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The Range of a Module Measure Defined on an Effect Algebra

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Abstract: Effect algebras are the main object of study in quantum mechanics. Module measures are those measures defined on an effect algebra with values on a topological module. Let R be a topological ring and M a topological R-module. Let L be an effect algebra. The range of a module measure $\mu : L \to M$ is studied. Among other results, we prove that if L is an sRDP σ-effect algebra with a natural basis and $\mu : L \to \mathbb{R}$ is a countably additive measure, then μ has bounded variation.

Keywords: topological module; topological ring; effect algebra; module measure

MSC: 46H25; 16W80; 54H13

1. Introduction

Effect algebras, as well as Boolean algebras, are examples of universal algebras lying in between Monoid Theory and Order Theory. They have diverse origins. For example, Boolean algebras have historically been involved in topology, measure theory, electronics, computer sciences, etc. However, effect algebras originated from quantum mechanics [1]. More particularly, the first example of effect algebra is constituted by the positive selfadjoint operators on an infinite-dimensional separable complex Hilbert space that lies below the identity. These operators correspond to observable magnitudes. One can easily see that every Boolean algebra can be endowed with the structure of effect algebra, but still effect algebras were not apparently conceived as a generalization of Boolean algebras. This generalization is strict in the sense that there can be found effect algebras which are not isomorphic (in the category of effect algebras) to a Boolean algebra. We refer the reader to [2,3] for a wide perspective on effect algebras and Boolean algebras, and to [4–12] for the most recent studies on effect algebras.

Classical Measure Theory [13] deals with measures defined on Boolean algebras of sets. In view of the famous Stone Representation Theorem for Boolean algebras [14], every Boolean algebra is isomorphic (in the category of Boolean algebras) to a Boolean algebra of sets. This way, Classical Measure Theory retained full generality when it came to the domain of definition of classical measures until the birth of effect algebras. However, these classical measures were always real or complex valued. In the remarkable book [15], vector measures are deeply studied. These measures are again defined on a Boolean algebra of sets; however, they are valued on a real or complex Banach space. After the irruption of effect algebras, modern measures were no longer studied in Boolean algebras. In [1], the authors consider measures on effect algebras with values on a commutative group, introducing the interesting concept of "universal group" for an effect algebra. In [16,17], measures on effect algebras with values on a commutative topological group are considered. Later, in [18–21], the authors study vector-valued measures on effect algebras, that is, with values on real or complex Hausdorff locally convex topological vector spaces. A few years later, measures on effect algebras with values on a real or complex normed space were considered in [22,23]. Finally, in the recent manuscript [24], measures on effect algebras with values on a commutative normed group are studied.

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2. Objectives

This manuscript is a continuation of previous works, focusing on the geometry of the range of a module measure, that is, measures defined on an effect algebra and valued on a topological module over a topological ring. Since commutative topological groups can be regarded as topological $\mathbb{Z}$-modules, and every ring and every module can be endowed with the discrete topology, our choice of studying measures on effect algebras valued on topological modules over topological rings seems to be the most general setting so far. This manuscript aims at achieving the following objectives:

1. Defining and studying measures on the most general setting possible: Defined on an effect algebra and valued on a topological module over a topological ring.
2. Studying the range of a module measure.
3. Proving that countably additive real-valued measures defined on sRDP σ-effect algebras with a natural basis have bounded variation (Corollary 4).
4. Endowing full unit segments with structure of effect algebra (Theorem 6).
5. Defining a measure with physical significance in a quantum mechanical system (Proposition 1).
6. Constructing new module measures for quantum mechanical systems (Lemma 4).

Throughout this work, connections with Associative Ring Theory [25], Tropical Theory [26–28], and Operad Algebras [29] are established.

3. Preliminaries

A universal algebra [30] is a set endowed with multiary internal or external operations. These operations can be partially or totally defined. Two universal algebras are compatible when they have the same amount of operations of the same nature and size. An operator of universal algebras is simply a map between compatible universal algebras preserving the operations. Effect algebras and Boolean algebras are examples of universal algebras.

An effect algebra is a universal algebra $(L, \oplus, 0, 1, {}^{\perp})$ where

$$\begin{array}{rccc} \oplus : & \Sigma \subseteq L \times L & \to & L \\ & (p,q) & \mapsto & p \oplus q \end{array}$$

is a partially defined binary internal operation,

$$\begin{array}{rccc} {}^{\perp} : & L & \to & L \\ & p & \mapsto & p^{\perp} \end{array}$$

is a unary internal operation (called orthocomplementation), and

$$\begin{array}{rccc} 0 : & L & \to & L \\ & p & \mapsto & 0 \end{array} \text{ and } \begin{array}{rccc} 1 : & L & \to & L \\ & p & \mapsto & 1 \end{array}$$

are nullary internal operations satisfying the following conditions for all $p, q, r \in L$:

- Commutativity: Σ is a symmetric binary relation on L and if $(p,q) \in \Sigma$, then $p \oplus q = q \oplus p$.
- Associativity: If $(q,r), (p, q \oplus r) \in \Sigma$, then $(p,q), (p \oplus q, r) \in \Sigma$ and $(p \oplus q) \oplus r = p \oplus (q \oplus r)$.
- Orthocomplementation: $p^{\perp}$ is the only element in L such that $(p, p^{\perp}) \in \Sigma$ and $p \oplus p^{\perp} = 1$.
- Zero-One Law: $1 \neq 0$ and 0 is the only element in L such that $(1,0) \in \Sigma$.

Effect algebra of sets are among the most representative examples of effect algebras. An effect algebra of set is a subset L of the power set $\mathcal{P}(X)$ of a given nonempty set X such that $\{\varnothing, X\} \subseteq L$ and $(L, \cup, \varnothing, X)$ has effect algebra structure under the partial operation $A \cup B$ defined on $\Sigma := \{(A,B) \in L \times L : A \cap B = \varnothing\}$.

Two elements p, q of an effect algebra L are said to be orthogonal if $p \oplus q$ exists; that is, $(p, q) \in \Sigma$. Whenever we write $p \oplus q$, we will be assuming that they are orthogonal. For every $p, q, r \in L$, it is easy to check that $1^\perp = 0$, $p \oplus 0 = p$, $\left(p^\perp\right)^\perp = p$, and $q = r$ whenever $p \oplus q = p \oplus r$. A partial order can be defined in an effect algebra L:

$$p \leq q \Leftrightarrow \exists r \text{ with } p \oplus r = q.$$

The following properties can be easily verified for $p, q, r \in L$:

- r is unique verifying $p \oplus r = q$ and thus it is denoted by $q \ominus p$.
- $0 = \min(L)$ and $1 = \max(L)$; that is, L is a bounded poset.
- $p^\perp = 1 \ominus p$.
- $p \leq q$ if and only if $1 \ominus q \leq 1 \ominus p$.
- If $p \leq q$ and $p \oplus r$ and $q \oplus r$ exist, then $p \oplus r \leq q \oplus r$.
- If $r \leq p \leq q$, then $p \ominus r \leq q \ominus r$.
- If $p \leq q \leq r$, then $r \ominus q \leq r \ominus p$.

Notice that if L, G are effect algebras and $f : L \to G$ is an effect algebra operator, then f is clearly increasing.

If $A = \{a_1, \ldots, a_k\}$ is a finite subset of orthogonal elements of an effect algebra L, then we will denote

$$\bigoplus A := \bigoplus_{n=1}^{k} a_n := a_1 \oplus \cdots \oplus a_k.$$

A subset $B \subseteq L$ is said to be orthogonal provided that for every finite subset $A \subseteq B$, $\bigoplus A$ exists. We will denote $\bigoplus B := \sup\{\bigoplus A : A \subseteq B \text{ is finite}\}$, provided that this supremum exists. It is easy to see that, if $(a_n)_{n\in\mathbb{N}}$ is an orthogonal sequence such that $\bigoplus\{a_n : n \in \mathbb{N}\}$ exists, then

$$\bigoplus\{a_n : n \in \mathbb{N}\} = \sup\left\{\bigoplus_{n=1}^{k} a_n : k \in \mathbb{N}\right\}.$$

We will often denote $\bigoplus\{a_n : n \in \mathbb{N}\} := \bigoplus_{n\in\mathbb{N}} a_n = \bigoplus_{n=1}^{\infty} a_n$. An effect algebra in which every orthogonal sequence $(a_n)_{n\in\mathbb{N}}$ verifies that $\bigoplus_{n\in\mathbb{N}} a_n$ exists is called an effect σ-algebra or σ-effect algebra.

An effect algebra L is said to enjoy the Riesz Decomposition Property (RDP) [31] if $c \leq a \oplus b$ implies that $c = c_1 \oplus c_2$ with $c_1 \leq a$ and $c_2 \leq b$. Notice that the previous condition easily extends to finite orthosums. In this sense, L is said to enjoy the strong Riesz Decomposition Property (sRDP) if $c \leq \bigoplus_{n\in\mathbb{N}} a_n$ implies that $c = \bigoplus_{n\in\mathbb{N}} c_n$ and $c_n \leq a_n$ for all $n \in \mathbb{N}$.

An orthogonal sequence $(b_n)_{n\in\mathbb{N}}$ of an effect algebra L is called a natural basis of L provided that $1 = \bigoplus_{n\in\mathbb{N}} b_n$ and b_n is minimal in $L \setminus \{0\}$ for every $n \in \mathbb{N}$. We will say that effect algebra L is natural if it has the sRDP and has a natural basis. In [24] Proposition 2.7, it was proved that if L is an effect algebra with the sRDP and $(b_n)_{n\in\mathbb{N}}$ a natural basis of L, then for every $a \in L$ there exists a subsequence $(b_{n_k})_{k\in\mathbb{N}}$ of $(b_n)_{n\in\mathbb{N}}$ such that $a = \bigoplus_{k\in\mathbb{N}} b_{n_k}$. In particular, $B := \left\{b_{i_1} \oplus \cdots \oplus b_{i_k} : i_1, \ldots, i_k \in \mathbb{N}\right\}$ is a generator of L (recall that a subset B of a poset L is called a generator of L if, for every $a \in L$, there exists an increasing sequence $(b_n)_{n\in\mathbb{N}} \subseteq B$ such that $a = \bigvee_{n\in\mathbb{N}} b_n$).

Module measures are those measures with values on a topological module. Let L be an effect algebra, M a topological module over a topological ring R, and $\mu : L \to M$ a map. We say that μ is a measure if it is an additive map; that is, $\mu(a \oplus b) = \mu(a) + \mu(b)$ for every orthogonal $a, b \in L$. It is trivial to check that $\mu(0) = 0$, $\mu(b \ominus a) = \mu(b) - \mu(a)$ for all $a, b \in L$, and if L is a distributive complemented lattice, then $\mu(a \vee b) = \mu(a) + \mu(b) - \mu(a \wedge b)$ for all $a, b \in L$. Observe that the set of all measures on an effect algebra L with values on a topological module M over a topological ring R is a submodule of M^L.

In addition, a measure $\mu : L \to M$ is said to be:

- σ-additive or countably additive if $\mu(\bigoplus_{n\in\mathbb{N}} a_n) \in \sum_{n\in\mathbb{N}} \mu(a_n)$ for every orthogonal sequence $(a_n)_{n\in\mathbb{N}} \subseteq L$ such that $\bigoplus_{n\in\mathbb{N}} a_n$ exists;
- Strongly additive if $\sum_{n\in\mathbb{N}} \mu(a_n)$ converges for every orthogonal sequence $(a_n)_{n\in\mathbb{N}} \subseteq L$;
- Exhaustive or strongly bounded if $0 \in \lim_{n\to\infty} \mu(a_n)$ for every orthogonal sequence $(a_n)_{n\in\mathbb{N}} \subseteq L$.

Notice that if μ is a σ-additive measure and M is Hausdorff, then $\sum_{n\in\mathbb{N}} \mu(a_n)$ is unconditionally convergent for all orthogonal sequences $(a_n)_{n\in\mathbb{N}}$ for which $\bigoplus_{n\in\mathbb{N}} a_n$ exists. Moreover, if μ is a σ-additive and L is a σ-effect algebra, then μ is strongly additive.

Suppose now that M is a seminormed module over a seminormed ring R. We say that a measure $\mu : L \to M$ is absolutely additive if $\sum_{n\in\mathbb{N}} \|\mu(a_n)\| < +\infty$ for every orthogonal sequence $(a_n)_{n\in\mathbb{N}} \subseteq L$.

Let L be an effect algebra, M a seminormed module over a seminormed ring R, and $\mu : L \to M$ a measure. We say that $\pi = \{e_1, e_2, \ldots, e_n\} \subseteq L$ is a decomposition of $e \in L$ if π is an orthogonal set and $e = e_1 \oplus \cdots \oplus e_n$. The variation of μ in $e \in L$ is defined as the map

$$\begin{array}{rccl} |\mu| : & L & \to & [0, +\infty] \\ & e & \mapsto & |\mu|(e) := \sup\limits_{\pi\in\Pi} \sum\limits_{e_i\in\pi} \|\mu(e_i)\|, \end{array}$$

where the supremum is taken over the family Π of all decompositions $\pi = \{e_1, e_2, \ldots, e_n\}$ of e. We say that μ is of bounded variation if $|\mu|(1) < +\infty$. The set of measures of bounded variation on L with values on M is denoted by $\mathrm{ba}(L, M)$. It is not difficult to check that $\mathrm{ba}(L, M)$ is a submodule of M^L.

4. Results

We will begin with a technical lemma, which extends classical properties of Boolean algebras to the scope of effect algebras.

Lemma 1. *Let L be an effect algebra. Consider a sequence $(a_n)_{n\in\mathbb{N}} \subseteq L$. Then:*

1. *If $(a_n)_{n\in\mathbb{N}}$ is increasing and $\bigvee_{n\in\mathbb{N}} a_n$ exists, then*
$$\bigvee_{n\in\mathbb{N}} a_n = a_1 \oplus \bigoplus_{n\geq 2} (a_n \ominus a_{n-1}).$$
2. *If $(a_n)_{n\in\mathbb{N}}$ is decreasing and $\bigwedge_{n\in\mathbb{N}} a_n$ exists, then $(1 \ominus a_n)_{n\in\mathbb{N}}$ is increasing, $\bigvee_{n\in\mathbb{N}}(1 \ominus a_n)$ exists, and*
$$\bigvee_{n\in\mathbb{N}} (1 \ominus a_n) = 1 \ominus \bigwedge_{n\in\mathbb{N}} a_n.$$

Proof.

1. Notice that $a_1 \leq \bigvee_{n\in\mathbb{N}} a_n$; therefore $(\bigvee_{n\in\mathbb{N}} a_n) \ominus a_1$ exists in L. Thus, it only remains to show that
$$\left(\bigvee_{n\in\mathbb{N}} a_n\right) \ominus a_1 = \bigoplus_{n\geq 2} (a_n \ominus a_{n-1}).$$
Observe that, for every $k \geq 2$,
$$\bigoplus_{n=2}^{k} (a_n \ominus a_{n-1}) = a_k \ominus a_1 \leq \left(\bigvee_{n\in\mathbb{N}} a_n\right) \ominus a_1.$$

Thus, $(\bigvee_{n\in\mathbb{N}} a_n) \ominus a_1$ is an upper bound of $\left\{\bigoplus_{n=2}^{k}(a_n \ominus a_{n-1}) : k \geq 2\right\}$. Consider another upper bound d for $\left\{\bigoplus_{n=2}^{k}(a_n \ominus a_{n-1}) : k \geq 2\right\}$; that is,

$$a_k \ominus a_1 = \bigoplus_{n=2}^{k}(a_n \ominus a_{n-1}) \leq d$$

for every $k \geq 2$. Then $a_k \leq d \oplus a_1$ for every $k \in \mathbb{N}$; hence

$$\bigvee_{n\in\mathbb{N}} a_n \leq d \oplus a_1,$$

and so

$$\left(\bigvee_{n\in\mathbb{N}} a_n\right) \ominus a_1 \leq d.$$

As a consequence,

$$\left(\bigvee_{n\in\mathbb{N}} a_n\right) \ominus a_1 = \sup_{k\geq 2} \bigoplus_{n=2}^{k}(a_n \ominus a_{n-1}) = \bigoplus_{n=2}^{k}(a_n \ominus a_{n-1}).$$

2. On the one hand, $\bigwedge_{n\in\mathbb{N}} a_n \leq a_n$ for all $n \in \mathbb{N}$; thus $1 \ominus a_n \leq 1 \ominus \bigwedge_{n\in\mathbb{N}} a_n$ for all $n \in \mathbb{N}$; that is, $1 \ominus \bigwedge_{n\in\mathbb{N}} a_n$ is an upper bound for $(1 \ominus a_n)_{n\in\mathbb{N}}$. Consider another upper bound d for $(1 \ominus a_n)_{n\in\mathbb{N}}$. Then $1 \ominus a_n \leq d$ for every $n \in \mathbb{N}$, so $1 \ominus d \leq a_n$ for all $n \in \mathbb{N}$; hence $1 \ominus d \leq \bigwedge_{n\in\mathbb{N}} a_n$, meaning that $1 \ominus \bigwedge_{n\in\mathbb{N}} a_n \leq d$. This shows that

$$\bigvee_{n\in\mathbb{N}} (1 \ominus a_n) = 1 \ominus \bigwedge_{n\in\mathbb{N}} a_n.$$

□

With the help of Lemma 1, we can prove the following theorem, which extends a classical property of measures on Boolean algebras to the scope of effect algebras.

Theorem 1. *Let L be an effect algebra, M a topological module over a topological ring R, and $\mu : L \to M$ a σ-additive measure. If $(a_n)_{n\in\mathbb{N}} \subseteq L$ is an increasing sequence such that $\bigvee_{n\in\mathbb{N}} a_n$ exists, then $(\mu(a_n))_{n\in\mathbb{N}}$ converges to*

$$\mu\left(\bigvee_{n\in\mathbb{N}} a_n\right).$$

Proof. On the one hand, in virtue of Lemma 1 (1),

$$\bigvee_{n\in\mathbb{N}} a_n = a_1 \oplus \bigoplus_{n\geq 2}(a_n \ominus a_{n-1}).$$

On the other hand, for every $k \geq 2$,

$$\sum_{n=2}^{k}(\mu(a_n) - \mu(a_{n-1})) = \mu(a_k) - \mu(a_1).$$

Finally,

$$\begin{aligned}\mu\left(\bigvee_{n\in\mathbb{N}} a_n\right) &= \mu\left(a_1 \oplus \bigoplus_{n\geq 2}(a_n \ominus a_{n-1})\right)\\ &= \mu(a_1) + \mu\left(\bigoplus_{n\geq 2}(a_n \ominus a_{n-1})\right)\\ &\in \mu(a_1) + \sum_{n=2}^{\infty}(\mu(a_n) - \mu(a_{n-1}))\\ &= \mu(a_1) + \lim_{n\to\infty}(\mu(a_n) - \mu(a_1))\\ &\subseteq \lim_{n\to\infty}\mu(a_n).\end{aligned}$$

□

Corollary 1. *Let L be an effect algebra, M a topological module over a topological ring R, and $\mu : L \to M$ a σ-additive measure. If $(a_n)_{n\in\mathbb{N}} \subseteq L$ is a decreasing sequence such that $\bigwedge_{n\in\mathbb{N}} a_n$ exists, then $(\mu(a_n))_{n\in\mathbb{N}}$ converges to*

$$\mu\left(\bigwedge_{n\in\mathbb{N}} a_n\right).$$

Proof. In accordance with Lemma 1 (2), $(1 \ominus a_n)_{n\in\mathbb{N}}$ is an increasing sequence such that $\bigvee_{n\in\mathbb{N}}(1 \ominus a_n) = 1 \ominus \bigwedge_{n\in\mathbb{N}} a_n$ exists. Therefore, by Theorem 1, $(\mu(1 \ominus a_n))_{n\in\mathbb{N}}$ converges to

$$\mu\left(\bigvee_{n\in\mathbb{N}}(1 \ominus a_n)\right) = \mu\left(1 \ominus \bigwedge_{n\in\mathbb{N}} a_n\right) = \mu(1) - \mu\left(\bigwedge_{n\in\mathbb{N}} a_n\right).$$

Finally, it suffices to observe that $\mu(1 \ominus a_n) = \mu(1) - \mu(a_n)$ for all $n \in \mathbb{N}$. □

Recall that a subset A of a topological module M over a topological ring R is said to be bounded provided that for each 0-neighborhood $U \subseteq M$ there exists an invertible $u \in \mathcal{U}(R)$ such that $A \subseteq uU$. It is not difficult to prove that if A is bounded, then $\mathrm{cl}(A)$ is bounded as well.

Corollary 2. *Let L be an effect algebra, M a topological module over a topological ring R, and $\mu : L \to M$ a σ-additive measure. Then $\mu(L)$ is bounded in M if and only if $\mu(B)$ is bounded in M, for B a generator of L.*

Proof.

- $\Rightarrow$ If $\mu(L)$ is bounded in M and B is a generator of L, then $\mu(B)$ is trivially bounded because $\mu(B) \subseteq \mu(L)$.
- $\Leftrightarrow$ Assume that there exists a generator B of L such that $\mu(B)$ is bounded in M. Fix an arbitrary $a \in L$ and choose, by hypothesis, an increasing sequence $(b_n)_{n\in\mathbb{N}} \subseteq B$ such that $a = \bigvee_{n\in\mathbb{N}} b_n$. In accordance with Theorem 1, we have that $(\mu(b_n))_{n\in\mathbb{N}}$ converges to $\mu(a)$. As a consequence, $\mu(a) \in \mathrm{cl}(\mu(B))$. The arbitrariness of $a \in L$ shows that $\mu(L) \subseteq \mathrm{cl}(\mu(B))$. Finally, since $\mu(B)$ is bounded in M, we have that $\mathrm{cl}(\mu(B))$ is bounded in M as well, hence so is $\mu(L)$.

□

The next theorem shows that, under somehow restrictive conditions on the seminormed module, σ-additive measures on σ-effect algebras are absolutely additive.

Theorem 2. *Let L be an effect algebra, M a seminormed module over a seminormed ring R, and $\mu : L \to M$ a measure. Then:*

1. *If L is a σ-effect algebra, $R = \mathbb{R}$, M is finite-dimensional and normed, and μ is σ-additive, then μ is absolutely additive.*
2. *If M is complete and μ is absolutely additive, then μ is strongly additive.*
3. *If μ is absolutely additive and L satisfies the sRDP and has a natural basis $(b_n)_{n\in\mathbb{N}}$, then $\mu(L)$ is seminorm-bounded.*

Proof.

1. Let $(a_n)_{n\in\mathbb{N}} \subseteq L$ be an orthogonal sequence. Since L is a σ-effect algebra, we have that $\bigoplus_{n\in\mathbb{N}} a_n$ exists in L. Next, μ is σ-additive and M is Hausdorff, so $\mu(\bigoplus_{n\in\mathbb{N}} a_n) = \sum_{n\in\mathbb{N}} \mu(a_n)$. Notice that $\sum_{n\in\mathbb{N}} \mu(a_n)$ is unconditionally convergent. Finally, since M is a finite-dimensional real normed space, we have that every unconditionally convergent series in M is absolutely convergent, meaning that $\sum_{n\in\mathbb{N}} \mu(a_n)$ is absolutely convergent.
2. Let $(a_n)_{n\in\mathbb{N}} \subseteq L$ be an orthogonal sequence. By hypothesis, $\sum_{n\in\mathbb{N}} \|\mu(a_n)\| < +\infty$. This means that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Since M is complete, $\sum_{n=1}^{\infty} a_n$ is convergent. This proves that μ is strongly additive.
3. Since $(b_n)_{n\in\mathbb{N}}$ is orthogonal, by hypothesis we can consider $K := \sum_{n\in\mathbb{N}} \|\mu(b_n)\| < +\infty$. In virtue of [24] Proposition 2.7,
$$B := \left\{b_{i_1} \oplus \cdots \oplus b_{i_k} : i_1, \ldots, i_k \in \mathbb{N}\right\}$$
is a generator of L. According to Corollary 2 adapted to seminorm boundedness, it only suffices to show that $\mu(B)$ is seminorm-bounded. For each $b_{i_1} \oplus \cdots \oplus b_{i_k} \in B$,
$$\|\mu(b_{i_1} \oplus \cdots \oplus b_{i_k})\| = \left\| \sum_{j=1}^{k} \mu\left(b_{i_j}\right) \right\| \leq \sum_{j=1}^{k} \left\| \mu\left(b_{i_j}\right) \right\| \leq K.$$
As a consequence, $\mu(B)$ is seminorm-bounded. □

Theorem 3. *Let L be an effect algebra, M a seminormed module over a seminormed ring R, and $\mu : L \to M$ a measure. Let $e, f \in L$ be orthogonal. Then:*

1. $|\mu|(e) + |\mu|(f) \leq |\mu|(e \oplus f)$.
2. *If L is RDP, then* $|\mu|(e \oplus f) \leq |\mu|(e) + |\mu|(f)$.

Proof.

1. Fix an arbitrary $\varepsilon > 0$ and let $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_m\}$ be decompositions of e and f, respectively, such that
$$|\mu|(e) - \sum_{i=1}^{n} \|\mu(e_i)\| < \frac{\varepsilon}{2}$$
and
$$|\mu|(f) - \sum_{i=1}^{m} \|\mu(f_i)\| < \frac{\varepsilon}{2}.$$
Then $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ is a decomposition of $e \oplus f$; therefore,
$$|\mu|(e) + |\mu|(f) < \sum_{i=1}^{n} \|\mu(e_i)\| + \sum_{i=1}^{m} \|\mu(f_i)\| + \varepsilon \leq |\mu|(e \oplus f) + \varepsilon.$$
The arbitrariness of ε implies that $|\mu|(e) + |\mu|(f) \leq |\mu|(e \oplus f)$.
2. If L is RDP, given a decomposition $e \oplus f = a_1 \oplus a_2 \oplus \cdots \oplus a_n$, since $e \leq e \oplus f = a_1 \oplus a_2 \oplus \cdots \oplus a_n$, there exist $e_1, e_2, \ldots, e_n \in L$ such that $e_1 \leq a_1$, $e_2 \leq a_2$, $\ldots$ $e_n \leq a_n$

and $e = e_1 \oplus e_2 \oplus \cdots \oplus e_n$. There exist $f_1, \ldots, f_n \in L$ such that $e_i \oplus f_i = a_i$ for all $i \in \{1, \ldots, n\}$. Then

$$\begin{aligned} e \oplus f &= a_1 \oplus a_2 \oplus \cdots \oplus a_n \\ &= (e_1 \oplus f_1) \oplus (e_2 \oplus f_2) \oplus \cdots \oplus (e_n \oplus f_n) \\ &= (e_1 \oplus e_2 \oplus \cdots \oplus e_n) \oplus (f_1 \oplus f_2 \oplus \cdots \oplus f_n), \end{aligned}$$

so $f = f_1 \oplus f_2 \oplus \cdots \oplus f_n$ and $f_i \leq f$ for all $i \in \{1, \ldots, n\}$. This shows that $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ are decompositions of e and f, respectively. Thus,

$$\begin{aligned} \sum_{i=1}^{n} \|\mu(a_i)\| &= \sum_{i=1}^{n} \|\mu(e_i \oplus f_i)\| = \sum_{i=1}^{n} \|\mu(e_i) + \mu(f_i)\| \\ &\leq \sum_{i=1}^{n} \|\mu(e_i)\| + \sum_{i=1}^{n} \|\mu(f_i)\| \\ &\leq |\mu|(e) + |\mu|(f), \end{aligned}$$

Since this inequality is valid for every decomposition of $e \oplus f$, we conclude that

$$|\mu|(e \oplus f) \leq |\mu|(e) + |\mu|(f).$$

□

A direct consequence of Theorem 3 (1) is the fact that, if $\mu : L \to M$ is a measure on an effect algebra L with values in a seminormed module M over a seminormed ring R, then $|\mu| : L \to [0, +\infty]$ is an increasing function; that is, if $a, b \in L$ with $a \leq b$, then $|\mu|(a) \leq |\mu|(b)$.

Corollary 3. *Let L be an effect algebra and M a seminormed module over a seminormed ring R. If L is an RDP effect algebra, then the variation of a measure $\mu : L \to M$ is also a measure on L.*

Theorem 4. *Let L be an effect algebra and M a seminormed module over a seminormed ring R. Let $\mu : L \to M$ be a measure. Then:*

1. *If μ has bounded variation, then $\mu(L)$ is seminormed bounded.*
2. *If M is complete and μ has bounded variation, then μ is absolutely additive and strongly additive.*
3. *If $M = \mathbb{R}$ and $\mu(L)$ is bounded, then μ has bounded variation.*

Proof.

1. Fix an arbitrary $a \in L$. Then

$$\|\mu(a)\| \leq \|\mu(a)\| + \left\|\mu\left(a^{\perp}\right)\right\| \leq |\mu|(1).$$

2. If $(a_n)_{n \in \mathbb{N}} \subseteq L$ is an orthogonal sequence, then

$$\sum_{n=1}^{k} \|\mu(a_n)\| \leq |\mu|\left(\bigoplus_{n=1}^{k} a_n\right) \leq |\mu|(1)$$

for each $k \in \mathbb{N}$. Thus, $\sum_{n \in \mathbb{N}} \|\mu(a_n)\| \leq |\mu|(1) < \infty$, so $\sum_{n \in \mathbb{N}} \mu(a_n)$ is absolutely convergent, hence convergent.

3. Fix an arbitrary decomposition $\pi = \{e_1, e_2, \ldots, e_n\}$ of 1. Let us denote $I^+ := \{i \in \{1, \ldots, n\} : \mu(e_i) \geq 0\}$ and $I^- := \{1, \ldots, n\} \setminus I^+$. Then

$$\begin{aligned} \sum_{i=1}^{n} |\mu(e_i)| &= \sum_{i \in I^+} \mu(e_i) - \sum_{i \in I^-} \mu(e_i) \\ &= \mu\left(\bigoplus_{i \in I^+} e_i\right) - \mu\left(\bigoplus_{i \in I^-} e_i\right) \\ &\leq \sup \mu(L) - \inf \mu(L). \end{aligned}$$

As a consequence, $|\mu|(1) \leq \sup \mu(L) - \inf \mu(L)$. □

Corollary 4. *Let L be an sRDP σ-effect algebra with a natural basis. If $\mu : L \to \mathbb{R}$ is a countably additive measure, then μ has bounded variation.*

Proof. Since L is a σ-effect algebra and μ is σ-additive, we conclude, by means of Theorem 2 (1), that μ is absolutely additive. According to Theorem 2 (3), $\mu(L)$ is bounded. Finally, Theorem 4 (3) assures that μ has bounded variation. □

We will conclude this manuscript by proving that the variation of a measure induces a seminorm.

Theorem 5. *Let L be an effect algebra and M a seminormed module over a seminormed ring R. The map*

$$\begin{aligned} \| \ \|_1 : \quad \mathrm{ba}(L, M) &\to [0, +\infty) \\ \mu &\mapsto \|\mu\|_1 := |\mu|(1) \end{aligned}$$

defines a seminorm in $\mathrm{ba}(L, M)$.

Proof. Take $\mu, \nu \in \mathrm{ba}(L, M)$ and $r \in R$. Fix an arbitrary partition $\{e_1, \ldots, e_n\}$ of 1. Notice that

$$\sum_{i=1}^{n} \|(\mu + \nu)(e_i)\| \leq \sum_{i=1}^{n} \|\mu(e_i)\| + \sum_{i=1}^{n} \|\nu(e_i)\| \leq |\mu|(1) + |\nu|(1).$$

The arbitrariness of the partition $\{e_1, \ldots, e_n\}$ of 1 shows that $|\mu + \nu|(1) \leq |\mu|(1) + |\nu|(1)$. This proves the triangular inequality. Next,

$$\sum_{i=1}^{n} \|(r\mu)(e_i)\| = \sum_{i=1}^{n} \|r\mu(e_i)\| \leq \|r\| \sum_{i=1}^{n} \|\mu(e_i)\| \leq \|r\| |\mu|(1).$$

Again, the arbitrariness of the partition $\{e_1, \ldots, e_n\}$ of 1 shows that $|r\mu|(1) \leq \|r\| |\mu|(1)$. We then conclude that $\| \ \|_1$ is a seminorm on $\mathrm{ba}(L, M)$. Finally, if R is absolutely semivalued, for every $\varepsilon > 0$, we can fix a partition $\{e_1, \ldots, e_n\}$ of 1 satisfying that

$$|\mu|(1) - \varepsilon < \sum_{i=1}^{n} \|\mu(e_i)\|,$$

meaning that

$$|r|(|\mu|(1) - \varepsilon) \leq |r| \sum_{i=1}^{n} \|\mu(e_i)\| = \sum_{i=1}^{n} \|r\mu(e_i)\| \leq |r\mu|(1),$$

which implies that $|r||\mu|(1) = |r\mu|(1)$ by bearing in mind the arbitrariness of $\varepsilon > 0$. □

5. Examples

In [25] Definition 3.2, a new concept in Associative Ring Theory was introduced, the notion of "unit segment". Let R be a topological ring. A subset $B^+ \subsetneq R$ is called a unit segment if it verifies that $0, 1 \in B^+$, $B^+B^+ = B^+$, $B^+ \cap B^- = \{0\}$, and $1 + B^- = B^+$. Here, $B^- := -B^+$ is called negative unit segment. The previous definition remains equivalent if the second condition is substituted by $B^+B^+ \subseteq B^+$ or if the last condition is changed to $1 + B^- \subseteq B^+$. A trivial example of unit segment is the real interval $[0, 1]$. In [30] Lemma 53, it is shown that a topological ring has a unit segment if and only if the ring has characteristic differences from 2. Basic properties of unit segments are presented in the next lemma.

Lemma 2. *Let R be a topological ring. Let $B^+ \subsetneq R$ be a unit segment. Then:*

1. $(1 + B^+) \cap B^+ = \{1\}$.
2. *If $r_1, \ldots, r_k \in B^+$ with $r_1 + \cdots + r_k = 1$, then $r_1 + \cdots + r_{k-1} \in B^+$.*

Proof.

1. If $a, b \in B^+$ satisfies that $1 + a = b$, then $1 - b = -a \in B^+ \cap B^- = \{0\}$.
2. Simply notice that $r_1 + \cdots + r_{k-1} = 1 - r_k \in 1 + B^- = B^+$.

□

Lemma 2 (2) motivates the following definition.

Definition 1. *Let R be a topological ring. A unit segment $B^+ \subsetneq R$ is called full if for every $r_1, \ldots, r_k \in B^+$ with $r_1 + \cdots + r_k \in B^+$, then $r_1 + \cdots + r_{k-1} \in B^+$.*

Notice that full unit segments satisfy that if $r_1, \ldots, r_k \in B^+$ with $r_1 + \cdots + r_k \in B^+$, then $\sum_{j \in J} r_j \in B^+$ for every $J \subseteq \{1, \ldots, k\}$. According to [25] Theorem 3.3, if R is a totally ordered ring, then $B^+ := [0, 1]$ is a unit segment of R. We will show next that full unit segments can be endowed with a partial order in such a way that they form an interval $[0, 1]$.

Lemma 3. *Let R be a topological ring. Let $B^+ \subsetneq R$ be a full unit segment. The binary relation on B^+ given by $a \leq b \Leftrightarrow \exists c \in B^+$ with $a + c = b$ is a partial order on B^+ whose maximum and minimum elements are 1 and 0, respectively.*

Proof. For every $a \in B^+$, $a + 0 = a$, so $a \leq a$ and the relation is transitive. Next, if $a, b \in B^+$ with $a \leq b$ and $b \leq a$, then we can find $c, d \in B^+$ such that $a + c = b$ and $b + d = a$; hence $a + c + d = a$, so $c = -d \in B^+ \cap B^- = \{0\}$, reaching that $a = b$; thus the relation is antisymmetric. Finally, if $a, b, c \in B^+$ with $a \leq b$ and $b \leq c$, then we can find $r, t \in B^+$ such that $a + r = b$ and $b + t = c$; hence $a + (r + t) = c$ with $r + t \in B^+$ due to the fullness of B^+, so $a \leq c$ and the relation is transitive. □

Notice that, in Lemma 3, the fullness of the unit segment has only been used to show the transitive property of the order relation. In fact, full unit segments can be endowed with structure of effect algebra. This is our next theorem.

Theorem 6. *If R is a topological ring and $B^+ \subsetneq R$ is a full unit segment, then the universal algebra $(B^+, +|_\Sigma, {}^c, 0, 1)$ becomes an effect algebra, where the ring addition has been restricted to the symmetric binary relation on B^+ given by $\Sigma := \{(r, s) \in B^+ \times B^+ : r + s \in B^+\}$ and the orthocomplementation is defined as*

$$\begin{array}{rccl} {}^c: & B^+ & \to & B^+ \\ & r & \mapsto & r^c := 1 - r. \end{array}$$

Proof. Notice that $+|_\Sigma$ is clearly associative whenever it is defined. Indeed, if $a, b, c \in B^+$ with $b + c \in B^+$ and $a + (b + c) \in B^+$, then $a + b \in B^+$ due to the fullness of B^+ and

$(a+b)+c = a+(b+c) \in B^+$. In a similar way, $+|_\Sigma$ is commutative whenever it is defined. The uniqueness of the orthocomplement is also inferred from the uniqueness of the opposite. Finally, the Zero-One Law is also trivially verified by bearing in mind Lemma 2 (1). $\square$

6. Quantum Mechanics

Let H be an infinite-dimensional separable complex Hilbert space. A linear operator $T \in \mathcal{B}(H)$ is called selfadjoint provided that $T^* = T$; that is, T coincides with its adjoint, which is equivalent to the fact that $(T(x)|y) = (x|T(y))$ for all $x, y \in H$. Selfadjoint operators trivially satisfy that $(T(x)|x) \in \mathbb{R}$ for all $x \in H$. A selfadjoint operator is called positive, written $T \geq 0$, provided that $(T(x)|x) \geq 0$ for all $x \in H$. A partial order can be easily defined on the set of all selfadjoint operators: $T \leq S \Leftrightarrow S - T \geq 0$.

Let us recall now the first two postulates of quantum mechanics [32]. We will not follow the classical quantum mechanics notation (the bra-ket notation), but the classical functional analysis notation to keep consistency with the notation in the rest of the paper:

1. First Postulate of Quantum Mechanics: To every quantum mechanical system corresponds an infinite-dimensional separable complex Hilbert space H.
2. Second Postulate of Quantum Mechanics: Every observable magnitude of the quantum mechanical system H is represented by a selfadjoint operator $T : H \to H$.

A pure state of a quantum mechanical system H in a fixed instant of time t is represented by a unit ray $\mathsf{S}_{\mathbb{C}}x$ with $x \in H$ and $\|x\| = 1$. An element of the previous ray is called a state vector or a ket.

On the other hand, the correspondence between observable magnitudes and selfadjoint operators is not in general bijective; that is, not all selfadjoint operators represent an observable magnitude. The existence of observable magnitudes represented by selfadjoint unbounded operators implies that the Hilbert space H representing the quantum mechanical system is infinite-dimensional, since every linear operator on a finite dimensional Banach space is compact and thus bounded. If an observable magnitude is represented by a selfadjoint bounded operator $T : H \to H$, then $\|T\|$ measures the intensity of the observable magnitude.

According to [33–36], in a quantum mechanical system H, a selfadjoint bounded operator $T \in \mathcal{B}(H)$ such that $0 \leq T \leq I$ corresponds to an effect for the quantum mechanical system. Effects are of significance in representing unsharp measurements or observations on the quantum mechanical system. Effect-valued measures play an important role in stochastic quantum mechanics [37–39]. The set of all effects for H can be organized into an effect algebra. In other words, as mentioned in the introduction, the first example of effect algebra is constituted by the positive selfadjoint operators on an infinite-dimensional separable complex Hilbert space that lie below the identity.

Example 1. *The first example of effect algebra [1] is* $E(H) := \{T \in \mathcal{B}(H) : T^* = T, 0 \leq T \leq I\}$ *with addition restricted to* $T \oplus S := T + S \Leftrightarrow T + S \in E(H)$ *and orthocomplementation defined by* $T^\perp := I - T$.

An important effect in a quantum mechanical system H is the probability density operator [40]. Recall that a probability density matrix represents a partial state of knowledge of a finite-dimensional quantum mechanical system [41–43] Section 6:

$$\rho(\bullet) = \sum_{i=1}^{n} w_i (\bullet|\psi_i)\psi_i.$$

Based on that information we conclude that with probability w_i the system may be in a pure state ψ_i. For quantum mechanical systems represented by infinite-dimensional separable complex Hilbert spaces (for instance, those with unbounded observable magnitudes), the probability density matrix is in fact an operator defined as follows:

$$\begin{array}{rccl} D: & H & \to & H \\ & x & \mapsto & D(x) := \sum_{i=1}^{\infty} \rho_n (x|e_n), \end{array} \tag{1}$$

where $(e_n)_{n\in\mathbb{N}}$ is a previously fixed orthonormal base of H and $\sum_{n=1}^{\infty} \rho_n$ is a convex series; that is, $\rho_n \geq 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \rho_n = 1$. In accordance with [40] Subsection 6.1, $D \in E(H)$.

As an application of our results, we introduce a new measure defined on $E(H)$: the weighted trace. Recall that the trace of an effect represents the expectation value of energy.

Definition 2. *Let H be an infinite-dimensional separable complex Hilbert space. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal base of H. The weighted trace is defined as the following map:*

$$\begin{array}{rccl} tr_w: & E(H) & \to & [0,1] \\ & T & \mapsto & tr_w(T) := \sum_{i=1}^{\infty} \frac{1}{2^n} (T(e_n)|e_n). \end{array} \tag{2}$$

Proposition 1. *Let H be an infinite-dimensional separable complex Hilbert space. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal base of H. The weighted trace is a measure on $E(H)$.*

Proof. In the first place, tr_w is well defined since $\sum_{i=1}^{\infty} \frac{1}{2^n} (T(e_n)|e_n)$ is a convergent convex series. Indeed, $0 \leq (T(e_n)|e_n) \leq \|T\|$ for all $n \in \mathbb{N}$, so

$$tr_w(T) = \sum_{i=1}^{\infty} \frac{1}{2^n} (T(e_n)|e_n) \leq \sum_{i=1}^{\infty} \frac{1}{2^n} \|T\| = \|T\| \sum_{i=1}^{\infty} \frac{1}{2^n} = \|T\|.$$

Next, for every $T, S \in E(H)$ such that $T + S \in E(H)$, we have that

$$\begin{aligned} tr_w(T+S) &= \sum_{i=1}^{\infty} \frac{1}{2^n} ((T+S)(e_n)|e_n) = \sum_{i=1}^{\infty} \frac{1}{2^n} [(T(e_n)|e_n) + (S(e_n)|e_n)] \\ &= \sum_{i=1}^{\infty} \frac{1}{2^n} (T(e_n)|e_n) + \sum_{i=1}^{\infty} \frac{1}{2^n} (S(e_n)|e_n) = tr_w(T) + tr_w(S). \end{aligned}$$

□

Another example of effect algebra in this setting is the effect algebra of orthogonal subspaces of a Hilbert space.

Example 2. *Let H be an infinite-dimensional separable complex Hilbert space. Then $V(H) := \{P \subseteq H : P \text{ is a closed subspace of } H\}$ with addition restricted to $P \oplus Q := P + Q \Leftrightarrow Q \subseteq P^{\perp}$ and orthocomplementation defined by the orthogonal subspace.*

A very interesting module measure can be defined on $V(H)$.

Lemma 4. *Let H be an infinite-dimensional separable complex Hilbert space. The following map is a module measure on $V(H)$:*

$$\begin{array}{rccl} \pi: & V(H) & \to & \mathcal{B}(H) \\ & P & \mapsto & \pi_P, \end{array} \tag{3}$$

where $\pi_P : H \to H$ is the orthogonal projection of range P.

Proof. Let $P, Q \in V(H)$ such that $P \oplus Q$ exists; that is, $Q \subseteq P^{\perp}$. We have to show that $\pi_{P+Q} = \pi_P + \pi_Q$. Bear in mind that $(P+Q)^{\perp} = P^{\perp} \cap Q^{\perp}$. Indeed, fix an arbitrary $x \in H$. Notice that $x = \pi_P(x) + \pi_{P^{\perp}}(x)$. Since $Q \subseteq P^{\perp}$ and $\pi_{P^{\perp}}(x) \in P^{\perp}$, $\pi_{P^{\perp}}(x)$ can be decomposed into the summation of an element of Q and an element of $P^{\perp} \cap Q^{\perp}$;

that is, $\pi_{P^\perp}(x) = \pi_Q(\pi_{P^\perp}(x)) + \pi_{(P+Q)^\perp}(\pi_{P^\perp}(x))$. Finally, $Q \subseteq P^\perp$, so $\pi_Q(\pi_{P^\perp}(x)) = \pi_Q(x)$; hence

$$\begin{aligned} x &= \pi_P(x) + \pi_{P^\perp}(x) = [\pi_P(x) + \pi_Q(\pi_{P^\perp}(x))] + \pi_{(P+Q)^\perp}(\pi_{P^\perp}(x)) \\ &= [\pi_P(x) + \pi_Q(x)] + \pi_{(P+Q)^\perp}(\pi_{P^\perp}(x)), \end{aligned}$$

meaning that $\pi_{P+Q}(x) = \pi_P(x) + \pi_Q(x)$. □

The physical interpretation of the module measure (3) is still unclear.

7. Discussion

The main novelty of this work is to consider measures defined on effect algebras and valued on a topological module over a topological ring. As we already mentioned in the introduction, the range of a measure has been deeply studied in less general settings, such as measures defined on Boolean algebras or with values in a vector space. In our work, we study module measures, that is, measures with values on a module, which makes it harder since the level of abstraction is even higher.

Effect algebras are a proper generalization of Boolean algebras. Indeed, there can be found examples of effect algebras which are not isomorphic (in the category of effect algebras) to any Boolean algebra. On the other hand, commutative groups can be seen as $\mathbb{Z}$-modules. Since $\mathbb{Z}$ is an absolutely valued ring, the triangular inequality allows to conclude that every normed commutative group trivially becomes a normed $\mathbb{Z}$-module ($\|ng\| \leq |n|\|g\|$), which is in fact a topological module. This way, studying measures defined on an effect algebra and valued on a topological module over a topological ring is the most general setting so far.

We have provided several examples of effect algebras and module measures that serve to compare our approach with the rest of the literature. The first one (Theorem 6) is original from this work and connects effect algebras with Associative Ring Theory. The second one (Example 1) is the classical example of effect algebra given in [1] but endowed with a particular measure (2) that provides an insight of all the possible applications to quantum mechanics. The third example (Lemma 4) is a module measure very natural to define, but whose physical meaning is still unclear.

8. Conclusions

The main conclusions derived from this work are the following:

1. Module measures are probably the most general version of a measure in mathematics. As far as we know, this work is pioneering in defining and studying measures on the most general setting possible, that is, with domain of definition on an effect algebra and valued on a topological module over a topological ring. This way, we have accomplished a series of results that can be hardly generalized to other settings.
2. The classical result that asserts that the measure of a countable union of measurable subsets is the limit of the measures of the sets still holds for module measures (Theorem 1 and Corollary 1).
3. The classical result that states that countably additive real-valued measures defined on a Boolean algebra of sets have bounded variation also holds for sRDP σ-effect algebras with a natural basis (Corollary 4).
4. The classical result stating that the variation of a measure induces a seminorm still holds for module measures (Theorem 5).
5. The concept of unit segment was already known [25]. Here we introduced a new class of unit segments, called full unit segments, which have been endowed with structure of effect algebra (Theorem 6). Furthermore, strong connections with Associative Ring Theory were established in Theorem 6.
6. Based on the linearity of the trace, a new real-valued measure with physical significance in a quantum mechanical system is introduced (Example 1 and Proposition 1). Moreover,

it is interesting to study connections between abstract generalizations of Example 1 to C^*-algebras and $*$-rings.

7. A new module measure for quantum mechanical systems has been constructed whose physical meaning is still unclear (Lemma 4). It is very interesting to deeply study other measures similar to the one defined in Equation (2), such as, for instance, measures provided by the probability density operator [40,44] that describes the quantum state of a physical system.

It remains to provide a general classification of effect algebras, or at least, a characterization of effect algebras isomorphic (in the category of effect algebras) to a Boolean algebra.

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