frac{y\lambda^4[2]_q!}{6}-\frac{xy\lambda^4[2]_q!}{24}+\frac{y\lambda^5[2]_q!}{24}+\frac{xy\lambda^5[2]_q!}{120},\\ \mathbb{S}^{(s)}_{5,q}(3,x,y,\lambda)=&\frac{q^3y^3}{120}-\frac{q^3y^3\lambda}{24}+\frac{q^3y^3\lambda^2}{12}-\frac{q^3y^3\lambda^3}{12}+\frac{q^3y^3\lambda^4}{24}-\frac{q^3y^3\lambda^5}{120}\\&+\frac{xy\lambda[3]_q!}{24}-\frac{y\lambda^2[3]_q!}{12}-\frac{xy\lambda^2[3]_q!}{6}+\frac{y\lambda^3[3]_q!}{4}+\frac{xy\lambda^3[3]_q!}{4}-\frac{y\lambda^4[3]_q!}{4}\\&-\frac{xy\lambda^4[3]_q!}{6}+\frac{y\lambda^5[3]_q!}{12}+\frac{xy\lambda^5[3]_q!}{24}-\frac{x^2y[3]_q!}{120[2]_q!}+\frac{y\lambda[3]_q!}{24[2]_q!}+\frac{x^2y\lambda[3]_q!}{24[2]_q!}\\&-\frac{y\lambda^2[3]_q!}{6[2]_q!}-\frac{x^2y\lambda^2[3]_q!}{12[2]_q!}+\frac{y\lambda^3[3]_q!}{4[2]_q!}+\frac{x^2y\lambda^3[3]_q!}{12[2]_q!}-\frac{y\lambda^4[3]_q!}{6[2]_q!}-\frac{x^2y\lambda^4[3]_q!}{24[2]_q!}\\&+\frac{y\lambda^5[3]_q!}{24[2]_q!}+\frac{x^2y\lambda^5[3]_q!}{120[2]_q!}.\end{aligned}$$

We investigated the beautiful zeros of the q-Sine λ-array-type polynomials $\mathbb{S}^{(s)}_{m,q}(n,x,y,\lambda)$ by using a computer. We plotted the zeros of the q-Cosine λ-array-type polynomials $\mathbb{S}^{(s)}_{m,q}(n,x,y,\lambda)=0$ for $n=40$ (Figure 4).

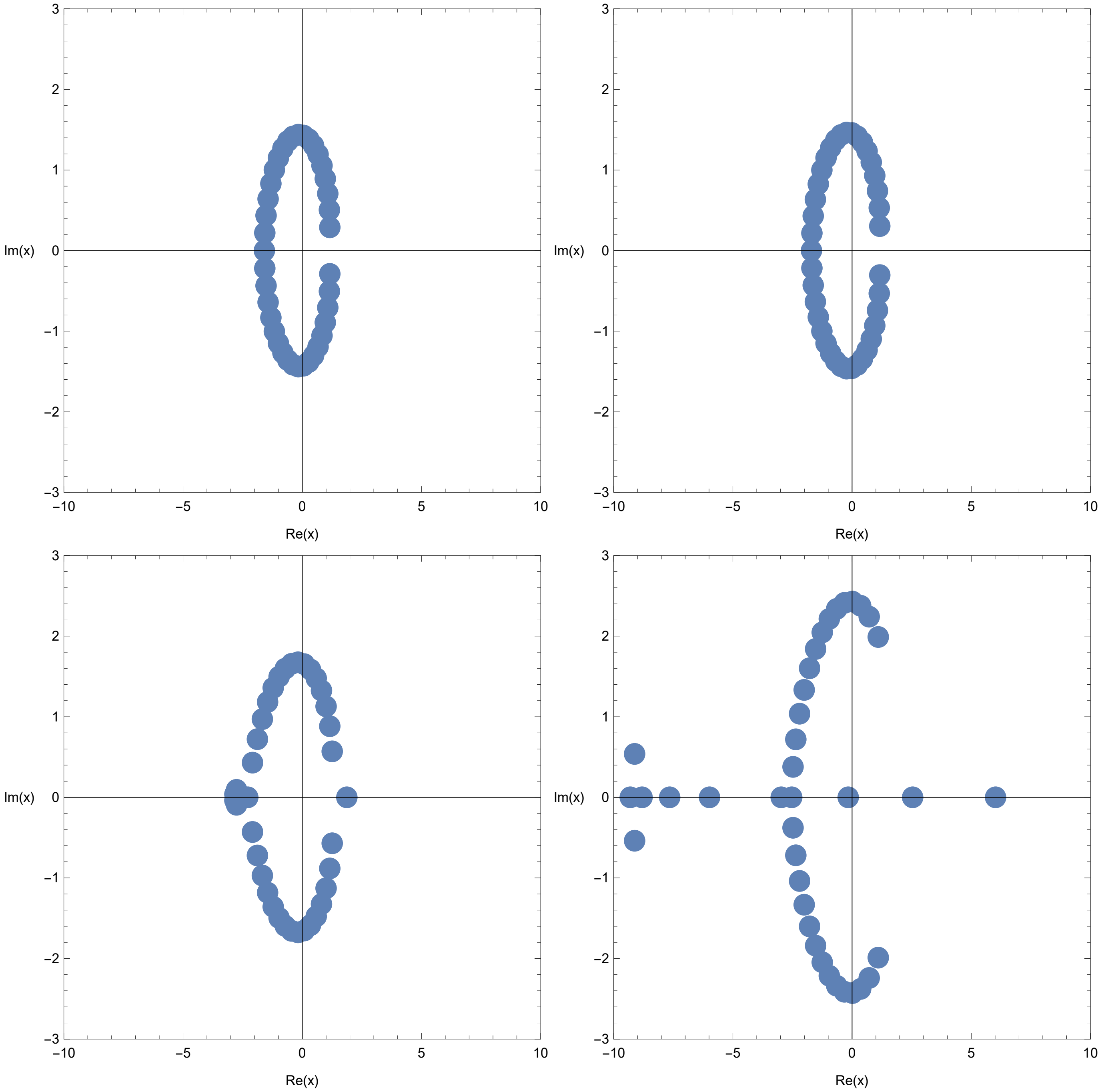

Figure 4. Zeros of $\mathbb{S}^{(s)}_{m,q}(n,x,y,\lambda)$.

In Figure 4 (top left), we chose $\lambda = 3, m = 5, y = 2$, and $q = \frac{1}{10}$. In Figure 4 (top right), we chose $\lambda = 3, m = 5, y = 2$, and $q = \frac{3}{10}$. In Figure 4 (bottom left), we chose $\lambda = 3, m = 5, y = 2$, and $q = \frac{7}{10}$. In Figure 4 (bottom right), we chose $\lambda = 3, m = 5, y = 2$, and $q = \frac{9}{10}$.

The stacks of zeros of the q-Sine λ-array-type polynomials $\mathbb{S}^{(s)}_{m,q}(n,x,y,\lambda) = 0$ for $1 \leq n \leq 40$, forming a 3D structure, are presented in Figure 5.

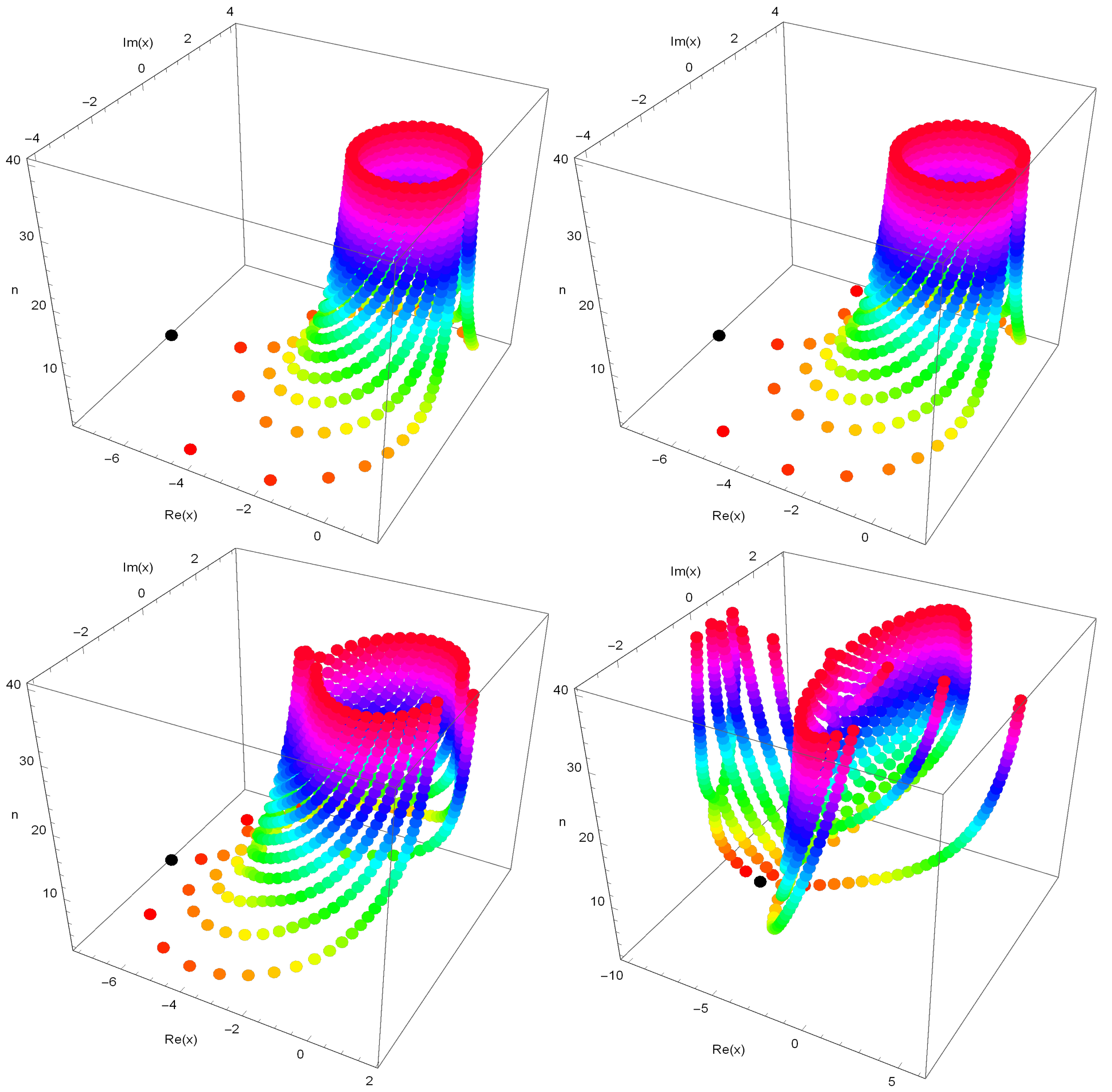

Figure 5. Zeros of $\mathbb{S}_{m,q}^{(s)}(n,x,y,\lambda)$.

In Figure 5 (top left), we chose $\lambda = 3, m = 5, y = 2$, and $q = \frac{1}{10}$. In Figure 5 (top right), we chose $\lambda = 3, m = 5, y = 2$, and $q = \frac{3}{10}$. In Figure 5 (bottom left), we chose $\lambda = 3, m = 5, y = 2$, and $q = \frac{7}{10}$. In Figure 5 (bottom right), we chose $\lambda = 3, m = 5, y = 2$, and $q = \frac{9}{10}$.

Next, we calculated an approximate solution satisfying the q-Sine λ-array-type polynomials $\mathbb{S}_{m,q}^{(s)}(n,x,y,\lambda) = 0$ for $q = \frac{9}{10}$. The results are given in Table 2.

Table 2. Approximate solutions of $S^{(c)}_{5,q}(n, x, 2, 3) = 0, x \in \mathbb{R}$.

Degree n	x
2	−7.5000
3	−8.3866, −5.8634
4	−8.9328, −6.6699, −4.7223
5	−9.3479, −7.0555, −5.5906, −3.7985
6	−9.6424, −7.5919, −2.9357
7	−9.8507, −7.9870, −6.1091 , −2.1017
8	−9.9922, −8.3262, −6.4196, −5.1020, −1.3181
9	−10.081, −8.6115, −6.7843, −0.59552
10	−10.127, −8.8539, −7.0942, −5.4156 , 0.065459
11	−10.137, −9.0620, −7.3707, −5.6846, −4.4208, 0.66754
12	−10.115, −9.2441, −7.6166, −5.9572, 1.2145

5. Conclusions

In this paper, using the q-Cosine polynomials and q-Sine polynomials, we introduced novel types of q-extensions of λ-array-type polynomials, and the features obtained multifarious homes and identities by using some collection manipulation techniques. Furthermore, we computed the q-quintessential representations and q-derivative operator policies for those polynomials. Moreover, we determined the approximate root movements of the brand new mentioned polynomials in a complicated plane, utilizing the Newton technique and illustrating them in figures. The shape of the approximate roots will pop out in diverse ways, depending on the circumstances of the variables, and there is a desire for new methods and theorems associated with this subject matter to be created and proven. We would like to continue to observe this line of study in the future.

Author Contributions: Conceptualization, W.A.K., C.S.R.; Formal analysis, M.S.A.; Funding acquisition, M.S.A.; Investigation, W.A.K.; Methodology, W.A.K., C.S.R. and M.S.A.; Project administration, C.S.R.; Software, W.A.K. and C.S.R.; Writing—original draft, W.A.K.; Writing—review & editing, W.A.K. and M.S.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No data were used to support this study.

Conflicts of Interest: The authors declare that they have no competing interests.

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