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Statistical Convergence of Δ-Spaces Using Fractional Order

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Abstract: The notion of fractional structures has been studied intensely in various fields. Using this concept, the main idea of this paper is to apply the Cesàro approach and introduce the new generalized Δ-structure of spaces on a fractional level. Also, the statistical notions will be studied using this new structure and some inclusion relations will be computed. In addition, the sequence space $\mathcal{W}_q(\Delta^{\kappa}_{\mathfrak{g}}, \mathfrak{f})$ will be introduced, and some fundamental inclusion relations and topological properties concerning it will be given.

Keywords: gamma function; convergence; fractional operator

1. Introduction

The condition of sequence convergence in analysis demands that almost all points from the sequence satisfy the convergence condition. For instance, in classical convergence, almost all elements of the sequence have to belong to an arbitrarily small neighborhood of the limit point. The main idea of statistical convergence is to relax this condition and demand validity of the convergence condition only for a majority of the points. Thus, statistical convergence shows a relaxing atmosphere on conventional convergence. The basic scenario of this convergence of a sequence l lies in the fact that most of the members of l converge and one does not worry about what is going on with other members. Early on, the idea of statistical convergence, which emerged in the first edition (published in Warsaw in 1935) of the monograph of Zygmund [1], stemmed not from statistics, but from problems of series summation. Formally the notion of this convergence was observed by Steinhaus [2] and Fast [3] and later by Schoenberg [4] and since then this field of study has become an active research area. Authors in different fields have shown its significance. For example, statistical convergence is studied in fields such as measure theory [5], trigonometric series, approximation theory [6], locally convex spaces [7], finitely additive set functions, Banach spaces [8], and so on [9–15].

Later on, the concept of statistical convergence and strong Cesàro summability were investigated from the sequence space point of view and linked with the summability theory by Akbas and Isik [16], Altin et al. [17], Aral and Et [18], Çinar et al. [19–21], Connor et al. [8], Dutta and Rhoades [22], Esi et al. [23], Et et al. [24–27], Ganie et al. [28–33], Mursaleen et al. [34–38], Schoenberg [4], and Sheikh et al. [39]. Several others connected the same structures to the summability applications.

The behavior of statistical convergence is analyzed via the density of subsets $\mathcal{B}$ of counting numbers and its natural density is defined as:

$$\delta(\mathcal{B}) = \lim_{r\to\infty} \frac{1}{r}\Big|\{i \le r : i \in \mathcal{B}\}\Big|.$$

Note that the number of entries of $\mathcal{B}$ that are not more than r is $\Big|\{i \le r : i \in \mathcal{B}\}\Big|$. Furthermore, for finite $\mathcal{B}$, $\delta(\mathcal{B}^c) = 1 - \delta(\mathcal{B})$. We consider (v_i) to be statistically convergent to L if for all $\epsilon > 0$,

$$\delta(\{i : |v_i - L| \ge \epsilon\}) = 0.$$

It will be written as $S - \lim v_i = L$, where S represents the set of all sequences that are statistically convergent.

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We consider a sequence (v_i) to be strongly Cesàro summable to λ if:

$$\lim_{r\to\infty}\frac{1}{r}\sum_{i=1}^{r}\left|v_i-\lambda\right|=0.$$

The set of sequences that is strongly Cesàro summable is denoted by $[C,\lambda]$ and is given as:

$$[C,\lambda]=\left\{v=(v_i):\lim_{r\to\infty}\frac{1}{r}\sum_{i=1}^{r}\left|v_i-\lambda\right|=0\text{ for some }\lambda\right\}.$$

The study of difference sequence spaces is a recent development in summability theory. As in [40], for $\mathcal{T}\in\{\ell_\infty,c,\mathfrak{C}_0\}$, define:

$$\mathcal{T}(\triangle)=\{v=(v_i)\in\Lambda:(\triangle v_i)\in\mathcal{T}\}$$

where $\triangle v_i=v_i-v_{i-1}$. This $\mathcal{T}(\triangle)$ was further studied in [25,29,32,41] and by many others.

It was later generalized in [26,28], where the authors defined the following:

$$\Delta^j(\mathcal{H})=\left\{v=(v_j):\left(\Delta^j v\right)\in\mathcal{H}\right\},\ for\ \mathcal{H}=\ell_\infty,c\ and\ \mathfrak{C}_0,$$

where $\Delta^j v_i=\Delta^{j-1}v_i-\Delta^{j-1}v_{i+1}$ for all $i\in\mathbf{N}$ where $j\geq 0$ is natural number and:

$$\Delta^j=\sum_{i=0}^{j}(-1)^i\binom{j}{i}v_{j+i}.$$

Later on Et and Esi [25] generalized these sequence spaces to the following sequence spaces. Let $g=(g_k)$ be any fixed sequence of nonzero complex numbers and let s be a non-negative integer. Then, $\Delta_g^0 v=(v_j g_j)$, $\Delta_g v=(v_j g_j-v_{j+1}g_{j+1})$, and:

$$\Delta_g^s(\mathcal{H})=\left\{v=(v_j)\in\Lambda:(\triangle_g^s v_j)\in\mathcal{H}\right\},\tag{1}$$

where $\mathcal{H}$ is any sequence space and:

$$\triangle_g^s v_j=\Delta_g^{s-1}v_j-\Delta_g^{s-1}v_{j+1}=\sum_{\mu=0}^{s}(-1)^\mu\binom{s}{\mu}g_{j+\mu}v_{j+\mu}\ \forall\ j\in\mathbb{N}.$$

For a real κ, let $\Gamma(\kappa)$ represent the Euler Gamma function with $\kappa\notin\{0,-1,-2,-3,\dots\}$ and given by:

$$\Gamma(\kappa)=\int_0^\infty t^{\kappa-1}e^{-t}dt.$$

In [6], the fractional operator $\Delta^\kappa:\Lambda\to\Lambda$ is defined as:

$$\Delta^\kappa v_k=\sum_{i=0}^{\infty}(-1)^i\frac{\Gamma(\kappa+1)}{i!\Gamma(\kappa-i+1)}v_{k+i}\tag{2}$$

This notion is more general than the Δ^m operator. Note that we assume that (2) holds throughout the paper. For κ to be a natural number, the sum in (2) can be written as a finite sum:

$$\sum_{i=0}^{\kappa}(-1)^i\frac{\Gamma(\kappa+1)}{i!\Gamma(\kappa-i+1)}v_{k+i}.\tag{3}$$

It was further studied in [6,7] and by many others.

2. Main Results

In this section, we introduce and study $\Delta_{\mathfrak{g}}^{\kappa}$-statistical convergence and the strong $(p, \Delta_{\mathfrak{g}}^{\kappa})$-Cesàro summability, for $\mathfrak{g} = (\mathfrak{g}_i)$ with $\mathfrak{g}_i \neq 0$ for all i. Furthermore, some new topological properties will be given.

Following the authors cited, we introduce the following fractional order difference spaces:

$$\begin{aligned} \ell_\infty(\Gamma, \Delta_{\mathfrak{g}}^{\kappa}, p) &= \{v = (v_k) \in \Lambda : sup_k |\Delta_{\mathfrak{g}}^{\kappa} v_k|^{p_k} < \infty\}, \\ c_0(\Gamma, \Delta_{\mathfrak{g}}^{\kappa}, p) &= \{v = (v_k) \in \Lambda : \lim_{k\to\infty} |\Delta_{\mathfrak{g}}^{\kappa} v_k|^{p_k} = 0\}, \\ c(\Gamma, \Delta_{\mathfrak{g}}^{\kappa}, p) &= \{v = (v_k) \in \Lambda : \lim_{k\to\infty} |\Delta_{\mathfrak{g}}^{\kappa} v_k - \lambda|^{p_k} = 0 \text{ for some } \lambda \in \mathbb{C}\} \end{aligned}$$

where $\Delta_{\mathfrak{g}}^{\kappa}$ is given in (1) and $p = (p_k)$ is a bounded sequence of positive reals.

Definition 1. *For a complex number λ, we call a sequence (v_k) to be $\Delta_{\mathfrak{g}}^{\kappa}$-statistically convergent if:*

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \le n : \left|\Delta_{\mathfrak{g}}^{\kappa} v_k - \lambda\right| \ge \epsilon \right\} \right| = 0.$$

In such a case, v is $\Delta_{\mathfrak{g}}^{\kappa}$-statistically convergent to λ and denoted by $S(\Delta_{\mathfrak{g}}^{\kappa}) - \lim v_k = \lambda$. By $S(\Delta_{g}^{\kappa})$, we designate all sequences that are Δ_{g}^{κ}-statistically convergent.

Theorem 1. *For sequences $v = (v_k)$, $w = (w_k)$ of real or complex numbers, we have:*

1. *If $S(\Delta_{\mathfrak{g}}^{\kappa}) - \lim v_k = v_0$ and $\mathfrak{c}$ is any complex number, then $S(\Delta_{\mathfrak{g}}^{\kappa}) - \lim \mathfrak{c} v_k = \mathfrak{c} v_0$.*
2. *If $S(\Delta_{\mathfrak{g}}^{\kappa}) - \lim v_k = v_0$ and $S(\Delta_{\mathfrak{g}}^{\kappa}) - \lim w_k = w_0$, then $S(\Delta_{\mathfrak{g}}^{\kappa}) - \lim(v_k + w_k) = v_0 + w_0$.*

Proof. (i) The result is trivial for $\mathfrak{c} = 0$, so we assume that $\mathfrak{c} \neq 0$, then:

$$\frac{1}{n} \left| \left\{ k \le n : \left|\Delta_{\mathfrak{g}}^{\kappa} \mathfrak{c} v_k - \mathfrak{c} v_0\right| \ge \epsilon \right\} \right| \le \frac{1}{n} \left| \left\{ k \le n : \left|\Delta_{\mathfrak{g}}^{\kappa} v_k - v_0\right| \ge \frac{\epsilon}{|\mathfrak{c}|} \right\} \right|,$$

thereby proving (i) of the result.

(ii) Now we see:

$$\begin{aligned} &\frac{1}{n} \left| \left\{ k \le n : \left|\Delta_{\mathfrak{g}}^{\kappa} (v_k + w_k) - (v_0 + w_0)\right| \ge \epsilon \right\} \right| \\ &\quad \le \frac{1}{n} \left| \left\{ k \le n : \left|\Delta_{\mathfrak{g}}^{\kappa} v_k - v_0\right| \ge \frac{\epsilon}{2} \right\} \right| + \frac{1}{n} \left| \left\{ k \le n : \left|\Delta_{\mathfrak{g}}^{\kappa} w_k - w_0\right| \ge \frac{\epsilon}{2} \right\} \right|. \end{aligned}$$

and hence result follows. □

Theorem 2. *The inclusion $c(\Gamma, \Delta_{\mathfrak{g}}^{\kappa}, p) \subset S(\Delta_{\mathfrak{g}}^{\kappa})$ is proper for $p_k = 1$.*

Proof. As c is subset of S, it follows that $c(\Gamma, \Delta_{\mathfrak{g}}^{\kappa}, p) \subset S(\Delta_{\mathfrak{g}}^{\kappa})$.

Next, we prove the proper containment part of the result. For this, we choose $\mathfrak{g}_k = e = (1, 1, \cdots)$ and define $v = (v_m)$ by:

$$\Delta_{\mathfrak{g}}^{\kappa} v_m = \begin{cases} 1, & m = r^3 \\ -1, & m + 1 = r^3 \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Then, we have $v \in S(\Delta_{\mathfrak{g}}^{\kappa})$, but $v \notin c(\Gamma, \Delta_{\mathfrak{g}}^{\kappa}, p)$, thereby proving the containment is proper. □

Theorem 3. *If $v = (v_m)$ is $\Delta_{\mathfrak{g}}^{\kappa}$-statistically convergent, then it is $\Delta_{\mathfrak{g}}^{\kappa}$-statistically Cauchy sequence.*

Proof. We suppose that v is $\Delta_{\mathfrak{g}}^{\kappa}$-statistically convergent to λ and $\epsilon > 0$. Then,

$$|\Delta_{\mathfrak{g}}^{\kappa} v_m - L| < \epsilon/2 \text{ for almost all } m.$$

We choose r in such a way that:

$$|\Delta_{\mathfrak{g}}^{\kappa} v_r - L| < \epsilon/2$$

holds. Then it is clear that:

$$|\Delta_{\mathfrak{g}}^{\kappa} v_m - \Delta_{\mathfrak{g}}^{\kappa} v_r| < |\Delta_{\mathfrak{g}}^{\kappa} v_m - L| + |\Delta_{\mathfrak{g}}^{\kappa} v_r - L| < \epsilon$$

for almost all m. Hence, we conclude that v is $\Delta_{\mathfrak{g}}^{\kappa}$-statistical Cauchy sequence. □

Theorem 4. *Neither $S(\Delta_{\mathfrak{g}}^{\kappa})$ nor $\ell_\infty(\Gamma, \Delta_{\mathfrak{g}}^{\kappa}, p)$ is included the other although $S(\Delta_{\mathfrak{g}}^{\kappa})$ and $\ell_\infty(\Gamma, \Delta_{\mathfrak{g}}^{\kappa}, p)$ overlap, for $p_k = 1$.*

Proof. Define $v = (v_m)$ as follows:

$$\Delta_{\mathfrak{g}}^{\kappa} v_m = \begin{cases} 1, & m = r^3 \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

It is clear that $v \in S(\Delta_{\mathfrak{g}}^{\kappa})$ but $v \notin \ell_\infty(\Gamma, \Delta_{\mathfrak{g}}^{\kappa}, p)$. Now consider:

$$v = (1, 0, 1, 0, 1, \dots) \text{ and } \mathfrak{g} = e = (1, 1, 1, \cdots),$$

then $\Delta_{\mathfrak{g}}^{\kappa} v_m = (-1)^m \, 2^{\kappa - 1}$ and $v \in \ell_\infty(\Gamma, \Delta_{\mathfrak{g}}^{\kappa}, p)$ but $v \notin S(\Delta_{\mathfrak{g}}^{\kappa})$. □

Theorem 5. *$S \cap S(\Delta_{\mathfrak{g}}^{\kappa}) \neq \emptyset$.*

Proof. Define $v = \mathfrak{g} = e$. As $v \in S$ and $\Delta_{\mathfrak{g}}^{\kappa} v_m = 0$, so $v \in S(\Delta_{\mathfrak{g}}^{\kappa})$, the intersection is nonempty. □

Definition 2. *For a positive real p, we call a sequence $v = (v_k)$ strongly $(p, \Delta_{\mathfrak{g}}^{\kappa})$-Cesàro summable if:*

$$\lim_{r \to \infty} \frac{1}{n} \sum_{i=1}^{r} \left| \Delta_{g}^{\kappa} v_i - \mathfrak{c} \right|^p = 0.$$

In such a case, v is strongly $(p, \Delta_{\mathfrak{g}}^{\kappa})$-Cesàro summable to $\mathfrak{c}$, and such sequences that are strongly $(p, \Delta_{\mathfrak{g}}^{\kappa})$-Cesàro summable will be abbreviated by $\mathcal{W}_p(\Delta_{\mathfrak{g}}^{\kappa})$, for a real or complex number $\mathfrak{c}$.

Theorem 6. *The inclusion $\mathcal{W}_q(\Delta_{\mathfrak{g}}^{\kappa}) \subset \mathcal{W}_p(\Delta_{\mathfrak{g}}^{\kappa})$ holds provided $0 < p < q < \infty$.*

The proof is trivial using Hölder's inequality.

Theorem 7. *Let $v = (v_i)$ be strongly $(p, \Delta_{\mathfrak{g}}^{\kappa})$-Cesàro summable to λ, then for $0 < p < \infty$, it is $\Delta_{\mathfrak{g}}^{\kappa}$-statistically convergent to $\mathfrak{c}$.*

Proof. Choose $v = (v_i)$ and $\epsilon > 0$, we see:

$$\sum_{i=1}^{r} \left|\Delta_{\mathfrak{g}}^{\kappa} v_i - \mathfrak{c}\right|^p = \sum_{\substack{i=1 \\ |v_i - L| \geq \epsilon}}^{r} \left|\Delta_{\mathfrak{g}}^{\kappa} v_i - \mathfrak{c}\right|^p + \sum_{\substack{i=1 \\ |v_i - \mathfrak{c}| < \epsilon}}^{r} \left|\Delta_{\mathfrak{g}}^{\kappa} v_i - \mathfrak{c}\right|^p$$
$$\geq \sum_{i=1}^{r} \left|\Delta_{\mathfrak{g}}^{\kappa} v_i - \mathfrak{c}\right|^p$$
$$\geq \left|\left\{ i \leq r : |\Delta_{\mathfrak{g}}^{\kappa} v_i - 1| \geq \epsilon \right\}\right| \cdot \epsilon^p$$

and so:

$$\frac{1}{r}\sum_{i=1}^{r} \left|\Delta_{\mathfrak{g}}^{\kappa} v_i - \mathfrak{c}\right|^p \geq \frac{1}{r}\left|\left\{ i \leq r : |\Delta_{\mathfrak{g}}^{\kappa} v_i - \mathfrak{c}| \geq \epsilon \right\}\right| \cdot \epsilon^p.$$

From this, if $v = (v_i)$ is $(p, \Delta_{\mathfrak{g}}^{\kappa})$-Cesàro summable to $\mathfrak{c}$, then it is $\Delta_{\mathfrak{g}}^{\kappa}$-statistically convergent to $\mathfrak{c}$. □

Corollary 1. *Let $v = (v_i)$ be $\Delta_{\mathfrak{g}}^{\kappa}$-bounded and $\Delta_{\mathfrak{g}}^{\kappa}$-statistically convergent to $\mathfrak{c}$, then it also strongly $(p, \Delta_{\mathfrak{g}}^{\kappa})$-Cesàro summable to $\mathfrak{c}$.*

3. New Statistical Convergence Using Modulus Function

In this section, we introduce some new scenarios of spaces by employing modulus functions.

In [42], modulus functions $\mathfrak{f} : [0, \infty) \to [0, \infty)$ are introduced as functions that satisfy the following properties:

1. $\mathfrak{f}(v) = 0$ iff $v = 0$,
2. $\mathfrak{f}(v + w) \leq \mathfrak{f}(v) + \mathfrak{f}(w)$ for all $v, w \geq 0$,
3. $\mathfrak{f}$ is a continuous function from the right at 0,
4. $\mathfrak{f}$ is increasing function.

Definition 3. *For a sequence of positive reals $p = (p_i)$, we define the following spaces:*

$$\mathcal{W}_p(\Delta_{\mathfrak{g}}^{\kappa}, \mathfrak{f}) = \left\{ v = (v_k) \in \Lambda : \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \left[\mathfrak{f}\left(\left|\Delta_{\mathfrak{g}}^{\kappa} v_k - \mathfrak{c}\right|\right)\right]^{p_k} = 0 \right\},$$

for $\mathfrak{f}$ a modulus function and $0 < h = \inf_k p_k \leq \sup_k p_k = H < \infty$.

Theorem 8. *The inclusion $\mathcal{W}_p(\Delta_{\mathfrak{g}}^{\kappa}, \mathfrak{f}) \subset S(\Delta_{\mathfrak{g}}^{\kappa})$ is proper for any modulus function $\mathfrak{f}$.*

Proof. For a modulus function $\mathfrak{f}$, let $v \in \mathcal{W}_p(\Delta_{\mathfrak{g}}^{\kappa}, \mathfrak{f})$ and choose $\epsilon > 0$. $\sum_1$ and $\sum_2$ over $i \leq r$ with $|\Delta_{\mathfrak{g}}^{\kappa} v_i - \mathfrak{c}| \geq \epsilon$ and $|\Delta_{\mathfrak{g}}^{\kappa} v_i - \mathfrak{c}| < \epsilon$, respectively. Then:

$$\frac{1}{r}\sum_{i=1}^{r} \left[\mathfrak{f}\left(\left|\Delta_{\mathfrak{g}}^{\kappa} v_i - 1\right|\right)\right]^{p_i} = \frac{1}{r}\sum_{1} \left[\mathfrak{f}\left(\left|\Delta_{\mathfrak{g}}^{\kappa} v_i - \mathfrak{c}\right|\right)\right]^{p_i}$$
$$\geq \frac{1}{r}\sum_{1} [\mathfrak{f}(\epsilon)]^{p_i}$$
$$\geq \frac{1}{r}\sum_{1} min\left([\mathfrak{f}(\epsilon)]^h, [\mathfrak{f}(\epsilon)]^H\right)$$
$$\geq \frac{1}{n}\left|\left\{ i \leq r : |\Delta_{\mathfrak{g}}^{\kappa} v_i - \mathfrak{c}| \geq \epsilon \right\}\right| min\left([\mathfrak{f}(\epsilon)]^h, [\mathfrak{f}(\epsilon)]^H\right).$$

Hence, $v \in S(\Delta_{\mathfrak{g}}^{\kappa})$. To establish proper containment, choose $\mathfrak{g} = e = (1, 1, \cdots)$ and define the sequence $v = (v_i)$ as:

$$\Delta_{\mathfrak{g}}^{\kappa} v_i = \begin{cases} 1/\sqrt{i}, & i \neq m^3 \\ 1, & i = m^3. \end{cases} \tag{6}$$

It is obvious that $v \in S(\Delta_{\mathfrak{g}}^{\kappa}) - \mathcal{W}_p(\Delta_{\mathfrak{g}}^{\kappa}, \mathfrak{f})$ when $p_i = 1$ and $\mathfrak{f}(v) = v$ is unbounded. □

Theorem 9. *$S(\Delta_{\mathfrak{g}}^{\kappa}) \subset \mathcal{W}_p(\Delta_{\mathfrak{g}}^{\kappa}, \mathfrak{f})$ where $\mathfrak{f}$ is bounded.*

Proof. For $\epsilon > 0$, we choose $\sum_1$ and $\sum_2$ as defined in previous theorem. If there exists an integer $\mathcal{M}$ with $\mathfrak{f}(v) < \mathcal{M}$, for every $v > 0$, then $\mathfrak{f}$ is bounded and hence we can see that:

$$\begin{aligned} \frac{1}{r}\sum_{i=1}^{r}\left[\mathfrak{f}\left(\left|\Delta_{\mathfrak{g}}^{\kappa} v_i - 1\right|\right)\right]^{p_i} &\leq \frac{1}{r}\left(\sum_1\left[\mathfrak{f}\left(\left|\Delta_{\mathfrak{g}}^{\kappa} v_i - \mathfrak{c}\right|\right)\right]^{p_i} + \sum_2\left[\mathfrak{f}\left(\left|\Delta^{\kappa} v_i - 1\right|\right)\right]^{p_i}\right) \\ &\leq \frac{1}{r}\sum_1 \max(\mathcal{M}^h, \mathcal{M}^H) + \frac{1}{r}\sum_2\left[\mathfrak{f}(\epsilon)\right]^{p_i} \\ &\leq \max(\mathcal{M}^h, \mathcal{M}^H)\frac{1}{r}\left|\left\{i \leq r : \left|\Delta_{\mathfrak{g}}^{\kappa} v_i - \mathfrak{c}\right| \geq \epsilon\right\}\right| \\ &+ \max(\mathfrak{f}(\epsilon)^h, f(\epsilon)^H). \end{aligned}$$

Consequently, $v \in \mathcal{W}_p(\Delta_{\mathfrak{g}}^{\kappa}, \mathfrak{f})$. Moreover, for a sequence as defined in (6), it is obvious that $S(\Delta_{\mathfrak{g}}^{\kappa}) \subset \mathcal{W}_p(\Delta_{\mathfrak{g}}^{\kappa}, \mathfrak{f})$ does not hold for an unbounded $\mathfrak{f}$. □

Theorem 10. *If modulus function $\mathfrak{f}$ is bounded, then $S(\Delta_{\mathfrak{g}}^{\kappa}) = \mathcal{W}_p(\Delta_{\mathfrak{g}}^{\kappa}, \mathfrak{f})$.*

Proof. As $\mathfrak{f}$ is bounded, we see that the equality $S(\Delta_{\mathfrak{g}}^{\kappa}) = \mathcal{W}_p(\Delta_{\mathfrak{g}}^{\kappa}, \mathfrak{f})$ holds by using Theorems (8) and (9). □

4. Conclusions

In this paper, we have studied the basic structure of some new sequence spaces by approaching the Cesàro notion and the generalized structure of the Δ-operator using statistical convergence. Furthermore, some inclusion relations have been given between the spaces studied in this article. Further, the spaces $\ell_\infty(\Gamma, \Delta_{\mathfrak{g}}^{\kappa}, p)$, $c(\Gamma, \Delta_{\mathfrak{g}}^{\kappa}, p)$ and $c_0(\Gamma, \Delta_{\mathfrak{g}}^{\kappa}, p)$ have been introduced and studied. Moreover, the notion of the space $\mathcal{W}_q(\Delta_{\mathfrak{g}}^{\kappa}, \mathfrak{f})$ has been presented, and its various topological structures have been given using the notion of fractional order. The consequences of the results obtained in this article are more general and extensive than the existing known results.

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