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Quantifying Complementarity via Robustness of Asymmetry

Xin Lü

School of Physics and Electrical Engineering, Liupanshui Normal University, Liupanshui 553000, China; lyuxinlps@outlook.com

Abstract: Complementarity plays a central role in the conceptual development of quantum mechanics, and also provides practical applications in quantum information technologies. How to properly quantify it is an important problem in quantum foundations, and there exists different types of complementarity relations. In this paper, a complementarity relation is established with the robustness of asymmetry. Specifically, the two complementary aspects are quantified by applying the robustness of asymmetry corresponding to two cyclic groups whose generators are linked by the Fourier matrix. This complementarity relation is compared with known results and considered in other perspectives, especially its operational meaning regarding quantum state discrimination. We conclude that the internal asymmetry of quantum states is closely related to other fundamental concepts, such as complementarity and coherence, and it is possible to quantitatively investigate complementarity and quantum state discrimination using the robustness of asymmetry.

Keywords: robustness of asymmetry; quantum coherence; complementarity relations; quantum state discrimination

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Citation: Lü, X. Quantifying Complementarity via Robustness of Asymmetry. *Symmetry* **2022**, *14*, 1738. https://doi.org/10.3390/sym14081738

Academic Editor: Valeriy Sbitnev

Received: 19 July 2022
Accepted: 17 August 2022
Published: 20 August 2022

1. Introduction

Wave particle duality [1] is the most familiar manifestation of the Bohr's principle of complementarity [2]. By quantifying wave particle duality, it is possible to establish various kinds of complementarity relations [3–16], where the two complementary aspects are specified as particleness and waveness. The most well-known such relation is established by Greenberger and Yasin [4], Jaeger et al. [5] and Englert [6] as

$$P^2(\rho) + V^2(\rho) \leq 1, \tag{1}$$

for any 2×2 density matrix ρ. The complementarity relation (1) refers exclusively to the wave particle duality in two path interferometers, where the function P, which is called Predictability, quantifies the particleness; while the other function V, which is called Visibility, quantifies the waveness. There are many options to generalize relation (1) to general n-path interferometers, and notably, we have the Dürr's relation [8], the one-bet relation [10] and relations based on quantum state discrimination [15]. The links between such complementarity relations and fundamental concepts in quantum mechanics, such as entanglement and coherence, have already been addressed in the literature [9,11–21]. In particular, it is possible to add a third term that quantifies the entanglement in relation (1) to make the inequality an equality [9,11,22].

In this paper, we would like to obtain a general complementarity relation by considering the robustness of asymmetry, which is a systematic way of quantifying the degree of asymmetry of quantum states [23,24]. The robustness measure is an option of quantification in any resource theory [25], e.g., we have the robustness of entanglement [26,27] that is a proper measure of entanglement. Similarly, by considering asymmetry as a resource, we have the robustness of asymmetry that automatically satisfies the axioms of being a proper resource measure. In 2016, Napoli et al. employ it to quantify coherence by considering the symmetric group $U(1)$ of the phases [24]. Since coherence is naturally related to the wave-like behavior, what is lacking for a complementarity relation is the quantifier of the particleness, i.e., the Predictability P, which is about the asymmetry of the paths. Consequently, we will consider the robustness of path asymmetry as the measure of Particleness in Section 4, and in this way establish a complementarity relation. For pure states, this complementarity relation is nothing but the familiar one-bet relation [10], but with a completely different physical meaning. For mixed states, this complementarity relation is different from any known relations, and it would be interesting to relate it to measures of entanglement as in [9,11,20,21]. By applying the robustness of asymmetry to derive complementarity relations, we associate both Predictability and Visibility with clear physical significance in terms of the internal symmetry of the quantum state, and therefore add new insights into the problem of quantifying complementarity. Moreover, minimum-error quantum state discrimination plays an important role in the derivation. In particular, we will show in Corollary 1 that the computation of the robustness of asymmetry can be converted to compute the success probability of minimum-error discrimination of a certain equiprobable collection of quantum states, and vice versa. There are various experimental demonstrations of minimum-error discrimination [28–31], so that it is possible to experimentally verify the complementarity relation in terms of the robustness of asymmetry. Our work demonstrates that there are intimate links between the concepts of complementarity and asymmetry, and as tools of quantifying, quantum state discrimination and the robustness of asymmetry are also closely related. This connection employed in this paper may provide further applications in other fields.

The rest of this paper is organized as follows. A brief description of the robustness of asymmetry, including its definition and main properties, is given in Section 2. As an example, the robustness of coherence is also introduced in this section. The problem of minimum-error discrimination and its link with the robustness of asymmetry is discussed in Section 3. We summarize the main result as Corollary 1, and use it to give a new proof of the fact that the robustness of coherence is the same as the l_1-norm of coherence for pure states. Examples of known complementarity relations are presented in Section 4, where we also define the Visibility and Predictability in terms of the robustness of asymmetry, and in this way construct the corresponding complementarity relation. We then conclude this paper with a summary.

2. Robustness of Asymmetry

Robustness of asymmetry, which measures the asymmetry of quantum states with respect to a unitary representation $\{U_g, g \in G\}$ of a symmetric group G, is introduced in [23]. By definition, for any quantum state ρ in a given Hilbert space $\mathcal{H}$, the robustness of asymmetry of ρ is

$$R_G(\rho) = \min_{\tau \in \mathcal{D}(\mathcal{H})} \left\{ s \geq 0 \,\middle|\, \frac{\rho + s\tau}{1+s} \in \mathcal{I}_G \right\}, \tag{2}$$

where $\mathcal{D}(\mathcal{H})$ denotes the set of all quantum states in $\mathcal{H}$, and the invariance set $\mathcal{I}_G$ is the collection of all symmetric states with respect to $\{U_g, g \in G\}$, i.e., for any state $\sigma \in \mathcal{D}(\mathcal{H})$,

$$\sigma \in \mathcal{I}_G \iff \frac{1}{|G|} \sum_{g \in G} U_g \sigma U_g^\dagger = \sigma. \tag{3}$$

Alternatively, it can also be defined as

$$R_G(\rho) = \min_{\sigma \in \mathcal{I}_G} \left\{ s \geq 0 \,\middle|\, \rho \leq (1+s)\sigma \right\}, \tag{4}$$

which is often useful in actual computations.

As proved in [23], the robustness of asymmetry satisfies the following desirable properties:

(R1) R_G is bounded, i.e., $0 \leq R_G(\rho) \leq \dim(\mathcal{H}) - 1$ for any $\rho \in \mathcal{D}(\mathcal{H})$;

(R2) R_G is faithful, i.e., $R_G(\rho) = 0 \iff \rho \in \mathcal{I}_G$;
(R3) R_G is strongly monotone, i.e., for any measurement $\{M_i\}$ such that $M_i \mathcal{I}_G M_i^\dagger \in \mathcal{I}_G$, $R_G(\rho) \geq \sum_i p_i R_G(\rho_i)$ with $p_i = \mathrm{Tr}(M_i \rho M_i^\dagger)$ and $\rho_i = M_i \rho M_i^\dagger / p_i$;
(R4) R_G is convex, i.e., $R_G(\sum_i w_i \rho_i) \leq \sum_i R_G(w_i \rho_i)$, for any collection of states $\{\rho_i\}$ and probability distribution $\{w_i\}$.

These properties suggest that the robustness of asymmetry fits well into the framework of resource theory, in particular, the requirements of vanishing for free states, monotonicity and convexity for a valid quantifier of resources [25] are automatically satisfied with the robustness of asymmetry. Furthermore, the computation of the robustness of asymmetry can be achieved by semidefinite programming [32–34], so that an optimal asymmetry witness W with respect to the symmetric group G such that

$$\mathrm{Tr}(\sigma W) \geq 0 \iff \sigma \in \mathcal{I}_G, \text{ and } R_G(\rho) = \mathrm{Tr}(\rho W) \tag{5}$$

always exists, which is similar as the witness of entanglement [35–39].

An important application of the robustness of asymmetry is to quantify quantum coherence [24]. Specifically, for a n-dimensional Hilbert space $\mathcal{H} = \mathcal{C}^n$, consider the cyclic group $G_Z = \langle Z \rangle$ generated by the unitary matrix

$$Z = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega^{n-1} \end{pmatrix}, \ \omega = e^{2\pi \mathrm{i}/n}, \tag{6}$$

then R_Z (by abusing the notation, we denote $R_{\langle Z \rangle}$ simply by R_Z), which is called the robustness of coherence, is a valid measure of quantum coherence satisfying the following property [23]

$$\frac{c_{l_1}(\rho)}{n-1} \leq R_Z(\rho) \leq c_{l_1}(\rho) \text{ for any } \rho \in \mathcal{D}(\mathbb{C}^n), \tag{7}$$

where c_{l_1} denotes the l_1-norm of coherence [40], i.e.,

$$c_{l_1}(\rho) = \sum_{k \neq j} \sum_{j} |\rho_{jk}|, \tag{8}$$

and the upper bound in (7) is attained with any pure states. Properties (R2) to (R4) for any robustness of asymmetry guarantee that the robustness of coherence satisfies the requirement for a quantifier in the resource theory of coherence [40], and the maximum $n-1$ in property (R1) is reached by the pure states $\rho = \sum_{j,k} |j\rangle\langle k| e^{\mathrm{i}(\phi_j - \phi_k)}/n$ with arbitrary phase factors $\{\phi_j, j = 0, \ldots, n-1\}$.

3. Minimum-Error Discrimination

Given a collection of n quantum states $\{\rho_j, j = 0, \ldots, n-1\}$ with the prior probability distribution $\{w_j, j = 0, \ldots, n-1\}$, the problem of minimum-error discrimination is finding the optimal measurement $\mathfrak{M} = \{M_j, j = 0, \ldots, n-1\}$ such that the success probability

$$p_s = \sum_{j=1}^{n} w_j \mathrm{Tr}(M_j \rho_j M_j^\dagger) \tag{9}$$

is maximized. It can be referred to in the following scenario: Alice sends Bob a quantum state from a collection of states according to an *a priori* probability distribution. Bob knows both the collection of the states and the probability distribution, and he would like to guess the received state by performing quantum measurement, which is described by the Klaus operators $\mathfrak{M} = \{M_j\}$. As shown in Figure 1, when the sending state is $\rho_?$, the probability p_j for the j-th outcome is $p_j = \mathrm{Tr}(M_j \rho_? M_j^\dagger)$, and the state afterwards is $\tilde{\rho}_j = M_j \rho_? M_j^\dagger / p_j$.

Such guessing is correct only when $? = j$, consequently the success probability p_s defined in (9) is indeed the average probability of correct guessing. The problem of minimum-error discrimination is unsolved except in very few situations [41–51]. For example, it is known that the optimal measurement for discriminating arbitrary two states ρ_1 and ρ_2 with the prior probabilities w_1 and w_2 is

$$M_1 = (w_1\rho_1 - w_2\rho_2)_+^{1/2}, M_2 = (w_1\rho_1 - w_2\rho_2)_-^{1/2}, \tag{10}$$

where the subscripts $+$ and $-$ denote the positive and negative part of the matrix, respectively. The resulting success probability is then

$$p_s = \frac{1}{2} + \frac{1}{2}\mathrm{Tr}|w_1\rho_1 - w_2\rho_2|, \tag{11}$$

which is known as the Helstrom bound [43]. On the other hand, if there are more than two states to be discriminated, the problem of minimum-error discrimination is solved only when the states satisfy certain symmetry properties [45–50].

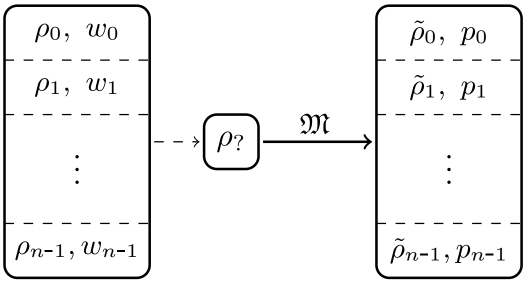

Figure 1. Scheme diagram of minimum-error discrimination. Alice selects one quantum state $\rho_?$ from a collection of quantum states $\{\rho_j\}$ according to the prior probabilities $\{w_j\}$, and sends it to Bob. With the knowledge of the collection of states and prior probabilities, Bob performs a measurement $\mathfrak{M} = \{M_j\}$ on the receiving state $\rho_?$, so that when the j-th outcome happens, he will guess that state to be ρ_j. The objective of minimum-error discrimination is to find the optimal measurement $\mathfrak{M}$ such that the probability p_s of correct guess, or the success probability, is maximized.

Minimum-error discrimination is closely related to the robustness of asymmetry introduced in the last section. Specifically, we have the following theorem that bounded the success probability p_s of minimum-error discrimination for a specified collection of quantum states with the corresponding robustness of asymmetry:

Theorem 1 ([23], Theorem 3)**.** *For any state ρ and unitary representation $\{U_g, g \in G\}$ of an arbitrary group G, the success probability p_s of minimum-error discrimination of the collection of states $\{U_g\rho U_g^\dagger, g \in G\}$ with the prior probability distribution $\{w_g, g \in G\}$ satisfies*

$$\max\left\{\frac{1}{|G|}(1 + R_G(\rho)), \max_{g\in G} w_g\right\} \le p_s \le (1 + R_G(\rho)) \max_{g\in G} w_g. \tag{12}$$

If we further restrict the prior probabilities to be all equal, i.e., $w_g = 1/|G|$ for any $g \in G$, then the computation of robustness of asymmetry is converted exactly to the computation of the success probability of the minimum-error discrimination of the associated collection of states, and vice versa. This fact is summarized in the following corollary, which plays a central role in the following discussions.

Corollary 1. *The success probability p_s of minimum-error quantum state discrimination of the equiprobable collection of states $\{U_g\rho U_g^\dagger, g \in G\}$ is*

$$p_s = \frac{1}{|G|}(1 + R_G(\rho)), \tag{13}$$

where $R_G(\rho)$ is the robustness of asymmetry of ρ with respect to the symmetric group $\{U_g, g \in G\}$.

Proof. The states are equiprobable implies that $w_g = 1/|G|$, $\forall g \in G$. Substituting this $\{w_g, g \in G\}$ into Theorem 1, and note that $\max_{g\in G} w_g = 1/|G|$, then both the lower bound and upper bound of p_s equal $(1 + R_G(\rho))/|G|$, and therefore the conclusion holds. □

As an example, we apply the Corollary 1 to show that for an arbitrary n-level pure state $\rho = |\phi\rangle\langle\phi| \in \mathcal{D}(\mathbb{C}^n)$, with

$$|\phi\rangle = \sum_{j=0}^{n-1} |j\rangle \sqrt{p_j} e^{\mathrm{i}\theta_j}, \text{ such that } \sum_j p_j = 1, \tag{14}$$

the robustness of coherence $R_Z(\rho)$ is the same as the l_1-norm of coherence defined in (8), i.e.,

$$R_Z(|\phi\rangle\langle\phi|) = \sum_{k\neq j}\sum_j |\rho_{jk}| = \sum_{k\neq j}\sum_j \sqrt{p_j p_k}. \tag{15}$$

Corollary 1 implies that $R_Z(\rho)$ can be related to the success probability p_s of minimum-error discrimination of the equiprobable collection of states

$$\{Z^l|\phi\rangle\langle\phi|Z^{-l}, l = 0, \ldots, n-1\}. \tag{16}$$

Such a discrimination problem is one of the few exceptions that have been rigorously solved; it is shown in [45,50] that the optimal measurement is the least squares measurement (LSM) [52–54]. More precisely, we have the following theorem

Theorem 2 ([45], Proposition 1)**.** *The optimal measurement for minimum-error discrimination of the equiprobable collection $\{|\phi_j\rangle = U^j|\phi\rangle, U^n = \mathbb{1}\}$ of n states generated by an arbitrary pure state $|\phi\rangle$ and a unitary matrix U of order n is the LSM given by the following Kraus measurement operators*

$$\{M_j = \Phi^{-1/2}|\phi_j\rangle\langle\phi_j|\Phi^{-1/2}, j = 0, \ldots, n-1\}, \tag{17}$$

with the Hermitian operator Φ defined as

$$\Phi = \sum_{j=0}^{n-1} |\phi_j\rangle\langle\phi_j|, \tag{18}$$

so that $[\Phi, U] = 0$. The corresponding success probability is then

$$p_s = \langle\phi|\Phi^{-1/2}|\phi\rangle^2. \tag{19}$$

Accordingly, the operator Φ defined in (18) for the ensemble (16) with the generator $|\phi\rangle$ of the form (14) and the unitary Z defined in (6) is

$$\Phi = \sum_{l=0}^{n-1} Z^l|\phi\rangle\langle\phi|Z^{-l} = \sum_{j,k=0}^{n-1} |j\rangle\langle k| \sqrt{p_j p_k} e^{\mathrm{i}(\theta_j - \theta_k)} \sum_{l=0}^{n-1} \omega^{l(j-k)} \tag{20}$$

$$= \sum_{j,k} |j\rangle\langle k| \sqrt{p_j p_k} e^{\mathrm{i}(\theta_j - \theta_k)} n\delta_{jk} = \sum_j |j\rangle\langle j| n p_j. \tag{21}$$

Substituting (14) and (21) back into (19), we obtain that

$$p_s = \frac{1}{n}\Big(\sum_j \sqrt{p_j}\Big)^2 = \frac{1}{n}\Big(1 + \sum_{k\neq j}\sum_j \sqrt{p_j p_k}\Big)$$
$$= \frac{1}{n}(1 + c_{l_1}(|\phi\rangle\langle\phi|)) \tag{22}$$

Comparing (13) and (22), we immediately have that $R_Z(|\phi\rangle\langle\phi|) = c_{l_1}(|\phi\rangle\langle\phi|)$, which is the desired conclusion. Although this fact has already been proven, the above proof is significantly simpler, and provides more physical insights: it suggests that there are deeper connections between the robustness of asymmetry and minimum-error discrimination. For example, since the robustness of asymmetry can often be easily computed through semidefinite programming [23], Corollary 1 can also be applied to compute the success probability p_s of minimum-error discrimination of certain equiprobable collections of states. Actually, there exists different ways to quantify asymmetry of quantum states, e.g., the relative entropy of asymmetry [55,56] or the asymmetry weight [57]. We choose the robustness of asymmetry, which is exactly because of its link to minimum-error discrimination, which provides it a clear operational meaning.

4. Quantifying Complementarity

In this section, we construct a complementarity relation using the robustness of asymmetry by considering the well-known wave particle duality. In such a relation, one quantifies the particleness and waveness using proper quantifiers, and establishes inequalities to manifest the complementarity principle. Specifically, using the orthonormal basis

$$\{|j\rangle, j = 0, \ldots, n-1\} \tag{23}$$

in the particle mode (the ket $|j\rangle$ is the state when the particle taking the j-th path) as the computational basis, then the quantifier P of the particleness, which is named as Predictability, is defined as a function of the diagonal entries $\{\rho_{jj}, j = 0, \ldots, n-1\}$ of the density matrix ρ; and the quantifier V of the waveness, which is named as Visibility, is defined as a function of the off-diagonal entries $\{\rho_{jk}, j \neq k\}$ of ρ in such basis. There exists different ways to define those quantifiers in the n-path interferometers, e.g., in the one-bet relation [10], the functions P and V are defined as

$$P_{\text{bet}}(\rho) = \frac{n \max_j \rho_{jj} - 1}{n-1}, \tag{24}$$
$$V_{\text{bet}}(\rho) = \max_H P_{\text{bet}}(H\rho H^\dagger), \tag{25}$$

where the maximum of V_{bet} is taken over all $n \times n$ Hadamard matrices, which directly manifests the complementarity between P and V; while in Dürr's relation [8]

$$P_{\text{dur}}(\rho) = \left[\frac{n}{n-1}\Big(\sum_j \rho_{jj}^2 - \frac{1}{n}\Big)\right]^{1/2}, \tag{26}$$
$$V_{\text{dur}}(\rho) = \left(\frac{n}{n-1}\sum_{k\neq j}\sum_j |\rho_{jk}|^2\right)^{1/2}. \tag{27}$$

Both the Predictability functions in (24) and (26) are defined by considering the path information, and both the Visibility functions in (25) and (27) are about the interference strength. The positivity of any quantum state $\rho \in \mathcal{D}(\mathbb{C}^n)$ implies immediately that

$$P_{\text{bet}}(\rho)^2 + V_{\text{bet}}(\rho)^2 \leq 1, \tag{28}$$
$$P_{\text{dur}}(\rho)^2 + V_{\text{dur}}(\rho)^2 \leq 1, \tag{29}$$

which are the complementarity relations corresponding to the pairs of quantifiers Equations (24) and (25) and Equations (26) and (27) respectively. It is also possible to establish complementarity relations in linear terms of P and V instead of the quadratic terms, as in (28) and (29). For example, by considering quantum state discrimination [15], one can define

$$P_{\text{qsd}}(\rho) = 1 - \frac{1}{n-1} \sum_{k \neq j} \sum_{j} \sqrt{\rho_{jj}\rho_{kk}}, \tag{30}$$

$$V_{\text{qsd}}(\rho) = \frac{1}{n-1} \sum_{k \neq j} \sum_{j} |\rho_{jk}|, \tag{31}$$

then the positivity of $\rho \in \mathcal{D}(\mathbb{C}^n)$ gives

$$P_{\text{qsd}}(\rho) + V_{\text{qsd}}(\rho) \leq 1. \tag{32}$$

Note that the Visibility defined in (31) is precisely the l_1-norm of coherence discussed in [40]

$$V(\rho) = \frac{1}{n-1} c_{l_1}(\rho), \ \rho \in \mathcal{D}(\mathbb{C}^n), \tag{33}$$

and it has been argued in previous works [12–16] that coherence is a valid measure of Visibility, which invites us to define Visibility using the robustness of coherence $R_Z(\rho)$ [24]. In particular, the normalized function $R_Z(\rho)/(n-1)$ satisfies the required properties of Visibility listed in [8,10]:

(V1) V is normalized, and $V(\rho) = 1 \iff \rho = \sum_{j,k} |j\rangle\langle k| e^{\mathrm{i}(\theta_j - \theta_k)}/n$;
(V2) V is faithful, i.e., $V(\rho) = 0 \iff \rho \in \mathcal{I}_Z$;
(V3) V is invariant under relabeling of the paths;
(V4) V is convex.

We immediately notice that the similarities between the criteria of a valid function of Visibility and the properties of robustness of asymmetry listed in Section 2. Actually, properties (R2) and (R4) are exactly the same as (V2) and (V4), and (R1), which guarantees that normalization is always possible. In order to demonstrate that (V3) also holds, it is convenient to introduce the permutation matrix

$$X = \sum_{j=0}^{n-1} |j \oplus 1\rangle\langle j| = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}, \tag{34}$$

where $\oplus$ denotes the addition modulo n. Using (34), the requirement (V3) can be expressed equivalently as

$$V(X^l \rho X^{-l}) = V(\rho) \text{ for any } l \in \mathbb{Z}. \tag{35}$$

Since $Z^l X^m = \omega^{lm} X^m Z^l$ with ω is defined in (6), i.e., the operators Z and X commute up to a phase factor, discriminating the equiprobable collection of states

$$\{Z^l X^m \rho X^{-m} Z^{-l}, l = 0, \ldots, n-1\} \tag{36}$$

is the same as discriminating the equiprobable collection $\{Z^l \rho Z^{-l}, l = 0, \ldots, n-1\}$. It is then obvious from the Corollary 1 that the robustness of coherence satisfies (35), or equivalently (V3). In conclusion, it is justified to define Visibility by the normalized robustness of coherence as

$$V_{\text{roa}}(\rho) = \frac{1}{n-1} R_Z(\rho), \ \rho \in \mathcal{D}(\mathbb{C}^n). \tag{37}$$

In order to define Predictability also with the robustness of asymmetry, one needs first a proper symmetric group. A natural choice is the cyclic group generated by X and defined in (34), since it is the generator obtained from Z through the Fourier transform

$$X = F^{\dagger} Z F, \tag{38}$$

where F is the matrix for the Fourier transform

$$F = \frac{1}{\sqrt{n}} \sum_{j,k=0}^{n-1} |j\rangle\langle k| \omega^{jk} = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \cdots & \omega^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \cdots & \omega^{(n-1)^2} \end{pmatrix} \tag{39}$$

with ω defined in (6). The corresponding invariant set is then

$$\mathcal{I}_X = \left\{ \rho \in \mathcal{D}(\mathbb{C}^n) \middle| \frac{1}{n} \sum_l X^l \rho (X^l)^{\dagger} = \rho \right\}. \tag{40}$$

Explicitly, the matrix element ρ_{jk} of $\rho \in \mathcal{I}_X$ satisfies that

$$\rho_{jk} = \rho_{j \oplus l, k \oplus l} \text{ with } l = 0, \ldots, n-1, \tag{41}$$

or in other words, any matrix $\rho \in \mathcal{I}_X$ is circulant [58]. In particular, all the diagonal entries of $\rho \in \mathcal{I}_X$ are the same, i.e.,

$$\rho_{jj} = \frac{1}{n} \text{ for any index } j. \tag{42}$$

Besides this formal consideration, we note that the symmetric group $\langle X \rangle$ also bears a clear physical significance: it represents all possible relabeling of the paths, and states that $\mathcal{I}_X$ are invariant under any permutations of the path labels, or in other words, those states are path symmetric. With that symmetric group, we similarly define Predictability as in (37)

$$P_{\text{roa}}(\rho) = \frac{1}{n-1} R_X(\rho_{\text{diag}}),\ \rho \in \mathcal{D}(\mathbb{C}^n), \tag{43}$$

where ρ_{diag} denotes the matrix obtained from ρ by setting all its off-diagonal entries to zero, i.e.,

$$\rho_{\text{diag}} = \sum_j |j\rangle\langle j|\rho|j\rangle\langle j|. \tag{44}$$

We consider ρ_{diag} instead of the original ρ because by setting the particle mode basis as the computational basis, the particle-like behavior depends solely on the diagonal entries of the density matrix ρ. It is then straightforward to verify that the function P_{roa} defined in (43) satisfies the requirements for Predictability [8,10]:

(P1) P is normalized, and $P(\rho) = 1 \iff \rho_{jj} = 1$ for one j;
(P2) P is faithful, i.e., $P(\rho) = 0 \iff \rho_{\text{diag}} \in \mathcal{I}_X$;
(P3) P is invariant under relabeling of the paths;
(P4) P is convex.

Again, properties (P2) and (P4) hold for any robustness of asymmetry, and (P3) holds trivially. Actually, the fact that P_{roa} defined in (43) satisfies all the above requirements from (P1) to (P4) can also be observed directly by relation (38) and V_{roa}, which satisfies (V1) to (V4). In order to obtain an explicit expression of P_{roa}, we apply the Corollary 1 that links the value of $R_X(\rho)$ to the success probability for discriminating the equiprobable collection of states $\{X^l \rho_{\text{diag}} X^{-l}, l = 0, \ldots, n-1\}$. Such a problem has already been solved in [15], and we have the following answer of its success probability

$$p_s = \max_j \rho_{jj}. \tag{45}$$

The Corollary 1 then implies that $R_X(\rho_{\text{diag}}) = n \max_j \rho_{jj} - 1$, or by definition (43)

$$P_{\text{roa}}(\rho) = \frac{n \max_j \rho_{jj} - 1}{n - 1}, \tag{46}$$

so that it coincides with the Predictability P_{bet} (24) in the one-bet relation.

As a result, we now have the pair $(V_{\text{roa}}, P_{\text{roa}})$ of Visibility and Predictability defined with the robustness of asymmetry as in (37) and (43), respectively. The Predictability P_{roa} defined in (43) is exactly the same as the Predictability P_{bet} in the one-bet relation discussed in various contexts [4,5,13–15]. Consequently, we have associated P_{bet} another physical significance in terms of the path asymmetry. As discussed before, the Visibility V_{roa} defined in (37) reduces to the l_1-norm of coherence (8) for pure states, so that for $\rho = |\phi\rangle\langle\phi|$

$$P_{\text{roa}}(\rho)^2 + V_{\text{roa}}(\rho)^2 = \frac{1}{(n-1)^2} R_X(\rho_{\text{diag}})^2 + \frac{1}{(n-1)^2} R_Z(\rho)^2 \tag{47}$$

$$= \frac{1}{(n-1)^2}\Big(n \max_j \rho_{jj} - 1\Big)^2 + \frac{1}{(n-1)^2}\Big(\sum_{k \neq j}\sum_j |\rho_{jk}|\Big)^2 \tag{48}$$

$$\leq 1, \tag{49}$$

where the maximal value of (48) happens with the following state (without loss of generality, it is assumed that the first diagonal entry p_{00} is the greatest, i.e., $\max_j \rho_{jj} = \rho_{00}$)

$$|\phi\rangle = |0\rangle p_{00} + \sum_{j \neq 0} |j\rangle \sqrt{\frac{1 - p_{00}}{n - 1}}. \tag{50}$$

An explicit calculation then demonstrates that (48) is upper bounded by 1, so that inequality (49) holds for any pure states. Moreover, the convexity properties (V4) and (P4) imply that for any mixed states $\rho = \sum_j w_j |\phi_j\rangle\langle\phi_j|$ with some probability distribution $\{w_j\}$,

$$P_{\text{roa}}(\rho)^2 + V_{\text{roa}}(\rho)^2 \leq \Big(\sum_j w_j P_{\text{roa}}(|\phi_j\rangle\langle\phi_j|)\Big)^2 + \Big(\sum_j w_j V_{\text{roa}}(|\phi_j\rangle\langle\phi_j|)\Big)^2 \tag{51}$$

$$\leq \sum_{j,k} w_j w_k = 1, \tag{52}$$

where inequality (52) is implied by applying the Cauchy–Schwartz inequality to the following pairs of vectors

$$\begin{pmatrix} P_{\text{roa}}(|\phi_j\rangle\langle\phi_j|) \\ V_{\text{roa}}(|\phi_j\rangle\langle\phi_j|) \end{pmatrix} \text{ and } \begin{pmatrix} P_{\text{roa}}(|\phi_k\rangle\langle\phi_k|) \\ V_{\text{roa}}(|\phi_k\rangle\langle\phi_k|) \end{pmatrix}, \tag{53}$$

and evoking the inequality (49) for pure states $\{|\phi_j\rangle\langle\phi_j|\}$. Accordingly, the inequality $P_{\text{roa}}(\rho)^2 + V_{\text{roa}}(\rho)^2 \leq 1$ with the quantifiers P_{roa} and V_{roa} defined in (43) and (37), respectively, holds for any states, and therefore is a proper complementarity relation. Although the physical motivation is completely different, this complementarity relation is very similar to the one-bet relation [10]; actually, it can be proved that they are exactly the same for pure states. For mixed states, the computation of the Visibility V_{bet} (33) has to consider Hadamard matrices, whose full parametrization is unknown in most dimensions, which makes this computation a notoriously difficult task. On the other hand, the computation of V_{roa} (37) can be easily carried out by semidefinite programming [23], which is the practical advantage of the new complementarity relation. It is hardly a surprise that the framework of the robustness of asymmetry can be applied to quantify wave-particle duality, since any knowledge of the particle aspect comes from the asymmetry of the paths, and the ability to interfere relies on the asymmetry of phases.

5. Conclusions

We have derived the following complementarity relation expressed by the robustness of asymmetry discussed in [23]

$$R_X(\rho_{\text{diag}})^2 + R_Z(\rho)^2 \leq (d-1)^2. \tag{54}$$

The fact that X and Z are linked by Fourier transform (38) suggests the complementarity between the two cyclic symmetric groups generated by these two generators. Consequently, although in deriving (54), we focused exclusively on the wave-particle duality as a concrete example; it holds for any pair of complementary observables that can be linked by the Fourier transform. It would be interesting if such relation can be generalized to a complete set of mutually unbiased bases. We have also proved the Corollary 1, which links minimum-error discrimination and the robustness of asymmetry; the usefulness of this Corollary is demonstrated by proving, for pure states, that the robustness of coherence coincides with the l_1-norm of coherence. As a tool of relating minimum-error discrimination and the robustness of asymmetry, we note that the Corollary 1 provides new methods of computing the success probability p_s of discriminating any equiprobable collection of states $\{U_g \rho U_g^\dagger, g \in G\}$ for some group G without explicitly constructing the optimal measurement, and makes it possible to consider the concept of robustness of asymmetry from the perspective of minimum-error discrimination. Our results suggest that it is possible to quantify complementarity by considering the internal asymmetry of quantum states, and that there exists deeper connections between minimum-error discrimination and the robustness of asymmetry.

Funding: This research was funded by the National Natural Science Foundation of China (Grant No. 12165012) and Research Fund of Liupanshui Normal University (Grant No. LPSSYKYJJ201813).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

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