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# Several Dynamic Properties for the gkCH Equation

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**Abstract:** In this paper, we focus on a generalized Camassa–Holm equation (also known as a gkCH equation), which includes both the Camassa–Holm equation and Novikov equation as two special cases. Because of the potential applications in physics, we will further investigate the properties of the equation from a mathematical point of view. More precisely, firstly, we give a new wave-breaking phenomenon. Then, we present the theorem of existence and uniqueness of global weak solutions for the equation, provided that the initial data satisfy certain sign conditions. Finally, we prove the Hölder continuity of a solution map for the equation.

**Keywords:** wave-breaking; Hölder continuity; global weak solutions

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## 1. Introduction

In this paper, we consider the Cauchy problem for a shallow water wave equation

$$u_t - u_{txx} + (k+2)u^k u_x - (k+1)u^{k-1}u_x u_{xx} - u^k u_{xxx} = 0, \tag{1}$$

$$u(0,x) = u_0(x), \tag{2}$$

for $t > 0$ and $x \in \mathbb{R}$, where the subscript denotes a partial derivative (such as $u_x = \frac{\partial u}{\partial x}$), $k \in N_0$ ($N_0$ denotes the set of nonnegative integers ), and $u = u(t,x)$ is connected with the average of horizontal velocity. Equation (1) first is found in [1] where it is regarded as a generalization of Camassa–Holm equation and is known as gkCH equation. It possesses many special properties, including single peakon, multi-peakon traveling wave solutions and conserved law

$$E(u(t)) = \int_{\mathbb{R}} (u^2 + u_x^2)dx = \int_{\mathbb{R}} (u_0^2 + u_{0x}^2)dx. \tag{3}$$

It is shown in [2] that the equation is well-posedness of global weak solutions in Sobolev spaces $H^s$ with $1 \leq s \leq 3/2$ given that initial value satisfies an associated sign condition. Guo and Wang [3] investigate blow-up criteria and blow-up phenomena under some conditions with different initial data. Himonas and Holliman [4] prove that the solution map of the equation is not uniformly continuous on the circle and on the line.

For $k = 1$, we obtain the integrable equation with quadratic nonlinear terms

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{4}$$

which was derived by Camassa and Holm [5] and by Fokas and Fuchssteiner [6]. It was called the Camassa–Holm equation, which describes the motion of shallow water waves. It admits many properties, including a Lax pair, a bi-Hamiltonian structure, and infinitely many conserved integrals [5]. In addition, it can be solved by the inverse scattering method. One of the remarkable features of the CH equation is that it has the single peakon solutions

$$u(t,x) = ce^{-|x-ct|}, c \in \mathbb{R}$$

and the multi-peakon solutions

$$u(t,x) = \sum_{i=1}^{N} p_i(t) e^{-|x-q_i(t)|},$$

where $p_i(t)$, $q_i(t)$ satisfy the Hamilton system [5]

$$\begin{cases} \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} = \sum_{i\neq j} p_i p_j sign(q_i - q_j) e^{|q_i - q_j|}, \\ \frac{dq_i}{dt} = -\frac{\partial H}{\partial p_i} = \sum_j p_j e^{|q_i - q_j|} \end{cases}$$

with the Hamiltonian $H = \frac{1}{2}\sum_{i,j=1}^{N} p_i p_j e^{|q_i|}$. It is shown that those peaked solitons were orbitally stable in the energy space [7]. Another remarkable feature of the CH equation is the so-called wave breaking phenomena, that is, the wave profile remains bounded while its slope becomes unbounded in finite time [8]. Hence, Equation (4) has attracted the attention of lots of researchers. The dynamic properties related to the equation can be found in [9–25] and the references therein.

For $k = 2$, we obtain the integrable equation with cubic nonlinear terms

$$u_t - u_{txx} + 4u^2 u_x - 3uu_x u_{xx} - u^2 u_{xxx} = 0, \tag{5}$$

which was derived by Vladimir Novikov in a symmetry classification of nonlocal PDEs [26] and was known as the Novikov equation. It is shown in [26] that, like the Camassa–Holm equation, Equation (5) possesses soliton solutions, infinitely many conserved quantities, a Lax pair in matrix form, and a bi-Hamiltonian structure. The conserved quantities

$$H_1[u(t)] = \int_{\mathbb{R}} (u^2 + u_x^2) dx$$

and

$$H_2[u(t)] = \int_{\mathbb{R}} (u^4 + 2u^2 u_x^2 - \frac{1}{3} u_x^4) dx$$

play an important role in the study of the dynamic properties related to Equation (5). More information about the Novikov equation can be found in Tiglay [27], Ni and Zhou [28], Wu and Yin [29,30], Yan, Li, and Zhang [31] Mi and Mu [32], and the references therein.

In this paper, because of the potential applications in physics, we will further investigate the properties of problems (1) and (2) from a mathematical point of view. Up to now, global weak solutions and the Hölder continuity seem not to have been investigated yet. The aim of this paper is to discuss the properties. More precisely, we first give a new wave-breaking phenomenon by using the inequality $\parallel u_x \parallel_{L^{2k}} \leq e^{c_0 t} \parallel u_{0x} \parallel_{L^{2k}} + c_0 t$ (see Lemma 1 ). Then, we rely on the approximation of the initial data $u_0^n$ , $L^{2k}$ estimate, and Helly's theorem to study the existence and uniqueness of global weak solutions for problems (1) and (2), provided that the initial data satisfy certain sign conditions. Finally, we prove the Hölder continuity of a solution map for the Cauchy problems (1) and (2). In our analysis, one problematic issue is that we have to deal with high order nonlinear term $u^{k-2}u_x^3$ to obtain an accurate estimate. Luckily, we overcome the problem by using some mathematical techniques and Lemma 2.

**Notations.** The space of all infinitely differentiable functions $\phi(t,x)$ with compact support in $[0,+\infty) \times \mathbb{R}$ is denoted by $C_0^\infty$. Let $L^p = L^p(\mathbb{R})(1 \leq p < +\infty)$ be the space of all measurable functions $h$ such that $\parallel h \parallel_{L^p}^p = \int_{\mathbb{R}} |h(t,x)|^p dx < \infty$. We define $L^\infty = L^\infty(\mathbb{R})$ with the standard norm $\parallel h \parallel_{L^\infty} = \inf_{m(e)=0} \sup_{x\in R\setminus e} |h(t,x)|$. For any real number $s$, $H^s = H^s(\mathbb{R})$ denotes the Sobolev space with the norm defined by

$$\parallel h \parallel_{H^s} = \left( \int_{\mathbb{R}} (1 + |\xi|^2)^s |\hat{h}(t,\xi)|^2 d\xi \right)^{\frac{1}{2}} < \infty,$$

where $\hat{h}(t,\xi) = \int_{\mathbb{R}} e^{-ix\xi} h(t,x) dx$.

We denote by $*$ the convolution. Let $c$ denote an arbitrary positive constant, and $f(x) \lesssim g(x)$ means that $f(x) \leq cg(x)$. Note that, if $G(x) := \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$, then $(1-\partial_x^2)^{-1}f = G * f$ for all $f \in L^2(\mathbb{R})$ and $G * (u - u_{xx}) = u$. Using this identity, we rewrite problems (1) and (2) as follows:

$$u_t + u^k u_x + \partial_x G * (u^{k+1} + \frac{2k-1}{2}u^{k-1}u_x^2) + G * (\frac{k-1}{2}u^{k-2}u_x^3) = 0, \tag{6}$$

$$u(0,x) = u_0(x), \tag{7}$$

for $t > 0$ and $x \in \mathbb{R}$, which is equivalent to

$$y_t + (k+1)u^{k-1}u_x y + u^k y_x = 0, \quad y = u - u_{xx}, \tag{8}$$

$$u(0,x) = u_0(x), \quad y_0 = u_0 - u_{0xx}. \tag{9}$$

## 2. Several Lemmas

In this section, several lemmas are collected as follows.

**Lemma 1** ([3] ($L^{2k}$ estimate))**.** *Given that $u_0 \in H^s$, $s \geq 3/2$, and $\| u_{0x} \|_{L^{2k}} < \infty$, $k = 2^p + 1$, $p$ is a nonnegative integer. Let $T$ be the lifespan of the solution to problems (1) and (2), then for $\forall t \in [0,T)$, the estimate*

$$\| u_x \|_{L^{2k}} \leq e^{c_0 t} \| u_{0x} \|_{L^{2k}} + c_0 t$$

*holds.*

**Lemma 2** ([3])**.** *Given that $u_0 \in H^s$, $s \geq 3/2$, and $\| u_{0x} \|_{L^{2k}} < \infty$, $k = 2^p + 1$, $p$ is nonnegative integer. Let $T$ be the lifespan of the solution to problem (1) and (2), then for $\forall t \in [0,T)$, the estimate*

$$\int_{\mathbb{R}} |u^{k-2}u_x^3| dx \leq \alpha := (e^{c_0 t} \| u_{0x} \|_{L^{2k}} + c_0 t)^3 \| u_0 \|_{H^1}^{k-2}$$

*holds.*

**Lemma 3** ($L^4$ estimate)**.** *Given that $u_0 \in H^s$, $s \geq 3/2$ and $\| u_0 \|_{L^4} < \infty$, $k = 2^p + 1$, $p$ is a nonnegative integer. Let $T$ be the lifespan of the solution to problems (1) and (2), then for $\forall t \in [0,T)$, the estimate*

$$\| u \|_{L^4} \leq c \| u_0 \|_{L^4} \tag{10}$$

*holds.*

**Proof.** Multiplying the first equation by $4u^3$ and integrating the resultant over $\mathbb{R}$, we obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}} u^4 dx + \int_{\mathbb{R}} 4u^{k+3}u_x dx \\ &+ \int_{\mathbb{R}} 4u^3 \left( \partial_x G * (u^{k+1} + \frac{2k-1}{2}u^{k-1}u_x^2) + G * (\frac{k-1}{2}u^{k-2}u_x^3) \right) dx = 0. \end{aligned} \tag{11}$$

Using Lemma 2, Yong's inequality, and Hölder's inequality, we arrive at

$$\frac{d}{dt}\int_{\mathbb{R}} u^4 dx \leq c \int_{\mathbb{R}} u^4 dx. \tag{12}$$

The Gronwall's inequality leads to

$$\| u \|_{L^4} \leq c \| u_0 \|_{L^4} . \tag{13}$$

□

## 3. Wave Breaking Phenomenon

Wave-breaking is defined as a particular type of blow-up, that is, the wave profile remains bounded while its slope becomes unbounded in finite time [8]. The blow-up mechanism not only presents fundamental importance from the mathematical point of view but also is of great physical interest, since it would help provide a key-mechanism for localizing energy in conservative systems by forming one or several small-scale spots. The purpose of this section is to study a wave-breaking phenomenon for problems (1) and (2).

The characteristic $q(t,x)$ relating to (1) and (2) is governed by

$$q_t(t,x) = u^k(t,q(t,x)), \quad t \in [0,T)$$

$$q(0,x) = x, \quad x \in \mathbb{R}.$$

Applying the classical results to the theory of ordinary differential equations, one can obtain that the characteristics $q(t,x) \in C^1([0,T) \times \mathbb{R})$ with $q_x(t,x) > 0$ for all $(t,x) \in [0,T) \times \mathbb{R}$. Furthermore, it is shown in [2] that the potential $y = u - u_{xx}$ satisfies

$$y(t,q(t,x))q_x^2(t,x) = y_0(x)e^{\int_0^t (k-1)u^{k-1}u_x(\tau,q(\tau,x))d\tau}. \tag{14}$$

**Lemma 4** ([3]). *Let $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$. Let $T > 0$ be the maximum existence time of the solution $u$ to the problem (1) and (2) with the initial data $u_0$. Then, the wave breaking occurs in finite time if and only if*

$$\lim_{t \to T^-} \inf_{x \in \mathbb{R}} u^{k-1}u_x = -\infty.$$

The main result of this section is stated as follows:

**Theorem 1.** *Let $\| u_{0x} \|_{L^{2k}} < \infty$, $k = 2^p + 1$, $p$ is a nonnegative integer, and $u_0 \in H^s(\mathbb{R})$ for $s > \frac{3}{2}$. Suppose that there exist some $x_2 \in \mathbb{R}$ such that $u_0^{k-1}u_{0x}(x_2) < -\sqrt{2b}$ Then, the wave breaking occurs in finite time $T^*$ with*

$$T^* = \frac{1}{\sqrt{2b}} \log\left(\frac{u^{k-1}u_{0x}(x_2) - \sqrt{2b}}{u^{k-1}u_{0x}(x_2) + \sqrt{2b}}\right), \tag{15}$$

*where $b = b(\alpha, k, \| u_0 \|_{H^1})$.*

**Proof.** Now, we prove the wave breaking phenomenon along the characteristics $q(t,x_2)$. From (6), it follows that

$$\begin{aligned} u'(t) = &-\partial_x(1-\partial_x^2)^{-1}\left(u^{k+1} + \frac{2k-1}{2}u^{k-1}u_x^2\right) \\ &-(1-\partial_x^2)^{-1}\left(\frac{k-1}{2}u^{k-2}u_x^3\right), \end{aligned} \tag{16}$$

and

$$\begin{aligned} u_x'(t) = &\frac{-1}{2}u^{k-1}u_x^2 + u^{k+1} - (1-\partial_x^2)^{-1}\left(u^{k+1} + \frac{2k-1}{2}u^{k-1}u_x^2\right) \\ &-\partial_x(1-\partial_x^2)^{-1}\left(\frac{k-1}{2}u^{k-2}u_x^3\right). \end{aligned} \tag{17}$$

Then, we deduce that

$$
\begin{aligned}
(u^{k-1}u_x)'(t) &= (k-1)u^{k-2}u_x u' + u^{k-1}u_x' \\
&= (k-1)u^{k-2}u_x\Big[-\partial_x(1-\partial_x^2)^{-1}\Big(u^{k+1}+\frac{2k-1}{2}u^{k-1}u_x^2\Big) \\
&\qquad -(1-\partial_x^2)^{-1}\Big(\frac{k-1}{2}u^{k-2}u_x^3\Big)\Big] \\
&\quad +u^{k-1}\Big[\frac{-1}{2}u^{k-1}u_x^2+u^{k+1}-(1-\partial_x^2)^{-1}\Big(u^{k+1}+\frac{2k-1}{2}u^{k-1}u_x^2\Big) \\
&\qquad -\partial_x(1-\partial_x^2)^{-1}\Big(\frac{k-1}{2}u^{k-2}u_x^3\Big)\Big] \\
&= \frac{-1}{2}u^{2(k-1)}u_x^2 - u^{k-1}\Big[(1-\partial_x^2)^{-1}\Big(u^{k+1}+\frac{2k-1}{2}u^{k-1}u_x^2\Big) \\
&\qquad +\partial_x(1-\partial_x^2)^{-1}\Big(\frac{k-1}{2}u^{k-2}u_x^3\Big)\Big] + u^{2k} \\
&\quad +(k-1)u^{k-2}u_x\Big[-\partial_x(1-\partial_x^2)^{-1}\Big(u^{k+1}+\frac{2k-1}{2}u^{k-1}u_x^2\Big) \\
&\qquad -(1-\partial_x^2)^{-1}\Big(\frac{k-1}{2}u^{k-2}u_x^3\Big)\Big].
\end{aligned} \tag{18}
$$

Setting $M(t) = u^{k-1}u_x(t, q(t, x_2))$ and using the Young's inequality, Lemmas 1 and 2, from (18), we obtain

$$M'(t) \le -\frac{1}{2}M^2 + b. \tag{19}$$

where $b = b(\alpha, k, \| u_0 \|_{H^1})$.

It is observed from assumption of Theorem 1 that $u_0^{k-1}u_{0x}(x_2) < -\sqrt{2b}$. Solving (19) results in

$$M \to -\infty \quad as \quad t \to T^*, \tag{20}$$

where $T^* = \frac{1}{\sqrt{2b}} \log\left(\frac{u_0^{k-1}u_{0x}(x_2)-\sqrt{2b}}{u_0^{k-1}u_{0x}(x_2)+\sqrt{2b}}\right)$. □

**Remark 1.** *Theorem 1 is different from that in [3]; it is a new wave-breaking phenomenon.*

## 4. Weak Solution

Because the solitons do not belong to the spaces $H^s(\mathbb{R})$ with $s > \frac{3}{2}$, it motivates us to carry out the study of weak solutions to the problem (1) and (2). Next, we first give the definition of a weak solution.

**Definition 1.** *Given initial data $u_0 \in H^s$, $s > \frac{3}{2}$, the function $u$ is said to be a weak solution to the initial-value problems (1) and (2), if it satisfies the following identity*

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}} u\varphi_t + \frac{1}{k+1}u^{k+1}\varphi_x + G*(u^{k+1}+\frac{2k-1}{2}u^{k-1}u_x^2)\varphi_x \\
&-G*(\frac{k-1}{2}u^{k-2}u_x^3)\varphi dxdt + \int_R u_0(x)\varphi(0,x)dx = 0
\end{aligned} \tag{21}
$$

*for any smooth test function $\varphi(t,x) \in C_c^\infty([0,T)\times\mathbb{R})$. If $u$ is a weak solution on $[0,T)$ for every $T > 0$, then it is called a global weak solution.*

Before proceeding with the proof, let us first present some lemmas that will be of use in our approach.

**Lemma 5.** *Let $u_0 \in H^3(\mathbb{R})$, and $y_0 = u_0 - u_{0xx}$ satisfies*

$$y_0 \le 0, \qquad x \in (-\infty, x_0) \quad \text{and} \quad y_0 \ge 0, \qquad x \in (x_0, +\infty),$$

*for some point $x_0 \in \mathbb{R}$, then the corresponding solution to problem (1) exists globally in time.*

**Lemma 6.** *Let $u_0 \in H^s(\mathbb{R}), s \ge 3$, and there exists $x_0 \in \mathbb{R}$ such that*

$$\begin{cases} y_0 \le 0, & \text{if} \quad x \le x_0, \\ y_0 \ge 0, & \text{if} \quad x \ge x_0. \end{cases}$$

*Let $u$ be the corresponding solution to problems (1) and (2) and set $y(t,x) = u(t,x) - u_{xx}(t,x)$. Then,*
*(i) $u_x(t,\cdot) \ge -\mid u(t,\cdot) \mid$ and $\| u \|^2_{H^1} = \| u_0 \|^2_{H^1}$.*
*(ii) $\| u_x \|_{L^\infty} \le \frac{1}{2} \| y(t,\cdot) \|_{L^1}$, $\| u \|_{L^1} \le \| y(t,\cdot) \|_{L^1}$ and $\| u_x \|_{L^1} \le \| y(t,\cdot) \|_{L^1}$.*
*(iii) If $y_0 \in L^1(\mathbb{R})$, then $y \in C^1(\mathbb{R}_+; L^1(\mathbb{R}))$ and*

$$\| y(t,\cdot) \|_{L^1} \le e^{\|u_0\|^k_{H^1} t} \| y_0(t,\cdot) \|_{L^1}.$$

**Proof.** Here, we omit the proof of Lemmas 5 and 6. The detailed proofs are referred to [10,33]. □

**Theorem 2.** *Let $u_0 \in H^1$ and $\| u_{0x} \|_{L^{2k}} < \infty$, $k = 2^p + 1$, $p$ is nonnegative integer, and $y_0 = (u_0 - u_{0xx}) \in M(\mathbb{R})$, assume that there is a $x_0 \in \mathbb{R}$ such that $\mathrm{supp} y_0^- \subset (-\infty, x_0)$ and $\mathrm{supp} y_0^+ \subset (x_0, \infty)$. Then, the Cauchy problem (1) has a unique solution $u \in W^{1,\infty}_{loc}(\mathbb{R}_+ \times \mathbb{R}) \bigcap L^\infty_{loc}(\mathbb{R}_+; H^1(\mathbb{R}))$ with initial data $u(0) = u_0$.*

**Proof of Existence.** The method of existence proof mainly depends on the approximation of the initial data $u_0^n$, $L^{2k}$ estimate, and Helly's theorem. Since there exists similarity between our proof and the previous proof in [10,33], here, we only give the frame of existence proof. The detailed proof of existence and uniqueness of global weak solution is referred to [10,33].

Step 1. A suitable approximation of the initial data $u_0 \in H^1(R)$ by smooth functions $u_0^n$ produces a sequence of global solutions $u^n(t,\cdot)$ of problem (6) and (7) in $H^3(\mathbb{R})$;

Step 2. A suitable priori estimate

$$\int_0^T \int_{\mathbb{R}} ([u^n(t,x)]^2 + [u^n_x(t,x)]^2 + [u^n_t(t,x)]^2) dxdt \le M,$$

where $M$ is a positive constant depending only on $\| G \|_{H^1(\mathbb{R})}$, $\| G \|_{L^2(\mathbb{R})}$, $\| G_x \|_{L^{2k}(\mathbb{R})}$ $\| y_0 \|_M$ and $T$, implies that there is a subsequence of $\{u^n\}$ which converges pointwise a.e. to a function $u \in H^1_{loc}(R_+ \times \mathbb{R})$ that satisfies problems (6) and (7) in the sense of distributions;

Step 3. An application of Lemmas 1–3 and 6 and the Arzela–Ascoli theorem shows that $u \in C_w(\mathbb{R}_+; H^1(\mathbb{R}))$, the space of continuous functions from $\mathbb{R}_+$ with values in $H^1(\mathbb{R})$ when the latter space is equipped with its weak topology;

Step 4. Establishing the strong continuity of the solution with respect to the temporal variable, As $u \in C_w(\mathbb{R}_+; H^1(\mathbb{R}))$, to conclude that $u \in C(\mathbb{R}_+; H^1(\mathbb{R}))$, it is enough to show that the functional $E(u(t)) = \int_{\mathbb{R}} (u^2 + u_x^2) dx$ is conserved in time by a regularization technique. □

**Remark 2.** *There exist many differences between weak solutions we discuss and that in [10,33]. First, the equation we discuss is more complex. Secondly, the assumptions of Theorem 3 is more difficult, such as it is necessary that $\| u_{0x} \|_{L^{2k}} < \infty$, $k = 2^p + 1$, and $p$ is a nonnegative integer.*

## 5. Hölder Continuity

It is shown in [4] that problems (1) and (2) are well-posed in Sobolev spaces $H^s$ on both the line and the circle for $s > 3/2$ and its data-to-solution map is continuous but not uniformly continuous. In this section, we will study the Hölder continuity of the solution map for the generalized Camassa–Holm Equation (1) in $H^r$ -topology for all $0 \leq r < s$. The main result is stated as follows.

**Theorem 3.** *Assume $s > \frac{3}{2}$ and $0 \leq r < s$. Then, the solution map to problems (1) and (2) is Hölder continuous on the space $H^s$ equipped with the $H^r$ norm, More precisely, for initial data $u(0)$ and $w(0)$ in a ball $B(0,h) := \{u \in H^s : \| u \|_{H^s} \leq h\}$ of $H^s$, the corresponding solution $u(t)$, $w(t)$ satisfies the inequality*

$$\| u(t) - w(t) \|_{C([0,T];H^r)} \leq c \| u(0) - w(0) \|_{H^r}^{\alpha}$$

*where $\alpha$ is given by*

$$\alpha = \begin{cases} 1, & \text{if} \quad (s,r) \in \Omega_1, \\ \frac{2(s-1)}{s-r}, & \text{if} \quad (s,r) \in \Omega_2, \\ s-r, & \text{if} \quad (s,r) \in \Omega_3 \end{cases}$$

*and the regions $\Omega_1$, $\Omega_2$, and $\Omega_3$ in the sr-plane are defined by*

$$\begin{aligned} \Omega_1 &= \{(s,r) : \quad s > \frac{3}{2}, \quad 0 \leq r \leq s-1, \quad r+s \geq 2\}, \\ \Omega_2 &= \{(s,r) : \quad \frac{3}{2} < s < 2, \quad 0 \leq r \leq 2-s\}, \\ \Omega_3 &= \{(s,r) : \quad s > \frac{3}{2}, \quad s-1 \leq r \leq s\}. \end{aligned}$$

*The lifespan $T$ and the constant $c$ depend on s,r, and h.*

To prove Theorem 3, we need the following lemmas.

**Lemma 7** ([4])**.** *Let $s > \frac{3}{2}$. If $u_0$ belongs to the Sobolev space $H^s$ on the circle or the line, then there exists $T = T(u_0) > 0$ and a unique solution $u \in C([0,T]; H^s)$ of the Cauchy problem for the gkCH Equation (1), which depends continuously on the initial data $u_0$. Furthermore, we have the following solution size and lifespan estimate*

$$\| u(t) \|_{H^s} \leq 2 \| u(0) \|_{H^s}, \text{for} \quad 0 \leq t \leq T \leq \frac{1}{2kc_s \| u(0) \|_{H^s}^k},$$

*where $c_s > 0$ is a constant, depending on s.*

**Lemma 8** ([34])**.** *For any $s > 0$, there is $c_s > 0$ such that*

$$\| fg \|_{H^s} \leq c_s \| f \|_{H^s} \| g \|_{L^\infty} + \| f \|_{L^\infty} \| g \|_{H^s} .$$

**Lemma 9** ([35])**.** *If $r + 1 \geq 0$, then*

$$\| [\Lambda^r \partial_x, f] v \|_{L^2} \leq c \| f \|_{H^s} \| v \|_{H^r},$$

*provided that $s > \frac{3}{2}$ and $r + 1 \leq s$.*

**Lemma 10** ([36])**.** *Let $r > \frac{1}{2}$, then*

$$\| fg \|_{H^{r-1}} \leq c_r \| f \|_{H^r} \| g \|_{H^{r-1}} .$$

**Lemma 11** ([36])**.** *Let $0 \leq r \leq 1$, $s > \frac{3}{2}$, and $r + s \geq 2$, then*

$$\| fg \|_{H^{r-1}} \leq c_{r,s} \| f \|_{H^{s-1}} \| g \|_{H^{r-1}} .$$

**Proof of Theorem 3.** We begin by writing the gkCH Equation (1) in the following convenient form:

$$u_t = -\frac{1}{k+1}\partial_x(u^{k+1}) - F(u), \tag{22}$$

$$F(u) = \partial_x\Lambda^{-2}(u^{k+1} + \frac{2k-1}{2}u^{k-1}u_x^2) + \Lambda^{-2}(\frac{k-1}{2}u^{k-2}u_x^3), \tag{23}$$

where $\Lambda = (1-\partial_x^2)^{\frac{1}{2}}$.

Let $u_0(x), w_0(x) \in B(0,h)$ and $u(t,x)$ and $w(t,x)$ be the two solutions to (22) and (23) with initial data $u_0(x)$ and $v_0(x)$, respectively. Define $v = u - w$, then $v$ satisfies

$$v_t = -\frac{1}{k+1}\partial_x(vf) - (F(u) - F(w)), \tag{24}$$

$$v(0,x) = u_0(x) - w_0(x), \tag{25}$$

where $f = \sum_{i=0}^{k} u^{k-i}w^i$. Applying $\Lambda^r$ to both sides of (24), then multiplying both sides by $\Lambda^r v$ and integrating, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt} \parallel v(t) \parallel_{H^r}^2 \\ &= -\frac{1}{k+1}\int \Lambda^r\partial_x(vf)\Lambda^r v dx - \int \Lambda^r(F(u)-F(w))\Lambda^r v dx,\end{aligned} \tag{26}$$

where the integral is taken over $\mathbb{R}$ of $\mathbb{T}$.

Let $0 \le r \le s-1$ and $r+s > 2$. To bound the first term on the right-hand side of (26), we commute $\Lambda^r\partial_x$ with $f$. From Lemma 9, we arrive at

$$\begin{aligned}&\mid \frac{1}{k+1}\int \Lambda^r\partial_x(vf)\Lambda^r v dx \mid \\ &\lesssim \mid \int [\Lambda^r\partial_x, f]v\Lambda^r v dx \mid + \mid \int f\Lambda^r\partial_x v\Lambda^r v dx \mid \\ &\lesssim \parallel [\Lambda^r\partial_x, f]v \parallel_{L^2} \parallel v \parallel_{H^r} + \parallel \partial_x f \parallel_{L^\infty} \parallel v \parallel_{H^r}^2 \\ &\lesssim \parallel f \parallel_{H^s} \parallel v \parallel_{H^r}^2,\end{aligned} \tag{27}$$

where we used Sobolev embedding theorem $H^{\frac{1}{2}+} \hookrightarrow L^\infty$.

Using Lemmas 7 and 8, the algebra property and the fact $\parallel u_0 \parallel_{H^s} < h$ and $\parallel w_0 \parallel_{H^s} < h$ yield

$$\parallel f \parallel_{H^s} \le (k+1)h^k. \tag{28}$$

Therefore,

$$\mid \frac{1}{k+1}\int \Lambda^r\partial_x(vf)\Lambda^r v dx \mid \lesssim (k+1)h^k \parallel v \parallel_{H^r}^2 . \tag{29}$$

For the non-local term of (26), applying the Cauchy–Schwarz inequality, we obtain

$$\int \Lambda^r(F(u)-F(w))\Lambda^r v dx \le \parallel F(u)-F(w) \parallel_{H^r} \parallel v \parallel_{H^r} . \tag{30}$$

For estimating $\parallel F(u) - F(w) \parallel_{H^r}$, we group the difference $F(u) - F(w)$ in three parts:

$$\begin{aligned}F(u) - F(w) = &\Lambda^{-2}\partial_x(vf) + \frac{2k-1}{2}(\Lambda^{-2}\partial_x[u^{k-1}v_x(u_x+w_x)] \\ &+\Lambda^{-2}\partial_x[w_x^2(vf_1)]) + \frac{k-1}{2}(\Lambda^{-2}[u^{k-2}v_x(u_x^2+u_xw_x+w_x^2)] \\ &+\Lambda^{-2}[vf_2w_x^3]),\end{aligned} \tag{31}$$

where $f_1 = \sum_{i=0}^{k-2} u^{k-i} w^i$ and $f_2 = \sum_{i=0}^{k-3} u^{k-i} w^i$.

Therefore, applying the triangle inequality, we deduce

$$\begin{aligned} \| F(u) - F(w) \|_{H^r} \lesssim & \| vf \|_{H^{r-1}} + \| u^{k-1} v_x (u_x + w_x) \|_{H^{r-1}} \\ & + \| w_x^2 (v f_1) \|_{H^{r-1}} + \| u^{k-2} v_x (u_x^2 + u_x w_x + w_x^2) \|_{H^{r-1}} \\ & + \| v f_2 w_x^3 \|_{H^{r-1}} . \end{aligned} \tag{32}$$

For $r > \frac{1}{2}$, applying Lemma 10, from (32), we obtain

$$\begin{aligned} & \| F(u) - F(w) \|_{H^r} \\ & \lesssim \| v \|_{H^r} \| f \|_{H^{r-1}} + \| u^{k-1} (u_x + w_x) \|_{H^r} \| v_x \|_{H^{r-1}} \\ & \quad + \| v \|_{H^r} \| w_x^2 f_1 \|_{H^{r-1}} + \| u^{k-2} (u_x^2 + u_x w_x + w_x^2) \|_{H^r} \| v_x \|_{H^{r-1}} \\ & \quad + \| v \|_{H^r} \| f_2 w_x^3 \|_{H^{r-1}} . \end{aligned} \tag{33}$$

Now, using the algebra property, the fact that $r \leq s - 1$ and Lemma 7, we have

$$\| F(u) - F(w) \|_{H^r} \lesssim (\| u_0 \|_{H^s} + \| w_0 \|_{H^s})^k \| v \|_{H^r} . \tag{34}$$

For $0 \leq r \leq \frac{1}{2}$, then from applying Lemma 11 to (32), we obtain

$$\begin{aligned} & \| F(u) - F(w) \|_{H^r} \\ & \lesssim \| v \|_{H^{r-1}} \| f \|_{H^{s-1}} + \| u^{k-1} (u_x + w_x) \|_{H^{s-1}} \| v_x \|_{H^{r-1}} \\ & \quad + \| v \|_{H^{r-1}} \| w_x^2 f_1 \|_{H^{s-1}} + \| u^{k-2} (u_x^2 + u_x w_x + w_x^2) \|_{H^{s-1}} \| v_x \|_{H^{r-1}} \\ & \quad + \| v \|_{H^{r-1}} \| f_2 w_x^3 \|_{H^{s-1}} . \end{aligned} \tag{35}$$

By applying the algebra property and Lemma 7, one has

$$\| F(u) - F(w) \|_{H^r} \lesssim (\| u_0 \|_{H^s} + \| w_0 \|_{H^s})^k \| v \|_{H^r} . \tag{36}$$

Combining (26), (28), (34) and (36), we obtain the energy inequality

$$\frac{1}{2} \frac{d}{dt} \| v(t) \|_{H^r}^2 \lesssim c \| v(t) \|_{H^r}^2, \tag{37}$$

where $c = c(s, r, k, h)$. This implies that

$$\| v(t) \|_{H^r} \lesssim e^{cT_0} \| v(0) \|_{H^r} \tag{38}$$

or equivalently

$$\| u(t) - w(t) \|_{H^r} \lesssim e^{cT_0} \| u(0) - w(t) \|_{H^r}, \tag{39}$$

which is the desired Lipschitz continuity in $\Omega_1$.

Since $0 \leq r \leq 2 - s$, using interpolating between $H^r$ and $H^s$ norms, we arrive at

$$\begin{aligned} \| u(t) - w(t) \|_{H^r} \leq & \| u(t) - w(t) \|_{H^{2-s}} \\ \leq & e^{cT_0} \| u(0) - w(0) \|_{H^{2-s}} \\ \leq & e^{cT_0} \| u(0) - w(0) \|_{H^r}^{\frac{2(s-1)}{s-r}} \| u(0) - w(0) \|_{H^s}^{\frac{2-s-r}{s-r}} \\ \lesssim & e^{cT_0} \| u(0) - w(0) \|_{H^r}^{\frac{2(s-1)}{s-r}} , \end{aligned} \tag{40}$$

which shows the Hölder Continuity in $\Omega_2$.

Since $s-1 \le r \le s$, using interpolating between $H^{s-1}$ and $H^s$ norms, we obtain

$$\begin{aligned} \| u(t) - w(t) \|_{H^r} &\le e^{cT_0} \| u(0) - w(t) \|_{H^r} \\ &\le e^{cT_0} \| u(0) - w(t) \|_{H^{s-1}}^{s-r} \| u(0) - w(t) \|_{H^s}^{r-s+1} \\ &\lesssim e^{cT_0} \| u(0) - w(t) \|_{H^{s-1}}^{s-r} \\ &\lesssim e^{cT_0} \| u(0) - w(t) \|_{H^r}^{s-r}, \end{aligned} \tag{41}$$

which implies the Hölder Continuity in $\Omega_3$.

This completes the proof of Theorem 3. □

## 6. Conclusions

In this paper, we focus on several dynamic properties of problems (1) and (2). We first employ the $L^{2k}$ estimate of $u_x$ (Lemma 1) to obtain a new wave breaking phenomenon, namely, the solution remains bounded while its slope becomes unbounded in finite time. Then, we depend on the $L^{2k}$ estimate of $u_x$ to present the theorem of existence and uniqueness of the global weak solution. Since there exists similarity between our proof and the previous proof in [10,33], we thus provide an outline of the proof. Finally, we study the Hölder continuity of the solution map for the generalized Camassa–Holm Equations (1) and (2) in $H^r$-topology for all $0 \le r < s$. The properties of problems (1) and (2) not only present a fundamental importance from a mathematical point of view but also are of great physical interest.

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