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Subordination Results on the q-Analogue of the Sălăgean Differential Operator

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Abstract: Aspects related to applications in the geometric function theory of q-calculus are presented in this paper. The study concerns the investigation of certain q-analogue differential operators in order to obtain their geometrical properties, which could be further developed in studies. Several interesting properties of the q-analogue of the Sălăgean differential operator are obtained here by using the differential subordination method.

Keywords: analytic functions; q-derivative; q-analogue of the Sălăgean differential operator; differential subordination; best dominant

MSC: 30C45

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1. Introduction

q-calculus has become interesting to many researchers due to its various applications in mathematics, engineering sciences, and physics. Jackson [1,2] initiated the application of q-calculus by defining the q-derivative and q-integral. The idea of using the geometric function theory of q-calculus was first employed for introducing and studying an extension of the set of starlike functions in 1990 by Ismail et al. [3]. However, it was the book chapter written by Srivastava in 1989 [4], which provided the basic context for applying q-calculus in geometric function theory. It was also Srivastava who recently wrote a comprehensive review article [5], where the applications in geometric function theory of q-calculus are highlighted, and the numerous q-operators defined by many researchers using convolutional and fractional calculus are mentioned.

The geometrical interpretation of q-analysis involves studies of different q-analogue differential operators. The q-analogue of the well-known Ruscheweyh differential operator was defined in [6], and following this idea, the q-analogue of the Sălăgean differential operator was defined in [7]. Those operators provided interesting results when they were used to introduce new sets of univalent functions as seen in [8–10].

The differential subordination theory initiated by Miller and Mocanu [11,12] is introduced to obtain the main results of this article.

Following are the notations and definitions used in the investigations.

Let $\mathcal{A}_n$ be the set of analytic and univalent functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ written as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad a_k \in \mathbb{C},$$

and note that $\mathcal{A}_1 := \mathcal{A}$.

The class of starlike functions is defined as:

$$S^* = \left\{ f \in \mathcal{A} : \mathrm{Re}\frac{zf'(z)}{f(z)} > 0 \right\}.$$

For two functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ analytic in the open unit disc U, the Hadamard product (or convolution) of f and g, written as $(f * g)(z)$ is defined by

$$f(z) * g(z) = (f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

The analytic function f_1 is subordinate to the analytic function f_2, written $f_1 \prec f_2$, if there is an analytic Schwartz function w in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$ such that $f_1(z) = f_2(w(z))$, for $z \in \mathbb{U}$.

When the function f_2 is univalent in $\mathbb{U}$, there is the equivalence relation: $f_1 \prec f_2 \Leftrightarrow f_1(0) = f_2(0)$ and $f_1(\mathbb{U}) \subset f_2(\mathbb{U})$.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h be an univalent function in U. If p is analytic in U and satisfies the second order differential subordination

$$\psi(p(z), zp'(z), z^2 p''(z); z) \prec h(z), \quad z \in U, \tag{1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1). A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (1) is said to be the best dominant of (1). The best dominant is unique up to a rotation of U.

Following are the notions and notations of q-calculus.

For $0 < q < 1$, $n \in \mathbb{N}$, we denote

$$[n]_q = \frac{1 - q^n}{1 - q},$$

and

$$[n]_q! = \begin{cases} \prod_{k=1}^{n} [k]_q, & n \in \mathbb{N}^*, \\ 1, & n = 0. \end{cases}$$

The q-derivative operator $\mathcal{D}_q$ is defined for a function $f \in \mathcal{A}$ by [2]

$$\mathcal{D}_q(f(z)) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}$$

It can be observed that

$$\lim_{q \to 1} \mathcal{D}_q(f(z)) = \lim_{q \to 1} \frac{f(z) - f(qz)}{(1-q)z} = f'(z)$$

for f, a differentiable function.

For $f(z) = z^k$, $\mathcal{D}_q(f(z)) = \mathcal{D}_q\left(z^k\right) = \frac{1-q^k}{1-q} z^{k-1} = [k]_q z^{k-1}$.

The Sălăgean differential operator [13] can be written as $S^m f(z) = z + \sum_{k=2}^{\infty} k^m a_k z^k$ when $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{A}$, $z \in U$, $a_k \in \mathbb{C}$.

Definition 1 ([7])**.** *We denote by $\mathcal{S}_q^m$ the q-analogue of the Sălăgean differential operator*

$$\mathcal{S}_q^m f(z) = z + \sum_{k=2}^{\infty} [k]_q^m a_k z^k,$$

when $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{A}$, $z \in U$.

We notice that $\lim_{q \to 1} \mathcal{S}_q^m f(z) = \lim_{q \to 1} \left(z + \sum_{k=2}^{\infty} [k]_q^m a_k z^k\right) = z + \sum_{k=2}^{\infty} k^m a_k z^k = S^m f(z)$.

We can write $\mathcal{D}_q\left(\mathcal{S}_q^m f(z)\right) = 1 + \sum_{k=2}^{\infty}[k]_q^{m+1}a_k z^{k-1}$, and $z\mathcal{D}_q\left(\mathcal{S}_q^m f(z)\right) = z + \sum_{k=2}^{\infty}[k]_q^{m+1}a_k z^k$; therefore, the following identity holds for the operator $\mathcal{S}_q^m$:

$$z\mathcal{D}_q\left(\mathcal{S}_q^m f(z)\right) = z + \sum_{k=2}^{\infty}[k]_q^{m+1}a_k z^k = \mathcal{S}_q^{m+1} f(z).$$

Inspired by the results obtained in [14] using the q-analogue of Ruscheweyh operator, in this investigation, the differential subordination theory is used to obtain results involving the q-analogue of the Sălăgean differential operator. In the next section, we recall the results established by other researchers involved in the proofs of the original results of this paper. Then, in the main results section, the new subordination results involving the q-analogue of the Sălăgean differential operator are contained in three theorems and a corollary.

2. Preliminaries

The following lemmas are used for the proof of the original results of this paper.

Lemma 1 ([12])**.** *Let h be an analytic and convex univalent function in $\mathbb{U}$ with $h(0) = 1$ and $g(z) = 1 + b_1 z + b_2 z^2 + \dots$ analytic in $\mathbb{U}$. If*

$$g(z) + \frac{z\mathcal{D}_q(g(z))}{c} \prec h(z),\ z \in \mathbb{U},\ c \neq 0,$$

then

$$g(z) \prec \frac{c}{z^c}\int_0^z t^{c-1}h(t)dt,$$

for $Re(c) \geq 0$.

Lemma 2 ([15])**.** *Let u be an univalent function in $\mathbb{U}$ and θ, ϕ be analytic functions in a domain $D \supset q(\mathbb{U})$ with $\phi(w) \neq 0$ for $w \in q(\mathbb{U})$. Consider $Q(z) = z\mathcal{D}_q(u(z))\phi(u(z))$ and $h(z) = \theta(Q(z) + u(z))$ supposing that $Q(z)$ is a starlike univalent function in $\mathbb{U}$ and $Re\left(\frac{z\mathcal{D}_q(h(z))}{Q(z)}\right) = Re\left(\frac{\mathcal{D}_q(\theta(u(z)))}{\phi(u(z))}\right) + \frac{z\mathcal{D}_q(Q(z))}{Q(z)} > 0,\ z \in \mathbb{U}$.*

If $p(z)$ is an analytic function in $\mathbb{U}$ such that $p(\mathbb{U}) \subset D$, $p(0) = q(0)$ and

$$z\mathcal{D}_q(p(z))\phi(p(z)) + \theta(p(z)) \prec z\mathcal{D}_q(u(z))\phi(u(z)) + \theta(u(z)) = h(z),$$

then $p \prec u$, and the best dominant is u.

Lemma 3 ([16])**.** *The function $(1-z)^\gamma = e^{\gamma\log(1-z)}$, $\gamma \neq 0$, is univalent in $\mathbb{U}$ if and only if $|\gamma - 1| \leq 1$ or $|\gamma + 1| \leq 1$.*

Lemma 4 ([17])**.** *Consider the analytic functions f_i in $\mathbb{U}$ of the form $1 + b_1 z + b_2 z^2 + \dots$ that satisfy the inequality $Re(f_i) > \beta_i$, $0 \leq \beta_i < 1$, $i = 1, 2$. Then, $f_1 * f_2$ is an analytic function in $\mathbb{U}$ of the form $1 + b_1 z + b_2 z^2 + \dots$ that satisfies the inequality $Re(f_1 * f_2) > 1 - 2(1-\beta_1)(1-\beta_2)$.*

Lemma 5 ([18])**.** *Consider the analytic function $f(z) = 1 + b_1 z + b_2 z^2 + \dots$ with the property $Re(f(z)) > \beta$, $0 \leq \beta < 1$. Then,*

$$Re(f(z)) > 2\beta - 1 + \frac{2(1-\beta)}{1+|z|},\ z \in \mathbb{U}.$$

3. Main Results

Theorem 1. *If $f \in \mathcal{A}$, and*

$$(1-\alpha)\frac{\mathcal{S}_q^m f(z)}{z} + \alpha\frac{\mathcal{S}_q^{m+1} f(z)}{z} \prec \frac{1+Az}{1+Bz}, \tag{2}$$

for $\alpha > 0$, $-1 \le B < A \le 1$, $z \neq 0$, *then*

$$Re\left(\left(\frac{\mathcal{S}_q^m f(z)}{z}\right)^{\frac{1}{n}}\right) > \left(\frac{1}{\alpha q}\int_0^1 u^{\frac{1}{\alpha q}-1}\frac{1-Au}{1-Bu}du\right)^{\frac{1}{n}}, \ \ n \ge 1, \tag{3}$$

and the result is sharp.

Proof. Denote $p(z) = \frac{\mathcal{S}_q^m f(z)}{z} = 1 + b_1 z + \ldots$ for $f \in \mathcal{A}$, analytic in $\mathbb{U}$. Applying the logaritmic q-differentiation, we obtain

$$\mathcal{D}_q(p(z)) = \mathcal{D}_q\left(\frac{\mathcal{S}_q^m f(z)}{z}\right) =$$

$$\frac{z\mathcal{D}_q\left(\mathcal{S}_q^m f(z)\right) - \mathcal{S}_q^m f(z)}{z \cdot qz} = \frac{\mathcal{S}_q^{m+1} f(z) - \mathcal{S}_q^m f(z)}{qz^2}$$

and

$$\frac{z\mathcal{D}_q(p(z))}{p(z)} = \frac{z}{\mathcal{S}_q^m f(z)} \cdot \frac{\mathcal{S}_q^{m+1} f(z) - \mathcal{S}_q^m f(z)}{qz} = \frac{1}{q}\left(\frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)} - 1\right).$$

We obtain

$$\frac{qz\mathcal{D}_q(p(z))}{p(z)} + 1 = \frac{\mathcal{S}_q^{m+1} f(z)}{\mathcal{S}_q^m f(z)} = \frac{\frac{\mathcal{S}_q^{m+1} f(z)}{z}}{p(z)};$$

so,

$$\frac{\mathcal{S}_q^{m+1} f(z)}{z} = qz\mathcal{D}_q(p(z)) + p(z),$$

and

$$\begin{aligned}(1-\alpha)\frac{\mathcal{S}_q^m f(z)}{z} + \alpha\frac{\mathcal{S}_q^{m+1} f(z)}{z} &= (1-\alpha)p(z) + \alpha\big(qz\mathcal{D}_q(p(z)) + p(z)\big) \\ &= p(z) + \alpha qz\mathcal{D}_q(p(z)).\end{aligned}$$

The differential subordination (2) can be written as

$$p(z) + \alpha qz\mathcal{D}_q(p(z)) \prec \frac{1+Az}{1+Bz},$$

and applying Lemma 1, we find

$$p(z) \prec \frac{1}{\alpha q} z^{-\frac{1}{\alpha q}} \int_0^z t^{\frac{1}{\alpha q}-1}\frac{1+At}{1+Bt}dt,$$

or using the subordination concept

$$\frac{\mathcal{S}_q^m f(z)}{z} = \frac{1}{\alpha q}\int_0^1 u^{\frac{1}{\alpha q}-1}\frac{Auw(z)+1}{Buw(z)+1}du.$$

Taking into account that $-1 \le B < A \le 1$, we obtain

$$\mathrm{Re}\left(\frac{\mathcal{S}_q^m f(z)}{z}\right) > \frac{1}{\alpha q}\int_0^1 u^{\frac{1}{\alpha q}-1}\frac{1-Au}{1-Bu}du,$$

using the inequality $\mathrm{Re}\left(w^{\frac{1}{n}}\right) \ge (\mathrm{Re}w)^{\frac{1}{n}}$, for $\mathrm{Re}w > 0$ and $n \ge 1$.

To prove the sharpness of (3), we define $f \in \mathcal{A}$ by

$$\frac{\mathcal{S}_q^m f(z)}{z} = \frac{1}{\alpha q}\int_0^1 u^{\frac{1}{\alpha q}-1}\frac{1+Auz}{1+Buz}du.$$

For this function, we obtain

$$(1-\alpha)\frac{\mathcal{S}_q^m f(z)}{z} + \alpha\frac{\mathcal{S}_q^{m+1} f(z)}{z} = \frac{1+Az}{1+Bz}$$

and

$$\frac{\mathcal{S}_q^m f(z)}{z} \to \frac{1}{\alpha q}\int_0^1 u^{\frac{1}{\alpha q}-1}\frac{1-Au}{1-Bu}du \ \text{ as } z \to -1,$$

which completes the proof. □

Corollary 1. *If $f \in \mathcal{A}$, and*

$$(1-\alpha)\frac{\mathcal{S}_q^m f(z)}{z} + \alpha\frac{\mathcal{S}_q^{m+1} f(z)}{z} \prec \frac{1+(2\beta-1)z}{1+z}, \tag{4}$$

for $0 \le \beta < 1$, $\alpha > 0$, then

$$Re\left(\left(\frac{\mathcal{S}_q^m f(z)}{z}\right)^{\frac{1}{n}}\right) > \left((2\beta-1) + \frac{2(1-\beta)}{\alpha q}\int_0^1 \frac{u^{\frac{1}{\alpha q}-1}}{1+u}du\right)^{\frac{1}{n}}, \ \ n \ge 1.$$

Proof. Using the same steps as the Theorem 1 proof for $p(z) = \frac{\mathcal{S}_q^m f(z)}{z}$, the differential subordination (4) passes into

$$p(z) + \alpha q z \mathcal{D}_q(p(z)) \prec \frac{1+(2\beta-1)z}{1+z}.$$

Therefore,

$$\mathrm{Re}\left(\left(\frac{\mathcal{S}_q^m f(z)}{z}\right)^{\frac{1}{n}}\right) > \left(\frac{1}{\alpha q}\int_0^1 u^{\frac{1}{\alpha q}-1}\frac{1+(2\beta-1)u}{1+u}du\right)^{\frac{1}{n}} =$$

$$\left(\frac{1}{\alpha q}\int_0^1 u^{\frac{1}{\alpha q}-1}\left((2\beta-1) + \frac{2(1-\beta)}{1+u}\right)du\right)^{\frac{1}{n}} =$$

$$\left((2\beta-1) + \frac{2(1-\beta)}{\alpha q}\int_0^1 \frac{u^{\frac{1}{\alpha q}-1}}{1+u}du\right)^{\frac{1}{n}}.$$

□

Example 1. *Let $f(z) = z + z^2$, $m = 1$, $\alpha = 2$, $\beta = \frac{1}{2}$, and $n = 2$. Then, $\mathcal{S}_q^1 f(z) = z + [2]_q z^2 = z + (1+q)z^2$, and $\mathcal{S}_q^2 f(z) = z + [2]_q^2 z^2 = z + (1+q)^2 z^2$.*

We have $(1-\alpha)\frac{\mathcal{S}_q^m f(z)}{z} + \alpha\frac{\mathcal{S}_q^{m+1} f(z)}{z} = -\frac{\mathcal{S}_q^1 f(z)}{z} + 2\frac{\mathcal{S}_q^2 f(z)}{z} = 1 + \left(2q^2 + 3q + 1\right)z$. Applying Corollary 1, we obtain

$$1 + \left(2q^2 + 3q + 1\right)z \prec \frac{1}{1+z}, \ \ z \in U,$$

which induces

$$Re\sqrt{1+(1+q)z} > \sqrt{\frac{1}{2q}\int_0^1 \frac{u^{\frac{1}{2q}-1}}{1+u}du}, \ \ z \in U.$$

Theorem 2. *Let* $0 \le \rho < 1$ *and* $\gamma \in \mathbb{C}\backslash\{0\}$ *such that* $\left|\frac{2(1-\rho)\gamma}{q} - 1\right| \le 1$ *or* $\left|\frac{2(1-\rho)\gamma}{q} + 1\right| \le 1$. *If* $f \in \mathcal{A}$ *satisfies the condition*

$$Re\left(\frac{\mathcal{S}_q^{m+1}f(z)}{\mathcal{S}_q^m f(z)}\right) > \rho, \ z \in \mathbb{U},$$

then

$$\left(\frac{\mathcal{S}_q^m f(z)}{z}\right)^{\gamma} \prec \frac{1}{(1-z)^{\frac{2\gamma(1-\rho)}{q}}}, \ \ z \in \mathbb{U},$$

and the best dominant is $\frac{1}{(1-z)^{\frac{2\gamma(1-\rho)}{q}}}$.

Proof. Taking $p(z) = \left(\frac{\mathcal{S}_q^m f(z)}{z}\right)^{\gamma}$ and applying logaritmic q-differentiation, we obtain

$$\mathcal{D}_q(p(z)) = \gamma\left(\frac{\mathcal{S}_q^m f(z)}{z}\right)^{\gamma-1} \frac{\mathcal{S}_q^{m+1}f(z) - \mathcal{S}_q^m f(z)}{qz^2}$$

and

$$\frac{z\mathcal{D}_q(p(z))}{p(z)} = \frac{\gamma}{q}\left(\frac{\mathcal{S}_q^{m+1}f(z)}{\mathcal{S}_q^m f(z)} - 1\right).$$

We obtain

$$\frac{\mathcal{S}_q^{m+1}f(z)}{\mathcal{S}_q^m f(z)} = 1 + \frac{q}{\gamma}\frac{z\mathcal{D}_q(p(z))}{p(z)}.$$

Relation $\mathrm{Re}\left(\frac{\mathcal{S}_q^{m+1}f(z)}{\mathcal{S}_q^m f(z)}\right) > \rho$ can be written as

$$\frac{\mathcal{S}_q^{m+1}f(z)}{\mathcal{S}_q^m f(z)} \prec \frac{1+(1-2\rho)z}{1-z},$$

which is equivalent with

$$1 + \frac{q}{\gamma}\frac{z\mathcal{D}_q(p(z))}{p(z)} \prec \frac{1+(1-2\rho)z}{1-z}, \ \ z \in \mathbb{U}.$$

Assuming

$$u(z) = \frac{1}{(1-z)^{\frac{2(1-\rho)\gamma}{q}}}, \ \ \phi(w) = \frac{q}{\gamma w}, \ \ \theta(w) = 1,$$

we find that $u(z)$ is univalent from Lemma 3. It is easy to show that u, θ, and ϕ meet the conditions from Lemma 2. The function $Q(z) = z\mathcal{D}_q(u(z))\phi(u(z)) = \frac{2(1-\rho)z}{1-z}$ is starlike univalent in $\mathbb{U}$, and $h(z) = \theta(Q(z) + u(z)) = \frac{1+(1-2\rho)z}{1-z}$.

Applying Lemma 2, we finish the proof. □

Theorem 3. *Let $\alpha < 1$, $-1 \le B_i < A_i \le 1$, and $i = 1,2$. If the functions $f_i \in \mathcal{A}$ serve the differential subordination*

$$(1-\alpha)\frac{\mathcal{S}_q^m f_i(z)}{z} + \alpha\frac{\mathcal{S}_q^{m+1} f_i(z)}{z} \prec \frac{1+A_i z}{1+B_i z}, i = 1,2, \tag{5}$$

then

$$(1-\alpha)\frac{\mathcal{S}_q^m (f_1 * f_2)(z)}{z} + \alpha\frac{\mathcal{S}_q^{m+1} (f_1 * f_2)(z)}{z} \prec \frac{1+(1-2\gamma)z}{1+z},$$

where $*$ means the convolution product of f_1 and f_2, and

$$\gamma = 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left(1 - \frac{1}{\alpha q}\int_0^1 \frac{u^{\frac{1}{\alpha q}-1}}{1+u}du\right).$$

Proof. Let $h_i(z) = (1-\alpha)\frac{\mathcal{S}_q^m f_i(z)}{z} + \alpha\frac{\mathcal{S}_q^{m+1} f_i(z)}{z}$. The differential subordination (5) can be written as $\mathrm{Re}(h_i(z)) > \frac{1-A_i}{1-B_i}$, $i = 1,2$.

By Theorem 1's proof, we obtain

$$\mathcal{S}_q^m f_i(z) = \frac{1}{\alpha q}\int_0^1 t^{\frac{1}{\alpha q}-1} h_i(t)dt,\ i = 1,2,$$

and

$$\mathcal{S}_q^m (f_1 * f_2)(z) = \frac{1}{\alpha q} z^{1-\frac{1}{\alpha q}} \int_0^1 t^{\frac{1}{\alpha q}-1} h_0(t)dt,$$

with

$$h_0(z) = (1-\alpha)\frac{\mathcal{S}_q^m (f_1 * f_2)(z)}{z} + \alpha\frac{\mathcal{S}_q^{m+1} (f_1 * f_2)(z)}{z} =$$

$$\frac{1}{\alpha q} z^{1-\frac{1}{\alpha q}} \int_0^1 t^{\frac{1}{\alpha q}-1} (h_1 * h_2)(t)dt.$$

Applying Lemma 4, we obtain that $h_1 * h_2$ is a function analytic in $\mathbb{U}$ written as $1 + b_1 z + b_2 z^2 + \ldots$ that satisfies the inequality $\mathrm{Re}(h_1 * h_2) > 1 - 2(1-\beta_1)(1-\beta_2) = \beta$.

By Lemma 5, we obtain

$$\mathrm{Re}(h_0(z)) = \frac{1}{\alpha q}\int_0^1 u^{\frac{1}{\alpha q}-1}\mathrm{Re}(h_1 * h_2)(uz)du \ge$$

$$\frac{1}{\alpha q}\int_0^1 u^{\frac{1}{\alpha q}-1}\left(2\beta - 1 + \frac{2(1-\beta)}{1+u|z|}\right)du >$$

(since $z \in U \Rightarrow |z| < 1$ and $\frac{2(1-\beta)}{1+u|z|} > \frac{2(1-\beta)}{1+u}$)

$$\frac{1}{\alpha q}\int_0^1 u^{\frac{1}{\alpha q}-1}\left(2\beta - 1 + \frac{2(1-\beta)}{1+u}\right)du =$$

$$\frac{2\beta-1}{\alpha q}\frac{u^{\frac{1}{\alpha q}}}{\frac{1}{\alpha q}}\Big|_0^1 + \frac{2(1-\beta)}{\alpha q}\int_0^1 \frac{u^{\frac{1}{\alpha q}-1}}{1+u}du =$$

$$2\beta - 1 + \frac{2(1-\beta)}{\alpha q}\int_0^1 \frac{u^{\frac{1}{\alpha q}-1}}{1+u}du =$$

(we have $2\beta - 1 = 2 - 4(1-\beta_1)(1-\beta_2) - 1 = 1 - 4\left(1 - \frac{1-A_1}{1-B_1}\right)\left(1 - \frac{1-A_2}{1-B_2}\right) = 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}$ and $2(1-\beta) = 2(1 - 1 + 2(1-\beta_1)(1-\beta_2)) = \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}$)

$$1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)} + \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)} \frac{1}{\alpha q} \int_0^1 \frac{u^{\frac{1}{\alpha q}-1}}{1+u} du =$$

$$1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)} \left(1 - \frac{1}{\alpha q} \int_0^1 \frac{u^{\frac{1}{\alpha q}-1}}{1+u} du\right) = \gamma,$$

and the assertion of Theorem 3 holds. □

4. Conclusions

The investigation from this article followed the line of study set by introducing q-calculus to the theory of complex analysis. The q-analogue of the Sălăgean differential operator given in Definition 1 was previously introduced by Govindaraj and Sivasubramanian [7] and was used mainly for introducing new sets of univalent functions. In this article, it was used to obtain some subordination results. A sharp subordination result was presented in Theorem 1 followed by a corollary obtained using another particular function with important geometric properties applied in the subordination. Theorem 2 was obtained considering certain conditions on the real part of an expression involving the q-analogue of the Sălăgean differential operator, and the last theorem involved a convex combination using the q-analogue of the Sălăgean differential operator.

The results obtained during this research could be further used for writing sandwich-type results if the dual theory of differential superordination is added to the study of the q-analogue of the Sălăgean differential operator as calculated for other q-operators seen in [19] or [20].

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