symmetry

Review

A Review on Some Linear Positive Operators Defined on Triangles

Teodora Cătinaş

Faculty of Mathematics and Computer Science, Babeş-Bolyai University, St. M. Kogălniceanu No. 1, RO-400084 Cluj-Napoca, Romania; tcatinas@math.ubbcluj.ro

Abstract: We consider results regarding Bernstein and Cheney–Sharma-type operators that interpolate functions defined on triangles with straight and curved sides and we introduce a new Cheney–Sharma-type operator for the triangle with one curved side, highlighting the symmetry between the methods. We present some properties of the operators, their products and Boolean sums and some results regarding the remainders of the corresponding approximation formulas, using modulus of continuity and Peano's theorem. Additionally, we consider some numerical examples to show the approximation properties of the given operators.

Keywords: Bernstein operator; Cheney–Sharma operator; product and Boolean sum operators; modulus of continuity; degree of exactness; error evaluation

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1. Introduction

Certain interpolation operators have been constructed for functions defined on triangles with straight sides (see, e.g., [1–10]) and for functions defined on domains with curved sides (see, e.g., [11–24]).

Different types of interpolation operators (Lagrange, Hermite, Birkhoff, Bernstein, Cheney–Sharma, Nielson, generalized Hermite) that match all the boundary information on curved domains (triangles, squares) have been constructed and studied by us (see, e.g., [13–15,17,18,21,22]). In these works we have studied the properties of the operators, their products and Boolean sums and the remainders of the corresponding approximation formulas, using modulus of continuity and Peano's theorem.

Here we consider two standards triangles. First we consider the standard triangle with all straight sides T_h (see Figure 1), for which if we consider the parallel lines to the coordinate axes through the point $(x, y) \in T_h$, they intersect the sides Γ_i, $i = 1, 2, 3$, of the triangle at the points $(0, y)$ and $(h - y, y)$, respectively $(x, 0)$ and $(x, h - x)$.

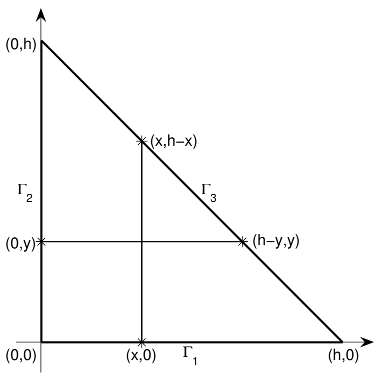

Figure 1. The standard triangle with all straight sides T_h.

We also consider the standard triangle with one curved side $\tilde{T}_h$ with vertices $V_1 = (0, h)$, $V_2 = (h, 0)$ and $V_3 = (0, 0)$, with two straight sides Γ_1, Γ_2, along the coordinate axes, and with the third side Γ_3 (opposite to the vertex V_3) defined by the one-to-one functions f and g, where g is the inverse of the function f, i.e., $y = f(x)$ and $x = g(y)$, with $f(0) = g(0) = h$, for $h > 0$. Additionally, we have $f(x) \leq h$ and $g(y) \leq h$, for $x, y \in [0, h]$. The functions f and g are defined as in [2]. Let F be a real-valued function defined on $\widetilde{T}_h$ and $(0, y)$, $(g(y), y)$, respectively, $(x, 0)$, $(x, f(x))$ be the points at which the parallel lines to the coordinate axes, passing through the point $(x, y) \in \widetilde{T}_h$, intersect the sides Γ_i, $i = 1, 2, 3$ (See Figure 2).

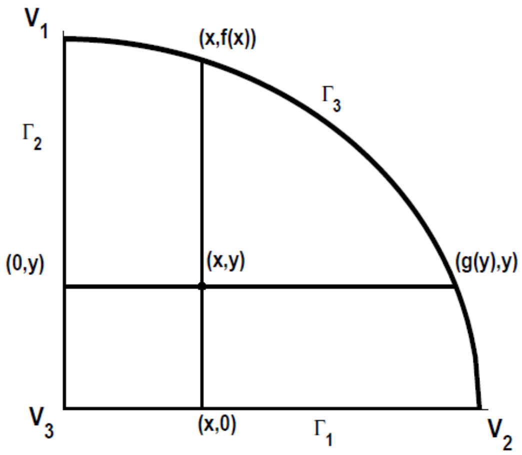

Figure 2. The standard triangle with one curved side $\tilde{T}_h$.

The aim of this paper is to survey results regarding Bernstein- and Cheney–Sharma-type operators that interpolate functions defined on triangles with straight sides and with one curved side, obtained in [3,15,18], and to introduce a new Cheney–Sharma-type operator defined on $\widetilde{T}_h$. There is a symmetrical connection between the methods proposed for the triangle with straight sides and the ones for the triangle with curved sides.

Using the interpolation properties of the operators, blending function interpolants can be constructed that exactly match the function on some sides of the given region. There are many important applications of these blending functions in computer-aided geometric design (see, e.g., [1,25–28]), in finite element method for differential equations (see, e.g., [23–25,29–32]), for construction of surfaces that satisfy some given conditions (see, e.g., [16,20]), in combination with the triangular Shepard method (see, e.g., [33,34]) or in numerical integration formulas (see, e.g., [35]).

The paper is structured in three main sections: Bernstein-type operators, Cheney–Sharma operators of the second kind and Cheney–Sharma operators of the first kind. The first section has two subsections regarding Bernstein-type operators defined on triangle with straight sides and on triangle with one curved side, respectively. The second section has also two subsections regarding the same two types of domains. The last sections contains some new results regarding Cheney–Sharma operators of the first kind defined on triangle with one curved side.

2. Bernstein Type Operator

Since the Bernstein-type operators interpolate a given function at the endpoints of the interval, these operators can also be used as interpolation operators both on triangles with straight sides and with curved sides.

2.1. Bernstein Operator on Triangle with All Straight Sides

Let f be a real-valued function defined on the standard triangle with all straight sides T_h (see Figure 1).

Let $\Delta_m^x = \left\{ i\frac{h-y}{m},\, i = 0,\dots,m \right\}$ and $\Delta_n^y = \left\{ j\frac{h-x}{n},\, j = 0,\dots,n \right\}$ be uniform partitions of the intervals $[0, h-y]$ and $[0, h-x]$.

One considers the Bernstein-type operators B_m^x and B_n^y defined by [3]

$$(B_m^x f)(x,y) = \sum_{i=0}^{m} p_{m,i}(x,y) f\left(i\frac{h-y}{m}, y\right),$$

where

$$p_{m,i}(x,y) = \frac{\binom{m}{i} x^i (h-x-y)^{m-i}}{(h-y)^m}, \quad 0 \leqslant x+y \leqslant h,$$

respectively

$$\left(B_n^y f\right)(x,y) = \sum_{j=0}^{n} q_{n,j}(x,y) f\left(x, j\frac{h-x}{n}\right),$$

with

$$q_{n,j}(x,y) = \frac{\binom{n}{j} y^j (h-x-y)^{n-j}}{(h-x)^n}, \quad 0 \leqslant x+y \leqslant h.$$

Theorem 1 ([3])**.** *If f is a real-valued function defined on T_h then:*

(i) $B_m^x f = f$ *on* $\Gamma_2 \cup \Gamma_3$,

(ii) $(B_m^x e_{i0})(x,y) = x^i,\ i = 0,1\ (\mathrm{dex}(B_m^x) = 1)$,

$(B_m^x e_{20})(x,y) = x^2 + \frac{x(h-x-y)}{m}$,

$$\left(B_m^x e_{ij}\right)(x,y) = \begin{cases} y^j x^i, & i = 0,1,\ j \in \mathbb{N}, \\ y^j \left(x^2 + \frac{x(h-x-y)}{m}\right), & i = 2,\ j \in \mathbb{N}, \end{cases}$$

where $e_{ij}(x,y) = x^i y^j$.

Proof. The interpolation property (i) follows from the relations (see [3])

$$p_{m,i}(0,y) = \begin{cases} 1, \text{ for } i = 0, \\ 0, \text{ for } i > 0, \end{cases}$$

and

$$p_{m,i}(h-y,y) = \begin{cases} 0, \text{ for } i < m, \\ 1, \text{ for } i = m. \end{cases}$$

The property (ii) follows directly. □

Remark 1. *In the same way there are proved similar results for the operator B_n^y.*

Product and Boolean Sum Operators

Let $P_{mn} = B_m^x B_n^y$ and $Q_{nm} = B_n^y B_m^x$ be given by

$$(P_{mn} f)(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} p_{m,i}(x,y) q_{n,j}\left(i\frac{h-y}{m}, y\right) f\left(i\frac{h-y}{m}, j\frac{(m-i)h+iy}{mn}\right),$$

$$(Q_{nm} f)(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} p_{m,i}\left(x, j\frac{h-x}{n}\right) q_{n,j}(x,y) f\left(i\frac{(n-j)h+jx}{mn}, j\frac{h-x}{n}\right).$$

Theorem 2 ([3])**.** *The operators P_{mn} and Q_{nm} satisfy the following relations:*

(i) $(P_{mn} f)(x,0) = (Q_{nm} f)(x,0) = (B_m^x f)(x,0)$,

(ii) $(P_{mn} f)(0,y) = (Q_{nm} f)(0,y) = \left(B_n^y f\right)(0,y)$,

(iii) $(P_{mn} f)(x,h-x) = (Q_{nm} f)(x,h-x) = f(x,h-x), \quad x,y \in [0,h]$.

The proofs follow by a straightforward computation.

Remark 2. *The product operator P_{mn} interpolates the function f at the vertex $(0,0)$ and on the hypothenuse $x+y=h$ of the triangle T_h.*

Let consider the Boolean sums of the Bernstein-type operators B_m^x and B_n^y, given by

$$S_{mn} := B_m^x \oplus B_n^y = B_m^x + B_n^y - B_m^x B_n^y,$$
$$T_{nm} := B_n^y \oplus B_m^x = B_n^y + B_m^x - B_n^y B_m^x.$$

Remark 3. *The Boolean sum is a transfinite (blending) operator.*

Theorem 3 ([3])**.** *If f is a real-valued function defined on T_h then*

$$S_{mn} f\Big|_{\partial T_h} = f\Big|_{\partial T_h}.$$

Proof. We have

$$S_{mn} f = \left(B_m^x + B_n^y - B_m^x B_n^y\right) f.$$

The result follows by the interpolation properties of B_m^x, B_n^y and Theorem 2. □

2.2. Bernstein Operator on Triangle with One Curved Side

Let F be a real-valued function defined on the standard triangle with a curved side $\widetilde{T}_h$ (see Figure 2). One considers the Bernstein-type operators B_m^x and B_n^y defined by [15]

$$(B_m^x F)(x,y) = \sum_{i=0}^{m} p_{m,i}(x,y) F\left(\tfrac{i}{m} g(y), y\right),$$

with

$$p_{m,i}(x,y) = \binom{m}{i}\left(\tfrac{x}{g(y)}\right)^i \left(1 - \tfrac{x}{g(y)}\right)^{m-i}, \quad 0 \leqslant x+y \leqslant g(y),$$

and

$$\left(B_n^y F\right)(x,y) = \sum_{j=0}^{n} q_{n,j}(x,y) F\left(x, \tfrac{j}{n} f(x)\right),$$

with

$$q_{n,j}(x,y) = \binom{n}{j}\left(\tfrac{y}{f(x)}\right)^j \left(1 - \tfrac{y}{f(x)}\right)^{n-j}, \quad 0 \le x+y \le f(x),$$

where

$$\Delta_m^x = \left\{ \tfrac{i}{m} g(y) \Big| \, i = \overline{0,m} \right\} \text{ and } \Delta_n^y = \left\{ \tfrac{j}{n} f(x) \Big| \, j = \overline{0,n} \right\}$$

are uniform partitions of the intervals $[0, g(y)]$ and $[0, f(x)]$, with $g(y) \neq 0$ and $f(x) \neq 0$, for $x, y \in [0, h]$.

Theorem 4 ([15])**.** *With the above notations, if F is a real-valued function defined on $\widetilde{T}_h$ then:*

(i) $B_m^x F = F$ *on* $\Gamma_2 \cup \Gamma_3$,
(ii) $B_n^y F = F$ *on* $\Gamma_1 \cup \Gamma_3$,
and
(iii) $(B_m^x e_{ij})(x,y) = x^i y^j, \;\; i = 0,1; j \in \mathbb{N},$
$(B_m^x e_{2j})(x,y) = \left[x^2 + \frac{x(g(y)-x)}{m}\right] y^j, \;\; j \in \mathbb{N},$
(iv) $\left(B_n^y e_{ij}\right)(x,y) = x^i y^j, \;\; i \in \mathbb{N}, j = 0,1,$
$\left(B_n^y e_{i2}\right)(x,y) = x^i \left[y^2 + \frac{y(f(x)-y)}{n}\right], \;\; i \in \mathbb{N}.$

Proof. The proof of (i) and (ii) is based on the relations:

$$p_{m,i}(0,y) = \begin{cases} 1, \text{ for } i = 0, \\ 0, \text{ for } i > 0, \end{cases}$$

$$p_{m,i}(g(y),y) = \begin{cases} 0, \text{ for } i < m, \\ 1, \text{ for } i = m, \end{cases}$$

and

$$q_{n,j}(x,0) = \begin{cases} 1, \text{ for } j = 0, \\ 0, \text{ for } j > 0, \end{cases}$$

$$q_{n,j}(x,f(x)) = \begin{cases} 0, \text{ for } j < n, \\ 1, \text{ for } j = n. \end{cases}$$

The properties (iii) and (iv) are obtained directly. □

Theorem 5 ([15])**.** *If $F(\cdot,y) \in C[0,g(y)]$ and $(R_m^x F)(x,y) = F - B_m^x F$ then*

$$\left|(R_m^x F)(x,y)\right| \le \left(1 + \tfrac{h}{2\delta\sqrt{m}}\right)\omega(F(\cdot,y);\delta), \quad y \in [0,h],$$

where $\omega(F(\cdot,y);\delta)$ is the modulus of continuity of the function F with regard to the variable x.

Moreover, if $\delta = 1/\sqrt{m}$ then

$$\left|(R_m^x F)(x,y)\right| \le \left(1 + \tfrac{h}{2}\right)\omega(F(\cdot,y);\tfrac{1}{\sqrt{m}}), \quad y \in [0,h]. \tag{1}$$

Proof. From the property $(B_m^x e_{00})(x,y) = 1$, it follows that

$$\left|(R_m^x F)(x,y)\right| \le \sum_{i=0}^{m} p_{m,i}(x,y)\left|F(x,y) - F(\tfrac{i}{m}g(y),y)\right|.$$

Using the inequality

$$\left|F(x,y) - F(\tfrac{i}{m}g(y),y)\right| \le \left(\tfrac{1}{\delta}\left|x - \tfrac{i}{m}g(y)\right| + 1\right)\omega(F(\cdot,y);\delta),$$

one obtains

$$\begin{aligned} |(R_m^x F)(x,y)| &\le \sum_{i=0}^{m} p_{m,i}(x,y)\left(\tfrac{1}{\delta}\left|x - \tfrac{i}{m}g(y)\right| + 1\right)\omega(F(\cdot,y);\delta) \\ &\le \left[1 + \tfrac{1}{\delta}\left(\sum_{i=0}^{m} p_{m,i}(x,y)\left(x - \tfrac{i}{m}g(y)\right)^2\right)^{1/2}\right]\omega(F(\cdot,y);\delta) \\ &= \left[1 + \tfrac{1}{\delta}\sqrt{\tfrac{x(g(y)-x)}{m}}\right]\omega(F(\cdot,y);\delta). \end{aligned}$$

Since,

$$\max_{0 \le x \le g(y)} \left[x(g(y)-x)\right] = \tfrac{g^2(y)}{4} \text{ and } \max_{0 \le y \le h} g^2(y) = h^2,$$

it follows that

$$\max_{(x,y) \in \widetilde{T}_h} \left[x(g(y)-x)\right] = \tfrac{h^2}{4},$$

hence

$$\left|(R_m^x F)(x,y)\right| \le \left(1 + \tfrac{h}{2\delta\sqrt{m}}\right)\omega(F(\cdot,y);\delta).$$

For $\delta = 1/\sqrt{m}$, one obtains (1). □

Theorem 6 ([15]). *If* $F(\cdot, y) \in C^2[0,h]$ *then*

$$(R_m^x F)(x,y) = \tfrac{x[x-g(y)]}{2m} F^{(2,0)}(\xi, y), \text{ for } \xi \in [0, g(y)]$$

and

$$\left|(R_m^x F)(x,y)\right| \le \tfrac{h^2}{8m} M_{20}F, \ (x,y) \in \widetilde{T}_h,$$

where

$$M_{ij}F = \max_{(x,y)\in \widetilde{T}_h} \left|F^{(i,j)}(x,y)\right|.$$

Proof. The proof is based on Peano's theorem, taking into account that $\text{dex}(B_m^x) = 1$. □

Product and Boolean Sum Operators

Let $P_{mn} = B_m^x B_n^y$ and $Q_{nm} = B_n^y B_m^x$ be the products of the operators B_m^x and B_n^y. We have [15]

$$(P_{mn}F)(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y) q_{n,j}\left(\tfrac{i}{m}g(y), y\right) F\left(\tfrac{i}{m}g(y), \tfrac{j}{n} f\left(\tfrac{i}{m}g(y)\right)\right)$$
$$(Q_{nm}F)(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}\left(x, \tfrac{j}{n} f(x)\right) q_{n,j}(x,y) F\left(\tfrac{i}{m}g\left(\tfrac{j}{n}f(x)\right), \tfrac{j}{n}f(x)\right).$$

Theorem 7 ([15]). *If F is a real-valued function defined on* $\widetilde{T}_h$ *then:*

(i) $(P_{mn}F)(V_3) = F(V_3)$,
$P_{mn}F = F$, *on* Γ_3
and

(ii) $(Q_{nm}F)(V_3) = F(V_3)$,
$Q_{nm}F = F$, *on* Γ_3.

Proof. It results from the properties

$$\begin{aligned}
(P_{mn}F)(x,0) &= (B_m^x F)(x,0),\\
(P_{mn}F)(0,y) &= (B_n^y F)(0,y),\\
(P_{mn}F)(x,f(x)) &= F(x,f(x)), \quad x,y \in [0,h]
\end{aligned}$$

and

$$\begin{aligned}
(Q_{nm}F)(x,0) &= (B_m^x F)(x,0),\\
(Q_{nm}F)(0,y) &= (B_n^y F)(0,y),\\
(Q_{nm}F)(g(y),y) &= F(g(y),y), \quad x,y \in [0,h],
\end{aligned}$$

which can be verified by a straightforward computation. □

Let us consider the approximation formula

$$F = P_{mn}F + R_{mn}^P F,$$

where R_{mn}^P is the corresponding remainder operator.

Theorem 8 ([15]). *If* $F \in C(\widetilde{T}_h)$ *then*

$$\left|\left(R_{mn}^P F\right)(x,y)\right| \le (1+h)\omega\left(F; \tfrac{1}{\sqrt{m}}, \tfrac{1}{\sqrt{n}}\right), \quad (x,y) \in \widetilde{T}_h$$

Proof. We have

$$\begin{aligned}\Big|(R^P_{mn}F)(x,y)\Big| \le \Big[&\tfrac{1}{\delta_1}\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\Big(\tfrac{i}{m}g(y),y\Big)\Big|x-\tfrac{i}{m}g(y)\Big| \\ &+\tfrac{1}{\delta_2}\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\Big(\tfrac{i}{m}g(y),y\Big)\Big|y-\tfrac{j}{n}f\Big(\tfrac{i}{m}g(y)\Big)\Big| \\ &+\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\Big(\tfrac{i}{m}g(y),y\Big)\Big]\omega(F;\delta_1,\delta_2).\end{aligned}$$

Since,

$$\begin{aligned}&\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\Big(\tfrac{i}{m}g(y),y\Big)\Big|x-\tfrac{i}{m}g(y)\Big| \le \sqrt{\tfrac{x(g(y)-x)}{m}},\\ &\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\Big(\tfrac{i}{m}g(y),y\Big)\Big|y-\tfrac{j}{n}f(\tfrac{i}{m}g(y))\Big| \le \sqrt{\tfrac{y(f(x)-y)}{n}},\\ &\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\Big(\tfrac{i}{m}g(y),y\Big) = 1,\end{aligned}$$

it follows that

$$\Big|(R^P_{mn}F)(x,y)\Big| \le \left(1+\tfrac{1}{\delta_1}\sqrt{\tfrac{x(g(y)-x)}{m}}+\tfrac{1}{\delta_2}\sqrt{\tfrac{y(f(x)-y)}{n}}\right)\omega(F;\delta_1,\delta_2).$$

But

$$x\big(g(y)-x\big) \le \tfrac{h^2}{4} \text{ and } y\big(f(x)-y\big) \le \tfrac{h^2}{4},$$

whence,

$$\Big|(R^P_{mn}F)(x,y)\Big| \le \Big(1+\tfrac{1}{\delta_1}\tfrac{h}{2\sqrt{m}}+\tfrac{1}{\delta_2}\tfrac{h}{2\sqrt{n}}\Big)\omega(F;\delta_1,\delta_2)$$

and

$$\Big|(R^P_{mn}F)(x,y)\Big| \le (1+h)\omega\left(F;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right).$$

□

We consider the Boolean sums of the operators B^x_m and B^y_n, i.e.,

$$\begin{aligned}S_{mn} &:= B^x_m \oplus B^y_n = B^x_m + B^y_n - B^x_m B^y_n,\\ T_{nm} &:= B^y_n \oplus B^x_m = B^y_n + B^x_m - B^y_n B^x_m.\end{aligned}$$

Theorem 9 ([15]). *If F is a real-valued function defined on $\widetilde{T}_h$ then*

$$\begin{aligned}S_{mn}F\big|_{\partial\widetilde{T}} &= F\big|_{\partial\widetilde{T}}\\ T_{nm}F\big|_{\partial\widetilde{T}} &= F\big|_{\partial\widetilde{T}}.\end{aligned}$$

Proof. The proof follows by a direct verification. □

For the remainder of the Boolean sum approximation formula,

$$F = S_{mn}F + R^S_{mn}F,$$

we have the following result.

Theorem 10 ([15]). *If* $F \in C(\widetilde{T}_h)$ *then*

$$\begin{aligned}\left|(R^S_{mn}F)(x,y)\right| \leq &(1+\tfrac{h}{2})\omega(F(\cdot,y);\tfrac{1}{\sqrt{m}}) + (1+\tfrac{h}{2})\omega(F(x,\cdot);\tfrac{1}{\sqrt{n}}) \\ &+ (1+h)\omega(F;\tfrac{1}{\sqrt{m}},\tfrac{1}{\sqrt{n}}),\ (x,y) \in \widetilde{T}_h.\end{aligned} \quad (2)$$

Proof. The identity

$$F - S_{mn}F = F - B^x_m F + F - B^y_n F - (F - P_{mn}F)$$

implies that

$$\left|(R^S_{mn}F)(x,y)\right| \leq \left|(R^x_m F)(x,y)\right| + \left|\left(R^y_n F\right)(x,y)\right| + \left|(R^P_{mn}F)(x,y)\right|$$

and the conclusion follows. □

3. Cheney–Sharma Operator of the Second Kind

Let $m \in \mathbb{N}$ and β be a nonnegative parameter. In [36], based on the following Jensen's identity,

$$(x+y)(x+y+m\beta)^{m-1} = \sum_{k=0}^{m}\binom{m}{k} x(x+k\beta)^{k-1} y[y+(m-k)\beta]^{m-k-1},\ (x,y) \in \mathbb{R}^2, \quad (3)$$

it was introduced the Cheney–Sharma operator of second kind $Q_m : C[0,1] \to C[0,1]$, given by

$$\begin{aligned}(Q_m f)(x) &= \sum_{i=0}^{m} q_{m,i}(x) f(\tfrac{k}{m}), \\ q_{m,i}(x) &= \tbinom{m}{i}\frac{x(x+i\beta)^{i-1}(1-x)[1-x+(m-i)\beta]^{m-i-1}}{(1+m\beta)^{m-1}}.\end{aligned} \quad (4)$$

We recall some results regarding these Cheney–Sharma-type operators.

Remark 4. *(1) Notice that for* $\beta = 0$*, the operator* Q_m *becomes the Bernstein operator.*

(2) In [37] it is proved that the Cheney–Sharma operator Q_m *interpolates a given function at the endpoints of the interval.*

(3) In [36,37], there have been proved that the Cheney–Sharma operator Q_m *reproduces the constant and the linear functions, so its degree of exactness is* 1 *(denoted* $\mathrm{dex}(Q_m) = 1$*).*

(4) In [36] it is given the following result

$$\begin{aligned}(Q_m e_2)(x) = &x(1+m\beta)^{1-m}[S(2,m-2,x+2\beta,1-x) \\ &- (m-2)\beta S(2,m-3,x+2\beta,1-x+\beta)],\end{aligned} \quad (5)$$

where $e_i(x) = x^i$*,* $i \in \mathbb{N}$*, and*

$$S(j,m,x,y) = \sum_{k=0}^{m}\tbinom{m}{k}(x+k\beta)^{k+j-1}[y+(m-k)\beta]^{m-k}, \quad (6)$$

$j = \overline{0,m}$, $m \in \mathbb{N}$, $x,y \in [0,1]$, $\beta > 0$.

Remark 5. *We may use the Cheney–Sharma operators of second kind* Q^x_m *and* Q^y_n *as interpolation operators, because they interpolate a given function at the endpoints of the interval.*

3.1. Cheney–Sharma Operator on Triangle with All Straight Sides

Let f be a real-valued function defined on the standard triangle with all straight sides T_h (see Figure 1).

Let $\Delta_m^x = \left\{ i\frac{h-y}{m},\, i = \overline{0,m} \right\}$ and $\Delta_n^y = \left\{ j\frac{h-x}{n},\, j = \overline{0,n} \right\}$ be uniform partitions of the intervals $[0, h-y]$ and $[0, h-x]$, for $m, n \in \mathbb{N}$.

In [19] we study the Cheney–Sharma operator of the second kind for the functions defined on T_h. We study their interpolation properties, the corresponding product and Boolean sum operators, and the remainders of the interpolation formulas. The operators are given by

$$(Q_m^x f)(x,y) = \sum_{i=0}^{m} q_{m,i}(x,y) f\left(i\tfrac{h-y}{m}, y\right),$$
$$(Q_n^y f)(x,y) = \sum_{j=0}^{n} q_{n,j}(x,y) f\left(x, j\tfrac{h-x}{n}\right),$$

with

$$q_{m,i}(x,y) = \binom{m}{i} \frac{1}{(1+m\beta)^{m-1}} \left(\frac{x}{h-y} (\frac{x}{h-y} + i\beta)^{i-1} (1 - \frac{x}{h-y}) [1 - \frac{x}{h-y} + (m-i)\beta]^{m-i-1} \right),$$
$$q_{n,j}(x,y) = \binom{n}{j} \frac{1}{(1+nb)^{n-1}} \left(\frac{y}{h-x} (\frac{y}{h-x} + jb)^{j-1} (1 - \frac{y}{h-x}) [1 - \frac{y}{h-x} + (n-j)b]^{n-j-1} \right),$$

where $b, \beta \in \mathbb{R}_+$.

3.2. Cheney–Sharma Operator on Triangle with One Curved Side

Let F be a real-valued function defined on standard triangle with a curved side $\widetilde{T}_h$ (see Figure 2). For $m, n \in \mathbb{N}$, $\beta, b \in \mathbb{R}_+$, we have considered the following extensions of the Cheney–Sharma operator of the second kind to the case of functions defined on $\widetilde{T}_h$, see [18]:

$$(Q_m^x F)(x,y) = \sum_{i=0}^{m} q_{m,i}(x,y) F\left(i\tfrac{g(y)}{m}, y\right), \tag{7}$$
$$(Q_n^y F)(x,y) = \sum_{j=0}^{n} q_{n,j}(x,y) F\left(x, j\tfrac{f(x)}{n}\right),$$

with

$$q_{m,i}(x,y) = \binom{m}{i} \frac{1}{(1+m\beta)^{m-1}} \left(\frac{x}{g(y)} (\frac{x}{g(y)} + i\beta)^{i-1} (1 - \frac{x}{g(y)}) [1 - \frac{x}{g(y)} + (m-i)\beta]^{m-i-1} \right),$$
$$q_{n,j}(x,y) = \binom{n}{j} \frac{1}{(1+nb)^{n-1}} \left(\frac{y}{f(x)} (\frac{y}{f(x)} + jb)^{j-1} (1 - \frac{y}{f(x)}) [1 - \frac{y}{f(x)} + (n-j)b]^{n-j-1} \right),$$

where
$$\Delta_m^x = \left\{ i\tfrac{g(y)}{m} \,\middle|\, i = \overline{0,m} \right\} \text{ and } \Delta_n^y = \left\{ j\tfrac{f(x)}{n} \,\middle|\, j = \overline{0,n} \right\}$$
are uniform partitions of the intervals $[0, g(y)]$ and $[0, f(x)]$, respectively.

Theorem 11 ([18])**.** *If F is a real-valued function defined on $\widetilde{T}_h$ then*

(i) $Q_m^x F = F$ on $\Gamma_1 \cup \Gamma_3$,
(ii) $Q_n^y F = F$ on $\Gamma_2 \cup \Gamma_3$.

Proof. (i) We may write

$$(Q_m^x F)(x,y) = \frac{1}{(1+m\beta)^{m-1}} \Big\{ (1 - \tfrac{x}{g(y)})[1 - \tfrac{x}{g(y)} + m\beta]^{m-1} F(0,y) \\ + \tfrac{x}{g(y)}(1 - \tfrac{x}{g(y)}) \sum_{i=1}^{m-1} \tbinom{m}{i} (\tfrac{x}{g(y)} + i\beta)^{i-1} \\ \cdot [1 - \tfrac{x}{g(y)} + (m-i)\beta]^{m-i-1} F\Big(i\tfrac{g(y)}{m}, y\Big) \\ + \tfrac{x}{g(y)} (\tfrac{x}{g(y)} + m\beta)^{m-1} F(g(y),y) \Big\}. \tag{8}$$

Considering (8), it follows (i).

(ii) Similarly, writing

$$(Q_n^y F)(x,y) = \frac{1}{(1+nb)^{n-1}} \Big\{ (1 - \tfrac{y}{f(x)})[1 - \tfrac{y}{f(x)} + nb]^{n-1} F(x,0) \\ + \tfrac{y}{f(x)}(1 - \tfrac{y}{f(x)}) \sum_{j=1}^{n-1} \tbinom{n}{j} (\tfrac{y}{f(x)} + jb)^{j-1} \\ \cdot [1 - \tfrac{y}{f(x)} + (n-j)b]^{n-j-1} F\Big(x, j\tfrac{f(x)}{n}\Big) \\ + \tfrac{y}{f(x)} (\tfrac{y}{f(x)} + nb)^{n-1} F(x,f(x)) \Big\},$$

we get (ii). □

Theorem 12 ([18])**.** *The operators Q_m^x and Q_n^y have the following properties:*

(i) $\big(Q_m^x e_{ij}\big)(x,y) = x^i y^j, \; i = 0,1; j \in \mathbb{N};$

(ii) $\Big(Q_n^y e_{ij}\Big)(x,y) = x^i y^j, \; i \in \mathbb{N}; j = 0,1,$ where $e_{ij}(x,y) = x^i y^j, \; i,j \in \mathbb{N}$.

Proof. The proof is based on the Remark 4. □

We consider the approximation formula

$$F = Q_m^x F + R_m^x F,$$

where $R_m^x F$ denotes the approximation error.

Theorem 13 ([18])**.** *If $F(\cdot,y) \in C[0,g(y)]$ then we have*

$$\big|(R_m^x F)(x,y)\big| \le (1 + \tfrac{1}{\delta}\sqrt{A_m - x^2})\omega(F(\cdot,y);\delta), \quad \forall \delta > 0, \tag{9}$$

where $\omega(F(\cdot,y);\delta)$ is the modulus of continuity and $A_m = x(1+m\beta)^{1-m}[S(2,m-2,x+2\beta,1-x) - (m-2)\beta S(2,m-3,x+2\beta,1-x+\beta)]$, with S given in (6).

Proof. By Theorem 12 we have that $\text{dex}(Q_m^x) = 1$, thus we may apply the following property of linear operators (see, e.g., [38])

$$\big|(Q_m^x F)(x,y) - F(x,y)\big| \le [1 + \delta^{-1}\sqrt{(Q_m^x e_{20})(x,y) - x^2}]\omega(F(\cdot,y);\delta), \;\; \forall \delta > 0,$$

and taking into account (5), we obtain (9). □

Theorem 14 ([18])**.** *If $F(\cdot,y) \in C^2[0,g(y)]$ then*

$$(R_m^x F)(x,y) = \tfrac{1}{2} F^{(2,0)}(\xi,y)\{x^2 - x(1+m\beta)^{1-m}[S(2,m-2,x+2\beta,1-x) \\ - (m-2)\beta S(2,m-3,x+2\beta,1-x+\beta)]\}, \tag{10}$$

for $\xi \in [0, g(y)]$ *and* $\beta > 0$.

Proof. Taking into account that $\text{dex}(Q^x_m) = 1$, by Theorem 12, and applying the Peano's theorem (see, e.g., [39]), it follows

$$(R^x_m F)(x,y) = \int_0^{g(y)} K_{20}(x,y;s) F^{(2,0)}(s,y) ds,$$

where

$$K_{20}(x,y;s) = (x-s)_+ - \sum_{i=0}^{m} q_{m,i}(x,y)\left(i\tfrac{g(y)}{m} - s\right)_+.$$

For a given $\nu \in \{1,\ldots,m\}$ one denotes by $K^\nu_{20}(x,y;\cdot)$ the restriction of the kernel $K_{20}(x,y;\cdot)$ to the interval $\left[(\nu-1)\frac{g(y)}{m}, \nu\frac{g(y)}{m}\right]$, i.e.,

$$K^\nu_{20}(x,y;\nu) = (x-s)_+ - \sum_{i=\nu}^{m} q_{m,i}(x,y)\left(i\tfrac{g(y)}{m} - s\right),$$

whence,

$$K^\nu_{20}(x,y;s) = \begin{cases} x - s - \sum\limits_{i=\nu}^{m} q_{m,i}(x,y)\left(i\frac{g(y)}{m} - s\right), & s < x \\ -\sum\limits_{i=\nu}^{m} q_{m,i}(x,y)\left(i\frac{g(y)}{m} - s\right), & s \geq x. \end{cases}$$

It follows that $K^\nu_{20}(x,y;s) \leq 0$, for $s \geq x$.
For $s < x$ we have

$$K^\nu_{20}(x,y;s) = x - s - \textstyle\sum_{i=0}^{m} q_{m,i}(x,y)\left[i\frac{g(y)}{m} - s\right] + \sum_{i=0}^{\nu-1} q_{m,i}(x,y)\left[i\frac{g(y)}{m} - s\right].$$

Applying Theorem 12, we obtain

$$\sum_{i=0}^{m} q_{m,i}(x,y)\left[i\tfrac{g(y)}{m} - s\right] = (Q^x_m e_{10})(x,y) - s(Q^x_m e_{00})(x,y) = x - s,$$

whence it follows that

$$K^\nu_{20}(x,y;s) = \sum_{i=0}^{\nu-1} q_{m,i}(x,y)\left[i\tfrac{g(y)}{m} - s\right] \leq 0.$$

So, $K^\nu_{20}(x,y;\cdot) \leq 0$, for any $\nu \in \{1,\ldots,m\}$, i.e., $K_{20}(x,y;s) \leq 0$, for $s \in [0, g(y)]$.
By the Mean Value Theorem, one obtains

$$(R^x_m F)(x,y) = F^{(2,0)}(\xi, y) \int_0^{g(y)} K_{20}(x,y;s) ds, \quad \text{for all } 0 \leq \xi \leq g(y),$$

with

$$\int_0^{g(y)} K_{20}(x,y;s) ds = \tfrac{1}{2}[x^2 - (Q^x_m e_{20})(x,y)],$$

and using (5) we obtain (10). □

Remark 6. *Analogous results with the ones in Theorems 13 and 14 can be obtained for the remainder* $R^y_n F$ *of the formula* $F = Q^y_n F + R^y_n F$.

Product and Boolean Sum Operators

Let $P^1_{mn} = Q^x_m Q^y_n$ and $P^2_{nm} = Q^y_n Q^x_m$ be the products of the operators Q^x_m and Q^y_n.

We have

$$\left(P^1_{mn}F\right)(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}q_{m,i}(x,y)q_{n,j}\left(i\tfrac{g(y)}{m},y\right)F\left(i\tfrac{g(y)}{m},j\frac{f(i\frac{g(y)}{m})}{n}\right),$$

and

$$\left(P^2_{nm}F\right)(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}q_{m,i}\left(x,j\tfrac{f(x)}{n}\right)q_{n,j}(x,y)F\left(i\frac{g(j\frac{f(x)}{n})}{m},j\tfrac{f(x)}{n}\right).$$

Theorem 15 ([18]). *If F is a real-valued function defined on $\widetilde{T}_h$ then*

(i) $(P^1_{mn}F)(V_i) = F(V_i), \quad i=1,2,3;$
$(P^1_{mn}F)(\Gamma_3) = F(\Gamma_3),$
(ii) $(P^2_{nm}F)(V_i) = F(V_i), \quad i=1,2,3;$
$(P^2_{nm}F)(\Gamma_3) = F(\Gamma_3),$

Proof. By a straightforward computation, we obtain the following properties

$$\begin{aligned}(P^1_{mn}F)(x,0) &= (Q^x_m F)(x,0),\\ (P^1_{mn}F)(0,y) &= (Q^y_n F)(0,y),\\ (P^1_{mn}F)(x,f(x)) &= F(x,f(x)), \quad x,y\in[0,h]\end{aligned}$$

and

$$\begin{aligned}(P^2_{nm}F)(x,0) &= (Q^x_m F)(x,0),\\ (P^2_{nm}F)(0,y) &= (Q^y_n F)(0,y),\\ (P^2_{nm}F)(g(y),y) &= F(g(y),y), \quad x,y\in[0,h],\end{aligned}$$

and, taking into account Theorem 11, they imply (i) and (ii). □

We consider the following approximation formula

$$F = P^1_{mn}F + R^{P1}_{mn}F,$$

where R^{P1}_{mn} is the corresponding remainder operator.

Theorem 16 ([18]). *If $F\in C(\widetilde{T}_h)$ then*

$$\left|(R^{P1}_{mn}F)(x,y)\right| \le (A_m+B_n-x^2-y^2+1)\omega\left(F;\tfrac{1}{\sqrt{A_m-x^2}},\tfrac{1}{\sqrt{B_n-y^2}}\right),\ \forall(x,y)\in\widetilde{T}_h, \qquad (11)$$

where

$$\begin{aligned}A_m =& x(1+m\beta)^{1-m}[S(2,m-2,x+2\beta,1-x)\\ &-(m-2)\beta S(2,m-3,x+2\beta,1-x+\beta)]\\ B_n =& y(1+nb)^{1-n}[S(2,n-2,y+2b,1-y)-(n-2)bS(2,n-3,y+2b,1-y+\beta)]\end{aligned} \qquad (12)$$

and $\omega(F;\delta_1,\delta_2)$, with $\delta_1>0$, $\delta_2>0$, is the bivariate modulus of continuity.

Proof. Using a basic property of the modulus of continuity we have

$$\begin{aligned}\left|(R_{mn}^{P1}F)(x,y)\right| \leq \Bigg[& \frac{1}{\delta_1}\sum_{i=0}^{m}\sum_{j=0}^{n} q_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}g(y),y\right)\left|x-\tfrac{i}{m}g(y)\right| \\ & + \frac{1}{\delta_2}\sum_{i=0}^{m}\sum_{j=0}^{n} q_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}g(y),y\right)\left|y-\tfrac{j}{n}f\left(\tfrac{i}{m}g(y)\right)\right| \\ & + \sum_{i=0}^{m}\sum_{j=0}^{n} q_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}g(y),y\right)\Bigg]\omega(F;\delta_1,\delta_2), \ \ \forall\delta_1,\delta_2>0.\end{aligned}$$

Since

$$\begin{aligned}&\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}g(y),y\right)\left|x-\tfrac{i}{m}g(y)\right| \leq \sqrt{(Q_m^x e_{20})(x,y)-x^2}, \\ &\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}g(y),y\right)\left|y-\tfrac{j}{n}f(\tfrac{i}{m}g(y))\right| \leq \sqrt{(Q_n^y e_{02})(x,y)-y^2}, \\ &\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}g(y),y\right) = 1,\end{aligned}$$

applying (5), we obtain

$$\begin{aligned}&\left|(R_{mn}^{P1}F)(x,y)\right| \leq \Big\{\tfrac{1}{\delta_1}[x(1+m\beta)^{1-m}]^{\frac{1}{2}} \\ &\cdot\left\{[S(2,m-2,x+2\beta,1-x)-(m-2)\beta S(2,m-3,x+2\beta,1-x+\beta)]-x^2\right\}^{\frac{1}{2}} + \tfrac{1}{\delta_2}[y(1+nb)^{1-n}]^{\frac{1}{2}} \\ &\cdot\left\{[S(2,n-2,y+2b,1-y)-(n-2)bS(2,n-3,y+2b,1-y+\beta)]-y^2\right\}^{\frac{1}{2}} + 1\Big\}\omega(F;\delta_1,\delta_2).\end{aligned}$$

Denoting

$$\begin{aligned}A_m &= x(1+m\beta)^{1-m}[S(2,m-2,x+2\beta,1-x)-(m-2)\beta S(2,m-3,x+2\beta,1-x+\beta)], \\ B_n &= y(1+nb)^{1-n}[S(2,n-2,y+2b,1-y)-(n-2)bS(2,n-3,y+2b,1-y+\beta)]\end{aligned}$$

and, taking $\delta_1 = \frac{1}{\sqrt{A_m-x^2}}$ and $\delta_2 = \frac{1}{\sqrt{B_n-y^2}}$, we get (11). □

We consider the Boolean sums of the operators Q_m^x and Q_n^y,

$$\begin{aligned}S_{mn}^1 &:= Q_m^x \oplus Q_n^y = Q_m^x + Q_n^y - Q_m^x Q_n^y, \\ S_{nm}^2 &:= Q_n^y \oplus Q_m^x = Q_n^y + Q_m^x - Q_n^y Q_m^x.\end{aligned}$$

Theorem 17 ([18]). *If F is a real-valued function defined on $\widetilde{T}_h$, then*

$$\begin{aligned}S_{mn}^1 F\Big|_{\partial\widetilde{T}_h} &= F\Big|_{\partial\widetilde{T}_h}, \\ S_{mn}^2 F\Big|_{\partial\widetilde{T}_h} &= F\Big|_{\partial\widetilde{T}_h}.\end{aligned}$$

Proof. We have

$$\begin{aligned}\left(Q^x_m Q^y_n F\right)(x,0) &= (Q^x_m F)(x,0),\\ \left(Q^y_n Q^x_m F\right)(0,y) &= \left(Q^y_n F\right)(0,y),\\ (Q^x_m F)(x,h-x) &= \left(Q^y_n F\right)(x,h-x)\\ &= (P^1_{mn}F)(x,h-x) = (P^2_{nm}F)(x,h-x) = F(x,h-x),\end{aligned}$$

and, taking into account Theorem 11, the conclusion follows. □

We consider the following approximation formula

$$F = S^1_{mn}F + R^{S^1}_{mn}F,$$

where $R^{S^1}_{mn}$ is the corresponding remainder operator.

Theorem 18 ([18])**.** *If $F \in C(\widetilde{T}_h)$ then*

$$\begin{aligned}&\left|(R^{S^1}_{mn}F)(x,y)\right| \le \\ &\le (1 + A_m - x^2)\omega(F(\cdot,y); \tfrac{1}{\sqrt{A_m - x^2}}) + (1 + B_n - y^2)\omega(F(x,\cdot); \tfrac{1}{\sqrt{B_n - y^2}})\\ &+ (A_m + B_n - x^2 - y^2 + 1)\omega(F; \tfrac{1}{\sqrt{A_m - x^2}}, \tfrac{1}{\sqrt{B_n - y^2}}),\end{aligned} \tag{13}$$

with A_m and B_n given in (12).

Proof. The identity

$$F - S^1_{mn}F = (F - Q^x_m F) + (F - Q^y_n F) - (F - P^1_{mn}F)$$

implies that

$$\left|(R^{S^1}_{mn}F)(x,y)\right| \le \left|(R^x_m F)(x,y)\right| + \left|(R^y_n F)(x,y)\right| + \left|(R^{P^1}_{mn}F)(x,y)\right|,$$

and, applying Theorems 13 and 16, we get (13). □

4. Cheney–Sharma Operator of the First Kind

Let $m \in N$ and β be a non-negative parameter. In [36], based on the following Jensen's identity

$$(x + y + m\beta)^m = \sum_{k=0}^{m} \binom{m}{k} x(x + k\beta)^{k-1}[y + (m-k)\beta]^{m-k}, \ (\forall)\ (x,y) \in \mathbb{R}^2,$$

it was introduced the Cheney–Sharma operators of the first kind $G_m : C[0,1] \to C[0,1]$, given by

$$(G_m f)(x) = \sum_{i=0}^{m} q_{m,i}(x) f(\tfrac{k}{m}),$$

with

$$q_{m,i}(x) = \tbinom{m}{i}\frac{x(x + i\beta)^{i-1}[1 - x + (m-i)\beta]^{m-i}}{(1 + m\beta)^m}.$$

For F a real-valued function defined on $\widetilde{T}_h$, $m, n \in \mathbb{N}$, $\beta, b \in \mathbb{R}_+$, we consider here the new extensions of the Cheney–Sharma operator of the first kind,

$$(G_m^x F)(x,y) = \sum_{i=0}^{m} r_{m,i}(x,y) F\left(i\frac{g(y)}{m}, y\right), \tag{14}$$
$$(G_n^y F)(x,y) = \sum_{j=0}^{n} r_{n,j}(x,y) F\left(x, j\frac{f(x)}{n}\right),$$

with

$$r_{m,i}(x,y) = \binom{m}{i} \frac{\frac{x}{g(y)}\left(\frac{x}{g(y)} + i\beta\right)^{i-1}\left[1 - \frac{x}{g(y)} + (m-i)\beta\right]^{m-i}}{(1+m\beta)^m},$$
$$r_{n,j}(x,y) = \binom{n}{j} \frac{\frac{y}{f(x)}\left(\frac{y}{f(x)} + jb\right)^{j-1}\left[1 - \frac{y}{f(x)} + (n-j)b\right]^{n-j}}{(1+nb)^n},$$

where

$$\Delta_m^x = \left\{ i\frac{g(y)}{m} \,\middle|\, i = \overline{0,m} \right\} \text{ and } \Delta_n^y = \left\{ j\frac{f(x)}{n} \,\middle|\, j = \overline{0,n} \right\}$$

are uniform partitions of the intervals $[0, g(y)]$ and $[0, f(x)]$.

Remark 7. *The new extensions of the Cheney–Sharma operator of the first kind, introduced here, have similar properties as the ones of the Cheney–Sharma operator of second kind from Section 3.*

5. Numerical Examples

In this section, we consider two test functions for which we plot the graphs of the approximants using the methods presented here, and also we study the maximum approximation errors for the corresponding approximants.

Example 1. *We consider the following test functions, generally used in the literature (see, e.g., [40]):*

$$\text{Gentle: } F_1(x,y) = \frac{1}{3}\exp\left[-\frac{81}{16}\left((x-0.5)^2 + (y-0.5)^2\right)\right],$$
$$\text{Saddle: } F_2(x,y) = \frac{1.25 + \cos 5.4y}{6 + 6(3x-1)^2}. \tag{15}$$

Using Matlab, in Figure 3 we plot the graphs of F_1, $B_m^x F_1$, $P_{mn}F_1$, $S_{mn}F_1$, *defined on* $\tilde{T}_h$, *considering* $h = 1, m = 5, n = 6$ *and* $f : [0,1] \to [0,1]$, $f(x) = \sqrt{1-x^2}$.

In Figure 4 we plot the graphs of $Q_m^x F_1$, $Q_n^y F_1$, $P_{mn}^1 F_1$, $S_{mn}^1 F_1$, on $\tilde{T}_h$, considering $h = 1, m = 5, n = 6, \beta = b = 1$.

Table 1 contains maximum errors for approximating the functions given in (15) using some Bernstein- and Cheney–Sharma-type operators.

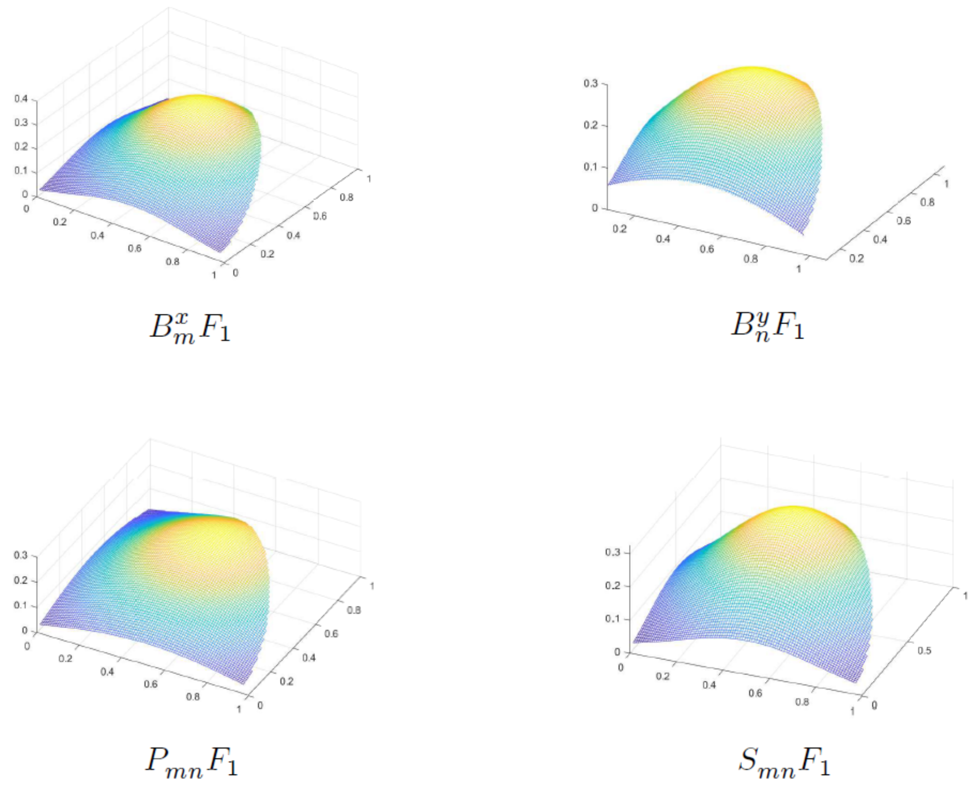

Figure 3. The Bernstein extensions for F_1.

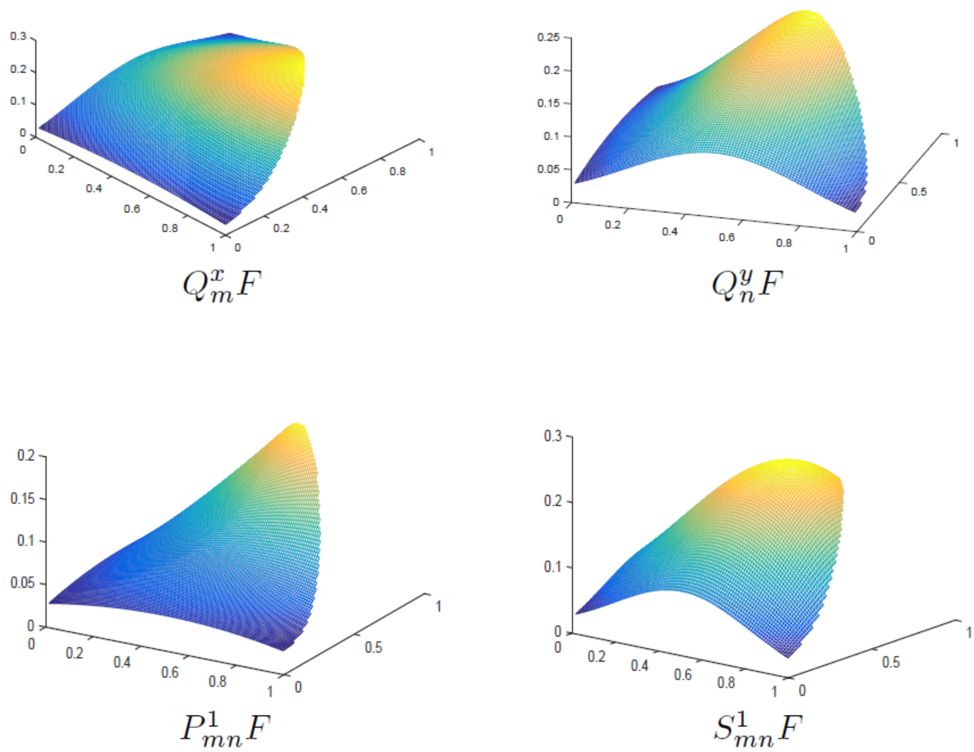

Figure 4. The Cheney–Sharma extensions for $\tilde{T}_h$.

Table 1. Maximum approximation errors.

Max Error	F_1	F_2
$B_m^x F$	0.0525	0.0821
$B_n^y F$	0.0452	0.0692
$P_{mn} F$	0.0858	0.0943
$Q_{nm} F$	0.0857	0.0944
$S_{mn} F$	0.0095	0.0144
$T_{nm} F$	0.0095	0.0112
$Q_m^x F$	0.1645	0.1921
$Q_n^y F$	0.1670	0.2266
$P_{mn}^1 F$	0.2403	0.1937
$S_{mn}^1 F$	0.0900	0.1125

6. Conclusions

By Table 1, Figures 3 and 4 we remark the good approximation properties of the considered Bernstein- and Cheney–Sharma-type operators, especially of the starting operators and the Boolean sum operators, the last being the ones that interpolate on the entire frontier of the domain.

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