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Conservation Laws and Travelling Wave Solutions for a Negative-Order KdV-CBS Equation in 3+1 Dimensions

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Abstract: In this paper, we study a new negative-order KdV-CBS equation in $(3+1)$ dimensions which is a combination of the Korteweg-de Vries (KdV) equation and Calogero–Bogoyavlenskii–Schiff (CBS) equation. Firstly, we determine the Lie point symmetries of the equation and conservation laws by using the multiplier method. The conservation laws will be used to obtain a triple reduction to a second order ordinary differential equation (ODE), which lead to line travelling waves and soliton solutions. Such solitons are obtained via the modified form of simple equation method and are displayed through three-dimensional plots at specific parameter values to lend physical meaning to nonlinear phenomena. It illustrates that these solutions might be extremely beneficial in understanding physical phenomena in a variety of applied mathematics areas.

Keywords: lie symmetries; exact solutions; invariant solutions

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1. Introduction

The analysis of higher-dimensional nonlinear systems, particularly integrable systems, has exploded in popularity in recent years. Solitary waves theory finds applications in a wide range of scientific domains, including telecommunications, transport phenomena, ocean waves, quantum mechanics, plasma physics, nonlinear fibre optics, and many more. In latest years, research into generating higher-dimensional integrable equations has gotten greater attention, with various integrable models established in the setting of (2+1) and (3+1)-dimensional equations [1,2]. The domain of integrable equations is crucial to investigate as it clarifies the true nature of nonlinearity in science disciplines and reveals its scientific nature.

A variety of chemical, biological and physical phenomena are modeled via nonlinear partial differential equations (PDEs) which contribute a key role in nonlinear science [3]. It provides an abundance of physical data and a deeper understanding of the problem's physical characteristics, resulting in additional applications [4–7]. Many tracks have been established for physical issues in recent years in order to come up with exact solutions for nonlinear PDEs using modern computer technologies. There are a number of effective approaches, for instance, the modified extended tanh-function technique [8], the extended modified auxiliary equation mapping method [9,10], the Riccati–Bernoulli Sub-ODE technique [11], the auxiliary equation technique [12], the (G'/G)-expansion technique [13], homotopy perturbation technique [14], the unified technique [15], the generalized unified technique [16], modified simple equation technique [17], the extended trial function technique [18], the homogeneous balance technique [19], the extended direct algebraic technique [20], the modified extended direct algebraic technique [21] and the collective variable technique [22].

Among the most powerful strategies for constructing nonlinear PDEs solutions is Lie group transformations theory. The investigation of the invariance property of a particular differential equation under a continuous group of transformations is the core principle behind this theory. The application of Lie symmetry transformation technique to PDEs, results in reductions and invariant solutions. Invariant solutions are frequently employed to investigate analytical features. Conservation laws are well acknowledged to participate actively in the solution of a PDE. Not all PDEs obeying conservation laws have physical interpretations, but they are crucial in investigating the integrability of PDEs.

In a wide range of scientific and technical domains, the classical Lie symmetry theory is frequently used [23,24]. This idea was originally suggested by a mathematician named Sophus Lie in the early nineteenth century [25,26]. It has significantly increased applications in nonlinear PDEs and has proven beneficial in various fields of differential equations [27]. The fundamental purpose underlying Lie group theory is to use the invariance requirement of nonlinear PDEs to achieve the similarity variable and reduction equation, and subsequently to derive similarity and solitary wave solutions [28]. These solutions explain a fundamental and significant physical phenomenon. The Chen–Lee–Liu equation, Sawada–Kotera equation, Boussinesq equation, and many more models have recently been studied using the Lie symmetry technique [29–32]. In 2017, Wazwaz investigated the (2+1)-dimensional modified KdV-Calogero–Bogoyavlenskii–Schiff equation to discover abundant solutions with various physical features [2].

We consider a new negative-order KdV-CBS model in $(3+1)$-dimensions given by

$$u_{xt} + u_{xxxy} + 4u_x u_{xy} + 2u_{xx}u_y + \lambda u_{xx} + \mu u_{xy} + \nu u_{xz} = 0, \tag{1}$$

where λ, μ and ν are unspecified coefficients. Equation (1) is the combination of the Korteweg–De Vries (KdV) equation and Calogero–Bogoyavlenskii–Schiff (CBS) equation. It should go without saying that for $\nu = 0$ and $\mu = 0$, Equation (1) will be simplified to the negative-order KdV equation. Although, for $\nu = 0$ and $\lambda = 0$, Equation (1) will be simplified to the negative-order CBS equation. In addition, the investigated model passes Painlevé test with no restrictions on the compatibility criteria or the variables used in the equation. The considered equation has been recently modeled by Wazwaz [33].

The aim of this paper is to determine the conservation laws, symmetries, line travelling waves and line soliton solutions admitted by the negative-order KDV-CBS equation in (3+1) dimensions. First, in Section 2 all Lie point symmetries are derived. In Section 3 the conservation laws arising from the low order multipliers are obtained. In Section 4 travelling wave reduction and first integrals are constructed. The line travelling waves $u = U(x + by + cz - at)$, where a, b and c are arbitrary constants, and ξ is a travelling wave transformation are considered for the negative-order KDV-CBS equation in (3+1) dimensions. The resulting fourth-order nonlinear differential equation for U is directly reduced to a first-order differential equation by using the multi-reduction method [34] to the invariant under translation conservation laws. These differential equation is then integrated to get the explicit form of the line travelling wave solutions. For some particular cases of the parameters the general solution is obtained in terms of the Weierstrass $\mathcal{P}$ function and the Weierstrass ζ function .

This paper then concentrates in Section 5 on the use of the extended simple equation method (ESEM) to extract solitons from the negative-order KdV-CBS equation in three dimensions. The ESEM method is a powerful mathematical technique for finding exact solutions to nonlinear partial differential equations (NLPDEs). Soliton solutions in three dimensions are established. Section 6 concludes with a few closing remarks.

2. Lie Point Symmetries

A one-parameter Lie group of transformations on (t, x, y, z, u) introduced by a vector field leaves the invariant solution space of the equation specified by Equation (1).

Each admitted point symmetry may be exploited to minimize the number of independent variables in Equation (1). For example, we might convert PDEs to ODEs. There may be symmetries of these ODEs that allow us to reduce the order of the equation and whose solutions appear to be related to invariant solutions $u(t, x, y, z)$ of Equation (1).

We consider a one-parameter Lie group of infinitesimals transformations acting on independent and dependent variables of the new $(3+1)$-dimensional negative-order KdV-CBS (nKdV-nCBS) Equation (1) given by

$$\begin{aligned} \tilde{t} &= t && + \varepsilon\tau(t,x,y,z,u) + \mathcal{O}(\varepsilon^2), \\ \tilde{x} &= x && + \varepsilon\xi^1(t,x,y,z,u) + \mathcal{O}(\varepsilon^2), \\ \tilde{y} &= y && + \varepsilon\xi^2(t,x,y,z,u) + \mathcal{O}(\varepsilon^2), \\ \tilde{z} &= y && + \varepsilon\xi^3(t,x,y,z,u) + \mathcal{O}(\varepsilon^2), \\ \tilde{u} &= u && + \varepsilon\eta(t,x,y,z,u) + \mathcal{O}(\varepsilon^2), \end{aligned}$$

where ε is the group parameter and the accompanying vector field looks like

$$X = \tau(t,x,y,z,u)\partial_t + \xi^1(t,x,y,z,u)\partial_x + \xi^2(t,x,y,z,u)\partial_y + \xi^3(t,x,y,z,u)\partial_z + \eta(t,x,y,z,u)\partial_u. \tag{2}$$

The resulting transformation group will be a point symmetry if and only if

$$pr^{(4)}X\big(u_{xt} + u_{xxxy} + 4u_xu_{xy} + 2u_{xx}u_y + \lambda u_{xx} + \mu u_{xy} + \nu u_{xz}\big)|_{\varepsilon} = 0, \tag{3}$$

where $pr^{(4)}X$ is the fourth prolongation of the vector field Equation (2), and ε denotes the solution space of Equation (1). This determining equation Equation (3) splits with respect to derivatives of u, yielding an over-determined linear system of 168 equations for the infinitesimals $\tau(t,x,y,u)$, $\xi(t,x,y,u)$, $\phi(t,x,y,u)$ and $\eta(t,x,y,u)$ called determining system. The following are the outcomes of solving the determining system:

$$\begin{aligned} &\tau = f_2(z-\nu t)t^2 + f_3(z-\nu t) + f_4(z-\nu t), \\ &\xi^1 = -\tfrac{1}{2}f_1(z-\nu t)x + \tfrac{1}{2}f_2(z-\nu t)tx + \tfrac{1}{2}f_4(z-\nu t)x + F_8(t,z), \\ &\xi^2 = f_1(z-\nu t)y + f_2(z-\nu t)ty + f_6(z-\nu t) + f_7(z-\nu t)t, \\ &\xi^3 = f_2(z-\nu t)\nu t^2 + f_4(z-\nu t)\nu t + f_5(z-\nu t), \\ &\eta = -\tfrac{1}{2}f_2(z-\nu t)tu - \tfrac{1}{2}f_4(z-\nu t)u + \tfrac{1}{2}f_1(z-\nu t)u \\ &+\tfrac{1}{4}f_2(z-\nu t)xy - \tfrac{1}{4}f_2(z-\nu t)\mu xt - \tfrac{1}{4}f_4(z-\nu t)\mu x + \tfrac{1}{4}f_1(z-\nu t)\mu x \\ &+\tfrac{1}{4}f_7(z-\nu t)x - \tfrac{3}{4}f_2(z-\nu t)\lambda yt + \tfrac{1}{2}F_{8t}y + \tfrac{1}{2}F_{8z}\nu y - \tfrac{1}{4}f_4(z-\nu t)\lambda y \\ &-\tfrac{1}{4}f_1(z-\nu t)\lambda y + F_{11}(t,z), \end{aligned} \tag{4}$$

where $f_i(z-\nu t)(i = 1,2,\ldots,7)$, $F_8(t,z)$ and $F_{11}(t,z)$ are arbitrary functions.

Theorem 1. *(i) The point symmetries admitted by the $(3+1)$-dimensional negative-order KdV-CBS Equation (1) [33] are generated by:*

$$\begin{aligned} X_1 &= t\partial_t + x\partial_x + (2\mu t - x)\partial_y + \nu t\partial_z - u\partial_u, \\ X_2 &= t\partial_t, \\ X_3 &= \partial_z, \\ X_4 &= 4t\partial_y + x\partial u, \\ X_5 &= \partial_y, \\ X_F &= F(t,z)\partial_x + (\tfrac{1}{2}F_t y + \tfrac{1}{2}F_z\nu y)\partial_u, \\ X_G &= G(t,z)\partial_u. \end{aligned}$$

(ii) These symmetries comprise a five-dimensional algebra

$$X_1, X_2, X_3, X_4, X_5. \tag{5}$$

(iii) These symmetries comprise two infinite-dimensional generators

$$X_F = F(t,z)\partial_x + (\tfrac{1}{2}F_t y + \tfrac{1}{2}F_z \nu y)\partial_u,$$

$$X_G = G(t,z)\partial_u.$$

Their commutator is given by

$$[X_1, X_2] = -2\mu X_5 - \nu X_3, \tag{6}$$

$$[X_1, X_4] = 2X_4, \tag{7}$$

$$[X_1, X_5] = -X_5, \tag{8}$$

$$[X_2, X_4] = 4X_5. \tag{9}$$

3. Conservation Laws

A local conservation law of a scalar PDE

$$G(t, x, y, z, u, u_t, u_x, u_y, u_z, \ldots) = 0,$$

for $u(t, x, y, z)$ is the equation of continuity

$$D_t T + D_x \Phi^x + D_y \Phi^y + D_z \Phi^z = 0, \tag{10}$$

keeping on the space $\mathcal{E}$ of solutions of the PDE, where T is represents the conserved density and $\Phi = (\Phi^x, \Phi^y, \Phi^z)$ is the spatial flux vector, which are functions of t, x, y, z, u, and derivatives of u. While, (T, Φ) indicates the conserved current.

Every non-trivial conservation law of the PDE $G = 0$ derives from a multiplier, and there establish one-to-one correspondence between non-trivial conserved currents $(T, \Phi)|_{\mathcal{E}}$ modulo trivial ones and non-zero multipliers $Q|_{\mathcal{E}}$, with $QG = D_t T + D_x \Phi^x + D_y \Phi^y + {} + D_z \Phi^z$ retaining as an identity. Here, Q is a function of t, x, y, z, u, with u derivatives, such that $Q|_{\mathcal{E}}$ is non-singular. Several explicit approaches may be used to produce a conserved current $(T, \Phi)|_{\mathcal{E}}$ for each solution Q.

For the negative-order KdV-CBS equation in $(3+1)$-dimensions (1), conservation laws have the characteristic form

$$D_t T + D_x \Phi^x + D_y \Phi^y + D_z \Phi^z = (u_{xt} + u_{xxxy} + 4u_x u_{xy} + 2u_{xx} u_y + \lambda u_{xx} + \mu u_{xy} + \nu u_{xz})Q. \tag{11}$$

Taking into account the following form of low-order multipliers:

$$Q(x, z, u, u_t, u_x, u_y, u_z, u_{xx}, u_{xy}, u_{xxx}, u_{xxy}), \tag{12}$$

all the previous form low-order multipliers are found by requiring that the divergence condition

$$E_u((u_{xt} + u_{xxxy} + 4u_x u_{xy} + 2u_{xx} u_y + \lambda u_{xx} + \mu u_{xy} + \nu u_{xz})Q) = 0, \tag{13}$$

is satisfied, where E_u represents the Euler operator with respect to u [35–37]. The divergence condition for the multipliers (12) splits with respect to the derivatives of $u(t, x, y, z)$ leading to an overdetermined system of 476 equations for Q with $\lambda \neq 0$, $\mu \neq 0$, $\nu \neq 0$, which can be easily solved. This yield the following proposition

Proposition 1. *The low-order multipliers admitted by the* $(3+1)$*-dimensional KdV-CBS Equation (1), with* $\lambda \neq 0$, $\mu \neq 0$, $\nu \neq 0$, *are given by*

$$\begin{aligned} &Q_1 = F(z),\\ &Q_2 = u_t,\\ &Q_3 = u_x,\\ &Q_4 = u_y,\\ &Q_5 = u_z,\\ &Q_6 = x - \tfrac{4zu_y}{\nu},\\ &Q_7 = 3u_x^2 + u_{xxx}. \end{aligned} \tag{14}$$

These multipliers generate non-trivial low-order conservation laws, which are listed below:

Theorem 2. *The conservation laws for the* $(3+1)$*-dimensional KdV-CBS Equation (1), with* $\lambda \neq 0$, $\mu \neq 0$, $\nu \neq 0$ *are given by:*

$$\begin{aligned} T_1 &= F(z)u_x,\\ X_1 &= -\nu F'(z)u + F(z)(\lambda + 2u_y)u_x,\\ Y_1 &= F(z)(\mu u_x + u_x^2 + u_{xxx}),\\ Z_1 &= \nu F(z)u_x. \end{aligned} \tag{15}$$

$$\begin{aligned} T_2 &= -\tfrac{1}{2}(\lambda + 2u_y)u_x^2 - \tfrac{1}{2}(\mu u_y + \nu u_z)u_x + \tfrac{1}{2}(u_{xy}u_{xx}),\\ X_2 &= \tfrac{u_t^2}{2} + \tfrac{u_t}{2}((2\lambda + 4u_y)u_x + \nu u_y + \mu u_z + 2u_{xxy})u_t - \tfrac{1}{2}(u_{xy}u_{tx}),\\ Y_2 &= \tfrac{1}{2}u_t u_x(\mu + 2u_x) - \tfrac{1}{2}(u_{tx}u_{xx}),\\ Z_2 &= \tfrac{1}{2}\nu u_t u_x. \end{aligned} \tag{16}$$

$$\begin{aligned} T_3 &= \tfrac{1}{2}u_x^2,\\ X_3 &= \tfrac{1}{2}u_x(\lambda + 2u_y)u_x + 2u_{xxy},\\ Y_3 &= -\tfrac{1}{2}u_{xx}^2 + \tfrac{1}{2}\mu u_x^2 + u_x^3,\\ Z_3 &= \tfrac{1}{2}\nu u_x^2. \end{aligned} \tag{17}$$

$$\begin{aligned} T_4 &= \tfrac{1}{2}u_x u_y,\\ X_4 &= \tfrac{1}{2}(\mu + 4u_x)u_y^2 + \tfrac{1}{2}(2\lambda u_x + \nu u_z + u_t + 2u_{xxy}) - \tfrac{1}{2}u_{xy}^2,\\ Y_4 &= -\tfrac{1}{2}(\lambda u_x + \nu u_z + u_t)u_x,\\ Z_4 &= \tfrac{1}{2}\nu u_x u_y. \end{aligned} \tag{18}$$

$$\begin{aligned} T_5 &= \tfrac{1}{2}u_x u_z, \\ X_5 &= \tfrac{1}{2}\nu u_z^2 + \tfrac{1}{2}((2\lambda + 4u_y)u_x + \mu u_y + 2u_{xxy} + u_t)u_z - \tfrac{1}{2}u_{xz}u_{xy}, \\ Y_5 &= \tfrac{1}{2}(\mu + 2u_x)u_x u_z - \tfrac{1}{2}u_{xx}u_{xz}, \\ Z_5 &= -\tfrac{1}{2}(\lambda + 2u_y)u_x^2 - \tfrac{1}{2}(u_t + \mu u_y)u_x + \tfrac{1}{2}u_{xx}u_{xy}. \end{aligned} \tag{19}$$

$$\begin{aligned} T_6 &= \tfrac{1}{\nu}(\nu x - 2zu_y)u_x, \\ X_6 &= \nu x u_z + (2xu_x - 2zu_z)u_y + x\lambda u_x + xu_{xxy} - \lambda u - u_{xy} \\ &\quad + \tfrac{2z}{\nu}((-\mu - 4u_x)u_y^2 + (-2\lambda u_x - u_t - 2u_{xxy}u_y + u_{xy}^2), \\ Y_6 &= (\mu x + xu_x + 2zu_z) + \tfrac{2z}{\nu}(u_t + \lambda u_x)u_x, \\ Z_6 &= -2zu_x u_y - \nu u. \end{aligned} \tag{20}$$

$$\begin{aligned} T_7 &= -\tfrac{1}{2}u_{xx}^2 + u_x^3, \\ X_7 &= \tfrac{1}{2}(\lambda + 2u_y)u_{xx}^2 + \tfrac{1}{2}(2\mu u_{xy} + 2\nu u_{xz} - 4u_x u_{xy} + 2u_{tx})u_{xx} + (\lambda + 2u_y)u_x^3, \\ Y_7 &= 3u_x^2 u_{xxx} + \tfrac{1}{2}u_{xxx}^2 - \tfrac{1}{2}\mu u_{xx}^2 + u_x u_{xx}^2 + \mu u_x^3 + \tfrac{5}{2}u_x^4, \\ Z_7 &= -\tfrac{1}{2}\nu u_{xx}^2 + \nu u_x^3. \end{aligned} \tag{21}$$

4. Travelling Wave Reduction and First Integrals

It is well-known that the most popular application of symmetry reduction is the reduction to ODE.

A line travelling wave takes the following form:

$$u(t,x,y,z) = U(\xi), \quad \xi = x + by + cz - at, \tag{22}$$

where a, b and c are arbitrary constants, and ξ is a travelling wave transformation [38]. By plugging (22) into Equation (1) gives a nonlinear fourth-order ODE

$$bU'''' + (6bU' + \lambda + b\mu + c\nu - a)U'' = 0. \tag{23}$$

Since this ODE arises from a symmetry reduction under translations of equation (1), the corresponding conservations laws of the equation invariant under translations will similarly reduce to a first integral of the ODE [39–41]. Furthermore, all first integrals that arise from symmetry invariant conservation laws can be found directly, by using the symmetry, through the general multi-reduction method introduced in [34]. This reduction yields two first integrals. The resulting first integrals of ODE (25) are derived from the corresponding symmetry invariant multipliers

$$Q_1 = u_\xi, \quad Q_2 = 1,$$

and are given by

$$\begin{aligned} \Psi_1 &= -2bU'U''' + bU''^2 - 4bU'^3 + (a - b\mu - c\nu - \lambda)U'^2 = C_1, \\ \Psi_2 &= bU''' + 3bU'^2 + (\lambda + b\mu + c\nu - a)U' = C_2. \end{aligned} \tag{24}$$

Eliminating U''' yields a second-order ODE

$$(U'')^2 + 2(U')^3 + (\mu + \frac{c\nu - a - \lambda}{b})(U')^2 - \frac{2C_2}{b}U' + \frac{2C_1}{b} = 0. \tag{25}$$

Consequently we have a direct triple reduction from the fourth-order PDE (1) to a second-order ODE (25).

Setting $U' = V$ yields a first-order ODE

$$(V')^2 + 2V^3 + (\mu + \frac{cv - a - \lambda}{b})V^2 - \frac{2C_2}{b}V + \frac{2C_1}{b} = 0. \tag{26}$$

For $a = b\mu + cv + \lambda$, the general solution can be given in terms of the Weierstrass $\mathcal{P}$ function, i.e.the general solution is

$$V(\xi) = \mathcal{P}(\sqrt{-\frac{1}{2}}\xi, g2, g3),$$

with $g_2 = \frac{4C_2}{b}$ and $g_3 = -\frac{4C_1}{b}$. Consequently, integrating once with respect to ξ yields

$$U(\xi) = I\sqrt{2}\zeta(\frac{I\sqrt{2}}{2}\xi, \frac{4C_2}{b}, -\frac{4C_1}{b}).$$

where $\mathcal{P}(\xi; g2, g3)$ is the Weierstrass elliptic function, general solution of

$$\mathcal{P}'^2 = 4\mathcal{P}^3 - g_2\mathcal{P} - g_3$$

with g_2, g_3 arbitrary constants (cf. ref. [42]), and $\zeta(\xi; g_2, g_3)$ is the Wierstrass zeta function defined by $\zeta(\xi) = -\int^{\xi} \mathcal{P}(s)ds$, (cf. ref. [43], p. 641).
Setting $C_1 = C_2 = 0$, $\alpha = 2\beta$ and $\beta = \frac{1}{4b}(-b\mu - cv + a - \lambda)$ in (26) an exact solution is

$$V(\xi) = \alpha\,\mathrm{sech}(\sqrt{(\beta)}\xi)^2.$$

Consequently an exact solution for (25) is

$$U(\xi) = 2\sqrt{\beta}\tanh(\sqrt{(\beta)}\xi),$$

yielding the line-kink solution

$$u(t, x, y, z) = 2\sqrt{\beta}\tanh(\sqrt{(\beta)}(x + by + cz - at)),$$

where a, b and c are arbitrary constants, for the $(3+1)$-dimensional negative-order KdV-CBS Equation (1).

5. Extraction of Solitons from a Negative-Order KdV-CBS Equation

In this section, we find the optical solitary wave solutions for Equation (26) using the extended simple equation method (ESEM). Using this method, the initial solution takes the following form [44]:

$$U(\xi) = \sum_{i=-N}^{N} B_i \varphi^i(\xi), \tag{27}$$

where $B_i (i = -N, -N+1, \ldots, -1, 0, 1, \ldots, N-1, N)$ are the unknown coefficients to be found and N is the positive integer which can be calculated using balance principle on Equation (26). By applying balance principle, we get $N = 1$ and therefore, Equation (27) takes the form

$$U(\xi) = \frac{B_{-1}}{\varphi(\xi)} + B_0 + B_1\varphi(\xi), \tag{28}$$

which satisfies the ansatz equation given by

$$\varphi'(\xi) = b_0 + b_1\varphi(\xi) + b_2\varphi^2(\xi), \tag{29}$$

where b_0, b_1 and b_2 are arbitrary constants.

If we take $b_1 = 0$ in Equation (29), then the ansatz Equation (29) converts to Riccati equation, and we have

$$\varphi(\xi) = \frac{\sqrt{b_0 b_2}}{b_2} tan\left(\sqrt{b_0 b_2}(\xi + \xi_0)\right), \quad b_0 b_2 > 0, \tag{30}$$

$$\varphi(\xi) = -\frac{\sqrt{-b_0 b_2}}{b_2} tanh\left(\sqrt{-b_0 b_2}\xi - \frac{m \ln(\xi_0)}{2}\right), \quad \xi_0 > 0, b_0 b_2 < 0, m = \pm 1. \tag{31}$$

If we take $b_0 = 0$ in Equation (29), then the ansatz Equation (29) becomes Bernoulli equation, and we have

$$\varphi(\xi) = \frac{b_1 e^{b_1 \xi}}{b_1 \xi_0 - b_2 e^{b_1 \xi}}, \quad b_1 > 0, \tag{32}$$

$$\varphi(\xi) = -\frac{b_1 e^{b_1 \xi}}{b_1 \xi_0 + b_2 e^{b_1 \xi}}, \quad b_1 < 0. \tag{33}$$

If we take $b_0 = b_1 = 0$ in Equation (29), then the ansatz Equation (29) converts to sparable equation, and in this case we have

$$\varphi(\xi) = \frac{1}{-b_2 \xi + \xi_0}, \quad b_2 \neq 0. \tag{34}$$

Thus, following is the general solution to ansatz Equation (29):

$$\varphi(\xi) = -\frac{b_1 - \sqrt{4b_0 b_2 - b_1^2} tan\left(\frac{\sqrt{4b_0 b_2 - b_1^2}}{2}(\xi + \xi_0)\right)}{2b_2}, \quad 4b_0 b_2 > b_1^2, b_2 > 0, \tag{35}$$

$$\varphi(\xi) = \frac{b_1 + \sqrt{4b_0 b_2 - b_1^2} tan\left(\frac{\sqrt{4b_0 b_2 - b_1^2}}{2}(\xi + \xi_0)\right)}{2b_2}, \quad 4b_0 b_2 > b_1^2, b_2 < 0, \tag{36}$$

where τ_0 represents the constant of integration. Now, after putting Equation (28) along with Equation (29) into Equation (26) generates a system of algebraic equations and further doing some algebra calculations, we get the following cases:

Case 1. When $b_1 = 0$,

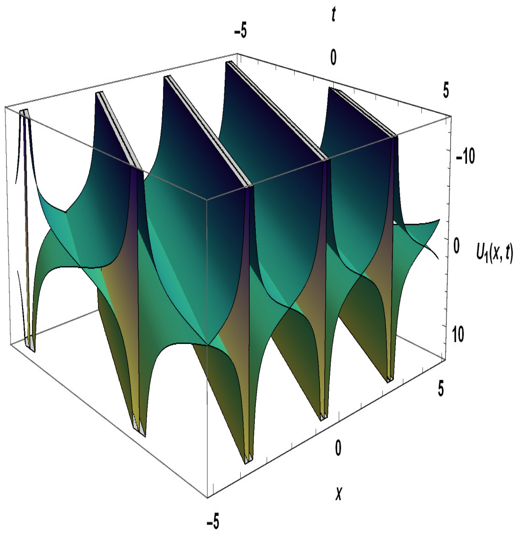
$$A_{-1} = \mp \frac{a - \lambda - cv - b\mu}{8bb_2}, \quad A_1 = \mp 2b_2, \quad b_0 = \frac{\lambda + cv + b\mu - a}{16b_2 b}. \tag{37}$$

By using parameters defined in Equation (37) along with the corresponding ansatz equation yields the following solution

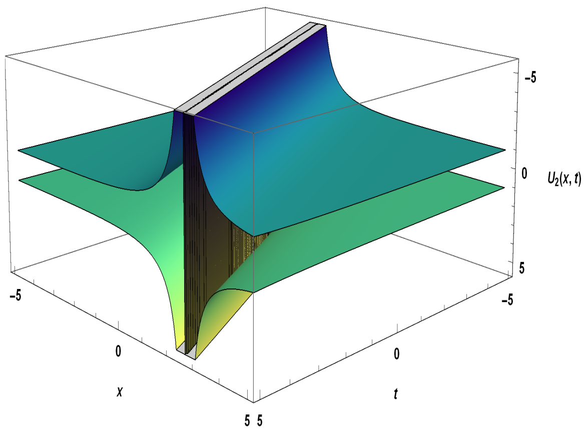
$$U_1(\xi) = \mp \frac{-a + \lambda + cv + b\mu + 8A_0 \tan\left(\sqrt{b_0 b_2}(\xi + \xi_0)\right)\sqrt{b_0 b_2} b - 16b_2 \tan\left(\sqrt{b_0 b_2}(\xi + \xi_0)\right)^2 b_0 b}{8b \tan\left(\sqrt{b_0 b_2}(\xi + \xi_0)\right)\sqrt{b_0 b_2}}, \tag{38}$$

$$U_2(\xi) = \mp \frac{2b_0 b_2 + A_0\sqrt{-b_0 b_2} \tanh\left(\frac{m \ln(\xi_0)}{2} - \sqrt{-b_0 b_2}\xi\right) + 2b_2 b_0 \tanh\left(\frac{m \ln(\xi_0)}{2} - \sqrt{-b_0 b_2}\xi\right)^2}{\sqrt{-b_0 b_2} \tanh\left(-\sqrt{-b_0 b_2}\xi + \frac{m \ln(\xi_0)}{2}\right)}, \tag{39}$$

where $b_0 = \frac{\lambda + cv + b\mu - a}{16b_2 b}$, $\xi = x + by + cz - at$ and $m = \pm 1$ (Figure 1).



(**a**)



(**b**)

Figure 1. Graphical representation of Equations (38) and (39) with suitable parameters $b = 0.2$, $c = 0.1, v = 0.4, \lambda = 0.1, \mu = 1, A_0 = 0.1, \xi_0 = 1, m = -1, y = 0, z = 0$: (**a**) $a = -0.5, b_2 = 0.2$ (**b**) $a = 0.5, b_2 = -0.2$.

Case 2. When $b_0 = 0$,

$$A_{-1} = 0, \quad A_1 = \mp 2b_2, \quad b_1 = \sqrt{\frac{a - b\mu - \lambda - cv}{b}}. \tag{40}$$

By using paramors defined in Equation (40) along with the corresponding ansatz equation yields the following solution

$$U_3(\xi) = \frac{A_0\sqrt{\frac{a-b\mu-\lambda-cv}{b}}\xi_0 - \left(A_0 b_2 - A_1\sqrt{\frac{a-b\mu-\lambda-cv}{b}}\right)e^{\sqrt{\frac{a-b\mu-\lambda-cv}{b}}\xi}}{\sqrt{\frac{a-b\mu-\lambda-cv}{b}}\xi_0 - b_2 e^{\sqrt{\frac{a-b\mu-\lambda-cv}{b}}\xi}}, \quad b \neq 0; \tag{41}$$

where $A_1 = \mp 2b_2$ and $\xi = x + by + cz - at$ (Figure 2).

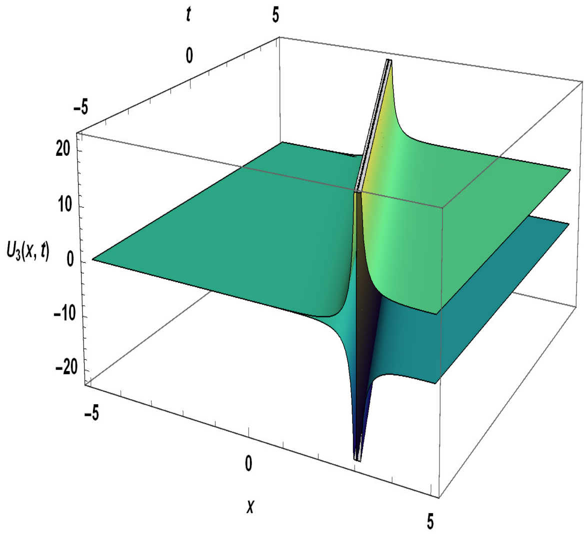


Figure 2. Graphical representation of Equation (41) with suitable parameters $b = -0.1, c = 0.3, v = 1.2, \lambda = 0.1, \mu = 1, A_0 = 0.1, \xi_0 = 1, m = -1, y = 0, z = 0, a = -0.4, b_2 = 0.2$.

Case 3. When $b_0 = b_1 = 0$,

$$A_{-1} = \mp\frac{a - \lambda - cv - b\mu}{6b_2 b}, \quad A_1 = \mp 2b_2. \tag{42}$$

By using paramors defined in Equation (42) along with the corresponding ansatz equation yields the following solution

$$U_4(\xi) = \frac{A_{-1}b_2^2\xi^2 - 2A_{-1}b_2\xi\xi_0 + A_{-1}\xi_0^2 - A_0 b_2\xi + A_0\xi_0 + A_1}{\xi_0 - b_2\xi}, \tag{43}$$

where $A_{-1} = \mp\frac{a-\lambda-cv-b\mu}{6b_2 b}$, $A_1 = \mp 2b_2$ and $\xi = x + by + cz - at$ (Figure 3).

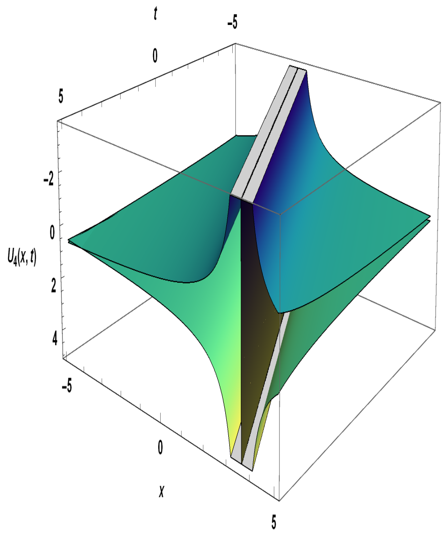


Figure 3. Graphical representation of Equation (43) with suitable parameters $a = 0.5, b = 0.1, c = 0.1, v = 0.2, \lambda = 0.4, \mu = 1, A_0 = 0.6, \xi_0 = 1, b_2 = 1, m = -1, y = 0, z = 0$.

Case 4.

$$\begin{aligned} &(\mathbf{i}): \quad A_{-1} = \frac{bb_1^2 + \lambda + cv + b\mu - a}{2b_2 b}, A_1 = 0, b_0 = \frac{bb_1^2 + \lambda + cv + b\mu - a}{4b_2 b}; \\ &(\mathbf{ii}): \quad A_{-1} = 0, \quad A_1 = -2b_2, \quad b_0 = \frac{bb_1^2 + \lambda + cv + b\mu - a}{4b_2 b}, \quad b, b_2 \neq 0. \end{aligned} \tag{44}$$

By using paramors defined in Equation (44) along with the corresponding anstaz equation yields the following solution

$$U_5(\xi) = \frac{2A_{-1}b_2 - A_0 b_1 + A_0 \tan\left(\frac{1}{2}\sqrt{\frac{\lambda+cv+b\mu-a}{b}}(\xi+\xi_0)\right)\sqrt{\frac{\lambda+cv+b\mu-a}{b}}}{-b_1 + \tan\left(\frac{1}{2}\sqrt{\frac{\lambda+cv+b\mu-a}{b}}(\xi+\xi_0)\right)\sqrt{\frac{\lambda+cv+b\mu-a}{b}}}, \quad 4b_0 b_2 > b_1^2, b_2 > 0, \tag{45}$$

$$U_6(\xi) = \frac{2A_{-1}b_2 + A_0 b_1 + A_0 \tan\left(\frac{1}{2}\sqrt{\frac{\lambda+cv+b\mu-a}{b}}(\xi+\xi_0)\right)\sqrt{\frac{\lambda+cv+b\mu-a}{b}}}{b_1 + \tan\left(\frac{1}{2}\sqrt{\frac{\lambda+cv+b\mu-a}{b}}(\xi+\xi_0)\right)\sqrt{\frac{\lambda+cv+b\mu-a}{b}}}, \quad 4b_0 b_2 > b_1^2, b_2 < 0, \tag{46}$$

and

$$U_7(\xi) = \frac{2A_0 b_2 - A_1 b_1 + A_1 \tan\left(\frac{1}{2}\sqrt{\frac{\lambda+cv+b\mu-a}{b}}(\xi+\xi_0)\right)\sqrt{\frac{\lambda+cv+b\mu-a}{b}}}{2b_2}, \quad b_2 > 0, \tag{47}$$

$$U_8(\xi) = \frac{2A_0 b_2 + A_1 b_1 + A_1 \tan\left(\frac{1}{2}\sqrt{\frac{\lambda+cv+b\mu-a}{b}}(\xi+\xi_0)\right)\sqrt{\frac{\lambda+cv+b\mu-a}{b}}}{2b_2}, \quad b_2 < 0, \tag{48}$$

where $A_{-1} = \mp\frac{a - bb_1^2 - \lambda - cv - b\mu}{2b_2 b}, A_1 = \mp 2b_2$ and $\xi = x + by + cz - at$ (Figures 4 and 5).

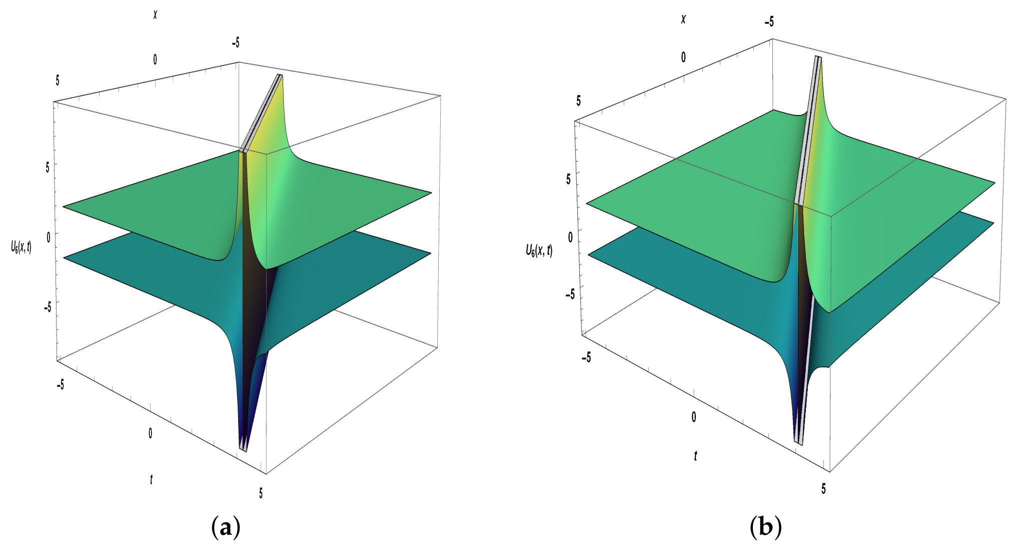
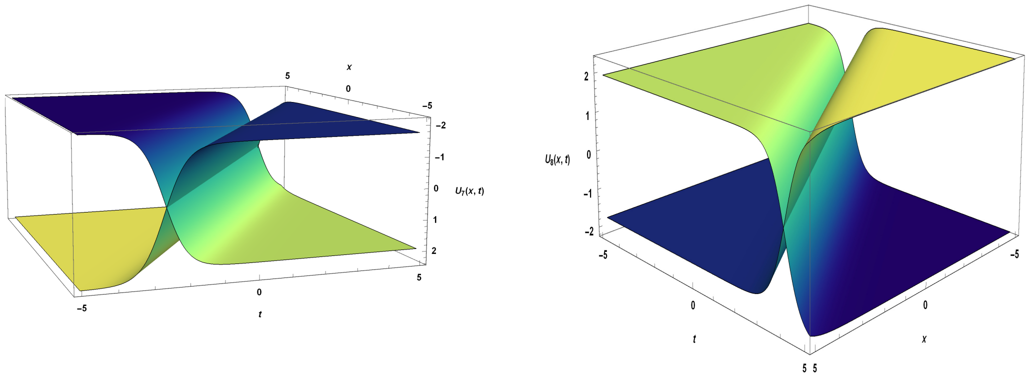


Figure 4. Graphical representation of Equations (45) and (46) with suitable parameters $a = 1.5, b = 0.2, c = 0.2, v = 0.3, \lambda = 0.4, \mu = 1, A_0 = 0.1, \xi_0 = 1, b_1 = 0.2, m = -1, y = 0, z = 0$: (**a**) $b_2 = 0.2$ (**b**) $b_2 = -0.2$.



(**a**) (**b**)

Figure 5. Graphical representation of Equations (47) and (48) with suitable parameters $a = 1.5, b = 0.2, c = 0.2, v = 0.3, \lambda = 0.4, \mu = 1, A_0 = 0.1, \xi_0 = 1, b_1 = 0.2, m = -1, y = 0, z = 0$: (**a**) $b_2 = 0.2$ (**b**) $b_2 = -0.2$.

6. Concluding Remarks

The present work has obtained several new results for the $(3+1)$ negative-order KdV-CBS equation. Firstly, Lie point symmetries and low-order conservation laws have been derived. From the translation symmetries, travelling wave reductions are obtained. Moreover, from the invariant conservation laws under translations two first integrals are obtained for the travelling wave ODE. It happens that these two first integrals are functionally independent and a triple reduction has been derived, yielding a first-order ODE. For some particular cases of the parameters, the general solution is obtained in terms of the Weierstrass function and a line-kink solution is obtained for the $(3+1)$ negative-order KdV-CBS Equation (1). These explicit invariant solutions can be expected to be important in understanding the asymptotics of more general solutions of (1).

Secondly, the extended simple equation approach has been used to obtain the exact solutions for the negative-order KdV-CBS equation, which is a powerful method to solve nonlinear PDEs. These solutions include solitary wave solutions, specifically singular and dark solitons. In addition, numerical solutions are represented in 3D plots as shown in Figures 1-5 to provide physical meaning of nonlinear phenomena. These solutions could be highly useful in comprehending physical phenomena in several disciplines of applied mathematics.

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