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Infinitely Many Solutions for the Discrete Boundary Value Problems of the Kirchhoff Type

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Abstract: In this paper, we study the existence and multiplicity of solutions for the discrete Dirichlet boundary value problem of the Kirchhoff type, which has a symmetric structure. By using the critical point theory, we establish the existence of infinitely many solutions under appropriate assumptions on the nonlinear term. Moreover, we obtain the existence of infinitely many positive solutions via the strong maximum principle. Finally, we take two examples to verify our results.

Keywords: discrete Kirchhoff-type problem; boundary value problems; infinitely many solutions; critical point theory

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1. Introduction

Let N be a positive integer and denote with $[1, N]$ the discrete set $\{1, \ldots, N\}$. In this paper, we consider the following discrete boundary value problem of the Kirchhoff type:

$$\begin{cases} -(a + b\sum\limits_{k=1}^{N+1} |\Delta u_{k-1}|^2)\Delta^2 u_{k-1} = \lambda f(k, u_k), & k \in [1, N], \\ u_0 = u_{N+1} = 0, \end{cases} \tag{1}$$

where a, b are two positive constants, and Δ is the forward difference operator defined by $\Delta u_k = u_{k+1} - u_k$. $\Delta^2 = \Delta(\Delta)$ and $f(k, \times) \in C(\mathbb{R}, \mathbb{R})$ for any $k \in [1, N]$ and $\lambda \in \mathbb{R}^+$. Problem (1) has a symmetric structure in the variable u_k; that is, if we replace u_{k-1} with u_{k+1}, and replace u_{k+1} with u_{k-1} in (1), then (1) is invariant since $\Delta^2 u_{k-1} = u_{k+1} + u_{k-1} - 2u_k$.

In the past two decades, there has been a lot of interest in the study of difference equations, such as in biology, economics, and other research fields [1–5]. Most results about the boundary value problems of difference equations are proved by using the method of upper and lower solutions as well as fixed-point methods; see [6–10] for more details. In 2003, Guo and Yu [11] discussed the second-order difference equation by using critical point theory, and they obtained the existence of periodic and subharmonic solutions. Since then, many researchers have studied difference equations via critical point theory, including boundary value problems [12–18], periodic solutions [19,20] as well as homoclinic solutions [21–24] and heteroclinic solutions [25].

Problem (1) is the discrete analogue of the following Kirchhoff-type problem:

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2 dx)\Delta u = \lambda f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2}$$

As to problem (2), Zou and He [26] established the existence of infinitely many positive solutions by using variational methods. In the case of $\lambda = 1$ in problem (2), Cheng and Wu [27] studied the two existence results, including at least one or no positive solution via variational methods. In 2016, Tang and Cheng [28] studied the existence of ground state sign-changing solutions when $\lambda = 1$ in problem (2) by applying the non-Nehari manifold method. As for Kirchhoff's changes and related applications, we refer the reader to [29,30] and the references therein.

Problem (2) is related to the stationary case of a nonlinear wave equation such as

$$u_{tt} - (a + b\int_\Omega |\nabla u|^2 dx)\Delta u = f(x, u),$$

which was proposed by Kirchhoff [31] as an extension of the classical D'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations.

As for the discrete case, when the parameter $\lambda = 1$ in problem (1) and f satisfies various assumptions, Yang and Liu [32] studied the existence of at least one nontrivial solution via variational methods and critical groups. A class of partial discrete Kirchhoff-type problems was discussed by Long and Deng [33] via invariant sets of descending flow and minimax methods, and some results on the existence of sign-changing solutions, positive solutions, and negative solutions were obtained.

To the best of our knowledge, although most of the previous works have been dedicated to boundary value problems, few have been studied in the discrete problems of the Kirchhoff type. Inspired by the above results, we intend to investigate the multiplicity of solutions for the discrete Kirchhoff-type problem with a Dirichlet boundary value condition by applying critical point theory.

2. Preliminaries

Let X be a reflexive real Banach space and $I_\lambda : X \to \mathbb{R}$ be a function satisfying the following structure hypothesis:

(Λ) *$I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ for all $u \in X$, where $\Phi, \Psi : X \to \mathbb{R}$ are two functions of class C^1 on X, and Φ is coercive, i.e., $\lim_{\|u\|\to\infty} \Phi(u) = +\infty$ and $\lambda \in \mathbb{R}^+$.*

Provided that $\inf_X \Phi < r$, put

$$\varphi(r) = \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\left(\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)\right) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma = \liminf_{r\to+\infty} \varphi(r), \quad \delta = \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Obviously, $\gamma \geq 0$ and $\delta \geq 0$. In the sequel, we agree to regard $\frac{1}{\gamma}$ (or $\frac{1}{\delta}$) as $+\infty$ when $\gamma = 0$ (or $\delta = 0$).

Moreover, recalling Theorem 2.5 of [34], we have the following lemma used to investigate problem (1).

Lemma 1. *Assuming that the condition (Λ) holds, one has the following:*

(a) If $\gamma < +\infty$, then for each $\lambda \in (0, \frac{1}{\gamma})$, the following alternatives hold:

(α_1) I_λ possesses a global minimum;

(α_2) There is a sequence $\{u_n\}$ of critical points (local minima) of I_λ, such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.

(b) If $\delta < +\infty$, then for each $\lambda \in (0, \frac{1}{\delta})$, the following alternatives hold:

(β_1) T is a global minimum of Φ, which is a local minimum of I_λ;

(β_2) *There is a sequence* $\{u_n\}$ *of pairwise distinct critical points (local minima) of* I_λ*, with* $\lim_{n\to+\infty}\Phi(u_n) = \inf_X \Phi$*, which weakly converges to a global minimum of* Φ*.*

Now we consider the N-dimensional Banach space S $=\{u : [0, N+1] \to \mathbb{R} : u_0 = u_{N+1} = 0\}$ and define the norm as follows:

$$||u|| := \left(\sum_{k=1}^{N+1} |\Delta u_{k-1}|^2\right)^{\frac{1}{2}}.$$

From ([35], Lemma 2.2), we have the following inequality:

$$\max_{k\in[1,N]} |u_k| \leq \frac{(N+1)^{\frac{1}{2}}}{2} ||u||, \quad \forall u \in S. \tag{3}$$

Let

$$\Phi(u) := \frac{a}{2}\sum_{k=1}^{N+1} |\Delta u_{k-1}|^2 + \frac{b}{4}\left(\sum_{k=1}^{N+1} |\Delta u_{k-1}|^2\right)^2,$$

$$\Psi(u) := \sum_{k=1}^{N} F(k, u_k) \qquad \text{and} \qquad I_\lambda(u) := \Phi(u) - \lambda\Psi(u) \tag{4}$$

where $F(k,\xi) := \int_0^\xi f(k,t)dt$ for every $(k,t) \in [1,N] \times \mathbb{R}$. Owing to Φ, $\Psi \in C^1(S,\mathbb{R})$, I_λ is also a class of $C^1(S,\mathbb{R})$. Using the summation by parts method and the boundary condition, one has

$$\begin{aligned}
I'(u)(v) &= \lim_{t\to 0}\frac{I(u+tv) - I(u)}{t} \\
&= a\sum_{k=1}^{N+1} \Delta u_{k-1}\Delta v_{k-1} + \left(b\sum_{k=1}^{N+1} |\Delta u_{k-1}|^2\right)\sum_{k=1}^{N+1} \Delta u_{k-1}\Delta v_{k-1} - \lambda\sum_{k=1}^{N} f(k,u_k)v_k \\
&= \left(a + b\sum_{k=1}^{N+1} |\Delta u_{k-1}|^2\right)\sum_{k=1}^{N+1} \Delta u_{k-1}\Delta v_{k-1} - \lambda\sum_{k=1}^{N} f(k,u_k)v_k \\
&= -\left(a + b\sum_{k=1}^{N+1} |\Delta u_{k-1}|^2\right)\sum_{k=1}^{N} \Delta^2 u_{k-1}v_k - \lambda\sum_{k=1}^{N} f(k,u_k)v_k \\
&= -\sum_{k=1}^{N}\left[\left(a + b\sum_{k=1}^{N+1} |\Delta u_{k-1}|^2\right)\Delta^2 u_{k-1} + \lambda f(k,u_k)\right]v_k
\end{aligned}$$

for any $u, v \in S$.

Thus, u is a critical point of I on S if and only if u is a solution of problem (1). Now we have reduced the existence of a solution for problem (1) to the existence of a critical point of I on S.

Finally, we point out the following two lemmas used to obtain positive solutions for our problem. The first is the following strong maximum principle.

Lemma 2. *Fix* $u \in S$*, such that either*

$$u_k > 0 \quad \text{or} \quad -(a + b\sum_{k=1}^{N+1} |\Delta u_{k-1}|^2)\Delta^2 u_{k-1} \geq 0$$

for each $k \in [1,N]$*. Then, either* $u \equiv 0$ *or* $u_k > 0$ *for each* $k \in [1,N]$*.*

Proof. Let $u_j = \min_{k\in[1,N]} u_k$. If $u_j > 0$, then $u_k > 0$ for each $k \in [1, N]$, and the conclusion follows. If $u_j \leq 0$, then we have

$$-(a + b\sum_{j=1}^{N+1} |\Delta u_{j-1}|^2)\Delta^2 u_{j-1} \geq 0.$$

Owing to $a, b > 0$, one has $\Delta^2 u_{j-1} \leq 0$. Considering the fact that u_j is the minimum, we obtain $u_{j+1} = u_{j-1} = u_j$. If $j + 1 = N + 1$, we have $u_j = 0$. Otherwise, $j + 1 \in [1, N]$. Replacing j with $j + 1$, we get $u_{j+2} = u_{j+1}$. Continuing this process $N + 1 - j$ times, we have $u_j = u_{j+1} = \cdots = u_N = u_{N+1} = 0$. In the same way, we also get $u_j = u_{j-1} = \cdots = u_1 = u_0 = 0$. Thus, we prove that $u \equiv 0$, and the proof is complete. □

Let

$$F^+(k, t) = \int_0^t f(k, s^+) ds, \quad (k, t) \in [1, N] \times \mathbb{R},$$

where $s^+ = max\{0, s\}$. Now we define $I_\lambda{}^+ = \Phi - \lambda\Psi^+$, where $\Psi^+(u) = \sum_{k=1}^{N} F^+(k, u_k)$ and Φ is defined as before. Similarly, the critical points of $I_\lambda{}^+$ are the solutions of the following problem:

$$\begin{cases} -(a + b\sum_{k=1}^{N+1} |\Delta u_{k-1}|^2)\Delta^2 u_{k-1} = \lambda f(k, u_k{}^+), & k \in [1, N], \\ u_0 = u_{N+1} = 0. \end{cases} \tag{5}$$

Lemma 3. *If $f(k, 0) \geq 0$ for each $k \in [1, N]$, then all the non-zero critical points of $I_\lambda{}^+$ are positive solutions of problem (1).*

Proof. From Lemma 2, it follows that all solutions of problem (5) are either zero or positive. Then, problem (1) admits positive solutions when problem (5) admits non-zero solutions. Therefore, the conclusion holds. □

3. Main Results

Let

$$H^\infty := \limsup_{t\to+\infty} \frac{\sum_{k=1}^{N} F(k, t)}{t^4} \quad \text{and} \quad H^0 := \limsup_{t\to 0^+} \frac{\sum_{k=1}^{N} F(k, t)}{t^2}.$$

Our main results are the following theorems.

Theorem 1. *Assume that there exist two real sequences $\{a_n\}$ and $\{b_n\}$, with $b_n > 0$ and $\lim_{n\to+\infty} b_n = +\infty$, such that*

$$|a_n| < \left[\left(\frac{2a \times b_n{}^2}{b(N+1)} + \frac{4b_n{}^4}{(N+1)^2} + \frac{a^2}{4b^2} \right)^{\frac{1}{2}} - \frac{a}{2b} \right]^{\frac{1}{2}}, \quad \forall n \in \mathbb{N} \tag{6}$$

and

$$G_\infty := \liminf_{n\to+\infty} \frac{\sum_{k=1}^{N} \max_{|t|\leq b_n} F(k, t) - \sum_{k=1}^{N} F(k, a_n)}{2b_n{}^2\left[a(N+1) + 2b \times b_n{}^2\right] - a_n{}^2(N+1)^2(a + b \times a_n{}^2)} < \frac{H^\infty}{b(N+1)^2}. \tag{7}$$

Then, for each $\lambda \in \left(\frac{b}{H^\infty}, \frac{1}{(N+1)^2 G_\infty} \right)$, problem (1) admits an unbounded sequence of solutions.

Proof. Fix $\lambda \in \left(\frac{b}{H^{\infty}}, \frac{1}{(N+1)^2 G_{\infty}}\right)$, and let S, Φ, Ψ, and I_λ be defined as in Section 2. Considering the fact that critical points of I_λ are solutions of problem (1), we will use Lemma 1 part (α) to prove our conclusion. Obviously, (Λ) holds. Thus, the conclusion holds provided that $\gamma < +\infty$ and I_λ is unbounded from below. To this end, write

$$\omega_n := \frac{2{b_n}^2\left[a(N+1) + 2b \times {b_n}^2\right]}{(N+1)^2}$$

for every $n \in \mathbb{N}$. From (3),

$$||u|| \leq \frac{2}{\sqrt{N+1}}\left[\left(\frac{(N+1)^2\omega_n}{4b} + \frac{a^2(N+1)^2}{16b^2}\right)^{\frac{1}{2}} - \frac{a(N+1)}{4b}\right]^{\frac{1}{2}},$$

then $|u_k| \ \leq b_n$ for every $k \in [1, N]$, and for each $n \in \mathbb{N}$, one has $\varphi(\omega_n) \leq (N+1)^2 \frac{\sum_{k=1}^{N}\max_{|t|\leq b_n} F(k,t) - \sum_{k=1}^{N} F(k,u_k)}{2{b_n}^2\left[a(N+1)+2b\times {b_n}^2\right] - (N+1)^2\left[\frac{a}{2}\sum_{k=1}^{N+1}|\Delta u_{k-1}|^2 + \frac{b}{4}\left(\sum_{k=1}^{N+1}|\Delta u_{k-1}|^2\right)^2\right]}$. Now, for each $n \in \mathbb{N}$, the sequence $\{\alpha_n\}$ taken from S is given by $(\alpha_n)_k := a_n$ for every $k \in [1, N]$, $(\alpha_n)_0 = (\alpha_n)_{N+1} = 0$. Moreover, $\Phi(\alpha_n) = {a_n}^2(a + b \times {a_n}^2)$, and from (6), we have $\Phi(\alpha_n) < \omega_n$. Therefore, we obtain

$$\varphi(\omega_n) \leq (N+1)^2 \frac{\sum_{k=1}^{N}\max_{|t|\leq b_n} F(k,t) - \sum_{k=1}^{N} F(k,a_n)}{2{b_n}^2\left[a(N+1) + 2b \times {b_n}^2\right] - {a_n}^2(N+1)^2(a + b \times {a_n}^2)}.$$

Hence, from (7), $\gamma \leq \liminf_{n\to+\infty} \varphi(\omega_n) \leq (N+1)^2 G_\infty < +\infty$ follows.

Now, we prove that I_λ is unbounded from below. Firstly, assuming that $H^\infty < +\infty$ and owing to $\lambda > \frac{b}{H^\infty}$, we can fix $\varepsilon > 0$, such that $H^\infty - \frac{b}{\lambda} > \varepsilon$. Thus, let $\{c_n\}$ be a real sequence, with $\lim_{n\to+\infty} c_n = +\infty$, such that

$$(H^\infty - \varepsilon){c_n}^4 < \sum_{k=1}^{N} F(k, c_n) < (H^\infty + \varepsilon){c_n}^4, \quad \forall n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, let $\{\beta_n\}$ be defined by $(\beta_n)_k := c_n$ for every $k \in [1, N]$, $(\beta_n)_0 = (\beta_n)_{N+1} = 0$. Clearly, $\{\beta_n\} \in S$. Therefore, we have

$$\begin{aligned} I_\lambda(\beta_n) &= \Phi(\beta_n) - \lambda\Psi(\beta_n) \\ &= {c_n}^2\left(a + b \times {c_n}^2\right) - \lambda \sum_{k=1}^{N} F(k, c_n) \\ &< {c_n}^2\left(a + b \times {c_n}^2\right) - \lambda(H^\infty - \varepsilon){c_n}^4 \\ &= a \ \times {c_n}^2 + [b - \lambda(H^\infty - \varepsilon)]{c_n}^4. \end{aligned}$$

Thus, $\lim_{n\to+\infty} I_\lambda(\beta_n) = -\infty$.

Next, assuming that $H^\infty = +\infty$, and taking $L \ > \ 0$ such that $L \ > \ \frac{b}{\lambda}$, we also put a real sequence $\{c_n\}$ with $\lim_{n\to+\infty} c_n = +\infty$, such that

$$\sum_{k=1}^{N} F(k, c_n) > L \times {c_n}^4, \quad \forall n \in \mathbb{N}.$$

Proving as before and selecting $\{\beta_n\}$ in S as above, one has

$$I_\lambda(\beta_n) < a \times {c_n}^2 + (b - \lambda \times L){c_n}^4.$$

Hence, $\lim_{n\to+\infty} I_\lambda(\beta_n) = -\infty$.

Therefore, we prove that $\gamma < +\infty$ and I_λ is unbounded from below in both cases. Bearing in mind Lemma 1 part (α), the proof is complete. □

Theorem 2. *Assume that there exist two real sequences $\{d_n\}$ and $\{e_n\}$, with $e_n > 0$ and $\lim_{n\to+\infty} e_n = 0$, such that*

$$|d_n| < \left[\left(\frac{2a \times {e_n}^2}{b(N+1)} + \frac{4{e_n}^4}{(N+1)^2} + \frac{a^2}{4b^2}\right)^{\frac{1}{2}} - \frac{a}{2b}\right]^{\frac{1}{2}}, \quad \forall n \in \mathbb{N} \tag{8}$$

and

$$G_0 := \liminf_{n\to+\infty} \frac{\sum_{k=1}^{N} \max_{|t|\le e_n} F(k,t) - \sum_{k=1}^{N} F(k,d_n)}{2{e_n}^2[a(N+1) + 2b \times {e_n}^2] - {d_n}^2(N+1)^2\left(a + b \times {d_n}^2\right)} < \frac{H^0}{a(N+1)^2}. \tag{9}$$

Then, for each $\lambda \in \left(\frac{a}{H^0}, \frac{1}{(N+1)^2 G_0}\right)$, problem (1) admits a sequence of non-zero solutions that converge to zero.

Proof. Let S, Φ, Ψ, and I_λ be defined as above and fix $\lambda \in \left(\frac{a}{H^0}, \frac{1}{(N+1)^2 G_0}\right)$. Now our goal is to use Lemma 1 part (β) to prove our conclusion as above. Clearly, (Λ) holds. Write

$$\overline{\omega}_n := \frac{2{e_n}^2\left[a(N+1) + 2b \times {e_n}^2\right]}{(N+1)^2}$$

for every n $\in \mathbb{N}$. Owing to (3), if

$$|u| \le \frac{2}{\sqrt{N+1}}\left[\left(\frac{(N+1)^2\overline{\omega}_n}{4b} + \frac{a^2(N+1)^2}{16b^2}\right)^{\frac{1}{2}} - \frac{a(N+1)}{4b}\right]^{\frac{1}{2}},$$

then $|u_k| \le e_n$ for every $k \in [1, N]$ and $n \in \mathbb{N}$, and we have

$$\varphi(\overline{\omega}_n) \le (N+1)^2 \frac{\sum_{k=1}^{N} \max_{|t|\le e_n} F(k,t) - \sum_{k=1}^{N} F(k,u_k)}{2{e_n}^2[a(N+1) + 2b \times {e_n}^2] - (N+1)^2\left[\frac{a}{2}\sum_{k=1}^{N+1} |\Delta u_{k-1}|^2 + \frac{b}{4}\left(\sum_{k=1}^{N+1} |\Delta u_{k-1}|^2\right)^2\right]}.$$

For each $n \in \mathbb{N}$, let $\{\gamma_n\}$ be defined by $(\gamma_n)_k := d_n$ for every $k \in [1, N]$, $(\gamma_n)_0 = (\gamma_n)_{N+1} = 0$. Obviously, $\{\gamma_n\} \in S$. Thus, one has

$$\varphi(\overline{\omega}_n) \le (N+1)^2 \frac{\sum_{k=1}^{N} \max_{|t|\le e_n} F(k,t) - \sum_{k=1}^{N} F(k,d_n)}{2{e_n}^2[a(N+1) + 2b \times {e_n}^2] - {d_n}^2(N+1)^2\left(a + b \times {d_n}^2\right)}.$$

Hence, by taking (9) into account, $\delta \le \liminf_{n\to+\infty} \varphi(\overline{\omega}_n) \le (N+1)^2 G_0 < +\infty$ follows.

Our aim is to verify if the global minimum of Φ is different from the local minimum of I_λ. As a matter of fact, it is easy to see that the global minimum of Φ is 0, and $\Phi = 0$ if and only if $u_k = 0$ for every $k \in [1, N]$. Therefore, our task is reduced to proving that 0 is not a local minimum of I_λ.

Using the same argument as in the proof of Theorem 1, we firstly assume that $H^0 < +\infty$. Since $\lambda > \frac{a}{H^0}$, we fix $\varepsilon > 0$, such that $H^0 - \frac{a}{\lambda} > \varepsilon$. Thus, we can take a real sequence $\{r_n\}$ with $\lim_{n\to+\infty} r_n = 0$, such that

$$\left(H^0 - \varepsilon\right){r_n}^2 < \sum_{k=1}^{N} F(k, r_n) < \left(H^0 + \varepsilon\right){r_n}^2, \quad \forall n \in \mathbb{N}.$$

Moreover, by taking in S the sequence $\{\mu_n\}$ that, for each $n \in \mathbb{N}$, is defined by $(\mu_n)_k := r_n$ for every $k \in [1, N]$, we have

$$\begin{aligned} I_\lambda(\mu_n) &= \Phi(\mu_n) - \lambda\Psi(\mu_n) \\ &= {r_n}^2\left(a + b \times {r_n}^2\right) - \lambda \sum_{k=1}^{N} F(k, r_n) \\ &< {r_n}^2\left(a + b \times {r_n}^2\right) - \lambda\left(H^0 - \varepsilon\right){r_n}^4 \\ &= \left[a - \lambda\left(H^0 - \varepsilon\right)\right]{r_n}^2 + b \times {r_n}^2. \end{aligned}$$

Thus, $I_\lambda(\mu_n) < 0$.

Next, assuming that $H^0 = +\infty$, we fix $M > 0$, such that $M > \frac{a}{\lambda}$; we also put a real sequence $\{r_n\}$ with $\lim_{n\to+\infty} r_n = 0$, such that

$$\sum_{k=1}^{N} F(k, r_n) > M \times {r_n}^2, \quad \forall n \in \mathbb{N}.$$

Choosing a real sequence $\{\mu_n\}$ from S in the same way as mentioned above, we have

$$I_\lambda(\mu_n) < (a - \lambda \times M){r_n}^2 + b \times {r_n}^4.$$

Therefore, $I_\lambda(\mu_n) < 0$.

Hence, the conclusion follows from part (β) of Lemma 1. $\square$

By setting particular conditions, we obtain the following consequences. Let

$$\overline{G}_\infty := \liminf_{t\to+\infty} \frac{\sum_{k=1}^{N} \max_{|\xi|\le t} F(k, \xi)}{2t^2[a(N+1) + 2b \times t^2]}.$$

Proposition 1. *Assume that*

$$\overline{G}_\infty < \frac{H^\infty}{b(N+1)^2}. \tag{10}$$

If $f(k, 0) \geq 0$ for all $k \in [0, N]$, then for each $\lambda \in \left(\frac{b}{H^\infty}, \frac{1}{(N+1)^2\overline{G}_\infty}\right)$, problem (1) admits an unbounded sequence of positive solutions.

Proof. Let $\{b_n\}$ be a real positive sequence with $\lim_{n\to+\infty} b_n = +\infty$, such that

$$\liminf_{n\to+\infty} \frac{\sum_{k=1}^{N} \max_{|t|\le b_n} F(k,t)}{2{b_n}^2\left[a(N+1)+2b\times {b_n}^2\right]} < \frac{1}{b(N+1)^2} \limsup_{n\to+\infty} \frac{\sum_{k=1}^{N} F(k,b_n)}{{b_n}^4}.$$

Conditions (6) and (7) of Theorem 1 follow when we take sequence $a_n = 0$ for each $n \in \mathbb{N}$. Let

$$f^+(k,t) = \begin{cases} f(k,t) & \text{if } t > 0 \\ f(k,0) & \text{if } t \le 0, \end{cases}$$

for each $k \in [1,N]$. From Lemma 3, our proof is complete. □

Proposition 2. *Assume that*

$$\liminf_{t\to+\infty} \frac{\max_{0\le\theta\le t} \int_0^\theta h(s)ds}{2t^2[a(N+1)+2b\times t^2]} < \frac{1}{b(N+1)^2} \limsup_{t\to+\infty} \frac{\int_0^t h(s)ds}{t^4}. \tag{11}$$

If $h : [0,+\infty) \to \mathbb{R}$ is a continuous function with $h(0) = 0$, and $\sigma : [1,N] \to \mathbb{R}$ is a non-negative and non-zero function. Then, for each

$$\lambda \in \frac{1}{\sum_{k=1}^{N}\sigma_k}\left(\frac{b(N+1)^2}{\limsup_{t\to+\infty}\frac{\int_0^t h(s)ds}{t^4}}, \frac{1}{\liminf_{t\to+\infty}\frac{\max_{0\le\theta\le t}\int_0^\theta h(s)ds}{2t^2[a(N+1)+2b\times t^2]}}\right),$$

the problem

$$\begin{cases} -(a+b\sum_{k=1}^{N+1}|\Delta u_{k-1}|^2)\Delta^2 u_{k-1} = \lambda\sigma_k h(u_k), & k \in [1,N], \\ u_0 = u_{N+1} = 0, \end{cases}$$

admits an unbounded sequence of positive solutions.

Proof. Let

$$f(k,t) = \begin{cases} \sigma_k h(t) & \text{if } t \ge 0 \\ 0 & \text{if } t < 0 \end{cases}$$

for each $k \in [1,N]$ and $t \in \mathbb{R}$. Therefore, we have $f(k,0) \ge 0$ for each $k \in [1,N]$, and the conclusion follows from Proposition 1. □

Remark 1. *If $f : [1,N]\times\mathbb{R} \to \mathbb{R}$ is a non-negative function in Proposition 1, condition (10) becomes*

$$\liminf_{t\to+\infty} \frac{\sum_{k=1}^{N} F(k,t)}{2t^2[a(N+1)+2b\times t^2]} < \frac{1}{b(N+1)^2} \limsup_{t\to+\infty}\left(\frac{\sum_{k=1}^{N} F(k,t)}{t^4}\right). \tag{12}$$

Then, the conclusion follows from Proposition 1.

Remark 2. *If $h : [0,+\infty) \to \mathbb{R}^+$ is a continuous function with $h(0) = 0$ in Proposition 2, then condition (11) shall be*

$$\liminf_{t\to+\infty} \frac{\int_0^t h(s)ds}{2t^2[a(N+1)+2b\times t^2]} < \frac{1}{b(N+1)^2} \limsup_{t\to+\infty}\left(\frac{\int_0^t h(s)ds}{t^4}\right). \tag{13}$$

Then, the solutions are also positive from Proposition 2.

Remark 3. *If we replace $t \to +\infty$ with $t \to 0^+$, we can also obtain the similar propositions and remarks in Theorem 2 in the same way.*

4. Examples

In this section, we present the following examples to illustrate our results.

Example 1. *Let ε be an arbitrarily positive constant, and let*

$$h(s) = \begin{cases} 2s^3[4 + 2\varepsilon + 4\sin(\varepsilon \ln(s)) + \varepsilon \cos(\varepsilon \ln s)] & \text{if } s > 0 \\ 0 & \text{if } s = 0, \end{cases}$$

with $\sigma_k = 1$ for each $k \in [1, N]$. Then, we have

$$\liminf_{t \to +\infty} \frac{\int_0^t h(s)\, ds}{2t^2[a(N+1) + 2b \times t^2]} = \liminf_{t \to +\infty} \frac{t^2[2 + \varepsilon + 2\sin(\varepsilon \ln t)]}{2a(N+1) + 4b \times t^2} = \frac{\varepsilon}{4b}$$

and

$$\limsup_{t \to +\infty} \frac{\int_0^t h(s)\, ds}{t^4} = 4 + \varepsilon.$$

It is easy to see that $h(s) \geq 0$, and when ε is sufficiently small,

$$\frac{\varepsilon}{4} < \frac{4 + \varepsilon}{(N+1)^2}.$$

Hence, condition (13) holds.

Then, from Remark 2, for each $\lambda \in \frac{1}{N}\left(\frac{b(N+1)^2}{4+\varepsilon}, \frac{4b}{\varepsilon}\right)$, the problem

$$\begin{cases} -(a + b\sum_{k=1}^{N+1} |\Delta u_{k-1}|^2)\Delta^2 u_{k-1} = 2\lambda {u_k}^3[4 + 2\varepsilon + 4\sin(\varepsilon \ln u_k) + \varepsilon \cos(\varepsilon \ln u_k)], k \in [1, N], \\ u_0 = u_{N+1} = 0, \end{cases}$$

admits an unbounded sequence of positive solutions.

Example 2. *Let a, b, N be such that*

$$\frac{b(N+1)}{2a} < \frac{3 + \sqrt{2}}{3 - \sqrt{2}}. \tag{14}$$

Then, for each $\lambda \in \frac{1}{N}\left(\frac{b(N+1)^2}{3+\sqrt{2}}, \frac{2a(N+1)}{3-\sqrt{2}}\right)$, the problem

$$\begin{cases} -(a + b\sum_{k=1}^{N+1} |\Delta u_{k-1}|^2)\Delta^2 u_{k-1} = \lambda[6u_k + u_k(\sin(\ln u_k) + 3\cos(\ln u_k))], k \in [1, N], \\ u_0 = u_{N+1} = 0, \end{cases}$$

admits a non-zero sequence of positive solutions that converge to zero.

In fact, let

$$h(s) = \begin{cases} 6s + s(\sin(\ln s) + 3\cos(\ln s)) & \text{if } s > 0 \\ 0 & \text{if } s = 0, \end{cases}$$

with $\sigma_k = 1$ for each $k \in [1, N]$. We then have

$$\liminf_{t \to 0^+} \frac{\int_0^t h(s)}{2t^2[a(N+1) + 2b \times t^2]} = \liminf_{t \to 0^+} \frac{\left[3 + \sqrt{2}\sin\left(\frac{\pi}{4} + \ln t\right)\right]}{2a(N+1) + 4b \times t^2} = \frac{3 - \sqrt{2}}{2a(N+1)}$$

and

$$\limsup_{t\to 0^+} \frac{\int_0^t h(s)\, ds}{t^2} = 3 + \sqrt{2}.$$

Therefore, from (14), one has

$$\liminf_{t\to 0^+} \frac{\int_0^t h(s)ds}{2t^2[a(N+1) + 2b \times t^2]} < \frac{1}{b(N+1)^2} \limsup_{t\to 0^+} \frac{\int_0^t h(s)ds}{t^2}.$$

By applying Remark 3, our aim is achieved and the conclusion holds.

5. Conclusions

In recent years, Kirchhoff-type problems have been widely studied in the continuous case, while few have been discussed in the discrete case. In this paper, we considered the multiplicity of solutions for the discrete Kirchhoff-type problem with a Dirichlet boundary value condition. In Section 2, we recalled critical point theory and showed some basic lemmas. In Section 3, we proved the existence of infinitely many solutions for problem (1) by using critical point theory. Moreover, we obtained the existence of infinitely many positive solutions by means of the strong maximum principle.

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