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An Efficient Technique to Solve Time-Fractional Kawahara and Modified Kawahara Equations

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Abstract: In this article, we analysed the approximate solutions of the time-fractional Kawahara equation and modified Kawahara equation, which describe the propagation of signals in transmission lines and the formation of nonlinear water waves in the long wavelength region. An efficient technique, namely the natural transform decomposition method, is used in the present study. Fractional derivatives are considered in Caputo, Caputo–Fabrizio, and Atangana–Baleanu operative in the Caputo manner. We have presented numerical results graphically to demonstrate the applicability and efficiency of derivatives with fractional order to depict the water waves in long wavelength regions. The symmetry pattern is a fundamental feature of the Kawahara equation and the symmetrical aspect of the solution can be seen from the graphical representations. The obtained outcomes of the proposed method are compared to those of other well-known numerical techniques, such as the homotopy analysis method and residual power series method. Numerical solutions converge to the exact solution of the Kawahara equations, demonstrating the significance of our proposed method.

Keywords: Caputo; Caputo–Fabrizio; Atangana–Baleanu in Caputo manner; natural transform decomposition; Kawahara and modified Kawahara equations

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1. Introduction

In recent years, the study of nonlinear physical processes has benefited tremendously from the exploration of travelling wave solutions for nonlinear equations. Numerous scientific and engineering disciplines, including fluid mechanics, plasma physics, optical fibres, solid state physics and geology, deal with nonlinear wave processes. One of the significant equations in physics and ocean engineering is the Kawahara equation. The purpose of this research is to investigate the analytical scheme and efficiency of using the natural transform decomposition method (NTDM) on finding the symmetric solutions of the time-fractional Kawahara equation (TFKE) and time fractional modified Kawahara equation (TFMKE), which are given below as follows

$$D_\tau^\mu V + VV_\zeta + V_{\zeta\zeta\zeta} - V_{\zeta\zeta\zeta\zeta\zeta} = 0, \ \ 0 < \mu \leq 1, \tag{1}$$

with

$$V(\zeta, 0) = f(\zeta), \tag{2}$$

$$D_\tau^\mu V + V^2 V_\zeta + aV_{\zeta\zeta\zeta} + bV_{\zeta\zeta\zeta\zeta\zeta} = 0, \ \ 0 < \mu \leq 1, \tag{3}$$

with

$$V(\zeta, 0) = g(\zeta), \tag{4}$$

where $a > 0$, $b < 0$ are nonzero arbitrary constants. Dispersive wave equations are important in both mathematics and physics. In the past few decades, the Kawahara equation (KE) and modified Kawahara equation (MKE) have been a popular and active study topic [1–3]. Kawahara proposed the KE for characterising solitary-wave propagation in media in 1972 [4]. Kawahara numerically investigated this kind of equation and found that it has monotone and oscillatory solitary wave solutions. The symmetry pattern and set of conservation laws are two further fundamental features of the Kawahara equation. Symmetries and conservation laws of a generalization of the Kawahara equation were examined in [5]. It can be seen in both plasma magneto-acoustic wave theory and shallow water waves with surface tension. Furthermore, the MKE has numerous applications in capillary-gravity water waves, plasma waves, and other fields [6–9].

Fractional calculus (FC) allows for the differentiation and integration of arbitrary orders and it has grown in popularity in recent decades in fields such as physics, fluid mechanics, electrical networks, groundwater problems, hepatitis B virus model, HIV dynamics model, biological sciences, diffusive transport, and electromagnetic theory [10–16]. Some of the applications of FC are control theory [17], dissipation [18], relaxation [19], modelling of processes such anomalous diffusion [20,21], and so on. Many scientists and engineers have worked to use fractional differential equations to examine various biological and physical systems. Solving these equations has proven to be a topic of study and interest for scientists from a wide range of disciplines. Numerous efficient approaches for dealing with such models have been established in the modern area of applied research and engineering. Homotopy analysis method (HAM) [22], variational iteration method (VIM) [23,24], monotone iterative technique [25], homotopy perturbation method (HPM) [26], reproducing kernel Hilbert space method [27–29], fractional Newton method [30], extended auxiliary equation mapping method, and extended direct algebraic mapping method [31], Laplace transform method for fuzzy partial differential equations [32], modified Adams–Bashforth method [33], $(m + \frac{1}{G'})-$expansion method [34], modified expansion function method and the sine–Gordon expansion method [35], the sine–Gordon expansion technique, and the modified $\exp(-\Omega(\zeta))$-expansion function technique [36], Laplace Adomian decomposition method [37], and several others are some of the most popular numerical and analytical approaches for solving linear and nonlinear fractional differential equations.

Several researchers have recently investigated the TFKE and TFMKE using various of methods and techniques, such as the iterative Laplace transform method [17], the HAM [38] and the new iterative method [39], and the residual power series method (RPSM) [40,41].

To the best of the author's knowledge, this is the first utilisation of NTDM for the study of Kawahara equations with three derivatives, where the Caputo (C) approach is singular and the Caputo–Fabrizio (CF) and Atangna–Baleanu operative in Caputo sense (ABC) approaches have non-singular kernels. The goal of this study is to solve the TFKE and TFMKE using NTDM. The NTDM developed using two powerful approaches, namely natural transform (NT) and Adomian decomposition method. Round-off errors are avoided with the NTDM since it does not require linearization, assumptions, perturbation or discretization. NTDM can transcend the preceding restrictions and limitations of perturbation techniques, allowing us to analyse strongly nonlinear problems. The limitation of the HPM is that it needs solving the functional equation in each iteration, which can be difficult and time-consuming. VIM has an inherent precision in finding the Lagrange multiplier, corrective function, and stationary conditions for fractional order. Unlike the classic Adomian process, the proposed approach does not include the calculation of the fractional derivative or fractional integrals in the recursive formula, which simplifies the estimation of the series terms. Therefore, this method is thought to be a useful tool for fast and easy solving specific classes of coupled nonlinear partial differential equations (PDEs). This method uses a fast convergence series to offer a solution that can be accurate or approximate. As a result, the NTDM is increasingly being used to solve a wide range of linear and nonlinear PDEs [42,43]. NTDM has been used to study a wide range of physical issues, including fractional order problems, such as the fractional system of ordinary differential equations [44], time fractional Klein–Gordon equation [45], time fractional-order coupled Burgers equations [46], and fractional-order Fisher's equation [47]. Recently, the Kaup–Kupershmidt equation [48] and Kuramoto–Sivashinsky equations [49] have been studied using the natural decomposition method.

The structure of the paper is summarised as follows. The NT of fundamental definitions as well as some additional results useful in the study of fractional differential equations are presented in Section 2. In Section 3, the basic idea for NTDM is to use fractional derivatives such as C, CF, and ABC. In Section 4, the solutions' uniqueness and convergence are investigated. Solutions of TFKE and TFMKE employing NTDM are included in Section 5. The numerical results and graphs for the TFKE and TFMKE are presented in Section 6. Lastly, in Section 7, we discuss our conclusions.

2. Basic Definitions

The fractional derivative definitions of C, CF, ABC, and some properties of NT are presented as follows.

Definition 1 ([50]). *In the Caputo manner, the fractional derivative of $f \in C_{-1}^{q}$ is shown as*

$$D_{\tau}^{\mu} f(\tau) = \begin{cases} \frac{d^q f(\tau)}{d\tau^q}, & \mu = q \in \mathbb{N}, \\ \frac{1}{\Gamma(q-\mu)} \int_0^{\tau} (\tau - \xi)^{q-\mu-1} f^q(\xi) d\xi, & q-1 < \mu < q,\ q \in \mathbb{N}. \end{cases} \tag{5}$$

Definition 2 ([51]). *Let $0 < \mu < 1$. Fractional CF derivative of order μ is denoted as*

$$^{CF}D_{\tau}^{\mu} f(\tau) = \frac{1}{1-\mu} \int_0^{\tau} f'(\zeta) \exp\Big(\frac{-\mu(\tau-\zeta)}{1-\mu}\Big) d\zeta,\ \tau \geq 0. \tag{6}$$

Definition 3 ([52]). *Fractional ABC derivative definition of f is as follows*

$$^{ABC}D_{\tau}^{\mu} f(\tau) = \frac{B[\mu]}{1-\mu} \int_0^{\tau} f'(\zeta) E_{\mu}\Big(\frac{-\mu(\tau-\zeta)^{\mu}}{1-\mu}\Big) d\zeta, \tag{7}$$

where $0 < \mu < 1$. The $B[\mu]$ is a normalization function and the Mittag-Leffler function is $E_{\mu} = \sum_{i=0}^{\infty} \frac{Z^i}{\Gamma(\mu i + 1)}$.

Definition 4 ([53,54]). *The NT of the function $f(\tau)$ is defined as*

$$N^{+}[f(\tau)] = R(s,u) = \frac{1}{u} \int_0^{+\infty} e^{(\frac{-s\tau}{u})} f(\tau) d\tau,\ \ u,s > 0. \tag{8}$$

Definition 5 ([55]). *NT of $D_{\tau}^{\mu} V(\tau)$ by means of C derivative is given as*

$$N^{+}[{}_0^C D_{\tau}^{\mu} V(\tau)] = (\frac{s}{v})^{\mu} \big(N^{+}[V(\tau)] - \frac{1}{s} V(0)\big). \tag{9}$$

Definition 6 ([56]). *NT of $D_{\tau}^{\mu} V(\tau)$ by means of CF derivative is defined as*

$$N^{+}[{}_0^{CF} D_{\tau}^{\mu} V(\tau)] = \frac{1}{1-\mu+\mu(\frac{v}{s})} \big(N^{+}[V(\tau)] - \frac{1}{s} V(0)\big). \tag{10}$$

Definition 7 ([57]). *NT of $D_{\tau}^{\mu} V(\tau)$ by means of ABC derivative is represented as*

$$N^{+}[{}_0^{ABC} D_{\tau}^{\mu} V(\tau)] = \frac{M[\mu]}{1-\mu+\mu(\frac{v}{s})^{\mu}} \big(N^{+}[V(\tau)] - \frac{1}{s} V(0)\big), \tag{11}$$

where $M[\mu]$ is the normalization function such that $M[0] = M[1] = 1$.

3. Basic Idea of NTDM

We consider the general inhomogeneous nonlinear equation as follows in this section, which contains the basic idea of NTDM.

$$D_\tau^\mu V(\zeta,\tau) = R[V(\zeta,\tau)] + F[V(\zeta,\tau)] + p(\zeta,\tau), \tag{12}$$

with the initial condition,

$$V(\zeta,0) = f(\zeta). \tag{13}$$

Here, R is linear, F is nonlinear and $p(\zeta,\tau)$ is the source term. Now, we can use the NT of Equation (12) by taking fractional derivatives of C, CF, and ABC definitions.

NTDM$_C$: By taking the NT of Equation (12) using C derivative, we get

$$\left(\frac{s}{v}\right)^\mu \left(N^+[V(\zeta,\tau)] - \frac{f(\zeta)}{s}\right) = N^+\Big[R[V(\zeta,\tau)] + F[V(\zeta,\tau)] + p(\zeta,\tau)\Big]. \tag{14}$$

By taking inverse NT on Equation (14), we get

$$V(\zeta,\tau) = N^{-1}\Big[\frac{f(\zeta)}{s} + \left(\frac{v}{s}\right)^\mu N^+\Big[R[V(\zeta,\tau)] + F[V(\zeta,\tau)] + p(\zeta,\tau)\Big]\Big]. \tag{15}$$

The nonlinear term can also be expressed as

$$F[V(\zeta,\tau)] = \sum_{k=0}^{\infty} A_k, \tag{16}$$

where A_k are the Adomian polynomials of $V_0, V_1, V_2, \ldots,$ and can be calculated with the given formula

$$A_k = \frac{1}{k!}\frac{d^k}{d\mu^k}\Big[F\Big(\sum_{k=0}^{\infty} \mu^k V_k\Big)\Big]_{\mu=0}, \ k = 0,1,2,\ldots\,. \tag{17}$$

Let the infinite series solution $V(\zeta,\tau)$ be of the form

$$V(\zeta,\tau) = \sum_{k=0}^{\infty} V_k(\zeta,\tau). \tag{18}$$

Now, we substitute Equations (16) and (18) into (15) to obtain

$$\begin{aligned}\sum_{k=0}^{\infty} V_k(\zeta,\tau) =& N^{-1}\Big[\frac{f(\zeta)}{s}\Big] + N^{-1}\Big[\left(\frac{v}{s}\right)^\mu N^+[p(\zeta,\tau)]\Big] \\ &+ N^{-1}\Big[\left(\frac{v}{s}\right)^\mu N^+\Big[R\sum_{k=0}^{\infty} V_k(\zeta,\tau) + \sum_{k=0}^{\infty} A_k\Big]\Big].\end{aligned} \tag{19}$$

By comparing the two sides of the Equation (19), we get

$$\begin{aligned}
{}^C V_0(\zeta,\tau) &= N^{-1}\Big[\frac{f(\zeta)}{s}\Big] + N^{-1}\Big[\left(\frac{v}{s}\right)^\mu N^+[p(\zeta,\tau)]\Big], \\
{}^C V_1(\zeta,\tau) &= N^{-1}\Big[\left(\frac{v}{s}\right)^\mu N^+\Big[R[V_0(\zeta,\tau)] + A_0\Big]\Big], \\
{}^C V_2(\zeta,\tau) &= N^{-1}\Big[\left(\frac{v}{s}\right)^\mu N^+\Big[R[V_1(\zeta,\tau)] + A_1\Big]\Big], \\
{}^C V_3(\zeta,\tau) &= N^{-1}\Big[\left(\frac{v}{s}\right)^\mu N^+\Big[R[V_2(\zeta,\tau)] + A_2\Big]\Big], \\
&\vdots \\
{}^C V_{k+1}(\zeta,\tau) &= N^{-1}\Big[\left(\frac{v}{s}\right)^\mu N^+\Big[R[V_k(\zeta,\tau)] + A_k\Big]\Big], \ k \geq 0.
\end{aligned} \tag{20}$$

By substituting (20) into (18), we get the NTDM$_C$ by the series solutions of (12) and (13) as

$${}^C V(\zeta,\tau) = {}^C V_0(\zeta,\tau) + {}^C V_1(\zeta,\tau) + {}^C V_2(\zeta,\tau) + \ldots\,. \tag{21}$$

NTDM$_{CF}$: By taking NT of Equation (12) using CF derivative, we get

$$\frac{1}{1-\mu+\mu(\frac{v}{s})}\Big(N^{+}[V(\zeta,\tau)]-\frac{f(\zeta)}{s}\Big)=N^{+}\Big[R[V(\zeta,\tau)]+F[V(\zeta,\tau)]+p(\zeta,\tau)\Big]. \tag{22}$$

By taking inverse NT on Equation (22), we get

$$V(\zeta,\tau)=N^{-1}\Big[\frac{f(\zeta)}{s}+\Big(1-\mu+\mu\Big(\frac{v}{s}\Big)\Big)N^{+}\Big[R[V(\zeta,\tau)]+F[V(\zeta,\tau)]+p(\zeta,\tau)\Big]\Big]. \tag{23}$$

Now, we substitute Equations (16) and (18) into (23) to obtain

$$\begin{aligned}\sum_{k=0}^{\infty}V_k(\zeta,\tau)&=N^{-1}\Big[\frac{f(\zeta)}{s}\Big]+N^{-1}\Big[\Big(1-\mu+\mu\Big(\frac{v}{s}\Big)\Big)N^{+}[p(\zeta,\tau)]\Big]\\&+N^{-1}\Big[\Big(1-\mu+\mu\Big(\frac{v}{s}\Big)\Big)N^{+}\Big[R\sum_{k=0}^{\infty}V_k(\zeta,\tau)+\sum_{k=0}^{\infty}A_k\Big]\Big].\end{aligned} \tag{24}$$

By comparing the two sides of the Equation (24), we get

$$\begin{aligned}{}^{CF}V_0(\zeta,\tau)&=N^{-1}\Big[\frac{f(\zeta)}{s}\Big]+N^{-1}\Big[\Big(1-\mu+\mu\Big(\frac{v}{s}\Big)\Big)N^{+}[p(\zeta,\tau)]\Big],\\{}^{CF}V_1(\zeta,\tau)&=N^{-1}\Big[\Big(1-\mu+\mu\Big(\frac{v}{s}\Big)\Big)N^{+}\Big[R[V_0(\zeta,\tau)]+A_0\Big]\Big],\\{}^{CF}V_2(\zeta,\tau)&=N^{-1}\Big[\Big(1-\mu+\mu\Big(\frac{v}{s}\Big)\Big)N^{+}\Big[R[V_1(\zeta,\tau)]+A_1\Big]\Big],\\{}^{CF}V_3(\zeta,\tau)&=N^{-1}\Big[\Big(1-\mu+\mu\Big(\frac{v}{s}\Big)\Big)N^{+}\Big[R[V_2(\zeta,\tau)]+A_2\Big]\Big],\\&\vdots\end{aligned}$$

$${}^{CF}V_{k+1}(\zeta,\tau)=N^{-1}\Big[\Big(1-\mu+\mu\Big(\frac{v}{s}\Big)\Big)N^{+}\Big[R[V_k(\zeta,\tau)]+A_k\Big]\Big],\ k\geq 0. \tag{25}$$

By substituting (25) into (18), we get the NTDM$_{CF}$ by the series solutions of (12) and (13) as

$${}^{CF}V(\zeta,\tau)={}^{CF}V_0(\zeta,\tau)+{}^{CF}V_1(\zeta,\tau)+{}^{CF}V_2(\zeta,\tau)+\dots. \tag{26}$$

NTDM$_{ABC}$: By taking NT of Equation (12) using ABC derivative, we get

$$\frac{M[\mu]}{1-\mu+\mu\Big(\frac{v}{s}\Big)^{\mu}}\Big(N^{+}[V(\zeta,\tau)]-\frac{f(\zeta)}{s}\Big)=N^{+}\Big[R[V(\zeta,\tau)]+F[V(\zeta,\tau)]+p(\zeta,\tau)\Big]. \tag{27}$$

By taking inverse NT on Equation (27), we get

$$V(\zeta,\tau)=N^{-1}\Big[\frac{f(\zeta)}{s}+\frac{1-\mu+\mu\Big(\frac{v}{s}\Big)^{\mu}}{M[\mu]}N^{+}\Big[R[V(\zeta,\tau)]+F[V(\zeta,\tau)]+p(\zeta,\tau)\Big]\Big]. \tag{28}$$

Now, we substitute Equations (16) and (18) into (28) to obtain

$$\begin{aligned}\sum_{k=0}^{\infty}V_k(\zeta,\tau)&=N^{-1}\Big[\frac{f(\zeta)}{s}\Big]+N^{-1}\Big[\frac{1-\mu+\mu\Big(\frac{v}{s}\Big)^{\mu}}{M[\mu]}N^{+}[p(\zeta,\tau)]\Big]\\&+N^{-1}\Big[\frac{1-\mu+\mu\Big(\frac{v}{s}\Big)^{\mu}}{M[\mu]}N^{+}\Big[R\sum_{k=0}^{\infty}V_k(\zeta,\tau)+\sum_{k=0}^{\infty}A_k\Big]\Big].\end{aligned} \tag{29}$$

By comparing the two sides of the Equation (29), we get

$$
\begin{aligned}
{}^{ABC}V_0(\zeta,\tau) &= N^{-1}\Big[\frac{f(\zeta)}{s}\Big] + N^{-1}\Big[\frac{1-\mu+\mu\left(\frac{v}{s}\right)^{\mu}}{M[\mu]}N^{+}[p(\zeta,\tau)]\Big],\\
{}^{ABC}V_1(\zeta,\tau) &= N^{-1}\Big[\frac{1-\mu+\mu\left(\frac{v}{s}\right)^{\mu}}{M[\mu]}N^{+}\Big[R[V_0(\zeta,\tau)]+A_0\Big]\Big],\\
{}^{ABC}V_2(\zeta,\tau) &= N^{-1}\Big[\frac{1-\mu+\mu\left(\frac{v}{s}\right)^{\mu}}{M[\mu]}N^{+}\Big[R[V_1(\zeta,\tau)]+A_1\Big]\Big],\\
{}^{ABC}V_3(\zeta,\tau) &= N^{-1}\Big[\frac{1-\mu+\mu\left(\frac{v}{s}\right)^{\mu}}{M[\mu]}N^{+}\Big[R[V_2(\zeta,\tau)]+A_2\Big]\Big],\\
&\vdots\\
{}^{ABC}V_{k+1}(\zeta,\tau) &= N^{-1}\Big[\frac{1-\mu+\mu\left(\frac{v}{s}\right)^{\mu}}{M[\mu]}N^{+}\Big[R[V_k(\zeta,\tau)]+A_k\Big]\Big],\ k\geq 0.
\end{aligned}
\tag{30}
$$

By substituting (30) into (18), we get the NTDM$_{ABC}$ by the series solutions of (12) and (13) as

$$
{}^{ABC}V(\zeta,\tau) = {}^{ABC}V_0(\zeta,\tau) + {}^{ABC}V_1(\zeta,\tau) + {}^{ABC}V_2(\zeta,\tau) + \dots . \tag{31}
$$

4. Convergence Analysis

In this section, we illustrate convergence and uniqueness of the NTDM$_C$, NTDM$_{CF}$, and NTDM$_{ABC}$.

Theorem 1 ([57]). *The* NTDM$_C$ *solution of (12) is unique when* $0 < (\delta_1+\delta_2)\frac{\tau^{\mu}}{\Gamma(1+\mu)} < 1$.

Proof. Assume that $H = (C[I], \|.\|)$ stands $\forall$ continuous mapping on the Banach space with the norm, specified on $I = [0, \mathbb{T}]$. For this, we propose the mapping $L : H \to H$, we have

$$
V^{C}_{k+1}(\zeta,\tau) = V^{C}_0 + N^{-1}\Big[\Big(\frac{v}{s}\Big)^{\mu}N^{+}[R(V_k(\zeta,\tau))]\Big] + N^{-1}\Big[\Big(\frac{v}{s}\Big)^{\mu}N^{+}[F(V_k(\zeta,\tau))]\Big], k\geq 0.
$$

Let us suppose $\mid R(V)-R(V^*)\mid < \delta_1 \mid V-V^*\mid$ and $\mid F(V)-F(V^*)\mid < \delta_2 \mid V-V^*\mid$, where δ_1 and δ_2 are Lipschitz constants, respectively, and V and V^* are two arbitrary values of the mapping.

$$
\begin{aligned}
\|L(V)-L(V^*)\| &= \max_{\tau\in I}\Big|N^{-1}\Big[\Big(\frac{v}{s}\Big)^{\mu}N^{+}[R(V)+F(V)]\Big] - N^{-1}\Big[\Big(\frac{v}{s}\Big)^{\mu}N^{+}[R(V^*)+F(V^*)]\Big]\Big|\\
&\leq \max_{\tau\in I}\Big|N^{-1}\Big[\Big(\frac{v}{s}\Big)^{\mu}N^{+}[R(V)-R(V^*)] + \Big(\frac{v}{s}\Big)^{\mu}N^{+}[F(V)-F(V^*)]\Big]\Big|\\
&\leq \max_{\tau\in I}\Big[\delta_1 N^{-1}\Big[\Big(\frac{v}{s}\Big)^{\mu}N^{+}|V-V^*|\Big] + \delta_2 N^{-1}\Big[\Big(\frac{v}{s}\Big)^{\mu}N^{+}|V-V^*|\Big]\Big]\\
&\leq \max_{\tau\in I}(\delta_1+\delta_2)\Big[N^{-1}\Big(\frac{v}{s}\Big)^{\mu}\Big[N^{+}|V-V^*|\Big]\Big]\\
&\leq (\delta_1+\delta_2)\Big[N^{-1}\Big[\Big(\frac{v}{s}\Big)^{\mu}N^{+}\|V-V^*\|\Big]\Big]\\
&= (\delta_1+\delta_2)\frac{\tau^{\mu}}{\Gamma(\mu+1)}\|V-V^*\|.
\end{aligned}
$$

The mapping is a contraction under the premise $0 < (\delta_1+\delta_2)\frac{\tau^{\mu}}{\Gamma(1+\mu)} < 1$. As a result of the Banach contraction fixed point theorem, there is a unique solution to (12). □

Theorem 2 ([57]). *When* $0 < (\delta_1+\delta_2)(1-\mu+\mu\tau) < 1$, *then* NTDM$_{CF}$ *solution to (12) is unique.*

Proof. This proof has been omitted since it is identical to Theorem 1. □

Theorem 3 ([57])**.** *When* $0 < (\delta_1 + \delta_2)(1 - \mu + \mu\frac{\tau^\mu}{\Gamma(\mu+1)}) < 1$, *the* $\mathrm{NTDM_{ABC}}$ *solution to (12) is unique.*

Proof. Because it is similar to Theorem 1, it has been omitted. □

Theorem 4 ([57])**.** *The general form of* $\mathrm{NTDM_C}$ *solution to (12) will be convergent.*

Proof. Assume that V_m is the mth parital sum and that $V_m = \sum_{k=0}^{m} V_k(\zeta, \tau)$. First, we demonstrate the V_m in Banach space in H is a Cauchy sequence. We obtain this by considering a new form Adomian polynomials.

Now,

$$\begin{aligned}
\|V_m - V_q\| &= \max_{\tau \in I} |V_m - V_q| \\
&= \max_{\tau \in I} \left| \sum_{r=q+1}^{m} V_r \right|, \; q = 1, 2, 3, \; \ldots . \\
&\leq \max_{\tau \in I} \left| N^{-1}\left[\left(\frac{v}{s}\right)^\mu N^+ \left[\sum_{r=q+1}^{m} (R(V_{r-1}) + F(V_{r-1})) \right]\right] \right| \\
&= \max_{\tau \in I} \left| N^{-1}\left[\left(\frac{v}{s}\right)^\mu N^+ \left[\sum_{r=q}^{m-1} R(V_r) + F(V_r) \right]\right] \right| \\
&\leq \max_{\tau \in I} \left| N^{-1}\left[\left(\frac{v}{s}\right)^\mu N^+ [R(V_{m-1}) - R(V_{q-1})] \right] \right| \\
&+ \max_{\tau \in I} \left| N^{-1}\left[\left(\frac{v}{s}\right)^\mu N^+ [F(V_{m-1}) - F(V_{q-1})] \right] \right| \\
&\leq \delta_1 \max_{\tau \in I} \left| N^{-1} \left(\frac{v}{s}\right)^\mu \left[N^+ [R(V_{m-1}) - R(V_{q-1})] \right] \right| \\
&+ \delta_2 \max_{\tau \in I} \left| N^{-1} \left(\frac{v}{s}\right)^\mu \left[N^+ [F(V_{m-1}) - F(V_{q-1})] \right] \right| \\
&= (\delta_1 + \delta_2) \frac{\tau^\mu}{\Gamma(\mu+1)} \|V_{m-1} - V_{q-1}\|.
\end{aligned}$$

Consider $m = q + 1$, then

$$\begin{aligned}
\|V_{q+1} - V_q\| &\leq \delta \|V_q - V_{q-1}\| \\
&\leq \delta^2 \|V_{q-1} - V_{q-2}\| \\
&\leq \cdots \leq \delta^q \|V_1 - V_0\|,
\end{aligned}$$

where $\delta = (\delta_1 + \delta_2)\frac{\tau^\mu}{\Gamma(\mu+1)}$. Analogously, we get the triangular inequality.

$$\begin{aligned}
\|V_m - V_q\| &\leq \|V_{q+1} - V_q\| + \|V_{q+2} - V_{q+1}\| + \cdots + \|V_m - V_{m-1}\| \\
&\leq (\delta^q + \delta^{q+1} + \cdots + \delta^{m-1}) \|V_1 - V_0\| \\
&\leq \delta^q \left(\frac{1 - \delta^{m-q}}{1 - \delta} \right) \|V_1\|.
\end{aligned}$$

As $0 < \delta < 1$, we get $1 - \delta^{m-q} < 1$. Therefore,

$$\|V_m - V_q\| \leq \frac{\delta^q}{1 - \delta} \max_{\tau \in I} \|V_1\|.$$

However, $\|V_1\| < \infty$. Thus, as $q \to \infty$, then $\|V_m - V_q\| \to 0$. Hence, V_m is a Cauchy sequence in H. As a result, the series V_m is convergent. □

Theorem 5 ([57]). *The (12)* NTDM$_{\text{CF}}$ *solution is convergent.*

Proof. The proof has been omitted since it is similar to Theorem 4. □

Theorem 6 ([57]). *The (12)* NTDM$_{\text{ABC}}$ *solution is convergent.*

Proof. The proof has been omitted since it is similar to Theorem 4. □

5. Solutions for TFKE and TFMKE

In this section, we obtain the following solutions of TFKE and TFMKE using NTDM by taking C, CF, and ABC derivatives.

Example 1. *Consider the TFKE as of the form*

$$D_\tau^\mu V + VV_\zeta + V_{\zeta\zeta\zeta} - V_{\zeta\zeta\zeta\zeta\zeta} = 0,$$

with initial condition,

$$V(\zeta,0) = \frac{105}{169}\operatorname{sech}^4\left(\frac{\zeta}{2\sqrt{13}}\right).$$

If $\mu = 1$, then the exact solution is [38],

$$V(\zeta,\tau) = \frac{105}{169}\operatorname{sech}^4\left(\frac{1}{2\sqrt{13}}\left(\zeta - \frac{36\tau}{169}\right)\right).$$

NTDM$_C$: We obtain the following solutions of NTDM$_C$ derivative as

$$\begin{aligned}
{}^C V_0(\zeta,\tau) =& \frac{105}{169}\operatorname{sech}^4\left(\frac{\zeta}{2\sqrt{13}}\right),\\
{}^C V_1(\zeta,\tau) =& \frac{7560\sqrt{13}\tau^\mu \sinh\left(\frac{\sqrt{13}\zeta}{26}\right)}{371{,}293\,\Gamma(1+\mu)\cosh^5\left(\frac{\sqrt{13}\zeta}{26}\right)},\\
{}^C V_2(\zeta,\tau) =& \frac{136{,}080\,\tau^{2\mu}\left(2\sinh\left(\frac{\sqrt{13}\zeta}{26}\right)-1\right)\left(2\sinh\left(\frac{\sqrt{13}\zeta}{26}\right)+1\right)}{62{,}748{,}517\,\Gamma(1+2\mu)\left(\sinh^2\left(\frac{\sqrt{13}\zeta}{26}\right)+1\right)^3},\\
&\vdots
\end{aligned}$$

by substituting ${}^C V_0(\zeta,\tau), {}^C V_1(\zeta,\tau), {}^C V_2(\zeta,\tau), \cdots$ values in (21), we obtain the approximate solution as

$$\begin{aligned}
{}^C V(\zeta,\tau) =& \frac{105}{169}\operatorname{sech}^4\left(\frac{\zeta}{2\sqrt{13}}\right) + \frac{7560\sqrt{13}\,\tau^\mu \sinh\left(\frac{\sqrt{13}\zeta}{26}\right)}{371{,}293\,\Gamma(1+\mu)\cosh^5\left(\frac{\sqrt{13}\zeta}{26}\right)}\\
&+ \frac{136{,}080\,\tau^{2\mu}\left(2\sinh\left(\frac{\sqrt{13}\zeta}{26}\right)-1\right)\left(2\sinh\left(\frac{\sqrt{13}\zeta}{26}\right)+1\right)}{62{,}748{,}517\,\Gamma(1+2\mu)\left(\sinh^2\left(\frac{\sqrt{13}\zeta}{26}\right)+1\right)^3} + \cdots.
\end{aligned}$$

NTDM$_{CF}$: We obtain the following solutions of NTDM$_{CF}$ derivative as

$$
\begin{aligned}
{}^{CF}V_0(\zeta,\tau) =& \frac{105}{169}\,\mathrm{sech}^4\left(\frac{\zeta}{2\sqrt{13}}\right),\\
{}^{CF}V_1(\zeta,\tau) =& \frac{7560\sqrt{13}\,\sinh\left(\frac{\sqrt{13}\,\zeta}{26}\right)(1-\mu+\mu\,\tau)}{371{,}293\cosh^5\left(\frac{\sqrt{13}\,\zeta}{26}\right)},\\
{}^{CF}V_2(\zeta,\tau) =& \frac{\left(2\cosh\left(\frac{\sqrt{13}\,\zeta}{13}\right)-3\right)}{62{,}748{,}517\left(\sinh^2\left(\frac{\sqrt{13}\,\zeta}{26}\right)+1\right)^3}\Big(136{,}080-272{,}160\,\mu+136{,}080\,\mu^2\\
&+272{,}160\,\mu\,\tau-272{,}160\,\mu^2\,\tau+68{,}040\,\mu^2\,\tau^2\Big),\\
&\vdots
\end{aligned}
$$

by substituting ${}^{CF}V_0(\zeta,\tau), {}^{CF}V_1(\zeta,\tau), {}^{CF}V_2(\zeta,\tau), \cdots$ values in (26), we obtain the approximate solution as

$$
\begin{aligned}
{}^{CF}V(\zeta,\tau) =& \frac{105}{169}\,\mathrm{sech}^4\left(\frac{\zeta}{2\sqrt{13}}\right)+\frac{7560\sqrt{13}\,\sinh\left(\frac{\sqrt{13}\,\zeta}{26}\right)(1-\mu+\mu\,\tau)}{371{,}293\cosh^5\left(\frac{\sqrt{13}\,\zeta}{26}\right)}\\
&+\frac{\left(2\cosh\left(\frac{\sqrt{13}\,\zeta}{13}\right)-3\right)}{62{,}748{,}517\left(\sinh^2\left(\frac{\sqrt{13}\,\zeta}{26}\right)+1\right)^3}\Big(136{,}080-272{,}160\,\mu+136{,}080\,\mu^2\\
&+272{,}160\,\mu\,\tau-272{,}160\,\mu^2\,\tau+68{,}040\,\mu^2\,\tau^2\Big)+\cdots.
\end{aligned}
$$

NTDM_{ABC}: We obtain the following solutions of NTDM_{ABC} derivative as

$$
\begin{aligned}
{}^{ABC}V_0(\zeta,\tau) =& \frac{105}{169}\,\mathrm{sech}^4\left(\frac{\zeta}{2\sqrt{13}}\right),\\
{}^{ABC}V_1(\zeta,\tau) =& \frac{7560\sqrt{13}\,\sinh\left(\frac{\sqrt{13}\,\zeta}{26}\right)(\Gamma(1+\mu)-\mu\,\Gamma(1+\mu)+\mu\,\tau^{\mu})}{371{,}293\,\Gamma(1+\mu)\cosh\left(\frac{\sqrt{13}\,\zeta}{26}\right)^5},\\
{}^{ABC}V_2(\zeta,\tau) =& \frac{136{,}080\left(2\sinh\frac{\sqrt{13}\zeta}{26}-1\right)\left(2\sinh\frac{\sqrt{13}\zeta}{26}+1\right)}{62{,}748{,}517\,\Gamma(1+\mu)\,\Gamma(1+2\mu)\left(\sinh^2\frac{\sqrt{13}\zeta}{26}+1\right)^3}\\
&\Big(\Gamma(1+\mu)\,\Gamma(1+2\mu)-2\,\mu\,\Gamma(1+\mu)\,\Gamma(1+2\mu)+2\,\mu\,\tau^{\mu}\,\Gamma(1+2\mu)\\
&+\mu^2\,\Gamma(1+\mu)\,\Gamma(1+2\mu)-2\,\mu^2\,\tau^{\mu}\,\Gamma(1+2\mu)+\mu^2\,\tau^{2\,\mu}\,\Gamma(1+\mu)\Big),\\
&\vdots
\end{aligned}
$$

by substituting ${}^{ABC}V_0(\zeta,\tau), {}^{ABC}V_1(\zeta,\tau), {}^{ABC}V_2(\zeta,\tau), \cdots$ values in (31), we obtain the approximate solution as

$$
\begin{aligned}
{}^{ABC}V(\zeta,\tau) =& \frac{105}{169}\,\mathrm{sech}^4\left(\frac{\zeta}{2\sqrt{13}}\right)+\frac{7560\sqrt{13}\,\sinh\left(\frac{\sqrt{13}\,\zeta}{26}\right)(\Gamma(1+\mu)-\mu\,\Gamma(1+\mu)+\mu\,\tau^{\mu})}{371{,}293\,\Gamma(1+\mu)\cosh\left(\frac{\sqrt{13}\,\zeta}{26}\right)^5}\\
&+\frac{136{,}080\left(2\sinh\frac{\sqrt{13}\zeta}{26}-1\right)\left(2\sinh\frac{\sqrt{13}\zeta}{26}+1\right)}{62{,}748{,}517\,\Gamma(1+\mu)\,\Gamma(1+2\mu)\left(\sinh^2\frac{\sqrt{13}\zeta}{26}+1\right)^3}\Big(\Gamma(1+\mu)\,\Gamma(1+2\mu)\\
&-2\,\mu\,\Gamma(1+\mu)\,\Gamma(1+2\mu)+2\,\mu\,\tau^{\mu}\,\Gamma(1+2\mu)+\mu^2\,\Gamma(1+\mu)\,\Gamma(1+2\mu)\\
&-2\,\mu^2\,\tau^{\mu}\,\Gamma(1+2\mu)+\mu^2\,\tau^{2\,\mu}\,\Gamma(1+\mu)\Big)+\cdots.
\end{aligned}
$$

Example 2. *Consider the TFMKE as of the form*

$$D_\tau^\mu V + V^2 V_\zeta + a\, V_{\zeta\zeta\zeta} + b\, V_{\zeta\zeta\zeta\zeta\zeta} = 0,$$

with initial condition,

$$V(\zeta,0) = \frac{3a}{\sqrt{-10b}}\,\mathrm{sech}^2(P\zeta).$$

If $\mu = 1$, then the exact solution is [38],

$$V(\zeta,\tau) = \frac{3a}{\sqrt{-10b}}\,\mathrm{sech}^2(P(\zeta - l\tau)),\;\; P = \frac{1}{2}\sqrt{\frac{-a}{5b}},\;\; l = \frac{25b-4a^2}{25b}.$$

NTDM_C: We obtain the following solutions of NTDM_C derivative as

$$\begin{aligned}
{}^C V_0(\zeta,\tau) =& \frac{3a}{\sqrt{-10b}}\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{-a}{5b}}\zeta\right),\\
{}^C V_1(\zeta,\tau) =& \frac{6\sqrt{2}\,a^3\,\tau^\mu\,\sinh\left(\frac{\sqrt{5}\,\zeta\sqrt{-\frac{a}{b}}}{10}\right)\sqrt{-\frac{a}{b}}}{125\,(-b)^{3/2}\,\Gamma(1+\mu)\cos\left(\frac{\sqrt{5}\,\sqrt{a}\,\zeta}{10\,\sqrt{b}}\right)^3},\\
{}^C V_2(\zeta,\tau) =& \frac{6\sqrt{10}a^{11/2}\tau^{2\mu}}{15{,}625(-b)^{\frac{7}{2}}\cos^8\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)\Gamma(1+2\mu)}\left(45\sqrt{a}\sinh^2\left(\frac{\sqrt{5}\zeta\sqrt{-\frac{a}{b}}}{10}\right) - 2\sqrt{a}\right.\\
&+51\sqrt{a}\sin^2\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right) - 51\sqrt{a}\sin^4\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right) + 2\sqrt{a}\sin^6\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)\\
&+6\sqrt{b}\sin\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)\sinh\left(\frac{\sqrt{5}\zeta\sqrt{-\frac{a}{b}}}{10}\right)\sqrt{-\frac{a}{b}} - 57\sqrt{b}\sin^3\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)\\
&\left.\times\sinh\left(\frac{\sqrt{5}\zeta\sqrt{-\frac{a}{b}}}{10}\right)\sqrt{-\frac{a}{b}} + 6\sqrt{b}\sin^5\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)\sinh\left(\frac{\sqrt{5}\zeta\sqrt{-\frac{a}{b}}}{10}\right)\sqrt{-\frac{a}{b}}\right),\\
&\vdots
\end{aligned}$$

by substituting ${}^C V_0(\zeta,\tau), {}^C V_1(\zeta,\tau), {}^C V_2(\zeta,\tau), \cdots$ values in (21), we obtain the approximate solution as

$$\begin{aligned}
{}^C V(\zeta,\tau) =& \frac{3a}{\sqrt{-10b}}\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{-a}{5b}}\zeta\right) + \frac{6\sqrt{2}a^3\tau^\mu\sinh\left(\frac{\sqrt{5}\zeta\sqrt{-\frac{a}{b}}}{10}\right)\sqrt{-\frac{a}{b}}}{125(-b)^{3/2}\Gamma(1+\mu)\cos\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)^3}\\
&+\frac{6\sqrt{10}a^{11/2}\tau^{2\mu}}{15{,}625(-b)^{\frac{7}{2}}\cos^8\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)\Gamma(1+2\mu)}\left(45\sqrt{a}\sinh^2\left(\frac{\sqrt{5}\zeta\sqrt{-\frac{a}{b}}}{10}\right) - 2\sqrt{a}\right.\\
&+51\sqrt{a}\sin^2\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right) - 51\sqrt{a}\sin^4\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right) + 2\sqrt{a}\sin^6\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)\\
&+6\sqrt{b}\sin\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)\sinh\left(\frac{\sqrt{5}\zeta\sqrt{-\frac{a}{b}}}{10}\right)\sqrt{-\frac{a}{b}} - 57\sqrt{b}\sin^3\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)\\
&\left.\times\sinh\left(\frac{\sqrt{5}\zeta\sqrt{-\frac{a}{b}}}{10}\right)\sqrt{-\frac{a}{b}} + 6\sqrt{b}\sin^5\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)\sinh\left(\frac{\sqrt{5}\zeta\sqrt{-\frac{a}{b}}}{10}\right)\sqrt{-\frac{a}{b}}\right)\\
&+\cdots.
\end{aligned}$$

NTDM$_{CF}$: We obtain the following solutions of NTDM$_{CF}$ derivative as

$$
\begin{aligned}
{}^{CF}V_0(\zeta,\tau) &= \frac{3a}{\sqrt{-10b}}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-a}{5b}}\zeta\right),\\
{}^{CF}V_1(\zeta,\tau) &= \frac{6\sqrt{2}a^3\sinh\left(\frac{\sqrt{5}\zeta\sqrt{-\frac{a}{b}}}{10}\right)\sqrt{-\frac{a}{b}}(1-\mu+\mu\tau)}{125(-b)^{3/2}\cos\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)^3},\\
{}^{CF}V_2(\zeta,\tau) &= \frac{3\sqrt{10}a^{11/2}\left(2-4\mu+2\mu^2+4\mu\tau-4\mu^2\tau+\mu^2\tau^2\right)}{15{,}625(-b)^{7/2}\cos^8\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)}\\
&\quad\left(45\sqrt{a}\sinh^2\left(\frac{\sqrt{5}\zeta\sqrt{-\frac{a}{b}}}{10}\right)-2\sqrt{a}+51\sqrt{a}\sin^2\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)\right.\\
&\quad-51\sqrt{a}\sin^4\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)+2\sqrt{a}\sin^6\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)+6\sqrt{b}\sin\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)\\
&\quad\times\sinh\left(\frac{\sqrt{5}\zeta\sqrt{-\frac{a}{b}}}{10}\right)\sqrt{-\frac{a}{b}}-57\sqrt{b}\sin^3\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)\sinh\left(\frac{\sqrt{5}\zeta\sqrt{-\frac{a}{b}}}{10}\right)\sqrt{-\frac{a}{b}}\\
&\quad\left.+6\sqrt{b}\sin^5\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)\sinh\left(\frac{\sqrt{5}\zeta\sqrt{-\frac{a}{b}}}{10}\right)\sqrt{-\frac{a}{b}}\right),\\
&\ \ \vdots
\end{aligned}
$$

by substituting ${}^{CF}V_0(\zeta,\tau), {}^{CF}V_1(\zeta,\tau), {}^{CF}V_2(\zeta,\tau),\cdots$ values in (26), we obtain the approximate solution as

$$
\begin{aligned}
{}^{CF}V(\zeta,\tau) &= \frac{3a}{\sqrt{-10b}}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-a}{5b}}\zeta\right)+\frac{6\sqrt{2}a^3\sinh\left(\frac{\sqrt{5}\zeta\sqrt{-\frac{a}{b}}}{10}\right)\sqrt{-\frac{a}{b}}(1-\mu+\mu\tau)}{125(-b)^{3/2}\cos\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)^3}\\
&\quad+\frac{3\sqrt{10}a^{11/2}\left(2-4\mu+2\mu^2+4\mu\tau-4\mu^2\tau+\mu^2\tau^2\right)}{15{,}625(-b)^{7/2}\cos^8\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)}\\
&\quad\left(45\sqrt{a}\sinh^2\left(\frac{\sqrt{5}\zeta\sqrt{-\frac{a}{b}}}{10}\right)-2\sqrt{a}+51\sqrt{a}\sin^2\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)\right.\\
&\quad-51\sqrt{a}\sin^4\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)+2\sqrt{a}\sin^6\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)+6\sqrt{b}\sin\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)\\
&\quad\times\sinh\left(\frac{\sqrt{5}\zeta\sqrt{-\frac{a}{b}}}{10}\right)\sqrt{-\frac{a}{b}}-57\sqrt{b}\sin^3\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)\sinh\left(\frac{\sqrt{5}\zeta\sqrt{-\frac{a}{b}}}{10}\right)\sqrt{-\frac{a}{b}}\\
&\quad\left.+6\sqrt{b}\sin^5\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)\sinh\left(\frac{\sqrt{5}\zeta\sqrt{-\frac{a}{b}}}{10}\right)\sqrt{-\frac{a}{b}}\right)+\cdots.
\end{aligned}
$$

NTDM$_{ABC}$: We obtain the following solutions of NTDM$_{ABC}$ derivative as

$$
\begin{aligned}
{}^{ABC}V_0(\zeta,\tau) &= \frac{3a}{\sqrt{-10b}}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-a}{5b}}\zeta\right),\\
{}^{ABC}V_1(\zeta,\tau) &= \frac{6\sqrt{2}\,a^3\sinh\left(\frac{\sqrt{5}\zeta\sqrt{-\frac{a}{b}}}{10}\right)\sqrt{-\frac{a}{b}}\,(\Gamma(1+\mu)-\mu\,\Gamma(1+\mu)+\mu\,\tau^{\mu})}{125\,(-b)^{3/2}\,\Gamma(1+\mu)\cos\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)^3},\\
{}^{ABC}V_2(\zeta,\tau) &= \frac{1}{15{,}625(-b)^{7/2}\Gamma(1+\mu)\cos^8\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)\Gamma(1+2\mu)}\\
&\quad\Big(6\sqrt{10}a^{11/2}\big(\Gamma(1+\mu)\Gamma(1+2\mu)-2\mu\Gamma(1+\mu)\Gamma(1+2\mu)+2\mu\tau^{\mu}\Gamma(1+2\mu)\\
&\quad+\mu^2\Gamma(1+\mu)\Gamma(1+2\mu)-2\mu^2\tau^{\mu}\Gamma(1+2\mu)+\mu^2\tau^{2\mu}\Gamma(1+\mu)\big)\\
&\quad\times\left(45\sqrt{a}\sinh\left(\frac{\sqrt{5}\zeta\sqrt{-\frac{a}{b}}}{10}\right)^2-2\sqrt{a}+51\sqrt{a}\sin^2\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)\right.\\
&\quad-51\sqrt{a}\sin^4\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)+2\sqrt{a}\sin^6\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)+\sinh\left(\frac{\sqrt{5}\zeta\sqrt{-\frac{a}{b}}}{10}\right)\sqrt{-\frac{a}{b}}\\
&\quad\left.\times\left(6\sqrt{b}\sin\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)-57\sqrt{b}\sin^3\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)+6\sqrt{b}\sin^5\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)\right)\right)\Big),\\
&\quad\vdots
\end{aligned}
$$

by substituting ${}^{ABC}V_0(\zeta,\tau), {}^{ABC}V_1(\zeta,\tau), {}^{ABC}V_2(\zeta,\tau),\cdots$ values in (31), we obtain the approximate solution as

$$
\begin{aligned}
{}^{ABC}V(\zeta,\tau) &= \frac{3a}{\sqrt{-10b}}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-a}{5b}}\zeta\right)\\
&\quad+\frac{6\sqrt{2}\,a^3\sinh\left(\frac{\sqrt{5}\zeta\sqrt{-\frac{a}{b}}}{10}\right)\sqrt{-\frac{a}{b}}\,(\Gamma(1+\mu)-\mu\,\Gamma(1+\mu)+\mu\,\tau^{\mu})}{125\,(-b)^{3/2}\,\Gamma(1+\mu)\cos\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)^3}\\
&\quad+\frac{1}{15{,}625(-b)^{7/2}\Gamma(1+\mu)\cos^8\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)\Gamma(1+2\mu)}\\
&\quad\Big(6\sqrt{10}a^{11/2}\big(\Gamma(1+\mu)\Gamma(1+2\mu)-2\mu\Gamma(1+\mu)\Gamma(1+2\mu)+2\mu\tau^{\mu}\Gamma(1+2\mu)\\
&\quad+\mu^2\Gamma(1+\mu)\Gamma(1+2\mu)-2\mu^2\tau^{\mu}\Gamma(1+2\mu)+\mu^2\tau^{2\mu}\Gamma(1+\mu)\big)\\
&\quad\times\left(45\sqrt{a}\sinh\left(\frac{\sqrt{5}\zeta\sqrt{-\frac{a}{b}}}{10}\right)^2-2\sqrt{a}+51\sqrt{a}\sin^2\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)\right.\\
&\quad-51\sqrt{a}\sin^4\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)+2\sqrt{a}\sin^6\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)+\sinh\left(\frac{\sqrt{5}\zeta\sqrt{-\frac{a}{b}}}{10}\right)\sqrt{-\frac{a}{b}}\\
&\quad\left.\times\left(6\sqrt{b}\sin\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)-57\sqrt{b}\sin^3\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)+6\sqrt{b}\sin^5\left(\frac{\sqrt{5}\sqrt{a}\zeta}{10\sqrt{b}}\right)\right)\right)\Big)\\
&\quad+\cdots.
\end{aligned}
$$

6. Numerical Results and Discussion

In this section, we illustrate the approximate solutions of TFKE and TFMKE using NTDM with unique space and time variables at various fractional-order values in Tables 1–4. To demonstrate the dynamical behaviour of the solutions, their animations are exploited using numerical simulation in Figures 1–4. Table 1 displays the absolute error of TFKE for various τ values at ζ = 10. Table 2 demonstrates the approximate solution of TFKE for various μ, τ values at fixed ζ = 10 using the current technique. Figure 1 shows the approximate solution of TFKE with different values of μ. Figure 2 shows the surface plot of the approximate solution of $V(\zeta, \tau)$ with different values of μ. Table 3 displays the approximate solutions of TFMKE for various τ and ζ values at $\mu = 1$. Table 4 displays the approximate solution of TFMKE with various values of μ, τ, and ζ. Figure 3 shows the approximate solution of TFMKE with different values of μ. Figure 4 shows the surface plot of the approximate solution of $V(\zeta, \tau)$ with different values of μ. The tables and graphs demonstrate the suggested techniques' accuracy and applicability. From the figures, we can observe that three derivative graphical patterns are similar and symmetric. Tables display the accuracy of the proposed method with existing techniques with various fractional-order values.

Table 1. Absolute error of TFKE in Example 1 with different values of τ at fixed $\zeta = 10$.

		$\mu = 1$		
τ	**NTDM$_C$**	**NTDM$_{CF}$**	**NTDM$_{ABC}$**	**RPSM [40]**
0.0	0	0	0	0
0.1	1.41553×10^{-15}	1.41553×10^{-15}	1.41553×10^{-15}	1.41553×10^{-15}
0.2	4.68063×10^{-14}	4.68063×10^{-14}	4.68063×10^{-14}	4.68063×10^{-14}
0.3	3.6391×10^{-13}	3.6391×10^{-13}	3.6391×10^{-13}	3.6391×10^{-13}
0.4	1.56886×10^{-12}	1.56886×10^{-12}	1.56886×10^{-12}	1.56886×10^{-12}
0.5	4.89617×10^{-12}	4.89617×10^{-12}	4.89617×10^{-12}	4.89617×10^{-12}
0.6	1.24542×10^{-11}	1.24542×10^{-11}	1.24542×10^{-11}	1.24542×10^{-11}
0.7	2.75069×10^{-11}	2.75069×10^{-11}	2.75069×10^{-11}	2.75069×10^{-11}
0.8	5.47829×10^{-11}	5.47829×10^{-11}	5.47829×10^{-11}	5.47829×10^{-11}
0.9	1.0081×10^{-10}	1.0081×10^{-10}	1.0081×10^{-10}	1.0081×10^{-10}
1.0	1.7428×10^{-10}	1.7428×10^{-10}	1.7428×10^{-10}	1.7428×10^{-10}

Table 2. Approximate solution of TFKE in Example 1 with different values of μ, τ at fixed $\zeta = 10$.

		$\mu = 0.25$		
τ	**NTDM$_C$**	**NTDM$_{CF}$**	**NTDM$_{ABC}$**	**RPSM [40]**
0.0	3.04206×10^{-2}	3.29806×10^{-2}	3.29806×10^{-2}	3.04206×10^{-2}
0.1	3.25021×10^{-2}	3.30723×10^{-2}	3.35567×10^{-2}	3.25033×10^{-2}
0.2	3.29225×10^{-2}	3.31643×10^{-2}	3.36675×10^{-2}	3.29247×10^{-2}
0.3	3.32093×10^{-2}	3.32564×10^{-2}	3.37421×10^{-2}	3.32123×10^{-2}
0.4	3.34339×10^{-2}	3.33488×10^{-2}	3.38000×10^{-2}	3.34378×10^{-2}
0.5	3.36214×10^{-2}	3.34414×10^{-2}	3.38480×10^{-2}	3.3626×10^{-2}
0.6	3.37839×10^{-2}	3.35342×10^{-2}	3.38893×10^{-2}	3.37893×10^{-2}
0.8	3.40585×10^{-2}	3.37206×10^{-2}	3.39586×10^{-2}	3.40655×10^{-2}
0.9	3.41778×10^{-2}	3.38141×10^{-2}	3.39885×10^{-2}	3.41855×10^{-2}
1.0	3.42882×10^{-2}	3.39078×10^{-2}	3.40160×10^{-2}	3.42966×10^{-2}

Table 2. *Cont.*

		$\mu = 0.50$		
0.0	3.04206×10^{-2}	3.20859×10^{-2}	3.20859×10^{-2}	3.04206×10^{-2}
0.1	3.15837×10^{-2}	3.22610×10^{-2}	3.27177×10^{-2}	3.15838×10^{-2}
0.2	3.20842×10^{-2}	3.24370×10^{-2}	3.29844×10^{-2}	3.20845×10^{-2}
0.3	3.24759×10^{-2}	3.26138×10^{-2}	3.31911×10^{-2}	3.24765×10^{-2}
0.4	3.28114×10^{-2}	3.27915×10^{-2}	3.33667×10^{-2}	3.28122×10^{-2}
0.5	3.31109×10^{-2}	3.29700×10^{-2}	3.35224×10^{-2}	3.31122×10^{-2}
0.6	3.33850×10^{-2}	3.31495×10^{-2}	3.36641×10^{-2}	3.33867×10^{-2}
0.7	3.36398×10^{-2}	3.33297×10^{-2}	3.37950×10^{-2}	3.36421×10^{-2}
0.8	3.38794×10^{-2}	3.35109×10^{-2}	3.39175×10^{-2}	3.38822×10^{-2}
0.9	3.41066×10^{-2}	3.36929×10^{-2}	3.40331×10^{-2}	3.41099×10^{-2}
1.0	3.43233×10^{-2}	3.38757×10^{-2}	3.41429×10^{-2}	3.43273×10^{-2}
		$\mu = 0.75$		
0.0	3.04206×10^{-2}	3.12331×10^{-2}	3.12331×10^{-2}	3.04206×10^{-2}
0.1	3.10417×10^{-2}	3.14837×10^{-2}	3.17206×10^{-2}	3.10417×10^{-2}
0.2	3.14737×10^{-2}	3.17362×10^{-2}	3.20580×10^{-2}	3.14737×10^{-2}
0.3	3.18582×10^{-2}	3.19905×10^{-2}	3.23572×10^{-2}	3.18583×10^{-2}
0.4	3.22162×10^{-2}	3.22466×10^{-2}	3.26348×10^{-2}	3.22163×10^{-2}
0.5	3.25565×10^{-2}	3.25047×10^{-2}	3.28978×10^{-2}	3.25567×10^{-2}
0.6	3.28839×10^{-2}	3.27645×10^{-2}	3.31501×10^{-2}	3.28842×10^{-2}
0.7	3.32014×10^{-2}	3.30263×10^{-2}	3.33941×10^{-2}	3.32019×10^{-2}
0.8	3.35111×10^{-2}	3.32900×10^{-2}	3.36315×10^{-2}	3.35117×10^{-2}
0.9	3.38144×10^{-2}	3.35556×10^{-2}	3.38633×10^{-2}	3.38153×10^{-2}
1.0	3.41124×10^{-2}	3.38231×10^{-2}	3.40906×10^{-2}	3.41135×10^{-2}
		$\mu = 1$		
0.0	3.04206×10^{-2}	3.04206×10^{-2}	3.04206×10^{-2}	3.04206×10^{-2}
0.1	3.07393×10^{-2}	3.07393×10^{-2}	3.07393×10^{-2}	3.07393×10^{-2}
0.2	3.10612×10^{-2}	3.10612×10^{-2}	3.10612×10^{-2}	3.10612×10^{-2}
0.3	3.13861×10^{-2}	3.13861×10^{-2}	3.13861×10^{-2}	3.13861×10^{-2}
0.4	3.17143×10^{-2}	3.17143×10^{-2}	3.17143×10^{-2}	3.17143×10^{-2}
0.5	3.20455×10^{-2}	3.20455×10^{-2}	3.20455×10^{-2}	3.20455×10^{-2}
0.6	3.23800×10^{-2}	3.23800×10^{-2}	3.23800×10^{-2}	3.23800×10^{-2}
0.7	3.27178×10^{-2}	3.27178×10^{-2}	3.27178×10^{-2}	3.27178×10^{-2}
0.8	3.30588×10^{-2}	3.30588×10^{-2}	3.30588×10^{-2}	3.30588×10^{-2}
0.9	3.34030×10^{-2}	3.34030×10^{-2}	3.34030×10^{-2}	3.34030×10^{-2}
1.0	3.37506×10^{-2}	3.37506×10^{-2}	3.37506×10^{-2}	3.37506×10^{-2}

Table 3. Approximate solution of TFMKE in Example 2 with different values of ζ, τ with $a = 0.001, b = -1$.

		$\mu = 1$			
ζ	τ	$\mathbf{NTDM}_C$	$\mathbf{NTDM}_{CF}$	$\mathbf{NTDM}_{ABC}$	**HAM [38]**
-20	0.0	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.299×10^{-4}
	0.2	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.299×10^{-4}
	0.4	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.299×10^{-4}
	0.6	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.299×10^{-4}
	0.8	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.299×10^{-4}
	1.0	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.299×10^{-4}
-10	0.0	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.439×10^{-4}
	0.2	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.439×10^{-4}
	0.4	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.439×10^{-4}
	0.6	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.439×10^{-4}
	0.8	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.439×10^{-4}
	1.0	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.439×10^{-4}
0	0.0	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.486×10^{-4}
	0.2	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.486×10^{-4}
	0.4	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.486×10^{-4}
	0.6	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.486×10^{-4}
	0.8	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.486×10^{-4}
	1.0	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.486×10^{-4}
10	0.0	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.439×10^{-4}
	0.2	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.439×10^{-4}
	0.4	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.439×10^{-4}
	0.6	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.439×10^{-4}
	0.8	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.439×10^{-4}
	1.0	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.439×10^{-4}
20	0.0	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.299×10^{-4}
	0.2	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.299×10^{-4}
	0.4	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.299×10^{-4}
	0.6	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.299×10^{-4}
	0.8	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.299×10^{-4}
	1.0	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.299×10^{-4}

Table 4. Approximate solution of TFMKE in Example 2 with different values of μ, τ and ζ with $a = 0.001, b = -1$.

ζ	τ	$\mu = 0.25$			$\mu = 0.50$			$\mu = 0.75$		
		$NTDM_C$	$NTDM_{CF}$	$NTDM_{ABC}$	$NTDM_C$	$NTDM_{CF}$	$NTDM_{ABC}$	$NTDM_C$	$NTDM_{CF}$	$NTDM_{ABC}$
−20	0.0	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}
	0.2	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}
	0.4	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}
	0.6	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}
	0.8	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}
	1.0	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}
−10	0.0	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}
	0.2	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}
	0.4	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}
	0.6	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}
	0.8	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}
	1.0	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}
0	0.0	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}
	0.2	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}
	0.4	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}
	0.6	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}
	0.8	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}
	1.0	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}	9.4868×10^{-4}
10	0.0	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}
	0.2	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}
	0.4	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}
	0.6	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}
	0.8	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}
	1.0	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}	9.4396×10^{-4}
20	0.0	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}
	0.2	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}
	0.4	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}
	0.6	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}
	0.8	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}
	1.0	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}	9.2996×10^{-4}

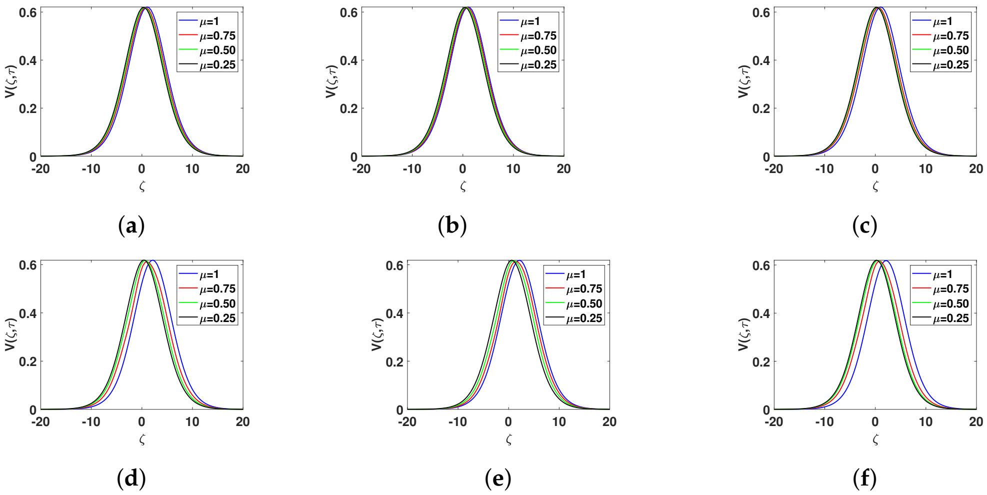

Figure 1. Approximate solution of Example 1 with different values μ. (**a**) NTDM$_C$, $\tau = 5$; (**b**) NTDM$_{CF}$, $\tau = 5$; (**c**) NTDM$_{ABC}$, $\tau = 5$; (**d**) NTDM$_C$, $\tau = 10$; (**e**) NTDM$_{CF}$, $\tau = 10$; (**f**) NTDM$_{ABC}$, $\tau = 10$.

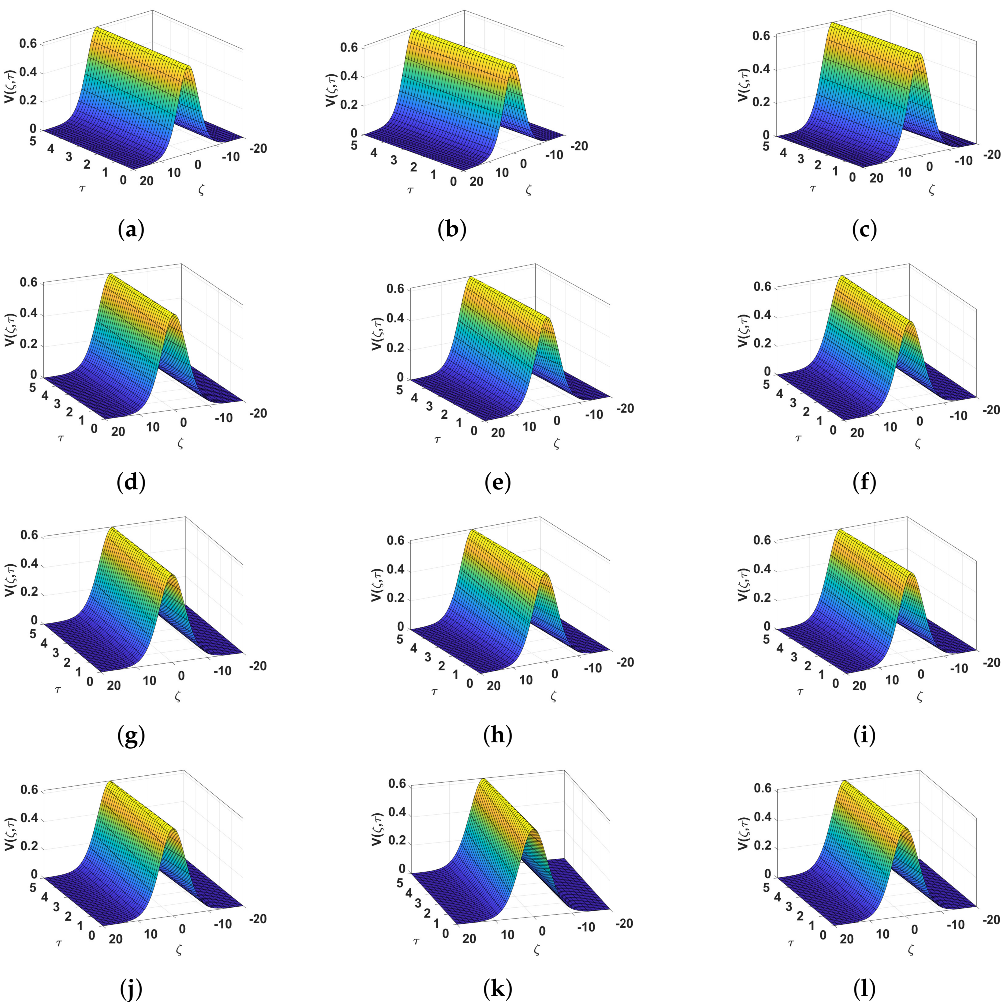

Figure 2. Surface plot of NTDM$_C$, NTDM$_{CF}$, NTDM$_{ABC}$ solution of $V(\zeta, \tau)$ for Example 1 with different values of μ. (**a**) NTDM$_C$, $\mu = 0.25$; (**b**) NTDM$_{CF}$, $\mu = 0.25$; (**c**) NTDM$_{ABC}$, $\mu = 0.25$; (**d**) NTDM$_C$, $\mu = 0.50$; (**e**) NTDM$_{CF}$, $\mu = 0.50$; (**f**) NTDM$_{ABC}$, $\mu = 0.50$; (**g**) NTDM$_C$, $\mu = 0.75$; (**h**) NTDM$_{CF}$, $\mu = 0.75$; (**i**) NTDM$_{ABC}$, $\mu = 0.75$; (**j**) NTDM$_C$, $\mu = 1$; (**k**) NTDM$_{CF}$, $\mu = 1$; (**l**) NTDM$_{ABC}$, $\mu = 1$.

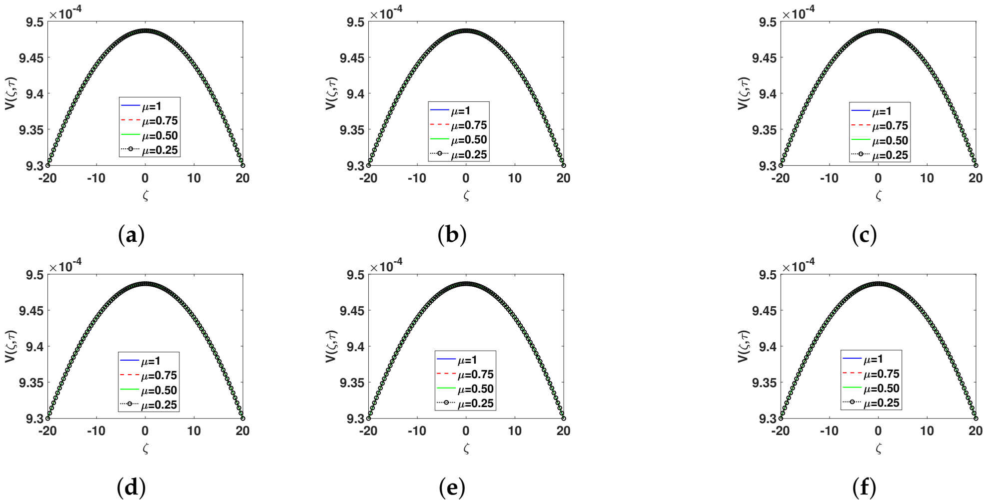

Figure 3. Approximate solution of Example 2 with different values μ and $a = 0.001, b = -1$. (**a**) NTDM$_C$, $\tau = 5$; (**b**) NTDM$_{CF}$, $\tau = 5$; (**c**) NTDM$_{ABC}$, $\tau = 5$; (**d**) NTDM$_C$, $\tau = 10$; (**e**) NTDM$_{CF}$, $\tau = 10$; (**f**) NTDM$_{ABC}$, $\tau = 10$.

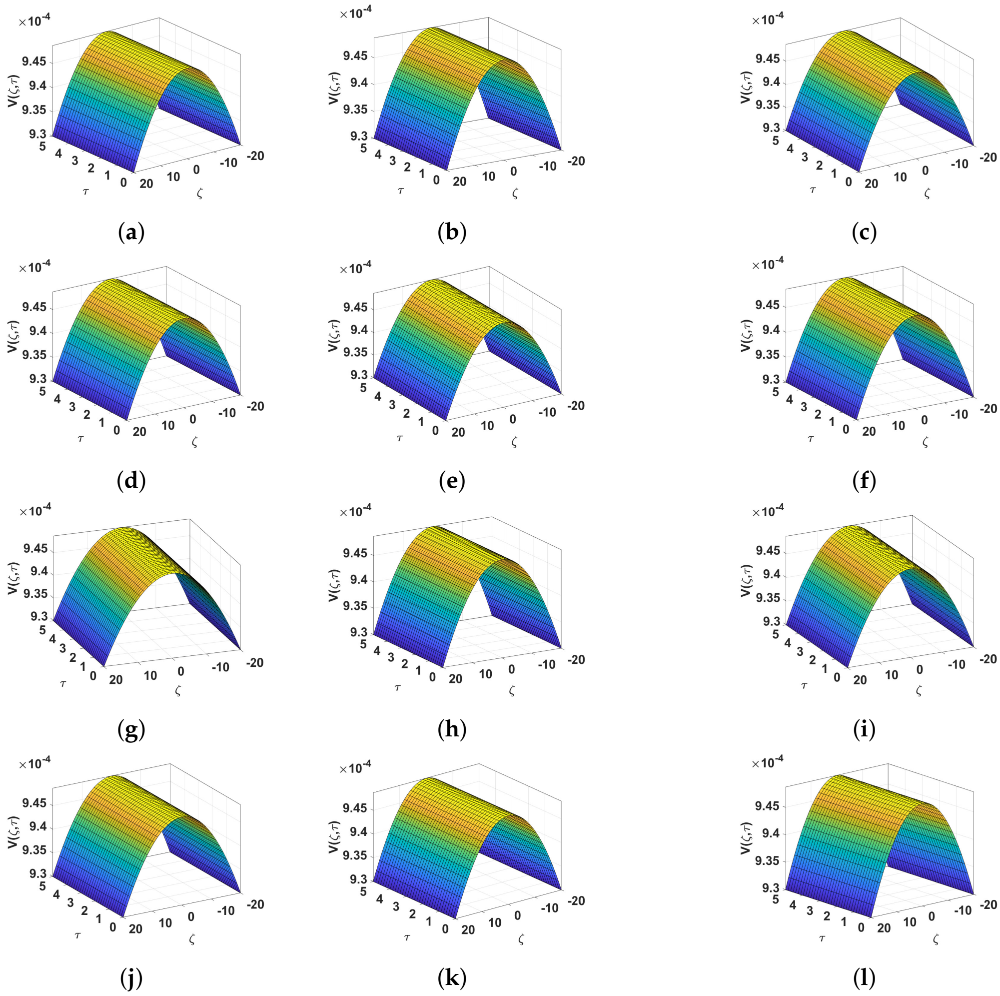

Figure 4. Surface plot of NTDM$_C$, NTDM$_{CF}$, NTDM$_{ABC}$ solution of $V(\zeta, \tau)$ for Example 2 with different values of μ with $a = 0.001, b = -1$. (**a**) NTDM$_C$, $\mu = 0.25$; (**b**) NTDM$_{CF}$, $\mu = 0.25$; (**c**) NTDM$_{ABC}$, $\mu = 0.25$; (**d**) NTDM$_C$, $\mu = 0.50$; (**e**) NTDM$_{CF}$, $\mu = 0.50$; (**f**) NTDM$_{ABC}$, $\mu = 0.50$; (**g**) NTDM$_C$, $\mu = 0.75$; (**h**) NTDM$_{CF}$, $\mu = 0.75$; (**i**) NTDM$_{ABC}$, $\mu = 0.75$; (**j**) NTDM$_C$, $\mu = 1$; (**k**) NTDM$_{CF}$, $\mu = 1$; (**l**) NTDM$_{ABC}$, $\mu = 1$.

7. Conclusions

In this work, we investigated the approximate solutions of TFKE and TFMKE based on the C, CF, and ABC fractional derivative operators using NTDM. The projected method is the amalgamation of two efficient techniques and it overcomes most of the limitations. The numerical simulation is shown to confirm the accuracy and to demonstrate that the fractional order goes to classical order. The derived solutions converge extremely fast to the actual solutions, indicating that approximate solutions are quite close to exact solutions. The numerical results suggest that the current technique is easy to use, effective, and precise. With the use of graphs and tables, the effect of all relevant parameters were discussed and presented. This is a fairly simple, dependable, and effective method for approximate solutions to several fractional physical models encountered in engineering and science such as the modified Korteweg–de Vries equation and it can also extended for fuzzy partial differential equations.

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