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Classification of Lorentzian Lie Groups Based on Codazzi Tensors Associated with Yano Connections

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Abstract: In this paper, we derive the expressions of Codazzi tensors associated with Yano connections in seven Lorentzian Lie groups. Furthermore, we complete the classification of three-dimensional Lorentzian Lie groups in which Ricci tensors associated with Yano connections are Codazzi tensors. The main results are listed in a table, and indicate that G_1 and G_7 do not have Codazzi tensors associated with Yano connections, G_2, G_3, G_4, G_5 and G_6 have Codazzi tensors associated with Yano connections.

Keywords: Codazzi tensors; Yano connectoins; Lorentzian Lie group

MSC: 53C40; 53C42

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1. Introduction

A Codazzi tensor is a symmetric 2-tensor whose covariant derivative is also symmetric in differential geometry. It is a powerful tool to study geometric properties of Riemannian manifolds with harmonic Weyl tensors or harmonic curvature. A natural example of Codazzi tensor is the second fundamental form of an immersed hypersurface in a space form. There are some typical works on Codazzi tensors. For example, in [1], Andrzej and Shen studied some geometric and topological consequences of the existence of a non-trivial Codazzi tensor on a Riemannian manifold. In [2], the detailed structure of certain Riemannian manifolds admitting Codazzi tensors was described. In [3], Liu, Simon and Wang introduced the notion of $(0, m)$-Codazzi tensors relative to an affine connection which then was used to study the topology of surfaces. In [4], the authors revealed the correspondence between Codazzi tensors that commute with its second fundamental form of a submanifold and Ribaucour transforms. In [5], Gabe constructed some examples of Codazzi tensors with two eigenvalue functions and classified this kind of Codazzi tensors on a Riemannian manifold. In [6], the authors proved structure theorem for Riemannian manifolds admitting a Codazzi tensors with exactly two distinct eigenvalues and then they classified three-dimensional gradient Ricci solitons. There are also some works on Codazzi tensors and complete Riemannian manifold, see [7,8].

In 2016, Etayo and Santamaria introduced the Yano connection on manifolds with a product structure or a complex structure based on Yano's work on the Levi Civita connection in [9]. The Yanno connection can be used to study some new properties of manifolds and Lie groups, which make the research for Yano connection significant geometrically as well. As we known that the authors classified three-dimensional Lorentzian Lie groups in [10,11]. Inspired by the above works, Wang defined a new kind of algebraic Ricci soliton associated with canonical connections on three-dimensional Lorentzian Lie groups with a product structure and classified the new algebraic Ricci soliton in [12]. In 2021, Wu and Wang studied affine Ricci solitons associated with the Bott connection on three-dimensional Lorentzian Lie groups in [13]. In 2021, Wu and Wang studied Codazzi tensors and the quasi-statistical structure associated with canonical connections on three-dimensional

Lorentzian Lie groups in [14]. There are also some works on Gauss Bonnet theorems on Lie groups, see [15–23]. However, very little is known about Codazzi tensors associated with Yano connections on Lorentzian Lie groups. This paper attempts to classify three-dimensional Lorentzian Lie groups on which Ricci tensors associated with Yano connections are Codazzi tensors.

To allow a useful study of Codazzi tensors associated with Yano connections in Lorentzian Lie groups, we derived the expressions of Yano connections in seven Lorentzian Lie groups. Then, we calculated the expressions of curvatures of the Yano connections. These expressions would be used to define the notions of Codazzi tensors associated with Yano connections. Furthermore, we derived the expressions of those Codazzi tensors in order to complete the classification of three-dimensional Lorentzian Lie groups on which Ricci tensors associated with Yano connections are Codazzi tensors. The main results of this paper are listed in Table 1 which shows the conditions that Ricci tensors associated with Yano connections is Codazzi tensors associated with Yano connections on $\{G_i\}_{i=1,2,\ldots,7}$.

Table 1. Codazzi tensors associated with Yano connections on three-dimensional Lorentzian Lie groups.

Lorentzian Lie Groups	Conditions of Codazzi Tensors Associated with Yano Connections
G_1	No solution
G_2	$\beta = 0, \gamma \neq 0$
G_3	$\beta\gamma^2 = 0$
G_4	$\alpha = \beta = 0, \eta = \pm 1$
G_5	permanent establishment
G_6	$\alpha = \beta = 0, \delta \neq 0$
G_7	No solution

We found that G_1 and G_7 do not have Codazzi tensors associated with Yano connections, G_2, G_3, G_4, G_5 and G_6 have Codazzi tensors associated with Yano connections.

The paper is organized in the following way. In Section 2, basic notions on three-dimensional Lorentzian Lie groups such as Yano connection, Riemannian curvature and Codazzi tensor are given. In Section 3, we derive the expressions of Yano connections and the associated curvatures in seven Lorentzian Lie groups and completed the classification of three-dimensional Lorentzian Lie groups on which Ricci tensors associated with Yano connections are Codazzi tensors. In Section 4, we summarize the main results and discuss further work for the future.

2. Basic Notions

In this section, we will introduce some basic notions on three-dimensional Lorentzian Lie groups such as Yano connection, Riemannian curvature and Codazzi tensor.

Let $\{G_i\}_{i=1,\ldots,7}$ be the connected, simply connected three-dimensional Lorentzian Lie group, and let $\{g_i\}_{i=1,\ldots,7}$ be the associated Lie algebra classified in [10,11]. The corresponding left-invariant Lorentzian metric is denoted by g. Let ∇^L be the Levi-Civita connection of G_i. The definition of the Yano connection ∇^Y is given as follows:

$$\nabla^Y_U V = \nabla^L_U V - \frac{1}{2}(\nabla^L_V J)JU - \frac{1}{4}[(\nabla^L_U J)JV - (\nabla^L_{JU} J)V] \tag{1}$$

where J is a product structure on $\{G_i\}_{i=1,\ldots,7}$ by $J\tilde{h}_1 = \tilde{h}_1$, $J\tilde{h}_2 = \tilde{h}_2$, $J\tilde{h}_3 = -\tilde{h}_3$.

The curvature of the Yano connection is defined by

$$R^Y(U,V)W = \nabla^Y_U \nabla^Y_V W - \nabla^Y_V \nabla^Y_U W - \nabla^Y_{[U,V]} W. \tag{2}$$

The Ricci tensor of (G_i, g) associated with the Yano connection ∇^Y is defined by

$$\rho^Y(U,V) = -g(R^Y(U,\tilde{h}_1)V,\tilde{h}_1) - g(R^Y(U,\tilde{h}_2)V,\tilde{h}_2) + g(R^Y(U,\tilde{h}_3)V,\tilde{h}_3), \tag{3}$$

where $\tilde{h}_1, \tilde{h}_2, \tilde{h}_3$ is a pseudo-orthonormal basis, with $\tilde{h}_3$ timelike.

Let

$$\tilde{\rho}^Y(U,V) = \frac{\rho^Y(U,V) + \rho^Y(V,U)}{2}. \tag{4}$$

Let M be a smooth manifold endowed with a linear connection ∇ and ω be a $(0,2)$ tensor field, then one can define

$$(\nabla_U \omega)(V,W) := U[\omega(V,W)] - \omega(\nabla_U V, W) - \omega(V, \nabla_U W) \tag{5}$$

for arbitrary vector fields U, V, W. The tensor fields ω is called a Codazzi tensor on (M, ∇) if it satisfies

$$f(U,V)W = (\nabla_U \omega)(V,W) - (\nabla_V \omega)(U,W) = 0, \tag{6}$$

where f is $C^\infty(M)$-linear for U, V, W. One can obtain the following proposition on conditions of Codazzi tensor.

Proposition 1. *The tensor ω is a Codazzi tensor on (M, ∇) if and only if*

$$f(U,V,W) = -f(V,U,W). \tag{7}$$

Then, we have that ω is a Codazzi tensor on $(G_i, \nabla)_{i=1,2,\dots,7}$ if and only if the following three equations hold:

$$\begin{cases} f(\tilde{h}_1, \tilde{h}_2, \tilde{h}_j) = 0, \\ f(\tilde{h}_1, \tilde{h}_3, \tilde{h}_j) = 0, \\ f(\tilde{h}_2, \tilde{h}_3, \tilde{h}_j) = 0, \end{cases} \tag{8}$$

where $1 \leq j \leq 3$.

3. Codazzi Tensors Associated with Yano Connections on Lorentzian Lie Groups

In this section, we will derive the expressions of Yano connection and the associated curvatures in seven Lorentzian Lie groups and complete the classification of three-dimensional Lorentzian Lie groups on which Ricci tensors associated with Yano connections are Codazzi tensors.

3.1. Codazzi Tensor Associated with Yano Connection of G_1

In this subsection, we consider the following Lie algebra of G_1 which satisfies

$$[\tilde{h}_1, \tilde{h}_2] = \alpha\tilde{h}_1 - \beta\tilde{h}_3, [\tilde{h}_1, \tilde{h}_3] = -\alpha\tilde{h}_1 - \beta\tilde{h}_2, [\tilde{h}_2, \tilde{h}_3] = \beta\tilde{h}_1 + \alpha\tilde{h}_2 + \alpha\tilde{h}_3, \alpha \neq 0,$$

where $\tilde{h}_1, \tilde{h}_2, \tilde{h}_3$ is a pseudo-orthonormal basis, with $\tilde{h}_3$ timelike. One can obtain the following two Lemmas on the expressions of Yano connections and the associated curvatures in the first Lorentzian Lie group.

Lemma 1. *The Yano connection ∇^Y of G_1 is given by*

$$\begin{aligned}
&\nabla^Y_{\tilde{h}_1}\tilde{h}_1 = -\alpha\tilde{h}_2,\ \nabla^Y_{\tilde{h}_1}\tilde{h}_2 = \alpha\tilde{h}_1 - \beta\tilde{h}_3,\ \nabla^Y_{\tilde{h}_1}\tilde{h}_3 = 0, \\
&\nabla^Y_{\tilde{h}_2}\tilde{h}_1 = \beta\tilde{h}_3,\ \nabla^Y_{\tilde{h}_2}\tilde{h}_2 = 0,\ \nabla^Y_{\tilde{h}_2}\tilde{h}_3 = \alpha\tilde{h}_3, \\
&\nabla^Y_{\tilde{h}_3}\tilde{h}_1 = \alpha\tilde{h}_1 + \beta\tilde{h}_2,\ \nabla^Y_{\tilde{h}_3}\tilde{h}_2 = -\beta\tilde{h}_1 - \alpha\tilde{h}_2,\ \nabla^Y_{\tilde{h}_3}\tilde{h}_3 = 0.
\end{aligned}$$

Lemma 2. *The curvature R^Y of the Yano connection ∇^Y of (G_1, g) is given by*

$$R^Y(\tilde{h}_1,\tilde{h}_2)\tilde{h}_1 = \alpha\beta\tilde{h}_1 + (\alpha^2+\beta^2)\tilde{h}_2,\ R^Y(\tilde{h}_1,\tilde{h}_2)\tilde{h}_2 = -(\alpha^2+\beta^2)\tilde{h}_1 - \alpha\beta\tilde{h}_2 + \alpha\beta\tilde{h}_3,$$
$$R^Y(\tilde{h}_1,\tilde{h}_2)\tilde{h}_3 = 0,\ R^Y(\tilde{h}_1,\tilde{h}_3)\tilde{h}_1 = -3\alpha^2\tilde{h}_2,$$
$$R^Y(\tilde{h}_1,\tilde{h}_3)\tilde{h}_2 = -\alpha^2\tilde{h}_1,\ R^Y(\tilde{h}_1,\tilde{h}_3)\tilde{h}_3 = \alpha\beta\tilde{h}_3,$$
$$R^Y(\tilde{h}_2,\tilde{h}_3)\tilde{h}_1 = -\alpha^2\tilde{h}_1,\ R^Y(\tilde{h}_2,\tilde{h}_3)\tilde{h}_2 = \alpha^2\tilde{h}_2,$$
$$R^Y(\tilde{h}_2,\tilde{h}_3)\tilde{h}_3 = -\alpha^2\tilde{h}_3.$$

One can prove the following theorem on Codazzi tensor in the first Lorentzian Lie group based on Lemma 2 and Lemma 4.

Theorem 1. *$\tilde{\rho}^Y$ is not a Codazzi tensor on (G_1, ∇^Y).*

Proof. By (3), we obtain

$$\rho^Y(\tilde{h}_1,\tilde{h}_1) = -\alpha^2-\beta^2,\ \rho^Y(\tilde{h}_1,\tilde{h}_2) = \alpha\beta,\ \rho^Y(\tilde{h}_1,\tilde{h}_3) = -\alpha\beta,$$
$$\rho^Y(\tilde{h}_2,\tilde{h}_1) = \alpha\beta,\ \rho^Y(\tilde{h}_2,\tilde{h}_2) = -(\alpha^2+\beta^2),\ \rho^Y(\tilde{h}_2,\tilde{h}_3) = \alpha^2,$$
$$\rho^Y(\tilde{h}_3,\tilde{h}_1) = 0,\ \rho^Y(\tilde{h}_3,\tilde{h}_2) = 0,\ \rho^Y(\tilde{h}_3,\tilde{h}_3) = 0.$$

Then

$$\tilde{\rho}^Y(\tilde{h}_1,\tilde{h}_1) = -(\alpha^2+\beta^2),\ \tilde{\rho}^Y(\tilde{h}_1,\tilde{h}_2) = \alpha\beta,\ \tilde{\rho}^Y(\tilde{h}_1,\tilde{h}_3) = -\frac{\alpha\beta}{2},$$
$$\tilde{\rho}^Y(\tilde{h}_2,\tilde{h}_2) = -(\alpha^2+\beta^2),\ \tilde{\rho}^Y(\tilde{h}_2,\tilde{h}_3) = \frac{\alpha^2}{2},\ \tilde{\rho}^Y(\tilde{h}_3,\tilde{h}_3) = 0.$$

By (5), we have

$$(\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_1) = -\frac{\alpha\beta^2}{2},\ (\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_1) = \alpha\beta^2,\ (\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_2) = 0,$$
$$(\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_2) = -\frac{\alpha^2\beta}{2},\ (\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_3) = \frac{\alpha^2\beta}{2},\ (\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_3) = \frac{\alpha^2\beta}{2},$$
$$(\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_1) = \frac{\alpha^3}{2},\ (\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_1) = 2\alpha^3,\ (\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_2) = 0,$$
$$(\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_2) = 0,\ (\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_3) = 0,\ (\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_3) = 0,$$
$$(\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_1) = \alpha\beta,\ (\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_1) = 0,\ (\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_2) = -\frac{\alpha^3}{2},$$
$$(\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_2) = -2\alpha^3,\ (\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_3) = 0,\ (\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_3) = \frac{\alpha^3-\alpha\beta^2}{2}.$$

Then, if $\tilde{\rho}^Y$ is a Codazzi tensor on (G_1, ∇^Y), by (6) and (7), we obtain the following five equations:

$$\begin{cases} \alpha\beta^2 = 0 \\ \alpha^2\beta = 0 \\ \alpha^3 = 0 \\ \alpha\beta = 0 \\ \alpha\beta^2 - \alpha^3 = 0. \end{cases} \tag{9}$$

By solving (9), we have $\alpha = 0$. This is a contradiction. □

3.2. Codazzi Tensor Associated with Yano Connection of G_2

In this subsection, we consider the following Lie algebra of G_2 which satisfies

$$[\tilde{h}_1,\tilde{h}_2] = \gamma\tilde{h}_2 - \beta\tilde{h}_3, [\tilde{h}_1,\tilde{h}_3] = -\beta\tilde{h}_2 - \gamma\tilde{h}_3, [\tilde{h}_2,\tilde{h}_3] = \alpha\tilde{h}_1, \gamma \neq 0,$$

where $\tilde{h}_1$, $\tilde{h}_2$, $\tilde{h}_3$ is a pseudo-orthonormal basis, with $\tilde{h}_3$ timelike. One can obtain the following two Lemmas on the expressions of Yano connection and the associated curvatures in the second Lorentzian Lie group.

Lemma 3. *The Yano connection ∇^Y of G_2 is given by*

$$\begin{aligned}
&\nabla^Y_{\tilde{h}_1}\tilde{h}_1 = 0,\ \nabla^Y_{\tilde{h}_1}\tilde{h}_2 = -\beta\tilde{h}_3,\ \nabla^Y_{\tilde{h}_1}\tilde{h}_3 = -2\beta\tilde{h}_2 - \gamma\tilde{h}_3,\\
&\nabla^Y_{\tilde{h}_2}\tilde{h}_1 = -\gamma\tilde{h}_2 + \beta\tilde{h}_3,\ \nabla^Y_{\tilde{h}_2}\tilde{h}_2 = \gamma\tilde{h}_1,\ \nabla^Y_{\tilde{h}_2}\tilde{h}_3 = 0,\\
&\nabla^Y_{\tilde{h}_3}\tilde{h}_1 = \beta\tilde{h}_2,\ \nabla^Y_{\tilde{h}_3}\tilde{h}_2 = -\alpha\tilde{h}_1,\ \nabla^Y_{\tilde{h}_3}\tilde{h}_3 = 0.
\end{aligned}$$

Lemma 4. *The curvature R^Y of the Yano connection ∇^Y of (G_2, g) is given by*

$$\begin{aligned}
&R^Y(\tilde{h}_1,\tilde{h}_2)\tilde{h}_1 = (\gamma^2-\beta^2)\tilde{h}_2 - \beta\gamma\tilde{h}_3,\ R^Y(\tilde{h}_1,\tilde{h}_2)\tilde{h}_2 = -(\gamma^2+\alpha\beta)\tilde{h}_1,\ R^Y(\tilde{h}_1,\tilde{h}_2)\tilde{h}_3 = 2\beta\gamma\tilde{h}_1,\\
&R^Y(\tilde{h}_1,\tilde{h}_3)\tilde{h}_1 = 0,\ R^Y(\tilde{h}_1,\tilde{h}_3)\tilde{h}_2 = (\beta\gamma-\alpha\gamma)\tilde{h}_1,\ R^Y(\tilde{h}_1,\tilde{h}_3)\tilde{h}_3 = -2\alpha\beta\tilde{h}_1,\\
&R^Y(\tilde{h}_2,\tilde{h}_3)\tilde{h}_1 = (\beta\gamma-\alpha\gamma)\tilde{h}_1,\ R^Y(\tilde{h}_2,\tilde{h}_3)\tilde{h}_2 = -\beta\gamma\tilde{h}_2 + \alpha\beta\tilde{h}_3,\ R^Y(\tilde{h}_2,\tilde{h}_3)\tilde{h}_3 = 2\alpha\beta\tilde{h}_2 + \alpha\gamma\tilde{h}_3.
\end{aligned}$$

Based on Lemmas 3 and 4, one can prove the following theorem on Codazzi tensor in the second Lorentzian Lie group.

Theorem 2. *$\tilde{\rho}^Y$ is a Codazzi tensor on (G_2, ∇^Y) if and only if $\beta = 0$, $\gamma \neq 0$.*

Proof. By (3), we have

$$\begin{aligned}
&\rho^Y(\tilde{h}_1,\tilde{h}_1) = \beta^2-\gamma^2,\ \rho^Y(\tilde{h}_1,\tilde{h}_2) = 0,\ \rho^Y(\tilde{h}_1,\tilde{h}_3) = 0,\\
&\rho^Y(\tilde{h}_2,\tilde{h}_1) = 0,\ \rho^Y(\tilde{h}_2,\tilde{h}_2) = -\gamma^2-2\alpha\beta,\ \rho^Y(\tilde{h}_2,\tilde{h}_3) = 2\beta\gamma-\alpha\gamma,\\
&\rho^Y(\tilde{h}_3,\tilde{h}_1) = 0,\ \rho^Y(\tilde{h}_3,\tilde{h}_2) = -\alpha\gamma,\ \rho^Y(\tilde{h}_3,\tilde{h}_3) = 0.
\end{aligned}$$

Then

$$\begin{aligned}
&\tilde{\rho}^Y(\tilde{h}_1,\tilde{h}_1) = \beta^2-\gamma^2,\ \tilde{\rho}^Y(\tilde{h}_1,\tilde{h}_2) = 0,\ \tilde{\rho}^Y(\tilde{h}_1,\tilde{h}_3) = 0,\\
&\tilde{\rho}^Y(\tilde{h}_2,\tilde{h}_2) = -\gamma^2-2\alpha\beta,\ \tilde{\rho}^Y(\tilde{h}_2,\tilde{h}_3) = \beta\gamma-\alpha\gamma,\ \tilde{\rho}^Y(\tilde{h}_3,\tilde{h}_3) = 0.
\end{aligned}$$

By (5), we obtain

$$\begin{aligned}
&(\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_1) = 0,\ (\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_1) = 0,\ (\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_2) = 2\beta^2\gamma - 2\alpha\beta\gamma,\\
&(\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_2) = -\alpha\beta\gamma - 2\beta^2\gamma,\ (\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_3) = -3\beta\gamma^2 - 4\alpha\beta^2 - \alpha\gamma^2,\\
&(\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_3) = \beta\gamma^2 - \alpha\gamma^2,\ (\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_1) = 0,\ (\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_1) = 0,\\
&(\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_2) = -\beta\gamma^2 - 4\alpha\beta^2 - \alpha\gamma^2,\ (\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_2) = \beta\gamma^2 + 3\alpha\beta^2 - \alpha\gamma^2,\\
&(\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_3) = 4\beta^2\gamma - 4\alpha\beta\gamma,\ (\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_3) = \alpha\beta\gamma - \beta^2\gamma,\\
&(\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_1) = \beta\gamma^2 - \alpha\gamma^2,\ (\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_1) = 3\alpha\beta^2 - \alpha\gamma^2 + \beta\gamma,\ (\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_2) = 0,\\
&(\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_2) = 0,\ (\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_3) = 0,\ (\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_3) = 0.
\end{aligned}$$

Then, if $\tilde{\rho}^Y$ is a Codazzi tensor on (G_2, ∇^Y), by (6) and (7), we obtain the following five equations:

$$\begin{cases}
2\beta^2\gamma - \alpha\beta\gamma = 0\\
\beta\gamma^2 + \alpha\beta = 0\\
2\beta\gamma^2 + 7\alpha\beta^2 = 0\\
\beta^2\gamma - \alpha\beta\gamma = 0\\
\alpha\beta^2 = 0.
\end{cases} \tag{10}$$

By solving (10), we obtain Theorem 2. □

3.3. Codazzi Tensor Associated with Yano Connection of G_3

In this subsection, we consider the following Lie algebra of G_3 which satisfies

$$[\tilde{h}_1, \tilde{h}_2] = -\gamma\tilde{h}_3, [\tilde{h}_1, \tilde{h}_3] = -\beta\tilde{h}_2, [\tilde{h}_2, \tilde{h}_3] = \alpha\tilde{h}_1,$$

where $\tilde{h}_1$, $\tilde{h}_2$, $\tilde{h}_3$ is a pseudo-orthonormal basis, with $\tilde{h}_3$ timelike. One can obtain the following two Lemmas on the expressions of Yano connection and the associated curvatures in the third Lorentzian Lie group.

Lemma 5. *The Yano connection ∇^Y of G_3 is given by*

$$\begin{aligned}
&\nabla^Y_{\tilde{h}_1}\tilde{h}_1 = 0,\ \nabla^Y_{\tilde{h}_1}\tilde{h}_2 = -\gamma\tilde{h}_3,\ \nabla^Y_{\tilde{h}_1}\tilde{h}_3 = 0,\\
&\nabla^Y_{\tilde{h}_2}\tilde{h}_1 = \gamma\tilde{h}_3,\ \nabla^Y_{\tilde{h}_2}\tilde{h}_2 = 0,\ \nabla^Y_{\tilde{h}_2}\tilde{h}_3 = -\gamma\tilde{h}_1,\\
&\nabla^Y_{\tilde{h}_3}\tilde{h}_1 = \beta\tilde{h}_2,\ \nabla^Y_{\tilde{h}_3}\tilde{h}_2 = -\alpha\tilde{h}_1,\ \nabla^Y_{\tilde{h}_3}\tilde{h}_3 = 0.
\end{aligned}$$

Lemma 6. *The curvature R^Y of the Yano connection ∇^Y of (G_3, g) is given by*

$$\begin{aligned}
&R^Y(\tilde{h}_1,\tilde{h}_2)\tilde{h}_1 = \beta\gamma\tilde{h}_2,\ R^Y(\tilde{h}_1,\tilde{h}_2)\tilde{h}_2 = -(\gamma^2+\alpha\gamma)\tilde{h}_1,\ R^Y(\tilde{h}_1,\tilde{h}_2)\tilde{h}_3 = 0,\\
&R^Y(\tilde{h}_1,\tilde{h}_3)\tilde{h}_1 = 0,\ R^Y(\tilde{h}_1,\tilde{h}_3)\tilde{h}_2 = 0,\ R^Y(\tilde{h}_1,\tilde{h}_3)\tilde{h}_3 = -\beta\gamma\tilde{h}_1,\\
&R^Y(\tilde{h}_2,\tilde{h}_3)\tilde{h}_1 = 0,\ R^Y(\tilde{h}_2,\tilde{h}_3)\tilde{h}_2 = 0,\ R^Y(\tilde{h}_2,\tilde{h}_3)\tilde{h}_3 = \beta\gamma\tilde{h}_2.
\end{aligned}$$

Based on Lemmas 5 and 6, one can prove the following theorem on Codazzi tensor in the third Lorentzian Lie group.

Theorem 3. *$\tilde{\rho}^Y$ is a Codazzi tensor on (G_3, ∇^Y) if and only if $\beta\gamma^2 = 0$.*

Proof. By (3), we have

$$\begin{aligned}
&\rho^Y(\tilde{h}_1,\tilde{h}_1) = -\beta\gamma,\ \rho^Y(\tilde{h}_1,\tilde{h}_2) = 0,\ \rho^Y(\tilde{h}_1,\tilde{h}_3) = 0,\\
&\rho^Y(\tilde{h}_2,\tilde{h}_1) = 0,\ \rho^Y(\tilde{h}_2,\tilde{h}_2) = -\gamma^2-\alpha\gamma,\ \rho^Y(\tilde{h}_2,\tilde{h}_3) = 0,\\
&\rho^Y(\tilde{h}_3,\tilde{h}_1) = 0,\ \rho^Y(\tilde{h}_3,\tilde{h}_2) = 0,\ \rho^Y(\tilde{h}_3,\tilde{h}_3) = 0.
\end{aligned}$$

Then

$$\begin{aligned}
&\tilde{\rho}^Y(\tilde{h}_1,\tilde{h}_1) = -\beta\gamma,\ \tilde{\rho}^Y(\tilde{h}_1,\tilde{h}_2) = 0,\ \tilde{\rho}^Y(\tilde{h}_1,\tilde{h}_3) = 0,\\
&\tilde{\rho}^Y(\tilde{h}_2,\tilde{h}_2) = -\gamma^2-\alpha\gamma,\ \tilde{\rho}^Y(\tilde{h}_2,\tilde{h}_3) = 0,\ \tilde{\rho}^Y(\tilde{h}_3,\tilde{h}_3) = 0.
\end{aligned}$$

By (5), we have

$$\begin{aligned}
&(\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_1) = 0,\ (\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_1) = 0,\ (\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_2) = 0,\\
&(\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_2) = 0,\ (\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_3) = 0,\ (\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_3) = -\beta\gamma^2,\\
&(\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_1) = 0,\ (\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_1) = 0,\ (\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_2) = 0,\\
&(\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_2) = \beta\gamma^2,\ (\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_3) = 0,\ (\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_3) = 0,\\
&(\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_1) = -\beta\gamma^2,\ (\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_1) = \beta\gamma^2,\ (\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_2) = 0,\\
&(\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_2) = 0,\ (\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_3) = 0,\ (\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_3) = 0.
\end{aligned}$$

Then, if $\tilde{\rho}^Y$ is a Codazzi tensor on (G_3, ∇^Y), by (6) and (7), we have the following one equation:

$$\beta\gamma^2 = 0. \tag{11}$$

By solving (11), it turns out Theorem 3. □

3.4. Codazzi Tensor Associated with Yano Connection of G_4

In this subsection, we consider the following Lie algebra of G_4 which satisfies

$$[\tilde{h}_1, \tilde{h}_2] = -\tilde{h}_2 + (2\eta - \beta)\tilde{h}_3, \eta = \pm 1, [\tilde{h}_1, \tilde{h}_3] = -\beta\tilde{h}_2 + \tilde{h}_3, [\tilde{h}_2, \tilde{h}_3] = \alpha\tilde{h}_1,$$

where $\tilde{h}_1$, $\tilde{h}_2$, $\tilde{h}_3$ is a pseudo-orthonormal basis, with $\tilde{h}_3$ timelike. One can obtain the following two Lemmas on the expressions of the Yano connection and the associated curvatures in the fourth Lorentzian Lie group.

Lemma 7. *The Yano connection ∇^Y of G_4 is given by*

$$\begin{aligned}
&\nabla^Y_{\tilde{h}_1}\tilde{h}_1 = 0,\ \nabla^Y_{\tilde{h}_1}\tilde{h}_2 = (2\eta - \beta)\tilde{h}_3,\ \nabla^Y_{\tilde{h}_1}\tilde{h}_3 = \tilde{h}_3,\\
&\nabla^Y_{\tilde{h}_2}\tilde{h}_1 = \tilde{h}_2 + (\beta - 2\eta)\tilde{h}_3,\ \nabla^Y_{\tilde{h}_2}\tilde{h}_2 = -\tilde{h}_1,\ \nabla^Y_{\tilde{h}_2}\tilde{h}_3 = 0,\\
&\nabla^Y_{\tilde{h}_3}\tilde{h}_1 = \beta\tilde{h}_2,\ \nabla^Y_{\tilde{h}_3}\tilde{h}_2 = -\alpha\tilde{h}_1,\ \nabla^Y_{\tilde{h}_3}\tilde{h}_3 = 0.
\end{aligned}$$

Lemma 8. *The curvature R^Y of the Yano connection ∇^Y of (G_4, g) is given by*

$$\begin{aligned}
&R^Y(\tilde{h}_1, \tilde{h}_2)\tilde{h}_1 = (\beta^2 - 2\beta\eta + 1)\tilde{h}_2,\ R^Y(\tilde{h}_1, \tilde{h}_2)\tilde{h}_2 = (2\alpha\eta - \alpha\beta - 1)\tilde{h}_1,\ R^Y(\tilde{h}_1, \tilde{h}_2)\tilde{h}_3 = 0,\\
&R^Y(\tilde{h}_1, \tilde{h}_3)\tilde{h}_1 = 0,\ R^Y(\tilde{h}_1, \tilde{h}_3)\tilde{h}_2 = (\alpha - \beta)\tilde{h}_1,\ R^Y(\tilde{h}_1, \tilde{h}_3)\tilde{h}_3 = 0,\\
&R^Y(\tilde{h}_2, \tilde{h}_3)\tilde{h}_1 = (\alpha - \beta)\tilde{h}_1,\ R^Y(\tilde{h}_2, \tilde{h}_3)\tilde{h}_2 = (\beta - \alpha)\tilde{h}_2,\ R^Y(\tilde{h}_2, \tilde{h}_3)\tilde{h}_3 = -\alpha\tilde{h}_3.
\end{aligned}$$

Based on Lemmas 7 and 8, one can prove the following theorem on Codazzi tensor in the fourth Lorentzian Lie group.

Theorem 4. *$\tilde{\rho}^Y$ is a Codazzi tensor on (G_4, ∇^Y) if and only if $\alpha = \beta = 0$, $\eta = \pm 1$.*

Proof. By (3), we obtain

$$\begin{aligned}
&\rho^Y(\tilde{h}_1, \tilde{h}_1) = 2\beta\eta - \beta^2 - 1,\ \rho^Y(\tilde{h}_1, \tilde{h}_2) = 0,\ \rho^Y(\tilde{h}_1, \tilde{h}_3) = 0,\\
&\rho^Y(\tilde{h}_2, \tilde{h}_1) = 0,\ \rho^Y(\tilde{h}_2, \tilde{h}_2) = 2\alpha\eta - \alpha\beta - 1,\ \rho^Y(\tilde{h}_2, \tilde{h}_3) = \alpha,\\
&\rho^Y(\tilde{h}_3, \tilde{h}_1) = 0,\ \rho^Y(\tilde{h}_3, \tilde{h}_2) = 0,\ \rho^Y(\tilde{h}_3, \tilde{h}_3) = 0.
\end{aligned}$$

Then

$$\begin{aligned}
&\tilde{\rho}^Y(\tilde{h}_1, \tilde{h}_1) = 2\beta\eta - \beta^2 - 1,\ \tilde{\rho}^Y(\tilde{h}_1, \tilde{h}_2) = 0,\ \tilde{\rho}^Y(\tilde{h}_1, \tilde{h}_3) = 0,\\
&\tilde{\rho}^Y(\tilde{h}_2, \tilde{h}_2) = 2\alpha\eta - \alpha\beta - 1,\ \tilde{\rho}^Y(\tilde{h}_2, \tilde{h}_3) = \frac{\alpha}{2},\ \tilde{\rho}^Y(\tilde{h}_3, \tilde{h}_3) = 0.
\end{aligned}$$

By (5), we have

$$(\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_1)=0,\ (\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_1)=0,\ (\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_2)=\alpha\beta-2\alpha\eta,$$
$$(\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_2)=-\alpha\eta+\frac{\alpha\beta}{2}+2\beta\eta-\beta^2,\ (\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_3)=-\frac{\alpha}{2},\ (\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_3)=-\frac{\alpha}{2},$$
$$(\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_1)=0,\ (\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_1)=0,\ (\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_2)=-\frac{\alpha}{2},$$
$$(\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_2)=\beta-\alpha,\ (\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_3)=0,\ (\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_3)=-\frac{\alpha\beta}{2},$$
$$(\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_1)=-\frac{\alpha}{2},\ (\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_1)=\beta-\alpha,\ (\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_2)=0,$$
$$(\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_2)=0,\ (\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_3)=0,\ (\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_3)=0.$$

Then, if $\tilde{\rho}^Y$ is a Codazzi tensor on (G_4,∇^Y), by (6) and (7), we obtain the following three equations:

$$\begin{cases}\alpha\beta-2\alpha\eta-4\beta\eta+\beta^2=0\\ \alpha-2\beta=0\\ \alpha\beta=0.\end{cases}\tag{12}$$

By solving (12), one can prove Theorem 4. □

3.5. Codazzi Tensor Associated with Yano Connection of G_5

In this subsection, we consider the following Lie algebra of G_5 which satisfies

$$[\tilde{h}_1,\tilde{h}_2]=0,[\tilde{h}_1,\tilde{h}_3]=\alpha\tilde{h}_1+\beta\tilde{h}_2,[\tilde{h}_2,\tilde{h}_3]=\gamma\tilde{h}_1+\delta\tilde{h}_2,\alpha+\delta\neq 0,\alpha\gamma+\beta\delta=0$$

where $\tilde{h}_1$, $\tilde{h}_2$, $\tilde{h}_3$ is a pseudo-orthonormal basis, with $\tilde{h}_3$ timelike. One can obtain the following two Lemmas on the expressions of Yano connection and the associated curvatures in the fifth Lorentzian Lie group.

Lemma 9. *The Yano connection ∇^Y of G_5 is given by*

$$\begin{aligned}&\nabla^Y_{\tilde{h}_1}\tilde{h}_1=0,\ \nabla^Y_{\tilde{h}_1}\tilde{h}_2=0,\ \nabla^Y_{\tilde{h}_1}\tilde{h}_3=0,\\ &\nabla^Y_{\tilde{h}_2}\tilde{h}_1=0,\ \nabla^Y_{\tilde{h}_2}\tilde{h}_2=0,\ \nabla^Y_{\tilde{h}_2}\tilde{h}_3=0,\\ &\nabla^Y_{\tilde{h}_3}\tilde{h}_1=-\alpha\tilde{h}_1+(\beta+\gamma)\tilde{h}_2,\ \nabla^Y_{\tilde{h}_3}\tilde{h}_2=-\gamma\tilde{h}_1-\delta\tilde{h}_2,\ \nabla^Y_{\tilde{h}_3}\tilde{h}_3=0.\end{aligned}$$

Lemma 10. *The curvature R^Y of the Yano connection ∇^Y of (G_5,g) is given by*

$$R^Y(\tilde{h}_1,\tilde{h}_2)\tilde{h}_j=R^Y(\tilde{h}_1,\tilde{h}_3)\tilde{h}_j=R^Y(\tilde{h}_2,\tilde{h}_3)\tilde{h}_j=0,$$

where $1\leq j\leq 3$.

Based on Lemmas 9 and 10, one can prove the following theorem on the Codazzi tensor in the fifth Lorentzian Lie group.

Theorem 5. *$\tilde{\rho}^Y$ is a Codazzi tensor on (G_5,∇^Y) .*

Proof. By (3), we have

$$\rho^Y(\tilde{h}_1,\tilde{h}_j)=\rho^Y(\tilde{h}_2,\tilde{h}_j)=\rho^Y(\tilde{h}_3,\tilde{h}_j)=0.$$

where $1\leq j\leq 3$.

Then

$$\tilde{\rho}^Y(\tilde{h}_1, \tilde{h}_1) = \tilde{\rho}^Y(\tilde{h}_1, \tilde{h}_2) = \tilde{\rho}^Y(\tilde{h}_1, \tilde{h}_3) = 0,$$
$$\tilde{\rho}^Y(\tilde{h}_2, \tilde{h}_2) = \tilde{\rho}^Y(\tilde{h}_2, \tilde{h}_3) = \tilde{\rho}^Y(\tilde{h}_3, \tilde{h}_3) = 0.$$

By (5), we have

$$(\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_2, \tilde{h}_j) = (\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_1, \tilde{h}_j) = (\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_3, \tilde{h}_j) = 0,$$
$$(\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_1, \tilde{h}_j) = (\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_3, \tilde{h}_j) = (\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_2, \tilde{h}_j) = 0.$$

where $1 \leq j \leq 3$. This means that $\tilde{\rho}^Y$ is a Codazzi tensor on (G_5, ∇^Y). □

3.6. Codazzi Tensor Associated with Yano Connection of G_6

In this subsection, we consider the following Lie algebra of G_6 which satisfies

$$[\tilde{h}_1, \tilde{h}_2] = \alpha\tilde{h}_2 + \beta\tilde{h}_3, [\tilde{h}_1, \tilde{h}_3] = \gamma\tilde{h}_2 + \delta\tilde{h}_3, [\tilde{h}_2, \tilde{h}_3] = 0, \alpha + \delta \neq 0, \alpha\gamma - \beta\delta = 0,$$

where $\tilde{h}_1$, $\tilde{h}_2$, $\tilde{h}_3$ is a pseudo-orthonormal basis, with $\tilde{h}_3$ timelike. One can obtain the following two Lemmas on the expressions of Yano connection and the associated curvatures in the sixth Lorentzian Lie group.

Lemma 11. *The Yano connection ∇^Y of G_6 is given by*

$$\nabla^Y_{\tilde{h}_1}\tilde{h}_1 = 0,\ \nabla^Y_{\tilde{h}_1}\tilde{h}_2 = \beta\tilde{h}_3,\ \nabla^Y_{\tilde{h}_1}\tilde{h}_3 = \delta\tilde{h}_3,$$
$$\nabla^Y_{\tilde{h}_2}\tilde{h}_1 = -\alpha\tilde{h}_2 - \beta\tilde{h}_3,\ \nabla^Y_{\tilde{h}_2}\tilde{h}_2 = \alpha\tilde{h}_1,\ \nabla^Y_{\tilde{h}_2}\tilde{h}_3 = 0,$$
$$\nabla^Y_{\tilde{h}_3}\tilde{h}_1 = -\gamma\tilde{h}_2,\ \nabla^Y_{\tilde{h}_3}\tilde{h}_2 = 0,\ \nabla^Y_{\tilde{h}_3}\tilde{h}_3 = 0.$$

Lemma 12. *The curvature R^Y of the Yano connection ∇^Y of (G_6, g) is given by*

$$R^Y(\tilde{h}_1, \tilde{h}_2)\tilde{h}_1 = (\beta\gamma + \alpha)\tilde{h}_2 - \beta\delta\tilde{h}_3,\ R^Y(\tilde{h}_1, \tilde{h}_2)\tilde{h}_2 = -\alpha^2\tilde{h}_1,\ R^Y(\tilde{h}_1, \tilde{h}_2)\tilde{h}_3 = 0,$$
$$R^Y(\tilde{h}_1, \tilde{h}_3)\tilde{h}_1 = (\alpha\gamma + \delta\gamma)\tilde{h}_2,\ R^Y(\tilde{h}_1, \tilde{h}_3)\tilde{h}_2 = -\alpha\gamma\tilde{h}_1,\ R^Y(\tilde{h}_1, \tilde{h}_3)\tilde{h}_3 = 0,$$
$$R^Y(\tilde{h}_2, \tilde{h}_3)\tilde{h}_1 = -\alpha\gamma\tilde{h}_1,\ R^Y(\tilde{h}_2, \tilde{h}_3)\tilde{h}_2 = \alpha\gamma\tilde{h}_2,\ R^Y(\tilde{h}_2, \tilde{h}_3)\tilde{h}_3 = 0.$$

Based on Lemmas 11 and 12, one can prove the following theorem on Codazzi tensor in the sixth Lorentzian Lie group.

Theorem 6. *$\tilde{\rho}^Y$ is a Codazzi tensor on (G_6, ∇^Y) if and only if $\alpha = \beta = 0$, $\delta \neq 0$.*

Proof. By (3), we have

$$\rho^Y(\tilde{h}_1, \tilde{h}_1) = -(\beta\gamma + \alpha),\ \rho^Y(\tilde{h}_1, \tilde{h}_2) = 0,\ \rho^Y(\tilde{h}_1, \tilde{h}_3) = 0,$$
$$\rho^Y(\tilde{h}_2, \tilde{h}_1) = 0,\ \rho^Y(\tilde{h}_2, \tilde{h}_2) = -\alpha^2,\ \rho^Y(\tilde{h}_2, \tilde{h}_3) = 0,$$
$$\rho^Y(\tilde{h}_3, \tilde{h}_1) = 0,\ \rho^Y(\tilde{h}_3, \tilde{h}_2) = 0,\ \rho^Y(\tilde{h}_3, \tilde{h}_3) = 0.$$

Then

$$\tilde{\rho}^Y(\tilde{h}_1, \tilde{h}_1) = -(\beta\gamma + \alpha),\ \tilde{\rho}^Y(\tilde{h}_1, \tilde{h}_2) = 0,\ \tilde{\rho}^Y(\tilde{h}_1, \tilde{h}_3) = 0,$$
$$\tilde{\rho}^Y(\tilde{h}_2, \tilde{h}_2) = -\alpha^2,\ \tilde{\rho}^Y(\tilde{h}_2, \tilde{h}_3) = 0,\ \tilde{\rho}^Y(\tilde{h}_3, \tilde{h}_3) = 0.$$

By (5), we have

$$
\begin{aligned}
&(\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_1)=0,\ (\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_1)=0,\ (\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_2)=0,\\
&(\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_2)=\alpha\beta\gamma-\alpha^2-\alpha^3,\ (\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_3)=0,\ (\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_3)=0,\\
&(\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_1)=0,\ (\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_1)=0,\ (\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_2)=0,\\
&(\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_2)=-\alpha^2\gamma,\ (\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_3)=0,\ (\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_3)=0,\\
&(\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_1)=0,\ (\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_1)=-\alpha^2\gamma,\ (\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_2)=0,\\
&(\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_2)=0,\ (\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_3)=0,\ (\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_3)=0.
\end{aligned}
$$

Then, if $\tilde{\rho}^Y$ is a Codazzi tensor on (G_6,∇^Y), by (6) and (7), we have the following two equations:

$$
\begin{cases}
\alpha^3+\alpha^2-\alpha\beta\gamma=0\\
\alpha^2\gamma=0.
\end{cases}
\tag{13}
$$

By solving (13), it turns out Theorem 6. □

3.7. Codazzi Tensor Associated with the Yano Connection of G_7

In this subsection, we consider the following Lie algebra of G_7 which satisfies

$$
[\tilde{h}_1,\tilde{h}_2]=-\alpha\tilde{h}_1-\beta\tilde{h}_2-\beta\tilde{h}_3,[\tilde{h}_1,\tilde{h}_3]=\alpha\tilde{h}_1+\beta\tilde{h}_2+\beta\tilde{h}_3,[\tilde{h}_2,\tilde{h}_3]=\gamma\tilde{h}_1+\delta\tilde{h}_2+\delta\tilde{h}_3,
$$

where $\tilde{h}_1$, $\tilde{h}_2$, $\tilde{h}_3$ is a pseudo-orthonormal basis, with $\tilde{h}_3$ timelike and $\alpha+\delta\neq 0,\alpha\gamma=0$. One can obtain the following two Lemmas on the expressions of Yano connection and the associated curvatures in the seventh Lorentzian Lie group.

Lemma 13. *The Yano connection ∇^Y of G_7 is given by*

$$
\begin{aligned}
&\nabla^Y_{\tilde{h}_1}\tilde{h}_1=\alpha\tilde{h}_2,\ \nabla^Y_{\tilde{h}_1}\tilde{h}_2=-\alpha\tilde{h}_1-\beta\tilde{h}_3,\ \nabla^Y_{\tilde{h}_1}\tilde{h}_3=\beta\tilde{h}_3,\\
&\nabla^Y_{\tilde{h}_2}\tilde{h}_1=\beta\tilde{h}_2+\beta\tilde{h}_3,\ \nabla^Y_{\tilde{h}_2}\tilde{h}_2=-\beta\tilde{h}_1,\ \nabla^Y_{\tilde{h}_2}\tilde{h}_3=\delta\tilde{h}_3,\\
&\nabla^Y_{\tilde{h}_3}\tilde{h}_1=-\alpha\tilde{h}_1-\beta\tilde{h}_2,\ \nabla^Y_{\tilde{h}_3}\tilde{h}_2=-\gamma\tilde{h}_1-\delta\tilde{h}_2,\ \nabla^Y_{\tilde{h}_3}\tilde{h}_3=0.
\end{aligned}
$$

Lemma 14. *The curvature R^Y of the Yano connection ∇^Y of (G_7,g) is given by*

$$
\begin{aligned}
&R^Y(\tilde{h}_1,\tilde{h}_2)\tilde{h}_1=-\alpha\beta\tilde{h}_1+\alpha^2\tilde{h}_2+\beta\tilde{h}_3,\ R^Y(\tilde{h}_1,\tilde{h}_2)\tilde{h}_2=-(\alpha^2+\beta^2+\beta\gamma)\tilde{h}_1-\beta\delta\tilde{h}_2+\beta\delta\tilde{h}_3,\\
&R^Y(\tilde{h}_1,\tilde{h}_2)\tilde{h}_3=(\beta\delta+\alpha\beta)\tilde{h}_3,\ R^Y(\tilde{h}_1,\tilde{h}_3)\tilde{h}_1=(2\alpha\beta+\alpha\gamma)\tilde{h}_1+(\alpha\delta-2\alpha^2)\tilde{h}_2,\\
&R^Y(\tilde{h}_1,\tilde{h}_3)\tilde{h}_2=(\beta^2+\beta\gamma+\alpha\delta)\tilde{h}_1+(\beta\delta-\alpha\gamma)\tilde{h}_2+(\beta\delta+\alpha\beta)\tilde{h}_3,\ R^Y(\tilde{h}_1,\tilde{h}_3)\tilde{h}_3=-(\alpha\beta+\beta\delta)\tilde{h}_3,\\
&R^Y(\tilde{h}_2,\tilde{h}_3)\tilde{h}_1=(\beta^2+\beta\gamma+\alpha\delta)\tilde{h}_1+(\beta\delta-\alpha\beta-\alpha\gamma)\tilde{h}_2-(\alpha\beta+\beta\delta)\tilde{h}_3,\\
&R^Y(\tilde{h}_2,\tilde{h}_3)\tilde{h}_2=(2\beta\delta-\alpha\beta+\alpha\gamma+\gamma\delta)\tilde{h}_1+(\delta-\beta\gamma-\beta^2)\tilde{h}_2,\ R^Y(\tilde{h}_2,\tilde{h}_3)\tilde{h}_3=-(\beta\gamma+\delta^2)\tilde{h}_3.
\end{aligned}
$$

One can prove the following theorem on the Codazzi tensor in the seventh Lorentzian Lie group.

Theorem 7. *$\tilde{\rho}^Y$ is not a Codazzi tensor on (G_7,∇^Y) .*

Proof. By (3), we have

$$
\begin{aligned}
&\rho^Y(\tilde{h}_1,\tilde{h}_1)=-\alpha^2,\ \rho^Y(\tilde{h}_1,\tilde{h}_2)=-\alpha\beta,\ \rho^Y(\tilde{h}_1,\tilde{h}_3)=\alpha\beta+\beta\delta,\\
&\rho^Y(\tilde{h}_2,\tilde{h}_1)=\beta\delta,\ \rho^Y(\tilde{h}_2,\tilde{h}_2)=-\alpha^2-\beta^2-\beta\gamma,\ \rho^Y(\tilde{h}_2,\tilde{h}_3)=\beta\gamma+\delta^2,\\
&\rho^Y(\tilde{h}_3,\tilde{h}_1)=\alpha\beta+\beta\delta,\ \rho^Y(\tilde{h}_3,\tilde{h}_2)=\alpha\delta+\delta,\ \rho^Y(\tilde{h}_3,\tilde{h}_3)=0.
\end{aligned}
\tag{14}
$$

Then

$$\tilde{\rho}^Y(\tilde{h}_1,\tilde{h}_1) = -\alpha^2,\ \tilde{\rho}^Y(\tilde{h}_1,\tilde{h}_2) = \frac{\beta\delta-\alpha\beta}{2},\ \tilde{\rho}^Y(\tilde{h}_1,\tilde{h}_3) = \alpha\beta+\beta\delta,$$
$$\tilde{\rho}^Y(\tilde{h}_2,\tilde{h}_2) = -\alpha^2-\beta^2-\beta\gamma,\ \tilde{\rho}^Y(\tilde{h}_2,\tilde{h}_3) = \frac{\delta^2+\delta+\alpha\delta+\beta\gamma}{2},\ \tilde{\rho}^Y(\tilde{h}_3,\tilde{h}_3) = 0. \quad (15)$$

By (5), we have

$$\begin{aligned}
&(\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_1) = 2\alpha\beta^2+\beta^2\delta+\alpha\beta\gamma,\\
&(\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_1) = -3\beta^2\delta-\alpha\beta^2,\\
&(\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_2) = 2\alpha\beta\delta-\alpha^2\beta+\beta\delta^2+\beta\delta+\beta^2\gamma,\\
&(\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_2) = \beta^3+\frac{1}{2}(\beta^2\gamma-\beta\delta^2-\beta\delta-\alpha\beta\delta),\\
&(\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_3) = \alpha^2\beta+\frac{1}{2}(\alpha\beta\delta-\beta\delta^2-\beta\delta-\beta^2\gamma),\\
&(\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_3) = -\frac{1}{2}(3\alpha\beta\delta+3\beta\delta^2+\beta\delta+\beta^2\gamma),\\
&(\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_1) = -\alpha\beta^2-\beta^2\delta-\frac{1}{2}\alpha\delta^2-\frac{1}{2}\alpha\beta\gamma-\frac{1}{2}\alpha\delta-\frac{1}{2}\alpha^2\delta,\\
&(\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_1) = -2\alpha^3+\beta^2\delta-\alpha\beta^2,\\
&(\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_2) = -\frac{1}{2}(\alpha\beta\delta+\beta\delta^2+\beta\delta+\beta^2\gamma),\\
&(\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_2) = -\frac{1}{2}\alpha^2\beta-\beta\alpha^2-\beta^3-\beta^2\gamma-\alpha^2\gamma+\frac{1}{2}\beta\delta^2,\\
&(\nabla^Y_{\tilde{h}_1}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_3) = 0,\ (\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_1,\tilde{h}_3) = \frac{1}{2}(3\alpha\beta\delta+\beta\delta^2+\beta\delta+\beta^2\gamma)+\alpha^2\beta,\\
&(\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_1) = -\frac{1}{2}(3\alpha\beta\delta+3\beta\delta^2+\beta\delta+\beta^2\gamma),\ (\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_1) = \gamma+\beta,\\
&(\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_2) = -\frac{1}{2}\delta^3-\frac{1}{2}\delta^2-\frac{1}{2}\alpha\delta^2-\frac{1}{2}\beta\gamma\delta+\alpha\beta^2+\beta^2\delta,\\
&(\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_2) = 2\beta\delta\gamma-\alpha\beta\gamma+\delta^3-\delta^2,\ (\nabla^Y_{\tilde{h}_2}\tilde{\rho}^Y)(\tilde{h}_3,\tilde{h}_3) = 2\delta,\\
&(\nabla^Y_{\tilde{h}_3}\tilde{\rho}^Y)(\tilde{h}_2,\tilde{h}_3) = \alpha\beta\gamma+\frac{1}{2}(3\beta\gamma\delta+\delta^3+\delta^2+\alpha\delta^2).
\end{aligned}$$

Then, if $\tilde{\rho}^Y$ is a Codazzi tensor on (G_7,∇^Y), by (6) and (7), we have the following nine equations:

$$\left\{\begin{array}{l}
3\alpha\beta^2+4\beta^2\delta+\alpha\beta\gamma=0\\
3\alpha\beta\delta-2\alpha^2\beta+3\beta\delta^2+3\beta\delta+\beta^2\gamma-2\beta^3=0\\
\alpha^2\beta+2\alpha\beta\delta+\beta\delta^2=0\\
4\beta^2\delta+\alpha\delta^2+\alpha\beta\gamma+\alpha\delta+\alpha^2\delta-4\alpha^3=0\\
\alpha\beta\delta+2\beta\delta^2+\beta\delta-\alpha^2\beta-2\alpha^2\beta-2\beta^3-2\alpha^2\gamma=0\\
3\alpha\beta\delta+\beta\delta^2+\beta\delta+\beta^2\gamma-2\alpha^2\beta=0\\
3\alpha\beta\delta+3\beta\delta^2+\beta\delta+\beta^2\gamma+2\gamma+2\beta=0\\
3\delta^3-\delta^2+\alpha\delta^2+5\beta\delta\gamma-2\alpha\beta^2-2\beta^2\delta-2\alpha\beta\gamma=0\\
4\delta-2\alpha\beta\gamma-3\beta\gamma\delta-\delta^3-\delta^2-\alpha\delta^2=0.
\end{array}\right. \quad (16)$$

By solving (16), we obtain $\alpha+\delta=0$. However, this is impossible. □

4. Conclusions

We derive the expressions of Yano connection and the associated curvatures in seven Lorentzian Lie groups and complete the classification of three-dimensional Lorentzian Lie groups on which Ricci tensors associated with Yano connections are Codazzi tensors.

The main results are listed in Table 1 which shows the conditions that Ricci tensors associated with Yano connections are Codazzi tensors associated with Yano connections on $\{G_i\}_{i=1,2,...,7}$. We found that G_1 and G_7 do not have Codazzi tensors associated with Yano connections, G_2, G_3, G_4, G_5 and G_6 have Codazzi tensors associated with Yano connections. In the future, we plan to proceed to study quasi-statistical structure associated with Yano connections and solitons on Lorentzian Lie Groups combined with the results in [24–35].

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