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Fuzzy Control Problem via Random Multi-Valued Equations in Symmetric F-n-NLS

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Abstract: To study an uncertain case of a control problem, we consider the symmetric F-n-NLS which is induced by a dynamic norm inspired by a random norm, distribution functions, and fuzzy sets. In this space, we consider a random multi-valued equation containing a parameter and investigate existence, and unbounded continuity of the solution set of it. As an application of our results, we consider a control problem with multi-point boundary conditions and a second order derivative operator.

Keywords: random multi-valued operator; random multi-valued equation; fuzzy control problem

MSC: 47C10

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1. Introduction

Consider the random operator Q_1. A natural generalization of parametric random equations of the form $\alpha = Q_1(\lambda, \gamma, \alpha)$, in which λ is a element of a probability measure space, is the multi-valued form [1],

$$\alpha \in Q_1(\lambda, \gamma, \alpha). \tag{1}$$

In regards to solutions, there are many approaches available in the literature, for example the principal eigenvalue-eigenvector method, the monotone minorant method [2,3] and topological degree. The idea in this paper is to use the topological degree for random multi-valued mappings and the method of evaluating solutions. The main idea is presenting an uncertain case of a control problem. To achieve this aim, we use a special space, i.e., symmetric F-n-NLS, that has a dynamic situation and a parameter τ, which can be time, which enable us to consider different cases. We note this kind of space induced by a dynamic norm which is inspired by random norms, probabilistic distances and fuzzy norms was studied; see [4] for details and applications. Our results can be applied in uncertainty problems, risk measures and super-hedging in finance [5].

For the random multi-valued operator Q_1, the following sets

$$\boldsymbol{U} = \{(\gamma, \alpha) : \alpha \in Q_1(\lambda, \gamma, \alpha)\}, \tag{2}$$

or

$$\boldsymbol{U} = \{\alpha : \exists\gamma, \alpha \in Q_1(\lambda, \gamma, \alpha)\}. \tag{3}$$

are solutions of (1). In this paper, we consider a control problem with multi-point boundary conditions and a second order derivative operator as

$$\begin{cases} \varphi''(\lambda,\iota)+\nu(\gamma,\iota)\mu(\varphi(\lambda,\iota))=0,\ \iota\in(0,1), \\ \nu(\gamma,\iota)\in Q_1(\lambda,\gamma,\varphi(\lambda,\iota)) \text{ a.e. on } [0,1] \\ \varphi(\lambda,0)=0, \varphi(\lambda,1)=\sum_{p=1}^{n}\omega_p\varphi(\lambda,\varsigma_p). \end{cases} \quad (4)$$

where $\varsigma_p \in (0,1)$, $0 \leq \omega_p$, $\sum_{p=1}^{n}\omega_p\varsigma_p < 1$ and $\lambda \in \digamma$. In Section 2, we introduce our special space, i.e., symmetric F-n-NLS and present some basic results which we need in the main section. In Section 3, we prove some properties of random multi-valued operator. In Section 4, we present an application of our results for a fuzzy control problem.

2. Preliminaries

Here, we let $E_1 = [0,1]$, $E_2 = (0,1]$, $E_3 = [0,\infty)$ and $E_4 = [0,\infty]$.

A mapping $\delta : \mathbb{R} \to E_1$, whose ϵ-level set is denoted by

$$[\delta]_\epsilon = \{\iota : \delta(\iota) \geq \epsilon\},$$

is said to be a fuzzy real number if it satisfies the following:

(i) δ is normal, i.e., there exists $\iota_0 \in \mathbb{R}$ such that $\delta(\iota_0) = 1$;
(ii) δ is upper semicontinuous;
(iii) δ is fuzzy convex, i.e., $\delta(\iota) \geq \min(\delta(\kappa), \delta(s))$, for each $\iota, \kappa \in \mathbb{R}$ such that $\kappa \leq \iota \leq s$ and $\epsilon \in E_2$;
(iv) For each $\epsilon \in E_2$, $[\delta]_\epsilon = [\delta^-_\epsilon, \delta^+_\epsilon]$, where $-\infty < \delta^-_\epsilon \leq \delta^+_\epsilon < +\infty$ and $[\delta]^0 = \overline{\{\delta \in \mathbb{R} : \delta(\iota) > 0\}}$ is compact.

Let the set $\mathbb{F}$ contain all upper semicontinuous normal convex fuzzy real numbers. $\mathbb{F}^+$ contains all non-negative fuzzy real numbers of $\mathbb{F}$. For each $\kappa \in \mathbb{R}$, we can define

$$\overline{\kappa}(\iota) = \begin{cases} 1, & \text{if } \iota = \kappa, \\ 0, & \text{if } \iota \neq \kappa, \end{cases}$$

so $\overline{\kappa} \in \mathbb{F}$ and $\mathbb{R}$ can be embedded in $\mathbb{F}$.

A partial order $\preceq$ in $\mathbb{F}$ is defined as follows: $\delta \preceq \sigma$ iff for each $\epsilon \in E_2$, $\delta^-_\epsilon \leq \sigma^-_\epsilon$ and $\delta^+_\epsilon \leq \sigma^+_\epsilon$ where $[\delta]_\epsilon = [\delta^-_\epsilon, \delta^+_\epsilon]$ and $[\sigma]_\epsilon = [\sigma^-_\epsilon, \sigma^+_\epsilon]$. The strict inequality in $\mathbb{F}$ is defined by $\delta \prec \sigma$ iff for each $\epsilon \in E_2$, $\delta^-_\epsilon < \sigma^-_\epsilon$ and $\delta^+_\epsilon < \sigma^+_\epsilon$ (see [6–8]).

The arithmetic operations $\oplus$, $\ominus$, $\odot$ and $\oslash$ on $\mathbb{F} \times \mathbb{F}$ are defined by

$$\begin{aligned} (\delta \oplus \sigma)(\iota) &= \sup_{\kappa\in\mathbb{R}} \min(\delta(\kappa), \sigma(\iota-\kappa)),\ \iota \in \mathbb{R}, \\ (\delta \ominus \sigma)(\iota) &= \sup_{\kappa\in\mathbb{R}} \min(\delta(\kappa), \sigma(\kappa-\iota)),\ \iota \in \mathbb{R}, \\ (\delta \odot \sigma)(\iota) &= \sup_{0\neq\kappa\in\mathbb{R}} \min\Big(\delta(\kappa), \sigma(\frac{\iota}{\kappa})\Big),\ \iota \in \mathbb{R}, \end{aligned}$$

$$(\delta \oslash \sigma)(\iota) = \sup_{\kappa\in\mathbb{R}} \min(\delta(\kappa\iota), \sigma(\kappa)),\ \iota \in \mathbb{R},\ \ \delta, \sigma(\succ 0) \in \mathbb{F}.$$

Definition 1. *Let $\mho$ be a real linear space over $\mathbb{R}$ with dim $\mho \geq n$. Suppose $\|\bullet, \ldots, \bullet\| : \mho^n \to \mathbb{F}^+$ is a mapping and $L, R : {E_1}^2 \to E_1$ are symmetric, nondecreasing mapping satisfying*

$$L(0,0) = 0 \ \ and \ \ R(1,1) = 1.$$

Write

$$[\|\vartheta_1, \vartheta_2, \ldots, \vartheta_n\|]_\epsilon = [\|\vartheta_1, \vartheta_2, \ldots, \vartheta_n\|^-_\epsilon, \|\vartheta_1, \vartheta_2, \ldots, \vartheta_n\|^+_\epsilon],$$

for $\vartheta_1, \vartheta_2, \ldots, \vartheta_n \in \mho$, $\boldsymbol{\epsilon} \in E_2$ and suppose that for every linearly independent vectors $\vartheta_1, \vartheta_2, \ldots, \vartheta_n \in \mho$, there exists $\boldsymbol{\epsilon}_0 \in E_2$ independent of $\vartheta_1, \vartheta_2, \ldots, \vartheta_n \in \mho$ such that for each $\boldsymbol{\epsilon} \leq \boldsymbol{\epsilon}_0$, one has

$$\inf \|\vartheta_1, \vartheta_2, \ldots, \vartheta_n\|^-_{\boldsymbol{\epsilon}} > 0, \quad \|\vartheta_1, \vartheta_2, \ldots, \vartheta_n\|^+_{\boldsymbol{\epsilon}} < \infty.$$

The quadruple $(\mho^n, \|\bullet, \ldots, \bullet\|, L, R)$ is said to be a symmetric fuzzy n-normed linear space (F-n-NLS) in the sense of Felbin [8] and $\|\bullet, \ldots, \bullet\|$ is a fuzzy n-norm if

(N1) $\|\vartheta_1, \vartheta_2, \ldots, \vartheta_n\| = \bar{0}$ *iff* $\vartheta_1, \vartheta_2, \ldots, \vartheta_n$ *are linearly dependent;*
(N2) $\|\vartheta_1, \vartheta_2, \ldots, \vartheta_n\|$ *is invariant under any permutation of* $\vartheta_1, \vartheta_2, \ldots, \vartheta_n \in \mho$*;*
(N3) $\|c\vartheta_1, \vartheta_2, \ldots, \vartheta_n\| = |c| \odot \|\vartheta_1, \vartheta_2, \ldots, \vartheta_n\|$ *for any* $c \in \mathbb{R}$*;*
(N4) $\|\vartheta_0 + \vartheta_1, \vartheta_2, \ldots, \vartheta_n\| \preceq \|\vartheta_0, \vartheta_2, \ldots, \vartheta_n\| \oplus \|\vartheta_1, \vartheta_2, \ldots, \vartheta_n\|$*;*
(i) *whenever* $\kappa \leq \|\vartheta_0, \vartheta_2, \ldots, \vartheta_n\|^-_1$, $\iota \leq \|\vartheta_1, \vartheta_2, \ldots, \vartheta_n\|^-_1$ *and* $\iota + \kappa \leq \|\vartheta_0 + \vartheta_1, \vartheta_2, \ldots, \vartheta_n\|^-_1$,

$$\|\vartheta_0 + \vartheta_1, \vartheta_2, \ldots, \vartheta_n\|(\kappa + \iota) \geq L(\|\vartheta_0, \vartheta_2, \ldots, \vartheta_n\|(\kappa), \|\vartheta_1, \vartheta_2, \ldots, \vartheta_n\|(\iota)),$$

(ii) *whenever* $\kappa \geq \|\vartheta_0, \vartheta_2, \ldots, \vartheta_n\|^-_1$, $\iota \geq \|\vartheta_1, \vartheta_2, \ldots, \vartheta_n\|^-_1$ *and* $\iota + \kappa \geq \|\vartheta_0 + \vartheta_1, \vartheta_2, \ldots, \vartheta_n\|^-_1$,

$$\|\vartheta_0 + \vartheta_1, \vartheta_2, \ldots, \vartheta_n\|(\kappa + \iota) \leq R(\|\vartheta_0, \vartheta_2, \ldots, \vartheta_n\|(\kappa), \|\vartheta_1, \vartheta_2, \ldots, \vartheta_n\|(\iota)).$$

Now, we consider a symmetric F-*n*-NLS in the sense of Narayanan-Vijayabalaji [9] and next we show a relationship between them.

Definition 2 ([9])**.** *Assume that $\mho$ is a linear space and $*$ is a continuous t-norm. Let the fuzzy subset η of $\mho^n \times \mathbb{R}$ with $\dim \mho \geq n$ satisfy*

(FN1) For all $\tau \in \mathbb{R}$ with $\tau \leq 0$, $\eta(\vartheta_1, \vartheta_2, \ldots, \vartheta_n, \tau) = 0$;
(FN2) For all $\tau \in \mathbb{R}$ with $\tau > 0$, $\eta(\vartheta_1, \vartheta_2, \ldots, \vartheta_n, \tau) = 1$ for $\tau \geq 0$ iff $\vartheta_1, \vartheta_2, \ldots, \vartheta_n$ are linearly dependent;
(FN3) $\eta(\vartheta_1, \vartheta_2, \ldots, \vartheta_n, \tau)$ is invariant under any permutation of $\vartheta_1, \vartheta_2, \ldots, \vartheta_n \in \mho$;
(FN4) For all $\tau \in \mathbb{R}$ with $\tau > 0$,

$$\eta(c\vartheta_1, \vartheta_2, \ldots, \vartheta_n, \tau) = \eta\left(\vartheta_1, \vartheta_2, \ldots, \vartheta_n, \frac{\tau}{|c|}\right) \text{ if } c \in \mathbb{R} \text{ with } c \neq 0;$$

(FN5) For all $\tau \in \mathbb{R}$ with $\tau, \theta > 0$,

$$\eta(\vartheta_0 + \vartheta_1, \vartheta_2, \ldots, \vartheta_n, \tau + \theta) \geq \eta(\vartheta_0, \vartheta_2, \ldots, \vartheta_n, \tau) * \eta(\vartheta_1, \vartheta_2, \ldots, \vartheta_n, \theta);$$

(FN6) $\eta(\vartheta_1, \vartheta_2, \ldots, \vartheta_n, .) : \mathring{E}_3 \to E_1$ is left continuous;
(FN7) $\lim_{\tau \to +\infty} \eta(\vartheta_1, \vartheta_2, \ldots, \vartheta_n, \tau) = 1$.

Thus, the triple $(\mho, \eta, *)$ is a symmetric F-n-NLS (see [10–12]).

A complete symmetric F-*n*-NLS is called symmetric F-*n*-BS.

Theorem 1 ([9,13–15])**.** *Let $(\mho, \eta, *)$ be a symmetric F-n-NLS in which $* = \min$ and*
(FN8) $\eta(\vartheta_1, \vartheta_2, \ldots, \vartheta_n, \tau) > 0$ for all $\tau > 0$ implies $\vartheta_1, \vartheta_2, \ldots, \vartheta_n$ are linearly dependent.

Define

$$\|\vartheta_1, \vartheta_2, \ldots, \vartheta_n\|_{\boldsymbol{\epsilon}} := \inf[\eta(\vartheta_1, \vartheta_2, \ldots, \vartheta_n, \tau)]_{\boldsymbol{\epsilon}}, \ \boldsymbol{\epsilon} \in \mathring{E}_1.$$

Then $\{\|\bullet, \cdots, \bullet\|_{\boldsymbol{\epsilon}} : \boldsymbol{\epsilon} \in \mathring{E}_1\}$ is an ascending family of fuzzy n-norms on $\mho$.

These fuzzy n-norms will be called the $\boldsymbol{\epsilon}$-n-norms on $\mho$ corresponding to the fuzzy n-norm on $\mho$.

We note that some applications can be found on [16,17].

Remark 1 ([18]). *Let $\eta_E : \mathbb{R} \times \mathring{E}_3 \to E_2$ be a Euclidean fuzzy norm (Euclidean fuzzy normed spaces were introduced by the authors in [18]). Then $\vartheta_1, \vartheta_2, \ldots, \vartheta_n \in \mho$ are linearly independent iff $\eta(\vartheta_1, \vartheta_2, \ldots, \vartheta_n, \tau) = \eta_E(1, \tau)$, for any $\tau > 0$.*

By the above remark, we have that, $\vartheta_1, \vartheta_2, \ldots, \vartheta_n \in \mho$ are linearly independent iff

$$\begin{aligned} \|\vartheta_1, \vartheta_2, \ldots, \vartheta_n\|_\epsilon &= \inf\{\tau : \eta(\vartheta_1, \vartheta_2, \ldots, \vartheta_n, \tau) \geq \boldsymbol{\epsilon},\ \boldsymbol{\epsilon} \in \mathring{E}_1\} \\ &= \inf\{\tau : \eta_E(1, \tau) \geq \boldsymbol{\epsilon},\ \boldsymbol{\epsilon} \in \mathring{E}_1\} \\ &= |1|_\epsilon. \end{aligned}$$

Consider the probability measure space $(\digamma, \mathring{E}_3, \xi)$ and let $(U, \boldsymbol{B}_U)$ and $(S, \boldsymbol{B}_S)$ be Borel measurable spaces, where U and S are symmetric F-n-BS. If $\{\lambda : \ \mathcal{F}(\lambda, \xi) \in B\} \in \mathring{E}_3$ for every ξ in U and $B \in \boldsymbol{B}_S$, we say $\mathcal{F} : \digamma \times U \to S$ is a random operator. Let 2^S be the family of all subsets of S. The mapping $\mathcal{F} : \digamma \times U \to 2^S$ is said to be random multi-valued operator. A random operator $\mathcal{F} : \digamma \times U \to S$ is said to be *linear* if $\mathcal{F}(\lambda, \mathbf{a}\xi_1 + \mathbf{b}\xi_2) = \mathbf{a}\mathcal{F}(\lambda, \xi_1) + \mathbf{b}\mathcal{F}(\lambda, \xi_2)$ almost everywhere for each ξ_1, ξ_2 in U and $\mathbf{a}, \mathbf{b}$ are scalers, and *bounded* if there exists a nonnegative real-valued random variable $M(\lambda)$ such that

$$\begin{aligned} & \eta(\mathcal{F}(\lambda_1, \xi_{1,1}) - \mathcal{F}(\lambda_1, \xi_{2,1}), \cdots, \mathcal{F}(\lambda_1, \xi_{1,n}) - \mathcal{F}(\lambda_1, \xi_{2,n}), M(\lambda)\tau) \\ \geq\ & \eta(\xi_{1,1} - \xi_{2,1}, \cdots, \xi_{1,n} - \xi_{2,n}, \tau), \end{aligned}$$

almost everywhere for each $\xi_{1,j} - \xi_{2,j}$ $(j = 1, 2, \ldots n)$ in U, $\tau \in \mathring{E}_3$ and $\lambda \in \digamma$.

Let $(\mathrm{Y}, \Omega, \|\bullet, \cdots, \bullet\|, L, R)$ be a symmetric F-n-BS over $\mathbb{R}$ with dim $\mathrm{Y} \geq n$ and ordering by the cone Ω, i.e., Ω is a closed convex subset of Y such that $\gamma\Omega \subset \Omega$ for $\gamma \geq 0$, $\Omega \cap (-\Omega) = \{0\}$, and $\alpha_p \leq \beta_p$ iff $\beta_p - \alpha_p \in \Omega$ for $\alpha, \beta \in \mathrm{Y}$ with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ and $1 \leq p \leq n$. For nonempty subsets Δ_1, Δ_2 of Y we write $\Delta_1 \succcurlyeq_2 \Delta_2$ (or, $\Delta_2 \preccurlyeq_2 \Delta_1$) iff for every $\alpha \in \Delta_1$, we can find a $\beta \in \Delta_2$ which $\alpha_p \geq \beta_p$ (or, $\beta_p \leq \alpha_p$) for $1 \leq p \leq n$. We say Ω is a normal cone if we can find a constant $K > 0$ where $0 \leq \alpha_p \leq \beta_p$ for $1 \leq p \leq n$ implies $\|\alpha_1, \alpha_2, \ldots, \alpha_n\|_\epsilon \leq K\|\beta_1, \beta_2, \ldots, \beta_n\|_\epsilon$. We note in this paper, we consider Ω as a normal cone with $K = 1$. Furthermore,

$$cc(\Delta_1) = \{G \subset \Delta_1 \subset \mathrm{Y} : \ \ G \text{ is nonempty closed convex}\}$$

Consider the open convex subset Ξ of Y, and let $\Xi_\Omega = \Omega \cap \Xi$, $\partial_\Omega\Xi = \Omega \cap \partial\Xi$ and $\dot{\Omega} = \Omega \backslash \{0\}$, where $\partial\Xi$ is boundary of Ξ in Y. The mapping $Q_3 : \digamma \times (\Omega \cap \overline{\Xi}) \to cc(\Omega)$ is said to be compact if $Q_3(\digamma \times \Delta_2)$ is relatively compact for any bounded subset Δ_2 of $\Omega \cap \overline{\Xi}$, where $Q_3(\digamma \times \Delta_2) = \cup_{\alpha \in \Delta_2} Q_3(\lambda, \alpha)$, for any $\lambda \in \digamma$. We say a random multi-valued operator Q_3 has the upper semi-continuity property (in short, u.s.c.rmvo) if

$$\{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \Omega \cap \overline{\Xi} : Q_3(\lambda, \alpha) \subset W\}$$

where $\Omega \cap \overline{\Xi} = (\Omega \cap \overline{\Xi})^\circ$ and $\lambda \in \digamma$. Further, if $\alpha \notin Q_3(\lambda, \alpha)$ for all $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \partial_\Omega\Xi$ and $\lambda \in \digamma$, the random fixed point index of Q_3 in Ξ with respect to Ω is defined which is an integer denoted by $i_\Omega(Q_3, \Xi)$.

Lemma 1. [2] *Let $Q_3 : \digamma \times (\Omega \cap \overline{\Xi}) \to cc(\Omega)$ be a compact u.s.c.rmvo. Then*

1. *$i_\Omega(Q_3, \Xi) = 0$ if there exists $\varphi \in \dot{\Omega}$ such that $\alpha \notin Q_3(\lambda, \alpha) + \varrho\varphi$ for all $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \partial_\Omega\Xi$, $\lambda \in \digamma$ and $\varrho \geq 0$.*
2. *$i_\Omega(Q_3, \Xi) = 1$ if $\varrho\alpha \notin Q_3(\lambda, \alpha)$ for all $\lambda \in \digamma$ and $\varrho \geq 1$.*

The following results are needed later to obtain a generalization of [19].

Lemma 2. [20] *Assume that $Q_3 : \digamma \times E_3 \subset \digamma \times \mathrm{Y} \to c(\mathrm{Y})$ is a u.s.c.rmvo, $(\alpha_\varepsilon, \beta_\varepsilon) \to (\alpha, \beta)$ with $\beta_\varepsilon \in Q_3(\lambda, \alpha_\varepsilon)$ and $\lambda \in \digamma$. Thus, $\beta \in Q_3(\lambda, \alpha)$.*

Lemma 3. [19] *Let* $\Psi : \digamma \times E_1 \times (\Omega \cap \overline{\Xi}) \to cc(\Omega)$ *be a compact u.s.c.rmvo with* $\alpha \notin \Psi(\lambda, \iota, \alpha)$ *for all* $(\iota, \alpha) \in E_1 \times \partial_\Omega \Xi$ *and* $\lambda \in \digamma$*. Then,* $i_\Omega(\Psi(\lambda, 0, .), \Xi) = i_\Omega(\Psi(\lambda, 1, .), \Xi)$.

3. Random Multi-Valued Operator

Lemma 4. *Let* $Q_3 : \digamma \times E_3 \times \Omega \to cc(\Omega)$ *be a compact u.s.c.rmvo and* $\Xi \subset \Upsilon$ *be open with* $0 \in \Xi$*. Additionally,*

1. $\iota\alpha \in Q_3(\lambda, 0, \alpha)$*, for any* $\lambda \in \digamma$*, for some* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \dot{\Omega}$ *implies* $\iota < 1$*,*
2. $i_\varrho(Q_3(\lambda, \gamma, .), \Xi) = 0$ *if* γ *is sufficiently large and* $\lambda \in \digamma$*.*

   Then $\{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \partial_\Omega \Xi : \exists \gamma > 0, \alpha \in Q_3(\lambda, \gamma, \alpha)\} \neq \emptyset$.

Proof. From the second condition in Lemma 4 we can find $\gamma_0 > 0$ such that $i_\varrho(Q_3(\lambda, \gamma, .), \Xi) = 0$ for all $\lambda \in \digamma$ and $\gamma \geq \gamma_0$. Define

$$\omega = \sup\{\gamma > 0 : i_\Omega(Q_3(\lambda, \gamma, .), \Xi) \neq 0\}.$$

We first observe that $\omega > 0$. Furthermore,

$$\forall \varepsilon > 0, \exists (\iota_\varepsilon, \alpha_\varepsilon) \in E_1 \times \partial_\Omega \Xi : \alpha_\varepsilon \in (1 - \iota_\varepsilon) Q_3(\lambda, \varepsilon, \alpha_\varepsilon) + \iota_\varepsilon Q_3(\lambda, 0, \alpha_\varepsilon). \tag{5}$$

for any $\lambda \in \digamma$. Since Q_3 is compact, without loss of generality we may assume that $\iota_\varepsilon \to \iota, \alpha_\varepsilon \to \alpha$ when $\varepsilon \to 0$ and $\lambda \in \digamma$. From (5) by Lemma 2 it follows that

$$\alpha \in (1 - \iota) Q_3(\lambda, 0, \alpha) + \iota Q_3(\lambda, 0, \alpha) \subset Q_3(\lambda, 0, \alpha).$$

This contradicts the first condition in Lemma 4. Thus, there exist $\varepsilon > 0$ such that $(\iota, \alpha) \notin \Psi(\lambda, \iota, \alpha)$ for all $\lambda \in \digamma$ and $(\iota, \alpha) \in E_1 \times \partial_\Omega \Xi$, where

$$\Psi(\lambda, \iota, \alpha) = (1 - \iota) Q_3(\lambda, \varepsilon, \alpha) + \iota Q_3(\lambda, 0, \alpha).$$

Using Lemma 3 we have

$$i_\Omega(Q_3(\lambda, 0, .), \Xi) = i_\Omega(Q_3(\lambda, \varepsilon, .), \Xi).$$

Using Lemma 1 implies that $i_\Omega(Q_3(\lambda, 0, .), \Xi) = 1$. Thus, $i_\Omega(Q_3(\lambda, \varepsilon, .), \Xi) = 1$, and we deduce $0 < \omega < \gamma_0$, for each $\lambda \in \digamma$.

Next, for every $\varepsilon \in (0, \omega)$ and $\lambda \in \digamma$, there exists $\gamma_\varepsilon \in (\omega - \varepsilon, \omega]$ with $i_\Omega(Q_3(\lambda, \gamma_\varepsilon, .), \Xi) \neq 0$. Consider the random multi-valued operator Ψ_ε defined by

$$\Psi_\varepsilon(\lambda, \iota, \alpha) = (1 - \iota) Q_3(\lambda, \gamma_\varepsilon, \alpha) + \iota Q_3(\lambda, \omega + \varepsilon, \alpha).$$

Now, we prove

$$\{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \partial_\Omega \Xi : \exists \gamma > 0, \alpha \in Q_3(\lambda, \gamma, \alpha)\} \neq \emptyset.$$

Assume the contrary, that

$$\{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \partial_\Omega \Xi : \exists \gamma > 0, \alpha \in Q_3(\lambda, \gamma, \alpha)\} = \emptyset. \tag{6}$$

Then, the random fixed point index of $Q_3(\lambda, \omega + \varepsilon)$ is well defined, for each $\lambda \in \digamma$. If

$$\alpha \notin \Psi_\varepsilon(\lambda, \iota, \alpha) \text{ for all } (\iota, \alpha) \in E_1 \times \partial_\Omega \Xi, \tag{7}$$

then, by Lemma 3 we obtain

$$i_\Omega(Q_3(\lambda, \gamma_\varepsilon, .), \Xi) = i_\Omega(Q_3(\lambda, \omega + \varepsilon, .), \Xi), \tag{8}$$

for each $\lambda \in \digamma$, a contradiction. Then, we can find a $(\iota_\varepsilon, \alpha_\varepsilon) \in E_1 \times \partial_\Omega \Xi$ satisfying

$$\alpha_\varepsilon \in (1-\iota_\varepsilon) Q_3(\lambda, \gamma_\varepsilon, \alpha_\varepsilon) + \iota_\varepsilon Q_3(\lambda, \omega + \varepsilon, \alpha_\varepsilon), \tag{9}$$

for each $\lambda \in \digamma$. Similarly, there is a $\alpha \in \partial_\Omega \Xi$ with $\alpha \in Q_3(\lambda, \omega, \alpha)$, which shows (6) is not true, and completes the proof. □

Let $(\Gamma, \Omega_\Gamma, \|\bullet, \cdots, \bullet\|^\Gamma, L, R)$ be asymmetric F-n-BS over $\mathbb{R}$ with dim $\mathrm{Y} \geq n$ ordered by the normal cone Ω_Γ. Suppose that $\mathrm{Y} \subset \Gamma, \Omega \subset \Omega_\Gamma \cap \mathrm{Y}$, the embedding $(\mathrm{Y}, \|\bullet, \cdots, \bullet\|_\epsilon) \hookrightarrow (\Gamma, \|\bullet, \cdots, \bullet\|_\epsilon^\Gamma)$ is continuous, and $Q_1 : \digamma \times E_3 \times \Omega \to cc(\Omega_\Gamma)$ is a compact u.s.c.rmvo. Assume $\Phi : \digamma \times \Gamma \to \mathrm{Y}$ is a compact random linear operator satisfying $\Phi(\lambda, \Omega_\Gamma) \subset \Omega$ such that $\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_n)$, for each $\lambda \in \digamma$.

Theorem 2. *Let*

1. *$\varrho\alpha \in \Phi \circ Q_1(\lambda, 0, \alpha)$, for any $\lambda \in \digamma$, for some $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \dot{\Omega}$ implies $\varrho < 1$;*
2. *we can find positive numbers a_1, a_2, a_3 and a random linear operator $Q_4 : \digamma \times \Gamma \to \mathbb{R}_+$ with $Q_4(\lambda, \beta) \neq 0$, for any $\lambda \in \digamma$, for some $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \Omega$ such that*
   - *(a) $Q_4\Phi(\lambda, \alpha) \succcurlyeq_2 \{a_1 Q_4(\lambda, \alpha)\}$ and*
     $$Q_4\Phi(\lambda, \alpha) \succcurlyeq_2 \{a_1 . \|(\Phi_1, \Phi_2, \ldots, \Phi_n)(\lambda, \alpha)\|_\epsilon^\Gamma\},$$
     for all $\lambda \in \digamma$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \Omega_\Gamma$,
   - *(b) $Q_4(\lambda, Q_1(\lambda, \gamma, \alpha)) \succcurlyeq_2 \{a_2 \gamma Q_4(\lambda, \alpha) - a_3\}$ for all $\lambda \in \digamma$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \Omega$, and*
   - *(c) we can find an increasing map (on the second part) $\varpi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that*
     $$\lim_{\gamma \to \infty} \varpi(\gamma, \frac{a_3}{a_1 a_2 \gamma - 1}) = 0 \tag{10}$$
     such that $(\varrho, \gamma, \alpha) \in E_1 \times E_3 \times \Omega$ with
     $$\alpha \in \varrho \Phi \circ Q_1(\lambda, \gamma, \alpha) + (1-\varrho) a_2 \gamma \Phi(\lambda, \alpha) \tag{11}$$
     implies
     $$\|(\alpha_1, \alpha_2, \ldots, \alpha_n)\|_\epsilon \leq \varpi(\gamma, \|(\alpha_1, \alpha_2, \ldots, \alpha_n)\|_\epsilon^\Gamma). \tag{12}$$

Then

$$\boldsymbol{U} = \{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \dot{\Omega} : \exists \gamma > 0, \alpha \in \Phi \circ Q_1(\lambda, \gamma, \alpha)\},$$

is an unbounded continuous branch emanating from 0, for each $\lambda \in \digamma$.

Proof. Suppose $\Xi \subset \mathrm{Y}$ is open and bounded where $0 \in \Xi$. We use Lemma 4 with $Q_3(\lambda, \gamma, \alpha) = \Phi \circ Q_1(\lambda, \gamma, \alpha)$ to show $\boldsymbol{U} \cap \partial_\Omega \Xi \neq \emptyset$, for any $\lambda \in \digamma$. Clearly, condition 1 of Lemma 4 holds. Assume that $(\varrho, \gamma, \alpha) \in E_1 \times E_3 \times \Omega$ satisfies (11), so $\alpha \in \Phi(\lambda, \varrho Q_1(\lambda, \gamma, \alpha) + (1-\varrho) a_2 \gamma \alpha)$, hence, $\alpha = \Phi(\lambda, \varrho \beta_\gamma + (1-\varrho) a_2 \gamma \alpha)$, for any $\lambda \in \digamma$, for some $\beta_\gamma \in Q_1(\lambda, \gamma, \alpha)$. By 2(a) and 2(b) we have

$$Q_4(\lambda, \alpha) \geq a_1 Q_4(\lambda, \varrho \beta_\gamma + (1-\varrho) a_2 \gamma \alpha) \geq a_1 (a_2 \gamma Q_4(\lambda, \alpha) - a_3), \tag{13}$$

and

$$\begin{aligned} Q_4(\lambda, \alpha) &\geq a_1 \|(\Phi_1, \Phi_2, \ldots, \Phi_n)(\lambda, \varrho \beta_\gamma + (1-\varrho) a_2 \gamma \alpha)\|_\epsilon^\Gamma \\ &= a_1 \|(\alpha_1, \alpha_2, \ldots, \alpha_n)\|_\epsilon^\Gamma, \end{aligned} \tag{14}$$

for each $\lambda \in \digamma$. For sufficiently large γ, (13) and (14) we conclude that

$$\|(\alpha_1, \alpha_2, \ldots, \alpha_n)\|_\epsilon^\Gamma \leq \frac{a_3}{a_1 a_2 \gamma - 1},$$

which combining with (12) gives

$$\|(\alpha_1, \alpha_2, \ldots, \alpha_n)\|_\epsilon \leq \varpi(\gamma, \frac{a_3}{a_1 a_2 \gamma - 1}), \tag{15}$$

for each $\lambda \in F$. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \partial_\Omega \Xi$, $0 < \varepsilon < a_2 \|(\alpha_1, \alpha_2, \ldots, \alpha_n)\|_\epsilon$ for some ε. From (15), (11) it follows that $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \notin \Psi(\lambda, \varrho, \alpha)$ for all $\lambda \in F$ and $(\varrho, \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)) \in E_1 \times \partial_\Omega \Xi$, where $\Psi(\lambda, \varrho, \alpha) = \varrho \Phi \circ Q_1(\lambda, \gamma, \alpha) + (1 - \varrho) a_2 \gamma \alpha$. Applying Lemma 1 we obtain $i_\Omega(Q_3(\lambda, \gamma, \Xi)) = i_\Omega(\phi, \Xi)$, here $\phi(\alpha) = a_2 \gamma \alpha$, and therefore $i_\Omega(Q_3(\lambda, \gamma, .), \Xi) = 0$, so condition 2 of Lemma 4 holds. The proof is complete. □

4. Applications

In this section, we study an uncertain case of a control problem. For it, we consider the compact u.s.c. rmvo $Q_1 : F \times E_3 \times \mathbb{R}_+ \to cc(\mathbb{R}_+)$ and the continuous map $\mu : \mathbb{R}_+ \to \mathbb{R}_+$. We consider the following control problem which contains a parameter:

$$\begin{cases} \varphi''(\lambda, \iota) + \nu(\gamma, \iota)\mu(\varphi(\lambda, \iota)) = 0, \ \iota \in \mathring{E}_1, \\ \nu(\gamma, \iota) \in Q_1(\lambda, \gamma, \varphi(\lambda, \iota)) \text{ a.e. on } E_1 \\ \varphi(\lambda, 0) = 0, \varphi(\lambda, 1) = \sum_{p=1}^n \omega_p \varphi(\lambda, \varsigma_p) \end{cases} \tag{16}$$

where $\varsigma_p \in \mathring{E}_1$, $0 \leq \omega_p$, $\sum_{p=1}^n \omega_p \varsigma_p < 1$ and $\lambda \in F$.

Denote $\Theta = \sum_{p=1}^n \omega_p \varsigma_p$ for every $(\iota, \kappa) \in E_1 \times E_1$, and let

$$\varpi(\iota, \kappa) = \begin{cases} \kappa(1 - \iota), \ \kappa \leq \iota, \\ \iota(1 - \kappa), \ \kappa > \iota \end{cases}$$

$$Q_5(\iota, \kappa) = \frac{\iota}{1 - \Theta} \sum_{p=1}^{n} \omega_p \varpi(\varsigma_p, \kappa) + \varpi(\iota, \kappa);$$

Let $C(E_1)$, resp., $C^1(E_1)$, be the symmetric F-n-BS of all continuous, resp., continuously differentiable, functions on E_1. Denote

$$\Upsilon = \Big\{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in C^1(E_1) : \alpha(0) = 0\Big\},$$

and

$$\Gamma = \{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in C(E_1) : \alpha(0) = 0\}.$$

Let $\Phi : F \times \Gamma \to \Upsilon$ be a random linear operator ($\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_n)$) defined by

$$\Phi(\lambda, \varphi)(\iota) = \int_0^1 Q_5(\iota, \kappa)\varphi(\lambda, \kappa) d\kappa, \tag{17}$$

for each $\iota \in E_1$ and $\lambda \in F$. We solve

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \Phi \circ Q_3(\lambda, \gamma, \alpha), \tag{18}$$

where the random multi-valued operator Q_3 is defined by

$$Q_3(\lambda, \gamma, \alpha)(\iota) = Q_1(\lambda, \gamma, \alpha(\iota))\mu(\alpha(\iota)),$$

for each $\iota \in E_1$ and $\lambda \in F$, since it is equivalent (17).

Theorem 3. *Let* $b_1 = \Big\{\sup_{\iota \in E_1} \int_0^1 Q_5(\iota, \kappa) d\kappa\Big\}^{-1}$. *Suppose we can find* $b_2 > 0, b_3 > 0, b_4 \in (0, b_1)$, $\lambda \in F$ *and* $s \in (0, 2)$ *such that*

1. $Q_1(\lambda, 0, \alpha)\mu(\alpha) \preccurlyeq_2 \gamma\alpha$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) > 0$,
2. $b_2 \gamma \alpha - b_3 \preccurlyeq_2 Q_1(\lambda, \gamma, \alpha)\mu(\alpha)$,

3. $Q_1(\lambda, \gamma, \alpha) \preccurlyeq_2 1 + \gamma^{\frac{s}{2}} |\alpha|^s$ *for all* $(\gamma, \alpha) \in \mathring{E}_3 \times \mathbb{R}_+$.

Thus, the positive fuzzy solution set $\boldsymbol{U}$ *for* (18) *is unbounded continuous in* $C^1(E_1)$*, originating from 0.*

Proof. Use Theorem 2 and cones

$$\Omega = \{\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \Upsilon : \alpha(\iota) \geq 0,\ \iota \in E_1\},$$

and

$$\Omega_\Gamma = \{\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \Gamma : \alpha(\iota) \geq 0,\ \iota \in E_1\}.$$

Then, Γ and Υ, resp., are symmetric F-n-BSs with the norms

$$\|(\alpha_1, \alpha_2, \dots, \alpha_n)\|^\Gamma_\epsilon = \sup_{\iota \in E_1} \|(\alpha_1, \alpha_2, \dots, \alpha_n)(\iota)\|_\epsilon,$$

and

$$\|(\alpha_1, \alpha_2, \dots, \alpha_n)\|_\epsilon = \|(\alpha_1, \alpha_2, \dots, \alpha_n)'\|^\Gamma_\epsilon.$$

Suppose $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \dot{\Omega}$, $\lambda \in \digamma$ and ϱ satisfies $\varrho\alpha \in \Phi \circ Q_3(\lambda, 0, \alpha)$, so we can find $\varphi(\lambda, \kappa) \in Q_1(\lambda, 0, \alpha(\kappa))$ such that

$$\begin{aligned}
|\varrho\alpha(\iota)| &= \left| \int_0^1 Q_5(\iota, \kappa)\varphi(\lambda, \kappa)\mu(\alpha(\kappa)) d\kappa \right| \\
&\leq b_4 \|(\alpha_1, \alpha_2, \dots, \alpha_n)\|^\Gamma_\epsilon \left| \int_0^1 Q_5(\iota, \kappa) d\kappa \right| \\
&\leq \|(\alpha_1, \alpha_2, \dots, \alpha_n)\|^\Gamma_\epsilon,
\end{aligned}$$

for each $\iota \in E_1$ and $\lambda \in \digamma$. Then $\varrho < 1$. By [21], we can conclude that the compact random linear operator Φ have an eigen-value $\varrho_0 > 0$ and a positive eigen-map φ_0. Define the random linear operator Q_4 on Γ, by $Q_4(\lambda, \alpha) = \int_0^1 \alpha(\kappa)\varphi_0(\kappa) d\kappa$. From condition 2. we have

$$\begin{aligned}
Q_4(\lambda, Q_3(\lambda, \gamma, \alpha)) &\succcurlyeq_2 \int_0^1 (b_2\gamma\alpha(\kappa) - b_3)\varphi_0(\kappa) d\kappa \\
&\geq b_2\gamma Q_4(\lambda, \alpha) - a_3,
\end{aligned}$$

for each $\lambda \in \digamma$; here $a_3 = b_3 \int_0^1 \varphi_0(\kappa) d\kappa$. When $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ in which $\beta(0) = 0$ and $\beta(1) \geq 0$, then we can find a number $b_5 > 0$ such that $\beta(\iota) \geq b_5 \|(\beta_1, \beta_2, \dots, \beta_n)\|^\Gamma_\epsilon \varphi_0(\iota)$ on E_1. For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \Omega_\Gamma$, $\Phi\alpha$ is a concave function with $\Phi\alpha(\lambda, 0) = 0$ and $\Phi\alpha(\lambda, 1) \geq 0$, and we have $\Phi\alpha(\lambda, \iota) \geq b_5 \|(\Phi_1, \Phi_2, \dots, \Phi_n)(\lambda, \alpha)\|^\Gamma_\epsilon \varphi_0(\iota)$, for each $\lambda \in \digamma$. From Fubini's Theorem it follows that

$$\begin{aligned}
Q_4(\lambda, \Phi(\lambda, \alpha)) &= \int_0^1 \left(\int_0^1 Q_5(\iota, \kappa)\alpha(\kappa) d\kappa \right) \varphi_0(\iota) d\iota \\
&= \int\int_{E_1 \times E_1} Q_5(\iota, \kappa)\alpha(\kappa)\varphi_0(\iota) d\kappa d\iota \\
&= \int_0^1 \left(\int_0^1 Q_5(\iota, \kappa)\varphi_0(\iota) d\iota \right) \alpha(\kappa) d\kappa \\
&= \int_0^1 \Phi\varphi_0(\lambda, \kappa)\alpha(\kappa) d\kappa \\
&= \varrho_0 \int_0^1 \varphi_0(\kappa)\alpha(\kappa) d\kappa \\
&= \varrho_0 Q_4(\lambda, \alpha),
\end{aligned}$$

for each $\lambda \in \digamma$. Consequently, there is constant $a_1 > 0$ satisfying

$$Q_4(\lambda, \Phi(\lambda, \alpha)) \geq a_1 Q_4(\lambda, \alpha) \text{ and } Q_4(\lambda, \Phi(\lambda, \alpha)) \geq a_1 \|(\Phi_1, \Phi_2, \ldots, \Phi_n)(\lambda, b_2)\|_\epsilon^\Gamma, \tag{19}$$

for each $\lambda \in \digamma$. Now, assume $(\varrho, \gamma, \alpha) \in E_1 \times E_3 \times \Omega$ with

$$\alpha \in \varrho\Phi \circ Q_3(\lambda, \gamma, \alpha) + (1-\varrho)b_2\gamma\Phi(\lambda, \alpha). \tag{20}$$

This implies

$$-\alpha'' \in \varrho Q_3(\lambda, \gamma, \alpha) + (1-\varrho)b_2\gamma\alpha, \tag{21}$$

for each $\lambda \in \digamma$. Now, $n_q, q = 0, 1, 2, \ldots, 6$ and n are constant, not depending on γ, α and $\iota \in E_1$. Using Theorem 2 implies that

$$\|(\alpha_1, \alpha_2, \ldots, \alpha_n)\|_\epsilon^\Gamma \leq \frac{a_3}{a_1 b_2 \gamma - 1}.$$

Therefore we can choose n_1 such that

$$\gamma\|(\alpha_1, \alpha_2, \ldots, \alpha_n)\|_\epsilon^\Gamma \leq n_1. \tag{22}$$

From (22), the well-known inequality

$$\left(\|(\alpha_1, \alpha_2, \ldots, \alpha_n)'\|_\epsilon^\Gamma\right)^2 \leq n_2\|(\alpha_1, \alpha_2, \ldots, \alpha_n)\|_\epsilon^\Gamma . \|(\alpha_1, \alpha_2, \ldots, \alpha_n)''\|_\epsilon^\Gamma \tag{23}$$

and (21) we obtain

$$\begin{aligned} \|(\alpha_1, \alpha_2, \ldots, \alpha_n)''\|_\epsilon^\Gamma &\leq n_3\left(1 + \gamma^{\frac{s}{2}}\left(\|(\alpha_1, \alpha_2, \ldots, \alpha_n)\|_\epsilon^\Gamma\right)^s\right) + b_2 n_1 \\ &\leq n_4\left(1 + \gamma^{\frac{s}{2}}\left(\|(\alpha_1, \alpha_2, \ldots, \alpha_n)\|_\epsilon^\Gamma\right)^s\right). \end{aligned} \tag{24}$$

Furthermore, for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \Omega$, we have

$$\|(\alpha_1, \alpha_2, \ldots, \alpha_n)\|_\epsilon^\Gamma \leq n_0\|(\alpha_1, \alpha_2, \ldots, \alpha_n)'\|_\epsilon^\Gamma. \tag{25}$$

Combining the inequalities, (22), (23), (24) and (25) we get

$$\|(\alpha_1, \alpha_2, \ldots, \alpha_n)''\|_\epsilon^\Gamma \leq n_5(1 + \left(\|(\alpha_1, \alpha_2, \ldots, \alpha_n)''\|_\epsilon^\Gamma\right)^{\frac{s}{2}}) \leq n_6. \tag{26}$$

From (23) we can choose n such that

$$\|(\alpha_1, \alpha_2, \ldots, \alpha_n)'\|_\epsilon^\Gamma \leq n\left(\|(\alpha_1, \alpha_2, \ldots, \alpha_n)\|_\epsilon^\Gamma\right)^{\frac{1}{2}}.$$

Since $\|(\alpha_1, \alpha_2, \ldots, \alpha_n)\|_\epsilon = \|(\alpha_1, \alpha_2, \ldots, \alpha_n)'\|_\epsilon^\Gamma$ we have condition (2c) of Theorem 2 satisfied with the function $\varpi(\gamma, \iota) = n\iota^{\frac{1}{2}}$. □

5. Conclusions

In this paper, using a generalized norm which has a dynamic case and is inspired by a random norm and fuzzy sets, we introduced a symmetric F-n-NLS to study the existence, and unbounded continuity of the solution set of random multi-valued equation containing a parameter. These results allow us to consider an uncertain control problem. The applied procedure can hopefully be useful in the future to consider other types of fuzzy control problems.

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