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A Symmetric Form of the Mean Value Involving Non-Isomorphic Abelian Groups

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Abstract: Let $a(n)$ be the number of non-isomorphic abelian groups of order n. In this paper, we study a symmetric form of the average value with respect to $a(n)$ and prove an asymptotic formula. Furthermore, we study an analogue of the well-known Titchmarsh divisor problem involving $a(n)$.

Keywords: finite abelian group; symmetric average value; asymptotic formula; Bombieri–Vinogradov theorem; Titchmarsh divisor problem

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1. Introduction

Let $a(n)$ denote the number of non-isomorphic abelian groups of order n. The Dirichlet series of $a(n)$ is

$$\sum_{n=1}^{\infty} a(n)n^{-s} = \zeta(s)\zeta(2s)\zeta(3s)\cdots\ (\Re s > 1),$$

where $\zeta(s)$ is the Riemann zeta function. It is well-known that the arithmetical function $a(n)$ is multiplicative and satisfies the equality $a(p^\alpha) = P(\alpha)$ for any prime p and integer $\alpha \geq 1$, where $P(\alpha)$ is the number of partitions of α. Hence, for each prime number p, we have $a(p) = 1, a(p^2) = 2, a(p^3) = 3, a(p^4) = 5, a(p^5) = 7$.

A vast amount of literature exists on the asymptotic properties of $a(n)$. See, e.g., refs. [1,2] for historical surveys. The classical problem is to study the summatory function

$$A(x) := \sum_{n \leq x} a(n).$$

In 1935, Erdös and Szekeres [3] proved that

$$A(x) = A_1 x + O(x^{1/2}), \tag{1}$$

where $A_1 = \prod_{v=2}^{\infty} \zeta(v)$. Schwarz [4] showed that

$$A(x) = A_1 x + A_2 x^{1/2} + A_3 x^{1/3} + R(x),$$

with $R(x) \ll x^{\frac{3}{10} - \frac{7}{30\cdot 23}} (\log x)^{21/23}$ and $A_j = \prod_{v \neq j} \zeta(v/j)\ (j = 1, 2, 3)$. Many authors have investigated the upper bound of $R(x)$. For later improvements, see [5–7]. The best result to date is

$$R(x) \ll x^{1/4+\varepsilon} \tag{2}$$

for every $\varepsilon > 0$, proved by O. Robert and P. Sargos [8].

For an arithmetic function $f : \mathbb{N} \to \mathbb{N}$, and any integer $r > 1$, one can define

$$f^{(r)}(n) = f(f(\cdots f(n)\cdots))$$

as the r-th iterate of f. If $r \geq 2$ is fixed, then two among the most natural problems concerning $f^{(r)}(n)$ are an evaluation of the sums of $f^{(r)}(n)$ and the determination of the maximal order of $f^{(r)}(n)$. In the case of $f(n) = d(n)$, representing the Dirichlet divisor function, these problems were investigated by Erdös and Kátai [9,10]. In [10] it was shown that

$$\sum_{n\leq x} d^{(r)}(n) = (1+o(1))D_r\, x\log_r x \quad (D_r > 0, x \to \infty)$$

holds for $r = 4$, which was proved earlier by I. Kátai to also be true for $r = 2, 3$. Additionally, there has been work on the analogue of this problem for $a(n)$. A. Ivić [11] considered the 2nd iterate of $a(n)$ and proved that

$$\sum_{n\leq x} a(a(n)) = Cx + O(x^{1/2}\log^4 x).$$

for a suitable $C > 0$.

In 1986, C. Spiro [12] studied a new iteration problem involving the divisor function and obtained

$$\sum_{n\leq x, d(n+d(n))=d(n)} 1 \gg \frac{x}{(\log x)^7}. \tag{3}$$

In view of the work of C. Spiro, one can conjecture that, for some $D > 0$,

$$\sum_{n\leq x} d(n+d(n)) = Dx\log x + O(x). \tag{4}$$

However, it seems very difficult at present to determine the rationality of (4). A result analogous to (4) is much less difficult if $d(n)$ is replaced by $a(n)$, or a suitable prime-independent multiplicative function $f(n)$ such that $f(p) = 1$. This is roughly due to the fact that $d(p) = 2$ and $a(p) = 1$.

Inspired by (3), A. Ivić [13] pointed out an asymptotic formula for the symmetric sum

$$Q(x) := \sum_{n\leq x} a(n+a(n)),$$

and derived that the result

$$Q(x) = C_1 x + O(x^{11/12+\varepsilon}) \tag{5}$$

holds, for a positive constant C_1. Recently, Fan and Zhai [14] improved Ivić's result (5) and got

$$Q(x) = C_1 x + O(x^{3/4+\varepsilon}). \tag{6}$$

In this paper, we shall use a different approach to improve (6).

Let

$$D_k(x) = \sum_{p\leq x} d(p-k), \tag{7}$$

where p runs through all prime numbers greater than k, and $k \geq 1$ is a fixed integer. The Titchmarsh divisor problem is to understand the behavior of $D_k(x)$ as $x \to \infty$. So far we know very little concerning the properties of $p-k$, such as whether $p-2$ contains an infinity of primes; therefore, a problem regarding $p-k$ for which we can give some sort of answer makes some sense.

Assuming the generalized Riemann hypothesis, Titchmarsh [15] showed that

$$D_k(x) \sim E_1 x \tag{8}$$

with

$$E_1 = \frac{\zeta(2)\zeta(3)}{\zeta(6)}\prod_{p|k}\left(1-\frac{p}{p^2-p+1}\right).$$

In 1963, Linik [16] proved (8) unconditionally. Subsequently, Fouvry [17] and Bombieri et al. [18] gave a secondary term,

$$D_k(x) = E_1 x + E_2 Lix + O\left(\frac{x}{(\log x)^c}\right), \tag{9}$$

for all $c > 1$ and

$$E_2 = E_1\left(\gamma - \sum_p \frac{\log p}{p^2 - p + 1} + \sum_{p|a} \frac{p^2 \log p}{(p-1)(p^2 - p + 1)}\right),$$

where γ denotes the Euler–Mascheroni constant and $Li(x)$ is the logarithmic integral function. Motivated by the above results, we shall study an analogue of the Titchmarsh divisor problem for the symmetric form with regard to $a(n)$.

Our main plan is as follows. In Section 2, we state some important lemmas, and in Section 3, we prove the symmetric form of the mean value concerning non-isomorphic abelian groups. The analogue of the Titchmarsh divisor problem for $a(n + a(n))$ is given in Section 4, with the help of the well-known Bombieri–Vinogradov theorem. We note that the proofs of the two results are analogous; however, there are also differences in some details.

Notation. In this paper, $\mathcal{P}$ denotes the set of all prime numbers, ε always denotes a small enough positive constant. $\mu(n)$ denotes the Möbius function, $\varphi(n)$ denotes Euler's totient function, and $d(n)$ denotes the Dirichlet divisor function.

2. Some Preliminary Lemmas

In this section, we quote some lemmas used in this paper.

Lemma 1. *We have*

$$\limsup_{n\to\infty} \log a(n) \cdot \frac{\log\log n}{\log n} = \frac{\log 5}{4}.$$

Proof. See, for example, Krätzel [2]. □

Lemma 2. *For a positive number $u > 0$, let $S(u)$ denote the number of square-full numbers not exceeding u, then we have*

$$S(u) \ll u^{1/2}.$$

Proof. P. T. Bateman and E. Grosswald [19] proved that

$$S(u) = \frac{\zeta(\frac{3}{2})}{\zeta(3)} u^{\frac{1}{2}} + \frac{\zeta(\frac{2}{3})}{\zeta(2)} u^{\frac{1}{3}} + O(u^{\frac{1}{6}}), \tag{10}$$

then Lemma 2 follows from (10) immediately. □

Suppose $m \geq 1, (a, m) = 1$, and $1 \leq a < m$. In the next Lemma, we care about the average distribution of primes in arithmetic progressions. Define

$$\pi(x; m, a) = \sum_{\substack{p \leq x \\ p \equiv a \ (mod\ m)}} 1$$

and

$$E(x; m, a) = \pi(x; m, a) - \frac{Lix}{\varphi(m)}. \tag{11}$$

Lemma 3. *Suppose $x \geq 3$. For any given positive number $A > 1$, we have the estimate*

$$\sum_{m \leq M} d^4(m)a(m) \max_{y \leq x} \max_{(a,m)=1} |E(y; m, a)| \ll \frac{x}{(\log x)^A},$$

where $M = x^{1/2}(\log x)^{-B}$ with $B = 2A + 272$, the implied constant depending on A.

Proof. Let $\lambda = A + 257$. We write

$$\sum_{m \leq M} d^4(m)a(m) \max_{y \leq x} \max_{(a,m)=1} |E(y; m, a)| = S_1 + S_2, \tag{12}$$

where

$$S_1 := \sum_{\substack{m \leq M \\ d^4(m)a(m) > (\log x)^\lambda}} d^4(m)a(m) \max_{y \leq x} \max_{(a,m)=1} |E(y; m, a)|,$$
$$S_2 := \sum_{\substack{m \leq M \\ d^4(m)a(m) \leq (\log x)^\lambda}} d^4(m)a(m) \max_{y \leq x} \max_{(a,m)=1} |E(y; m, a)|.$$

We estimate S_1 first. Trivially we have (note $y \leq x$)

$$|E(y; m, a)| \ll \frac{Lix}{\varphi(m)} + \sum_{\substack{n \leq x \\ n \equiv a (mod\ m)}} 1 \ll \frac{x}{\varphi(m) \log x} + \frac{x}{m} \ll \frac{x}{m},$$

where we used the estimate $\varphi(m) \gg m/ \log m$. Inserting the above bound into S_1, we see that

$$S_1 \ll \frac{x}{(\log x)^{\lambda-1}} \sum_{m \leq M} \frac{d^8(m)a^2(m)}{m}. \tag{13}$$

Suppose $s = \sigma + it$ with $\sigma > 1$. Since $d^8(m)a^2(m)$ is multiplicative, we have the following expression

$$\sum_{m=1}^{\infty} \frac{d^8(m)a^2(m)}{m^s} = \prod_p \left(1 + \frac{256}{p^s} + \frac{3^8 \cdot 2^2}{p^{2s}} + \cdots\right) = \zeta^{256}(s)G(s), \tag{14}$$

where $G(s)$ can be written as an infinite product, which is absolutely convergent for $\sigma > 1/2$. By the standard method of analytic number theory, we can obtain from (14) that

$$\sum_{m \leq M} \frac{d^8(m)a^2(m)}{m} \ll (\log M)^{256} \ll (\log x)^{256}. \tag{15}$$

From (13) and (15), we obtain

$$S_1 \ll \frac{x}{(\log x)^{\lambda-257}} \ll \frac{x}{(\log x)^A}. \tag{16}$$

Now, we estimate S_2. Let $A_1 > 1$ be any fixed real number. Then we have the estimate

$$\sum_{m \leq x^{1/2}(\log x)^{-B_1}} \max_{y \leq x} \max_{(a,m)=1} |E(y; m, a)| \ll \frac{x}{(\log x)^{A_1}}, \tag{17}$$

where $B_1 = A_1 + 15$. This is the well-known Bombieri–Vinogradov theorem. See Theorem 8.1 of [20].

Take $A_1 = 2A + 257$ and $B_1 = B = 2A + 272$ in (17). We have

$$S_2 \ll (\log x)^{\lambda} \sum_{m \le M} \max_{y \le x} \max_{(a,m)=1} |E(y;m,a)| \ll \frac{x}{(\log x)^{A_1 - \lambda}} \ll \frac{x}{(\log x)^{A}}. \tag{18}$$

Now, Lemma 3 follows from (12), (16) and (18). □

3. A Symmetric Form of Mean Value Concerning $a(n)$

In this section, we propose a symmetric form of mean value concerning $a(n)$. We have the following theorem.

Theorem 1. *For any $\varepsilon > 0$, we have the asymptotic formula*

$$Q(x) = C_1 x + O(x^{2/3+\varepsilon}), \tag{19}$$

where the O-constant relies only on ε.

Proof. We begin by noting that each natural number n can be uniquely written as $n = qs$ such that $(q,s) = 1$. We use this fact to obtain

$$Q(x) = \sum_{k_1 \le W(x)} \sum_{q_1 s_1 \le x,\, a(s_1) = k_1,\, (q_1, s_1) = 1} a(q_1 s_1 + k_1), \tag{20}$$

where q_1 is square-free, s_1 is square-full, and $W(x) := \max_{n \le x} a(n)$; the property that $a(n) = a(s(n))$ is also utilized, where $s(n)$ is the square-full part of n. Functions with this property were named s-functions; one can see [21] for more details. Taking advantage of the fact again, we have

$$Q(x) = \sum_{k_1 \le W(x)} \sum_{k_2 \le W(x + W(x))} k_2 \sum_{\substack{q_1 s_1 \le x,\, a(s_1) = k_1,\, (q_1, s_1) = 1 \\ q_1 s_1 + k_1 = q_2 s_2,\, a(s_2) = k_2,\, (q_2, s_2) = 1}} 1 \tag{21}$$

where q_2 is square-free and s_2 is square-full.

For convenience, we abbreviate the innermost sum of (21) as $S(k_1, k_2)$. Therefore the estimation of $Q(x)$ can be reduced to estimate $S(k_1, k_2)$.

3.1. Evaluation of the Sum $S(k_1, k_2)$

In this subsection, we shall study the sum $S(k_1, k_2)$. From the elementary relations

$$\mu^2(n) = \sum_{d^2 | n} \mu(d), \quad \sum_{d | n} \mu(d) = \begin{cases} 1 & n = 1 \\ 0 & n > 1, \end{cases} \tag{22}$$

we have

$$\begin{aligned} S(k_1, k_2) &= \sum_{q_1 s_1 \le x,\, a(s_1) = k_1,\, (q_1, s_1) = 1} 1 \sum_{q_1 s_1 + k_1 = q_2 s_2,\, a(s_2) = k_2,\, (q_2, s_2) = 1} 1 \\ &= \sum_{\substack{d_1^2 n_1 s_1 \le x,\, a(s_1) = k_1 \\ (d_1, s_1) = (n_1, s_1) = 1}} \mu(d_1) \sum_{\substack{d_1^2 n_1 s_1 + k_1 = d_2^2 n_2 s_2,\, a(s_2) = k_2 \\ (d_2, s_2) = (n_2, s_2) = 1}} \mu(d_2) \\ &= \sum_{\substack{d_1^2 d_1^* n_1^* s_1 \le x,\, a(s_1) = k_1 \\ (d_1, s_1) = 1,\, d_1^* | s_1}} \mu(d_1) \mu(d_1^*) \sum_{\substack{d_1^2 d_1^* n_1^* s_1 + k_1 = d_2^2 d_2^* n_2^* s_2,\, a(s_2) = k_2 \\ (d_2, s_2) = 1,\, d_2^* | s_2}} \mu(d_2) \mu(d_2^*). \end{aligned}$$

It follows that

$$S(k_1,k_2) = \sum_{m_1 n_1^* \le x} f_1(m_1) \sum_{m_1 n_1^* + k_1 = m_2 n_2^*} f_2(m_2),$$

where

$$f_1(m_1) := \sum_{\substack{m_1 = d_1^2 d_1^* s_1 \\ a(s_1)=k_1,\,(d_1,s_1)=1,\,d_1^* | s_1}} \mu(d_1)\,\mu(d_1^*), \tag{23}$$

and

$$f_2(m_2) := \sum_{\substack{m_2 = d_2^2 d_2^* s_2 \\ a(s_2)=k_2,\,(d_2,s_2)=1,\,d_2^* | s_2}} \mu(d_2)\,\mu(d_2^*). \tag{24}$$

Suppose $x^\varepsilon \ll y \ll x$ is a parameter to be determined later, we split the sum $S(k_1,k_2)$ into four parts:

$$S(k_1,k_2) = S_1 + S_2 + S_3 + S_4, \tag{25}$$

where

$$\begin{aligned}
S_1 &= \sum_{\substack{m_1 \le y,\, m_2 \le y \\ m_1 n_1^* \le x,\, m_1 n_1^* + k_1 = m_2 n_2^*}} f_1(m_1) f_2(m_2), \\
S_2 &= \sum_{\substack{y < m_1 \le \frac{x}{n_1^*},\, m_2 \le y \\ m_1 n_1^* + k_1 = m_2 n_2^*}} f_1(m_1) f_2(m_2) \\
S_3 &= \sum_{\substack{m_1 \le y,\, y < m_2 \le \frac{x+k_1}{n_2^*} \\ m_1 n_1^* \le x,\, m_1 n_1^* + k_1 = m_2 n_2^*}} f_1(m_1) f_2(m_2), \\
S_4 &= \sum_{\substack{y < m_1 \le \frac{x}{n_1^*},\, y < m_2 \le \frac{x+k_1}{n_2^*} \\ m_1 n_1^* + k_1 = m_2 n_2^*}} f_1(m_1) f_2(m_2).
\end{aligned}$$

Consider first the sum S_2. It is obviously seen that in the sum $S(k_1,k_2)$ m_1 and m_2 are both square-full; by noting that if d_1^* and d_2^* are not square-full, then $d_i^* s_i$ $(i = 1,2)$ is square-full due to $d_i^* | s_i$ $(i = 1,2)$. Let $d(a,b,c;n)$ denote the number of representations of an integer n in the form $n = n_1^a n_2^b n_3^c$. We know from the property of the 3-dimensional divisor problem

$$d(a,b,c;n) \ll n^\varepsilon. \tag{26}$$

Using (26) and the the definition of $f_i(m_i)$ $(i = 1,2)$, we obtain

$$f_i(m_i) \ll d(2,1,1;m_i) \ll m_i^\varepsilon \ll x^\varepsilon \ (i = 1,2). \tag{27}$$

From (27), Lemma 2, and partial summation, the sum S_2 can be estimated by

$$\begin{aligned}
&\ll x^\varepsilon \sum_{\substack{y < m_1 \le \frac{x}{n_1^*},\, m_2 \le y \\ m_1 n_1^* + k_1 = m_2 n_2^*}} 1 \\
&\ll \frac{x^{1+\varepsilon}}{y^{1/2}}.
\end{aligned} \tag{28}$$

Similar to the sum S_2, we can obtain

$$S_i \ll \frac{x^{1+\varepsilon}}{y^{1/2}} \ \ (i = 3,4). \tag{29}$$

Next, we evaluate the sum S_1. Recalling the definition of S_1, we have

$$S_1 = \sum_{\substack{m_1 \le y \\ m_2 \le y}} f_1(m_1) f_2(m_2) \sum_{\substack{m_1 n_1^* \le x \\ m_1 n_1^* + k_1 = m_2 n_2^*}} 1. \tag{30}$$

By observation, the innermost sum of (30) is equal to the solution of the following set of congruent equations

$$\begin{cases} n \equiv 0 \,(mod\, m_1) \\ n \equiv -k_1 \,(mod\, m_2) \end{cases} \tag{31}$$

for $n \le x$. The Chinese remainder theorem (for example, see [22]) reveals that (31) has a solution

$$n \equiv l \,(mod\, [m_1, m_2]) \tag{32}$$

for some l if and only if $(m_1, m_2) | k_1$. Let

$$\delta(k_1; m_1, m_2) = \begin{cases} 1, \ (m_1, m_2) \mid k_1, \\ 0, \ (m_1, m_2) \nmid k_1. \end{cases}$$

From (30) to (32),

$$S_1 = \sum_{\substack{m_1 \le y \\ m_2 \le y}} f_1(m_1) f_2(m_2) \delta(k_1; m_1, m_2) \sum_{\substack{n \le x \\ n \equiv l \,(mod\, [m_1, m_2])}} 1. \tag{33}$$

As for the innermost sum of (33), obviously we obtain

$$\sum_{\substack{n \le x \\ n \equiv l \,(mod\, [m_1, m_2])}} 1 = \frac{x}{[m_1, m_2]} + O(1). \tag{34}$$

From (33) and (34), we have by applying the fact both m_1 and m_2 are square-full and by using Lemma 2 that

$$S_1 = x \sum_{\substack{m_1 \le y \\ m_2 \le y}} \frac{f_1(m_1) f_2(m_2)}{[m_1, m_2]} \delta(k_1; m_1, m_2) + O(yx^{\varepsilon}). \tag{35}$$

Substituting (28), (29) and (35) into (25), we obtain

$$\begin{aligned} S(k_1, k_2) &= x \sum_{\substack{m_1 \le y \\ m_2 \le y}} \frac{f_1(m_1) f_2(m_2)}{[m_1, m_2]} \delta(k_1; m_1, m_2) + O(yx^{\varepsilon}) + O(x^{1+\varepsilon} y^{-1/2}) \\ &:= xB(y) + O(x^{2/3+\varepsilon}), \end{aligned} \tag{36}$$

for $y = x^{2/3}$ with

$$B(y) = \sum_{\substack{m_1 \le y \\ m_2 \le y}} \frac{f_1(m_1) f_2(m_2)}{[m_1, m_2]} \delta(k_1; m_1, m_2)$$

Now, we treat the sum $B(y)$. Unfolding variables, we obtain

$$
\begin{aligned}
B(y) &= \sum_{\substack{d_1^2 d_1^* s_1 \le y,\, a(s_1)=k_1 \\ (d_1,s_1)=1,\, d_1^* | s_1}} \sum_{\substack{d_2^2 d_2^* s_2 \le y,\, a(s_2)=k_2 \\ (d_2,s_2)=1,\, d_2^* | s_2}} \frac{\mu(d_1)\mu(d_1^*)\mu(d_2)\mu(d_2^*)\delta(k_1; d_1^2 d_1^* s_1, d_2^2 d_2^* s_2)}{[d_1^2 d_1^* s_1, d_2^2 d_2^* s_2]} \\
&= \sum_{\substack{d_1^2 d_1^* s_1 \le y,\, a(s_1)=k_1 \\ (d_1,s_1)=1,\, d_1^* | s_1}} \sum_{\substack{d_2^2 d_2^* s_2 \le y,\, a(s_2)=k_2 \\ (d_2,s_2)=1,\, d_2^* | s_2}} \frac{(d_1^2 d_1^* s_1, d_2^2 d_2^* s_2)\mu(d_1)\mu(d_1^*)\mu(d_2)\mu(d_2^*)}{d_1^2 d_1^* s_1 d_2^2 d_2^* s_2} \\
&\quad \times \delta(k_1; d_1^2 d_1^* s_1, d_2^2 d_2^* s_2) \\
&= \sum_{\substack{d_1^* s_1 \le y,\, a(s_1)=k_1 \\ d_1^* | s_1}} \frac{\mu(d_1^*)}{d_1^* s_1} \sum_{\substack{d_2^* s_2 \le y,\, a(s_2)=k_2 \\ d_2^* | s_2}} \frac{\mu(d_2^*)}{d_2^* s_2} \sum_{\substack{d_1 \le \sqrt{\frac{y}{d_1^* s_1}} \\ (d_1,s_1)=1}} \frac{\mu(d_1)}{d_1^2} \\
&\quad \times \sum_{\substack{d_2 \le \sqrt{\frac{y}{d_2^* s_2}} \\ (d_2,s_2)=1}} \frac{(d_1^2 d_1^* s_1, d_2^2 d_2^* s_2)\mu(d_2)\delta(k_1; d_1^2 d_1^* s_1, d_2^2 d_2^* s_2)}{d_2^2} \\
&= B_1 + O(B_2 x^{\varepsilon}) + O(B_3 x^{\varepsilon}),
\end{aligned}
\tag{37}
$$

where

$$
\begin{aligned}
B_1 &:= \sum_{\substack{d_1^* s_1 \le y,\, a(s_1)=k_1 \\ d_1^* | s_1}} \frac{\mu(d_1^*)}{d_1^* s_1} \sum_{\substack{d_2^* s_2 \le y,\, a(s_2)=k_2 \\ d_2^* | s_2}} \frac{\mu(d_2^*)}{d_2^* s_2} \sum_{\substack{d_1=1 \\ (d_1,s_1)=1}}^{\infty} \frac{\mu(d_1)}{d_1^2} \\
&\quad \times \sum_{\substack{d_2=1 \\ (d_2,s_2)=1}}^{\infty} \frac{(d_1^2 d_1^* s_1, d_2^2 d_2^* s_2)\mu(d_2)\delta(k_1; d_1^2 d_1^* s_1, d_2^2 d_2^* s_2)}{d_2^2} \qquad (38) \\
B_2 &:= \sum_{\substack{d_1^* s_1 \le y,\, a(s_1)=k_1 \\ d_1^* | s_1}} \frac{|\mu(d_1^*)|}{d_1^* s_1} \sum_{\substack{d_2^* s_2 \le y,\, a(s_2)=k_2 \\ d_2^* | s_2}} \frac{|\mu(d_2^*)|}{d_2^* s_2} \sum_{d_1 > \sqrt{\frac{y}{d_1^* s_1}}} \frac{|\mu(d_1)|}{d_1^2} \sum_{d_2=1}^{\infty} \frac{|\mu(d_2)|}{d_2^2} \\
B_3 &:= \sum_{\substack{d_1^* s_1 \le y,\, a(s_1)=k_1 \\ d_1^* | s_1}} \frac{|\mu(d_1^*)|}{d_1^* s_1} \sum_{\substack{d_2^* s_2 \le y,\, a(s_2)=k_2 \\ d_2^* | s_2}} \frac{|\mu(d_2^*)|}{d_2^* s_2} \sum_{d_1=1}^{\infty} \frac{|\mu(d_1)|}{d_1^2} \sum_{d_2 > \sqrt{\frac{y}{d_2^* s_2}}} \frac{|\mu(d_2)|}{d_2^2},
\end{aligned}
$$

and where in the last sum in B_2 and B_3, we use the fact $(d_1^2 d_1^* s_1, d_2^2 d_2^* s_2) < x^{\varepsilon}$, which follows from $(d_1^2 d_1^* s_1, d_2^2 d_2^* s_2) \mid k_1$.

Consider the sum B_2. From Lemma 2 and partial summation, we deduce that

$$
B_2 \ll \sum_{d_1^* s_1 \le y} \frac{1}{\sqrt{d_1^* s_1}\sqrt{y}} \sum_{d_2^* s_2 \le y} \frac{1}{d_2^* s_2} \ll y^{-\frac{1}{2}+\varepsilon}. \tag{39}
$$

Consider B_3. Following the same argument as (39), we obtain

$$
B_3 \ll y^{-\frac{1}{2}+\varepsilon}. \tag{40}
$$

We combine (37), (39) and (40), then

$$
B(y) = B_1 + O(y^{-\frac{1}{2}-\varepsilon}). \tag{41}
$$

On substituting (41) into (36), the result is

$$\begin{aligned} S(k_1,k_2) &= xB_1 + O(xy^{-\frac{1}{2}+\varepsilon}) + O(x^{\frac{2}{3}+\varepsilon}) \\ &= xB_1 + O(x^{\frac{2}{3}+\varepsilon}). \end{aligned} \tag{42}$$

3.2. *Proof of Theorem 1*

It follows from (21) and (42) that

$$Q(x) = xQ_1(x,y) + O(x^{2/3+\varepsilon}Q_2(x,y)), \tag{43}$$

where

$$\begin{aligned} Q_1(x,y) &:= \sum_{k_1 \le W(x)} \sum_{k_2 \le W(x+W(x))} k_2 B_1 \\ Q_2(x,y) &:= \sum_{k_1 \le W(x)} \sum_{k_2 \le W(x+W(x))} k_2. \end{aligned} \tag{44}$$

It is easy to prove

$$Q_2(x,y) \ll x^{\varepsilon}. \tag{45}$$

Hence, it remains to estimate $Q_1(x,y)$. Since $d_i^* s_i \le y\,(i=1,2)$, it holds that $s_i \le y\,(i=1,2)$ and $k_i \le W(y)\,(i=1,2)$, from which and also from (44), $Q_1(x,y)$ can be rewritten as

$$Q_1(x,y) := \sum_{k_1 \le W(y)} \sum_{k_2 \le W(y)} k_2 B_1. \tag{46}$$

Lemma 4. *Let y be a natural number, and $W(y) = \max_{n \le y} a(n)$. Denote by y_0 the smallest natural number not exceeding y such that $W(y) = a(y_0)$, then*

$$y_0 > y^{1-\varepsilon}.$$

Proof. From Lemma 1, we know that for any small positive constant $\varepsilon > 0$, the inequality

$$(1-0.1\varepsilon)\frac{\log 5}{4} \cdot \frac{\log y}{\log\log y} < \log W(y) < (1+0.1\varepsilon)\frac{\log 5}{4} \cdot \frac{\log y}{\log\log y}$$

holds for $y_0 > y^{1-\varepsilon}$. If $y_0 \le y^{1-\varepsilon}$, then

$$\log W(y) < (1+0.1\varepsilon)\frac{\log 5}{4}\frac{\log y^{1-\varepsilon}}{\log\log y^{1-\varepsilon}} = (1-0.9\varepsilon-0.1\varepsilon^2)\frac{\log 5}{4}\frac{\log y}{\log\log y + \log(1-\varepsilon)}.$$

The above two formulas imply that

$$(1-0.1\varepsilon)\frac{\log 5}{4} \cdot \frac{\log y}{\log\log y} < (1-0.9\varepsilon-0.1\varepsilon^2)\frac{\log 5}{4}\frac{\log y}{\log\log y}.$$

This is a contradiction if $\varepsilon > 0$ is small enough. So, we have $y_0 > y^{1-\varepsilon}$. □

Inserting (38) into (44) and expanding the range of k_1 and k_2 to infinity, we obtain

$$Q_1(x,y) = C_1 + O(\Sigma_1) + O(\Sigma_2), \tag{47}$$

where

$$C_1 = \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} k_2 \sum_{\substack{d_1^*s_1=1,\, a(s_1)=k_1\\ d_1^*|s_1}}^{\infty} \frac{\mu(d_1^*)}{d_1^*s_1} \sum_{\substack{d_2^*s_2=1,\, a(s_2)=k_2\\ d_2^*|s_2}}^{\infty} \frac{\mu(d_2^*)}{d_2^*s_2} \sum_{\substack{d_1=1\\ (d_1,s_1)=1}}^{\infty} \frac{\mu(d_1)}{d_1^2}$$
$$\times \sum_{\substack{d_2=1\\ (d_2,s_2)=1}}^{\infty} \frac{(d_1^2d_1^*s_1, d_2^2d_2^*s_2)\mu(d_2)\delta(k_1; d_1^2d_1^*s_1, d_2^2d_2^*s_2)}{d_2^2} \tag{48}$$
$$\Sigma_1 = \sum_{k_1>W(y)}\sum_{k_2=1}^{\infty} k_2 \sum_{d_1^*s_1>y_0} \frac{|\mu(d_1^*)|}{d_1^*s_1} \sum_{d_2^*s_2=1}^{\infty} \frac{|\mu(d_2^*)|}{d_2^*s_2} \sum_{d_1=1}^{\infty} \frac{|\mu(d_1)|}{d_1^2} \sum_{d_2=1}^{\infty} \frac{|\mu(d_2)|}{d_2^2},$$
$$\Sigma_2 = \sum_{k_1=1}^{\infty}\sum_{k_2>W(y)} k_2 \sum_{d_1^*s_1=1}^{\infty} \frac{|\mu(d_1^*)|}{d_1^*s_1} \sum_{d_2^*s_2>y_0} \frac{|\mu(d_2^*)|}{d_2^*s_2} \sum_{d_1=1}^{\infty} \frac{|\mu(d_1)|}{d_1^2} \sum_{d_2=1}^{\infty} \frac{|\mu(d_2)|}{d_2^2}.$$

We consider Σ_1. By using Lemma 4, we have

$$\begin{aligned}\Sigma_1 &\ll \sum_{k_1>W(y)}\sum_{k_2=1}^{\infty} k_2 \sum_{d_1^*s_1>y^{1-\varepsilon}} \frac{1}{d_1^*s_1} \sum_{d_2^*s_2=1}^{\infty} \frac{1}{d_2^*s_2}\\ &= \sum_{k_1>W(y)}\sum_{d_1^*s_1>y^{1-\varepsilon}} \frac{1}{d_1^*s_1} \cdot \sum_{k_2=1}^{\infty} k_2 \sum_{d_2^*s_2=1}^{\infty} \frac{1}{d_2^*s_2}\\ &= \Sigma_{11}\cdot\Sigma_{12},\end{aligned} \tag{49}$$

say. In view of the well-known upper bound $a(n) \ll n^{\varepsilon}$ and recalling that $d_i^*s_i$ $(i = 1, 2)$ is square-full, by partial summation, we have

$$\Sigma_{12} \ll \sum_{d_2^*s_2=1}^{\infty} \frac{s_2^{\varepsilon}}{d_2^*s_2} \ll 1. \tag{50}$$

However, we have by applying summation by parts and by using Lemma 2 that

$$\Sigma_{11} \ll y^{-\frac{1}{2}+\varepsilon}. \tag{51}$$

Gathering the three estimates above, we arrive at

$$\Sigma_1 \ll y^{-\frac{1}{2}+\varepsilon}. \tag{52}$$

As for Σ_2, we repeat the above argument to obtain

$$\Sigma_2 \ll y^{-\frac{1}{2}+\varepsilon}. \tag{53}$$

From (47), (52) and (53), we obtain

$$Q_1(x, y) = C_1 + O(y^{-\frac{1}{2}+\varepsilon}), \tag{54}$$

and by combining (43) and (45), we obtain

$$\begin{aligned}Q(x) &= C_1x + O(xy^{-\frac{1}{2}+\varepsilon}) + O(x^{2/3+\varepsilon})\\ &= C_1x + O(x^{2/3+\varepsilon})\end{aligned} \tag{55}$$

on recalling $y = x^{2/3}$. This completes the proof of Theorem 1. □

4. An Analogue of the Titchmarsh Divisor Problem

Let

$$P(x) := \sum_{p\le x} a(p-1+a(p-1)).$$

Motivated by the work of [18], we shall study the asymptotic behavior of $P(x)$. As an analogue of (9), we have the following.

Theorem 2. *Let $A>1$ be any fixed constant. We have*

$$P(x) = C_2\, Lix + O\left(\frac{x}{(\log x)^A}\right), \tag{56}$$

where C_2 is a constant defined by (75).

Proof. We start from the definition of $P(x)$. Since $p-1$ can be uniquely written as $p-1=q_1s_1$ such that $(q_1,s_1)=1$, we obtain

$$P(x) = \sum_{k_1\le W(x)} \sum_{\substack{q_1s_1\le x-1,\, a(s_1)=k_1,\, (q_1,s_1)=1\\ q_1s_1+1\in\mathcal{P}}} a(q_1s_1+k_1), \tag{57}$$

where q_1 is square-free, s_1 is square-full, $W(x) := \max_{p\le x} a(p-1)$, and $\mathcal{P}$ denotes the set of all primes. Here, the important property that $a(n)$ is an s-function is used. Using the fact that $q_1s_1+k_1$ can be uniquely written as $q_1s_1+k_1=q_2s_2$ such that $(q_2,s_2)=1$, we obtain

$$P(x) = \sum_{k_1\le W(x)} \sum_{k_2\le W(x+W(x))} k_2 \sum_{\substack{q_1s_1\le x-1,\, a(s_1)=k_1,\, (q_1,s_1)=1,\, q_1s_1+1\in\mathcal{P}\\ q_1s_1+k_1=q_2s_2,\, a(s_2)=k_2,\, (q_2,s_2)=1}} 1, \tag{58}$$

where q_2 is square-free and s_2 is square-full.

It suffices to consider the innermost sum of (58). For convenience, we abbreviate it as $T(k_1,k_2)$. By (22), we have

$$\begin{aligned} T(k_1,k_2) &= \sum_{\substack{q_1s_1\le x-1,\, a(s_1)=k_1,\, (q_1,s_1)=1\\ q_1s_1+1\in\mathcal{P}}} 1 \sum_{q_1s_1+k_1=q_2s_2,\, a(s_2)=k_2,\, (q_2,s_2)=1} 1 \\ &= \sum_{\substack{d_1^2n_1s_1\le x-1,\, a(s_1)=k_1\\ (d_1,s_1)=(n_1,s_1)=1,\, d_1^2n_1s_1+1\in\mathcal{P}}} \mu(d_1) \sum_{\substack{d_1^2n_1s_1+k_1=d_2^2n_2s_2,\, a(s_2)=k_2\\ (d_2,s_2)=(n_2,s_2)=1}} \mu(d_2) \\ &= \sum_{\substack{d_1^2d_1^*n_1^*s_1\le x-1,\, a(s_1)=k_1\\ (d_1,s_1)=1,\, d_1^*|s_1,\, d_1^2d_1^*n_1^*s_1+1\in\mathcal{P}}} \mu(d_1)\,\mu(d_1^*) \sum_{\substack{d_1^2d_1^*n_1^*s_1+k_1=d_2^2d_2^*n_2^*s_2,\, a(s_2)=k_2\\ (d_2,s_2)=1,\, d_2^*|s_2}} \mu(d_2)\,\mu(d_2^*). \end{aligned} \tag{59}$$

There holds that

$$T(k_1,k_2) = \sum_{\substack{m_1n_1^*\le x-1\\ m_1n_1^*+1\in\mathcal{P}}} f_1(m_1) \sum_{m_1n_1^*+k_1=m_2n_2^*} f_2(m_2). \tag{60}$$

where $f_1(m_1)$ and $f_2(m_2)$ were defined by (23) and (24), respectively.

Suppose $x^\varepsilon \ll z \ll x$ is a parameter to be determined later. We can write $T(k_1,k_2)$ as

$$T(k_1,k_2) = T_1+T_2+T_3+T_4, \tag{61}$$

where

$$T_1 = \sum_{\substack{m_1 \le z, m_2 \le z, m_1 n_1^* \le x-1 \\ m_1 n_1^* + 1 \in \mathcal{P}, m_1 n_1^* + k_1 = m_2 n_2^*}} f_1(m_1) f_2(m_2),$$
$$T_2 = \sum_{\substack{z < m_1 \le \frac{x-1}{n_1^*}, m_2 \le z \\ m_1 n_1^* + 1 \in \mathcal{P}, m_1 n_1^* + k_1 = m_2 n_2^*}} f_1(m_1) f_2(m_2),$$
$$T_3 = \sum_{\substack{m_1 \le z, z < m_2 \le \frac{x-1+k_1}{n_2^*}, m_1 n_1^* \le x-1 \\ m_1 n_1^* + 1 \in \mathcal{P}, m_1 n_1^* + k_1 = m_2 n_2^*}} f_1(m_1) f_2(m_2),$$
$$T_4 = \sum_{\substack{z < m_1 \le \frac{x-1}{n_1^*}, z < m_2 \le \frac{x-1+k_1}{n_2^*} \\ m_1 n_1^* + 1 \in \mathcal{P}, m_1 n_1^* + k_1 = m_2 n_2^*}} f_1(m_1) f_2(m_2).$$

By the same arguments as (28), we have

$$T_i \ll \frac{x^{1+\varepsilon}}{z^{1/2}} \quad (i = 2, 3, 4). \tag{62}$$

From (61) and (62), we obtain

$$T(k_1, k_2) = T_1 + O\left(\frac{x^{1+\varepsilon}}{z^{1/2}}\right). \tag{63}$$

Now, we evaluate the sum T_1. We have

$$T_1 = \sum_{\substack{m_1 \le z \\ m_2 \le z}} f_1(m_1) f_2(m_2) \sum_{\substack{m_1 n_1^* \le x-1, m_1 n_1^* + 1 \in \mathcal{P} \\ m_1 n_1^* + k_1 = m_2 n_2^*}} 1. \tag{64}$$

Note that the innermost sum in (64) is equal to the number of solutions of the following congruence equations

$$\begin{cases} p - 1 \equiv 0 \,(mod\, m_1) \\ p - 1 \equiv -k_1 \,(mod\, m_2) \end{cases} \tag{65}$$

for $p \le x$. By the Chinese remainder theorem, (65) has a solution

$$p \equiv t \,(mod\, [m_1, m_2]) \tag{66}$$

for some t satisfying $(t, [m_1, m_2]) = 1$ if and only if $(m_1, m_2) | k_1$. Let

$$\delta(k_1; m_1, m_2) = \begin{cases} 1, & (m_1, m_2) \mid k_1 \\ 0, & (m_1, m_2) \nmid k_1. \end{cases}$$

and

$$\lambda(t; m_1, m_2) = \begin{cases} 1, & (t, [m_1, m_2]) = 1 \\ 0, & (t, [m_1, m_2]) \neq 1. \end{cases}$$

Whence by (64)–(66) and (11), we have

$$\begin{aligned} T_1 &= \sum_{\substack{m_1 \le z \\ m_2 \le z}} f_1(m_1) f_2(m_2) \delta(k_1; m_1, m_2) \lambda(t; m_1, m_2) \sum_{\substack{p \le x \\ p \equiv t \,(mod\, [m_1, m_2])}} 1 \\ &= \sum_{\substack{m_1 \le z \\ m_2 \le z}} f_1(m_1) f_2(m_2) \delta(k_1; m_1, m_2) \lambda(t; m_1, m_2)\, \pi(x; [m_1, m_2], t) \\ &= \sum_{\substack{m_1 \le z \\ m_2 \le z}} f_1(m_1) f_2(m_2) \delta(k_1; m_1, m_2) \lambda(t; m_1, m_2) \frac{Lix}{\varphi([m_1, m_2])} \\ &\quad + \sum_{\substack{m_1 \le z \\ m_2 \le z}} f_1(m_1) f_2(m_2) \delta(k_1; m_1, m_2)\, \lambda(t; m_1, m_2)\, E(x; [m_1, m_2], t) \\ &:= T_{11} + T_{12}, \end{aligned} \tag{67}$$

say.

First, we consider the contribution of T_{11}. We can write

$$T_{11} = Lix \times G(z), \tag{68}$$

where

$$G(z) = \sum_{\substack{m_1 \le z \\ m_2 \le z}} \frac{f_1(m_1) f_2(m_2)}{\varphi([m_1, m_2])} \delta(k_1; m_1, m_2) \lambda(t; m_1, m_2). \tag{69}$$

Using the familiar bound $\varphi(q) \gg \frac{q}{\log q}$ for any $q > 0$, we compare (69) and (37) to obtain

$$G(z) = G_1 + O(G_2 x^{\varepsilon}) + O(G_3 x^{\varepsilon}), \tag{70}$$

where

$$\begin{aligned} G_1 &:= \sum_{\substack{d_1^* s_1 \le z, a(s_1) = k_1 \\ d_1^* | s_1, d_1^2 d_1^* s_1 n_1^* + 1 \in \mathcal{P}}} \mu(d_1^*) \sum_{\substack{d_2^* s_2 \le z, a(s_2) = k_2 \\ d_2^* | s_2}} \mu(d_2^*) \sum_{\substack{d_1 = 1 \\ (d_1, s_1) = 1}}^{\infty} \mu(d_1) \sum_{\substack{d_2 = 1 \\ (d_2, s_2) = 1}}^{\infty} \frac{\mu(d_2) \delta(k_1; d_1^2 d_1^* s_1, d_2^2 d_2^* s_2)}{\varphi([d_1^2 d_1^* s_1, d_2^2 d_2^* s_2])} \\ G_2 &:= \sum_{\substack{d_1^* s_1 \le z, a(s_1) = k_1 \\ d_1^* | s_1, d_1^2 d_1^* s_1 n_1^* + 1 \in \mathcal{P}}} \frac{|\mu(d_1^*)|}{d_1^* s_1} \sum_{\substack{d_2^* s_2 \le z, a(s_2) = k_2 \\ d_2^* | s_2}} \frac{|\mu(d_2^*)|}{d_2^* s_2} \sum_{d_1 > \sqrt{\frac{z}{d_1^* s_1}}} \frac{|\mu(d_1)|}{d_1^2} \sum_{d_2 = 1}^{\infty} \frac{|\mu(d_2)|}{d_2^2}, \\ G_3 &:= \sum_{\substack{d_1^* s_1 \le z, a(s_1) = k_1 \\ d_1^* | s_1, d_1^2 d_1^* s_1 n_1^* + 1 \in \mathcal{P}}} \frac{|\mu(d_1^*)|}{d_1^* s_1} \sum_{\substack{d_2^* s_2 \le z, a(s_2) = k_2 \\ d_2^* | s_2}} \frac{|\mu(d_2^*)|}{d_2^* s_2} \sum_{d_1 = 1}^{\infty} \frac{|\mu(d_1)|}{d_1^2} \sum_{d_2 > \sqrt{\frac{z}{d_2^* s_2}}} \frac{|\mu(d_2)|}{d_2^2}. \end{aligned} \tag{71}$$

We now have the following result by applying the same arguments as (39) and (40)

$$G_i \ll z^{-1/2+\varepsilon}\, (i = 2, 3). \tag{72}$$

Combining (68), (70) and (72),

$$T_{11} = Lix \times G_1 + O(xz^{-1/2+\varepsilon}). \tag{73}$$

From (58), (63), (67) and (73), we see that the contribution of T_{11} to $P(x)$ is

$$= Lix \sum_{k_1 \le W(x)} \sum_{k_2 \le W(x + W(x))} k_2\, G_1 + O(xz^{-1/2+\varepsilon}).$$

Note that the sum $\sum_{k_1\le W(x)}\sum_{k_2\le W(x+W(x))} k_2 G_1$ can be treated similarly to $xQ_1(x,y)$ in Section 3 (see (47)–(54)). So, we obtain that the contribution of T_{11} to $P(x)$ is

$$= C_2 Lix + O(xz^{-1/2+\varepsilon}), \tag{74}$$

which absorbed the effects of T_i $(i = 2,3,4)$ and

$$C_2 = \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} k_2 \sum_{\substack{d_1^* s_1=1,\, a(s_1)=k_1 \\ d_1^*|s_1, d_1^2 d_1^* s_1 n_1^*+1\in\mathcal{P}}}^{\infty} \mu(d_1^*) \sum_{\substack{d_2^* s_2=1,\, a(s_2)=k_2 \\ d_2^*|s_2}}^{\infty} \mu(d_2^*) \sum_{\substack{d_1=1 \\ (d_1,s_1)=1}}^{\infty} \mu(d_1) \tag{75}$$
$$\times \sum_{\substack{d_2=1 \\ (d_2,s_2)=1}}^{\infty} \frac{\mu(d_2)\delta(k_1; d_1^2 d_1^* s_1, d_2^2 d_2^* s_2)\lambda(t; d_1^2 d_1^* s_1, d_2^2 d_2^* s_2)}{\varphi([d_1^2 d_1^* s_1, d_2^2 d_2^* s_2])}.$$

Now, we study the contribution of T_{12}. Let $z = x^{1/5}$ and let $m = [m_1, m_2]$. It is easy to see that $m \le z^2 = x^{2/5}$ and

$$|f_1(m_1)| \le d_3(m_1) \le d^2(m_1) \le d^2(m), |f_2(m_2)| \le d_3(m_2) \le d^2(m_2) \le d^2(m).$$

So, by Lemma 3, we have

$$\begin{aligned}
&|\sum_{k_1\le W(x)}\sum_{k_2\le W(x+W(x))} k_2 T_{12}| \\
&\le \sum_{[m_1,m_2]\le z^2} d_3(m_1)d_3(m_2)a(m_2)\max_{y\le x}\max_{(t,[m_1,m_2])=1}|E(y;[m_1,m_2],t)| \qquad (76)\\
&\le \sum_{m\le x^{2/5}} d^4(m)a(m)\max_{y\le x}\max_{(t,m)=1}|E(y;m,t)| \\
&\le x(\log x)^{-A},
\end{aligned}$$

where $A > 1$ is any fixed number. Now, Theorem 2 follows from (61), (67), (74) and (76). □

5. Conclusions

In this paper, we established a symmetric form of the average value with regard to the non-isomorphic abelian groups based on the arithmetic structure of natural numbers. In addition, we studied an analogue of the Titchmarsh divisor problem for the symmetric form of $a(n)$ with the help of the modified Bombieri–Vinogradov theorem. We can easily generalize the results obtained for $a(n)$ to a class of functions that are "prime-independent".

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