symmetry

Article

Uncertain Stochastic Optimal Control with Jump and Its Application in a Portfolio Game

Chengyu Wu [1,†], Lu Yang [2,*,†] and Chengke Zhang [3,†]

1 School of Management, Guangdong University of Technology, Guangzhou 510520, China
2 School of Management, Guangdong Polytechnic Normal University, Guangzhou 510665, China
3 School of Economics and Commence, Guangdong University of Technology, Guangzhou 510520, China
* Correspondence: yanglu@mail2.gdut.edu.cn
† These authors contributed equally to this work.

Abstract: This article describes a class of jump-uncertain stochastic control systems, and derives an Itô–Liu formula with jump. We characterize an optimal control law, that satisfies the Hamilton–Jacobi–Bellman equation with jump. Then, this paper deduces the optimal portfolio game under uncertain stochastic financial markets with jump. The information of players is symmetrical. The financial market is constituted of a risk-free asset and a risky asset whose price process is subjected to the jump-uncertain stochastic Black–Scholes model. The game is formulated by two utility maximization problems, each investor tries to maximize his relative utility, which is the weighted average of terminal wealth difference between his terminal wealth and that of his competitor. Finally, the explicit expressions of equilibrium investment strategies and value functions for the constant absolute risk-averse and constant relative risk-averse utility function are derived by using the dynamic programming principle.

Keywords: jump-uncertain stochastic differential equation; the optimal equation of jump-uncertain stochastic process; portfolio game under symmetry information; power utility; exponential utility

check for updates

Citation: Wu, C.; Yang, L.; Zhang, C. Uncertain Stochastic Optimal Control with Jump and Its Application in a Portfolio Game. *Symmetry* **2022**, *14*, 1885. https://doi.org/10.3390/sym14091885

Academic Editor: Sergei D. Odintsov

Received: 16 August 2022
Accepted: 6 September 2022
Published: 9 September 2022

1. Introduction

In the real world, non-decisive information is everywhere. The emergence of information or events is affected by various accidental and unpredictable factors. Indecisive information includes objective indecisive information and subjective indecisive information. Objective indecisive information is random information. The effective tool to deal with random information is probability theory. In real life, there is another kind of subjective non-decisive information, namely fuzzy information. The effective tool for this kind of problem is fuzzy mathematics theory. Liu [1] first proposed the concept of the fuzzy set. So far, fuzzy mathematics has gradually become an important method to deal with subjective non-decisive information. In order to strictly measure the possibility of fuzzy events by mathematical methods, Zadeh [2] put forward the concepts of possibility measure and fuzzy variable, and developed a set of possibility theories. Later, scholars found that possibility theory can solve practical problems, then introduced the concept of necessity measure and dual part necessity measure. However, these two measures lack self-duality. Therefore, Liu [3] put forward the concept of credibility measure which solved the above problems in 2002, and then systematically put forth the credibility theory laid out in Liu [4], which further developed the fuzzy mathematics theory.

However, the complexity of the world makes the events we face uncertain in various forms. The uncertainty behaves neither like randomness nor like fuzziness was shown to do in many cases. In the real world, some information or knowledge is usually represented by human linguistic expressions like "about 1 km", "roughly 10 kg", and so on. To distinguish this phenomenon from randomness and fuzziness, an uncertainty theory was invented by Liu [5] and refined in 2009 based on normality, monotonicity, self-duality, countable subadditivity, and product measure axioms. It is especially suitable for the lack of historical data or unreliable historical data in which the required data are given subjectively by people, and it can be competent for all work of fuzzy mathematics, so it has been widely recognized and applied. Liu [6] first introduced the uncertain differential equation into the financial discipline, constructed the uncertain stock model, and derived its European option pricing formula. Later, Zhu [7] studied uncertain optimal control with application to a portfolio selection model.

However, the stock price may jump at scheduled or unscheduled times on account of economic crisis, war, announcements of economic statistics, announcements of monetary policy, and the release of major corporate events, etc. often leading to sharp jump fluctuations in securities prices. Sometimes this jump is very harmful. For example, the unpredictability and acuteness of jump in the return of financial assets will often lead to sudden huge losses or even bankruptcy of investment institutions or individuals. More serious jump events will lead to the collapse of stock market and even affect social stability (such as the U.S. stock market crisis in 1929 and 1987, the Asian financial crisis in 1998, the U.S. subprime mortgage crisis in 2007, the financial crisis in 2009, the 2019-nCoV and global inflation in 2021). These factors should be taken into account in the stock price models. Yao [8] first proposed a type of uncertain differential equation driven by both canonical process and renewal process. Yu [9] constructed an uncertain stock model with jumps. Deng [10] presented and dealt with an uncertain optimal control of linear quadratic models with jump by considering the effects of jumps on the optimal policies which is an extension of the model proposed by Zhu [7].

Because randomness and Liu uncertainty simultaneously appear in the financial market, we begin to consider the uncertain stochastic systems with jump. To describe this phenomenon, the concepts of chance distribution, expected value, and variance of uncertain random variables were introduced by Liu [11]. To deal with complex mathematical systems with uncertainty and randomness, Liu [11] designed opportunity theory, and Liu [12] defined opportunity theory as a mathematical methodology composed of uncertainty theory and probability theory. Uncertain stochastic analysis is a branch of pure mathematics that studies the integral and differential of uncertain stochastic processes. Fei [13] considered a class of uncertain backward stochastic differential equations driven by both an m-dimensional Brownian motion and a d-dimensional canonical process with uniform Lipschitzian coefficients. Fei [14] first described a class of uncertain stochastic control systems with Markov switching, and derived an Itô–Liu formula for Markov-modulated processes. Gao and Wu [15] studied the optimal investment strategy problem for a defined contribution pension fund under the jump-uncertain theory framework. Matenda and Chikodza [16] presented and examined uncertain stochastic differential equations and their important characteristics. Liu et al. [17] proposed a computational approach for value at the risk of uncertain random variables. Moreover, an uncertain stochastic differential equation with jump is a differential equation driven by a Brownian motion, a canonical Liu process, and a jump process. Based on the uncertain stochastic differential equation with jump, this paper suggests a stock model with jump for Itô–Liu financial markets. In the real world, there are many systems whose state processes follow uncertainty stochastic processes with jump. Hence, this paper presents an uncertain stochastic optimal control problem with jump. Based on the jump-uncertain stochastic stock model proposed by Matenda and Chikodza [16], using the ideas of jump-uncertain optimal control and uncertain stochastic optimal control in Deng and Zhu [18] and Fei [14], this paper gives the optimal principle of optimal control for a jump-uncertain stochastic system and deduces the optimal equation. In the real financial market, institutional investors not only pay attention to their own performance, but also tend to compare their performance with that of competitors. Take insurance as an example. Investors often choose the insurance companies that rank high to buy insurance. Therefore, insurance companies should not only strive to maximize their terminal wealth, but also widen the wealth gap with other insurance companies and enhance competitiveness. In order to increase their own investment performance, institutional investors play games and compete with each other, resulting in the interaction of investment strategies. At the same time, the differential game theory can study the financial decision-making problem well, and the financial model constructed by it can effectively describe the competition and strategic interaction between institutional investors. There are many articles on the portfolio game under the stochastic system, such as [19–21] and so on. The current portfolio literature under uncertain stochastic systems or uncertain systems with jump seems to ignore the study of optimal interactive decision-making of investors. Therefore, this paper studies the portfolio game problem under the jump-uncertain stochastic systems by using the equation of optimal.

The rest of this paper is structured as follows. Section 2 gives some necessary elementary concepts and theorems about jump-uncertain stochastic theory, proposes an uncertain stochastic optimal control problem with jump, and derives the principle of optimality through Bellman's dynamic programming principle. Then, as its applications, the portfolio game under the uncertain stochastic financial markets with jump is introduced by using the equation of optimality for the constant absolute risk averse (CARA) and constant relative risk averse (CRRA) utility functions in Section 3. Concluding remarks are presented in Section 4.

2. Preliminary

For convenience, we give some useful concepts first. Let Γ be a nonempty set and $\mathcal{L}$ is a σ- algebra; each element $\Lambda \in \mathcal{L}$ is called an event. The relevant definitions and properties of uncertain measure $\mathcal{M}$, probability space $(\Gamma, \mathcal{L}, \mathcal{M})$, and uncertain variable ξ can be referred to in Liu [5]. For concepts and properties of the uncertain random variable, refer to Fei [14], and for concepts and properties of the jump-uncertain variables, refer to Deng and Zhu [10]. In addition, this paper gives the notation of jump-uncertain random variables which are mainly involved.

We now give the concept of expected value for the jump-uncertain random variable.

Definition 1. *Let ξ be an uncertain random variable. Then the expected value is defined by $E[\xi] = E_P[E_U[\xi]]$, where the operators E_P and E_U stand for probability expectation and uncertain expectation, respectively.*

Definition 2. *(Jump Itô–Liu integral) Assume $X(t) = (Y(t), Z(t))$ is a jump-uncertain stochastic process, for any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{N+1} = b$, the mesh is written as $\Delta = \max\limits_{1<i<N} |t_{i+1} - t_i|$. Then the jump Itô–Liu integral of $X(t)$ with respect to $(W(t), C(t), N(t))$ is defined as follows:*

$$\begin{aligned} &\int_a^b X(s)d(W(t), C(t), N(t)) \\ &= \lim_{\Delta\to 0} \sum_{i=1}^{N} \left[Y(t_i)\left(W_{t_i+1} - W_{t_i}\right) + Z(t_i)\left(C_{t_i+1} - C_{t_i}\right) + \left(N_{t_i+1} - N_{t_i}\right)\right]. \end{aligned}$$

In this case, $X(t)$ is called jump Itô–Liu integrable. In particular, when $Y(t) \equiv 0$, $X(t)$ it is called a jump Liu integrable.

Definition 3. *Let $W(t)$, $C(t)$ and $N(t)$ be a one-dimensional Brownian motion, a one-dimensional canonical process, and a V jump-uncertain process with parameters r_1 and r_2, respectively. $f(X(t), t)$, $g_1(X(t), t)$, $g_2(X(t), t)$ and $g_3(X(t), t)$ are some given functions. Consequently, $X(t)$ is*

$$dX(t) = f(X(t), t)dt + g_1(X(t), t)dW(t) + g_2(X(t), t)dC(t) + g_3(X(t), t)dN(t), \quad (1)$$

a jump-uncertain stochastic differential equation.

Theorem 1. *If the coefficients of jump-uncertain stochastic differential Equation (1) satisfies*

$$|f(t,x) - g_1(t,y)| + |g_1(t,x) - g_1(t,y)| + |g_2(t,x) - g_2(t,y)| + |g_3(t,x) - g_3(t,y)| \leqslant L(1 + |x-y|), \forall x, y \in R, t > 0,$$

where L is a constant, and then Equation (1) has a unique solution.

Proof of Theorem 1. Similar to the discussion in [14], we omit it here. □

The optimal control model of jump-uncertain stochastic system is given as

$$\begin{cases} dx(t) = f[t, x(t), u(t)]dt + g_1[t, x(t), u(t)]dC(t) \\ \qquad +g_2[t, x(t), u(t)]dW(t) + g_3[t, x(t), u(t)]dN(t), \\ x(0) = x_0, \end{cases} \tag{2}$$

with initial condition x_0. The performance index function is

$$J(t,x) = E\left\{ \int_0^T g[t, x(t), u(t)]dt + Q(x(T)) \right\}, \tag{3}$$

where $f[t, x(t), u(t)]$ and $g_i[t, x(t), u(t)], i = 1, 2, 3$ are differentiable, $g[t, x(t), u(t)] \geqslant 0$ and $Q(x(T)) \geqslant 0$ are instantaneous utility function and terminal utility functional.

The so-called jump-uncertain stochastic optimal control problem means that the player looks for an optimal control strategy under the jump-uncertain stochastic dynamic system which is driven by Equation (2), to optimize the payment function (3).

Next, we explore jump uncertain stochastic optimal control problem. First, we derive the principle of optimality.

Theorem 2. *(Principle of optimality) For any $(t,x) \in [0,T] \times R^n$, we have*

$$V(t,x) = \inf_{u \in \mathcal{U}} E[g(t, x(t), u(t))\Delta t + V(t + \Delta t, x + \Delta x) + o(\Delta t)], \tag{4}$$

where $x(t) + \Delta x(t) = x_{t+\Delta t}$.

Proof of Theorem 2. Similar to Theorem 3.4 in Deng and Zhu [18] and Theorem 2 in Fei [14], we omit it here. □

Let $C^{1,2}([0,T] \times R^n)$ denote all functions $V(t,x)$ on a finite horizon $[0,T]$ that are continuously differentiable about t, and continuously twice differentiable about x. If $V(t,x) \in C^{1,2}([0,T] \times R^n)$, define operators $\mathcal{L}V(t,x)$, by

$$\mathcal{L}V(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f + \frac{1}{2}\frac{\partial^2 V}{\partial x^2} g_2^2 + \frac{3 - r_1 - r_2}{4} g_3 \frac{\partial V}{\partial x}. \tag{5}$$

In what follows, we give the optimal equation of the optimal control problem.

Theorem 3. *$V(t,x)$ is a solution of the following Hamilton–Jacobi–Bellman(HJB) equation*

$$\inf_{u \in \mathcal{U}} \{\mathcal{L}V(t,x) + g[t, x(t), u(t)]\} = 0, \tag{6}$$

with the terminal condition $V(T,x) = Q(T, x(T))$.

Before giving the proof of Theorem 3, we give the following Lemma 1.

Lemma 1. *Denote $\xi = b\varsigma + d\eta + cw$, where $\varsigma = \Delta C(t)$, $\eta = \Delta V(t)$, $w = \Delta W(t)$ and $b, d, c \in \mathbb{R}$. Let ς, η and w are independent. For any real number a, we have*

$$E\left[a\xi + \xi^2\right] = \frac{ad(3 - r_1 - r_2)}{4}\Delta t + c^2\Delta(t). \tag{7}$$

Proof of Lemma 1. To begin with, we have

$$\begin{aligned} & a\xi + \xi^2 \\ = & a(b\varsigma + d\eta + cw) + (b\varsigma + d\eta + cw)^2 \\ = & a(b\varsigma + d\eta + cw) + b^2\varsigma^2 + d^2\eta^2 + c^2w^2 + 2b\varsigma d\eta + 2d\eta cw + 2b\varsigma cw \\ \geqslant & a(b\varsigma + d\eta + cw) + c^2w^2 + 2b\varsigma d\eta + 2d\eta cw + 2b\varsigma cw. \end{aligned} \tag{8}$$

Because ς,η, w and w^2 are independent of each other, we get

$$\begin{aligned} & E\left[a(b\varsigma + d\eta + cw) + c^2w^2 + 2b\varsigma d\eta + 2d\eta cw + 2b\varsigma cw\right] \\ = & abE[\varsigma] + adE[\eta] + acE[w] + c^2E\left[w^2\right] + 2bdE[\varsigma]E[\eta] + 2cdE[w]E[\eta] + 2bdE[\varsigma]E[w]. \end{aligned}$$

It follows from the Definition 6 in [6] that $E(\varsigma) = 0$. According to the Definition 3.3 in [10], $E(\eta) = \frac{(3-r_1-r_2)}{4}\Delta t$ is obtained. Then the Equation (8) becomes

$$E\left[a\xi + \xi^2\right] \geqslant \frac{ad(3 - r_1 - r_2)}{4}\Delta t + c^2\Delta(t). \tag{9}$$

At the same time

$$\begin{aligned} & a\xi + \xi^2 = a(b\varsigma + d\eta + cw) + (b\varsigma + d\eta + cw)^2 \\ = & a(b\varsigma + d\eta + cw) + b^2\varsigma^2 + d^2\eta^2 + c^2w^2 + 2b\varsigma d\eta + 2d\eta cw + 2b\varsigma cw \\ = & \left(ab\varsigma + b^2\varsigma^2\right) + \left(ad\eta + d^2\eta^2\right) + \left(acw + c^2w^2\right) + 2b\varsigma d\eta + 2d\eta cw + 2b\varsigma cw, \end{aligned}$$

where ς,η, w, $ab\varsigma + b^2\varsigma^2$, $ad\eta + d^2\eta^2$ and $acw + c^2w^2$ are independent, so we get

$$\begin{aligned} E\left[a\xi + \xi^2\right] = & E\left[ab\varsigma + b^2\varsigma^2\right] + E\left[ad\eta + d^2\eta^2\right] + E\left[acw + c^2w^2\right] \\ & + 2bdE(\varsigma)E(\eta) + 2cdE(\eta)E(w) + 2bcE(\varsigma)E(w). \end{aligned}$$

According to the Theorem 3.4 in [18], we get $E\left[ab\varsigma + b^2\varsigma^2\right] = o(\Delta(t))$ and $E\left[ad\eta + d^2\eta^2\right] \leqslant \frac{ad(3-r_1-r_2)}{4}\Delta t$. From the properties of standard Brownian motion, we have $E\left[acw + c^2w^2\right] = E\left[\left(\frac{1}{2}a + cw\right)^2 - \frac{1}{4}a^2\right] = c^2\Delta(t)$. Hence we can get

$$E\left[a\xi + \xi^2\right] \leqslant \frac{ad(3 - r_1 - r_2)}{4}\Delta t + c^2\Delta(t). \tag{10}$$

Combining inequality (9) and inequality (10), the Equation (7) was obtained. □

The proof of Theorem 3 is given below.

Proof of Theorem 3. Similar to the Theorem 5.1 in [10], by using the Taylor series expansion, we get

$$\begin{aligned} V(t + \Delta t, x + \Delta x) = & V(t, x) + V_t(t, x)\Delta t + V_x(t, x)\Delta x + \frac{1}{2}V_{tt}(t, x)\Delta t^2 \\ & + \frac{1}{2}V_{xx}(t, x)\Delta x^2 + V_{tx}(t, x)\Delta x\Delta t + o(\Delta t). \end{aligned} \tag{11}$$

Substituting Equation (11) into Equation (6) yields

$$\begin{aligned} 0 = \inf_{u \in \mathcal{U}} & E[g(t, x(t), u(t))\Delta t + V_t(t,x)\Delta t + V_x(t,x)\Delta x \\ & + \frac{1}{2}V_{tt}(t,x)\Delta t^2 + \frac{1}{2}V_{xx}(t,x)\Delta x^2 + V_{tx}(t,x)\Delta x \Delta t + o(\Delta t). \end{aligned} \tag{12}$$

Suppose $\bar{\xi}$ is a jump stochastic uncertain variable such that $\Delta x = \bar{\xi} + f\Delta t$. It follows from Equation (12) that

$$\begin{aligned} 0 &= \inf_{u \in \mathcal{U}} \{ g(t, x(t), u(t))\Delta t + V_t(t,x)\Delta t + V_x(t,x) f \Delta t \\ & \quad + E\left[(V_x(t,x) + V_{xx}(t,x) f \Delta t + V_{tx}(t,x)\Delta t)\bar{\xi} + \frac{1}{2}V_{xx}(t,x)\bar{\xi}^2 \right] + o(\Delta t) \} \\ &= \inf_{u \in \mathcal{U}} [g(t, x(t), u(t))\Delta t + V_t(t,x)\Delta t + V_x(t,x) f \Delta t \\ & \quad + E\left(a_1 \bar{\xi} + b_1 \bar{\xi}^2 \right) + o(\Delta t)], \end{aligned} \tag{13}$$

where $a_1 = V_x(t,x) + V_{xx}(t,x) f \Delta t + V_{tx}(t,x)\Delta t, b_1 = \frac{1}{2}V_{xx}(t,x)$. According to the jump-uncertain stochastic differential Equation (2), $\bar{\xi} = \Delta x - f\Delta t = g_1(t, x(t), u(t))dW(t) + g_2(t, x(t), u(t))dC(t) + g_3(t, x(t), u(t))dN(t)$ is a jump-uncertain stochastic variable. Lemma 1 implies that Equation (13) becomes

$$\begin{aligned} 0 = \inf_{u \in \mathcal{U}} & [g(t, x(t), u(t))\Delta t + V_t(t,x)\Delta t + V_x(t,x) f \Delta t \\ & + \frac{1}{2}V_{xx}(t,x) g_2^2 \Delta t + V_x(t,x) g_3 \frac{(3 - r_1 - r_2)}{4}\Delta t + o(\Delta t)]. \end{aligned} \tag{14}$$

Dividing Equation (14) by Δt, and letting $\Delta t \to 0$, we can obtain the result (6). The Theorem 3 is proved. □

3. Jump-Uncertain Stochastic Financial Market

3.1. Financial Market

3.1.1. Model Formulation

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ be a complete filtered probability space with a filter $\{\mathcal{F}_t\}_{t\in[0,T]}$ satisfying the usual conditions.

3.1.2. Asset Price Process

We consider a financial market composed of one bond and one stock, where the risk-free money market account for dynamics

$$dS_0(t) = rS_0(t)dt, \quad S_0(0) = 1,$$

where r is risk-free interest rate. The price of the stock $S(t)$ satisfies the following jump-uncertain stochastic differential equation, that is jump-uncertain B-S model (see [16]),

$$dS(t) = S(t)[\mu dt + \sigma_1 dW(t) + \sigma_2 dC(t) + \lambda dN(t)], \quad S(0) = s_0,$$

where μ is appreciation of the stock price. σ_1 and σ_2 represent the volatility of stock price in the stochastic process and the volatility of the stock price in uncertain process, respectively, λ represents the volatility of the jump process, $N(t)$, $W(t)$ and $C(t)$ are independent jump-uncertain processes, Brownian motion, and canonical Liu processes.

3.1.3. Investors' Wealth Process

The proportions of money invested in the cash and stock at time t of investor k, $k \in \{1, 2\}$ are denoted by $X_k^{\pi_k}(t) - \pi_k(t)$ and $\pi_k(t)$, respectively. In addition, there are no transaction costs or taxes in the market, and short buying is also allowed. Then, the wealth $X_k^{\pi_k}(t)$ of the investor k is as follows

$$\begin{cases} dX_k^{\pi_k}(t) = \big(X_k^{\pi_k}(t) - \pi_k(t)\big)\dfrac{dS_0(t)}{S_0(t)} + \pi_k(t\dfrac{dS(t)}{S(t)} \\ = r\big(X_k^{\pi_k} - \pi_k(t)\big)(t)dt + \pi_k(t)[\mu dt + \sigma_1 dW(t) + \sigma_2 dC(t) + \lambda dN(t)], \\ X_k^{\pi_k}(0) = x_{k0} > 0, \end{cases}$$

where x_{k0} represents the initial wealth of investor k.

3.2. The Equilibrium Investment Strategies for CARA and CRRA Utility Functions

A non-zero-sum jump-uncertain stochastic differential game model with two investors who are competing with each other is established.

Denote $Z_k^{(\pi_k,\pi_m)}(t) = X_k(t) - \lambda_k X_m(t)$, and it readily follows that

$$\begin{aligned} dZ_k^{(\pi_k,\pi_m)}(t) &= dX_k(t) - \lambda_k dX_m(t) \\ &= \big[rZ_k^{\pi_k}(t) + (\mu - r)(\pi_k(t) - \lambda_k\pi_m(t))\big]dt \\ &\quad +(\pi_k(t) - \lambda_k\pi_m(t))[\sigma_1 dW(t) + \sigma_2 dC(t) + \lambda dN(t)], \end{aligned} \tag{15}$$

with $Z_k^{(\pi_k,\pi_m)}(0) = x_{k0} - \lambda_k x_{m0}$.

For convenience, $Z_k^{(\pi_k,\pi_m)}(t)$ is abbreviated as z_k. Investor k has a utility function denoted as $U_k : \mathbb{R}^+ \to \mathbb{R}$, where U_k is assumed to be increasing, strictly concave, and satisfies Inada conditions, i.e.,

$$U_k'(0+) = \lim_{z\to 0^+} U_k'(z_k) = +\infty, U_k''(+\infty) = \lim_{z\to\infty} U_k'(z_k) = 0.$$

Definition 4. *(Admissible strategy). Equilibrium investment strategies $\pi_1(t)$ and $\pi_2(t)$ are said to be admissible if the following conditions are satisfied:*

(i) $\pi_1(t)$ *and* $\pi_2(t)$ *are all* $\mathcal{F}_t-$ *measurable,*

(ii) $E\int_0^T \|\pi_k(t)\|^2 dt < \infty$,

(iii) *jump-uncertain stochastic differential Equation (15) has a unique solution* $\left\{Z_k^{\pi_k}(t)\right\}_{t\in[0,T]}$ *for all* $(t, z_k) \in O := [0, T] \times \mathbb{R}$.

Assume that the set of all admissible strategies is denoted by $\Pi_k, k = 1, 2$.

Similar to [16,19], we assume that the objective of investor k is to maximize the expected utility of his performance relative to his competitor at the terminal time T. That is, the optimization problems of investor k are

$$\sup_{\pi_k\in\Pi_k} E\left[U_k\left(X_k^{\pi_k}(T) - \lambda_k X_m^{\pi_m^*}(T)\right) \middle| Z_k^{(\pi_k,\pi_m^*)}(t) = z_k\right], \tag{16}$$

for $m \neq k \in \{1, 2\}$, $\lambda_k \in [0, 1]$ measures the sensitivity of investor k to the performance of his competitor.

According to [21], the problem of (16) can be transformed into the following non-zero-sum game problem:

Problem 1. *Find a Nash equilibrium* $(\pi_1^*, \pi_2^*) \in (\Pi_1, \Pi_2)$ *such that*

$$\begin{aligned} E\left[U_1\left(X_1^{\pi_1}(T) - \lambda_1 X_2^{\pi_2^*}(T)\right)\right] &\leqslant E\left[U_1\left(X_1^{\pi_1^*}(T) - \lambda_1 X_2^{\pi_2^*}(T)\right)\right], \\ E\left[U_2\left(X_2^{\pi_2}(T) - \lambda_2 X_1^{\pi_1^*}(T)\right)\right] &\leqslant E\left[U_2\left(X_2^{\pi_2^*}(T) - \lambda_2 X_1^{\pi_1^*}(T)\right)\right]. \end{aligned}$$

For $Z_k^{(\pi_k,\pi_m)}(t)$, where $0 \leqslant t \leqslant T$ and $m \neq k \in \{1,2\}$

$$W_k^{(\pi_k,\pi_m^*)}(t,z_k) \underset{\Delta}{=} \sup_{\pi_k \in \Pi_k} \mathrm{E}\Big[U_k\Big(X_k^{\pi_k}(T) - \lambda_k X_m^{\pi_m^*}(T)\Big)\Big],$$

is the value function.

Theorem 4. *(Verification Theorem) Let $J_k(t,z_k) \in C^{1,2}(\mathrm{O})$ be a function that satisfies a quadratic growth condition, i.e., there exists $M > 0$ such that $|J_k(t,z_k)| \leqslant M\Big(1 + |z_k|^2\Big)$, for all $(t,z_k) \in \mathrm{O}(k=1,2)$. For convenience, we abbreviate $J_k(t,z_k) \triangleq J_k$, define*

$$\mathcal{A}^{(\pi_k,\pi_m^*)} J_k \triangleq \frac{\partial J_k}{\partial t} + [rz_k + (\mu - r)(\pi_k(t) - \lambda_k \pi_m^*(t))]\frac{\partial J_k}{\partial z_k} + \frac{3(3 - r_1 - r_2)}{4}(\pi_k(t) - \lambda_k \pi_m^*(t))\lambda \frac{\partial J_k}{\partial z_k} + \frac{1}{2}(\pi_k(t) - \lambda_k \pi_m^*(t))^2 \sigma_1^2 \frac{\partial^2 J_k}{\partial z_k^2},$$

where

$$\pi_k^* = \arg\max_{\pi_k \in \Pi_k}\left\{\frac{\partial J_k}{\partial t} + [rz_k + (\mu - r)(\pi_k(t) - \lambda_k \pi_m^*(t))]\frac{\partial J_k}{\partial z_k} + \frac{3(3 - r_1 - r_2)}{4}(\pi_k(t) - \lambda_k \pi_m^*(t))\lambda \frac{\partial J_k}{\partial z_k} + \frac{1}{2}(\pi_k(t) - \lambda_k \pi_m^*(t))^2 \sigma_1^2 \frac{\partial^2 J_k}{\partial z_k^2}\right\}.$$

(1) Suppose that

$$-\frac{\partial J_k(t,z_k)}{\partial t} - \sup_{\pi_k \in \Pi_k}\left\{\mathcal{A}^{(\pi_k,\pi_m^*)} J_k(t,z_k)\right\} \geqslant 0,$$
$$J_k(T,z_k) \geqslant U_k(z_k),$$

then $J_k \geqslant W_k$.

(2) for $(\pi_k^*, \pi_m^*) \in (\Pi_k, \Pi_m)$, it follows that

$$-\frac{\partial J_k(t,z_k,l,v)}{\partial t} - \mathcal{A}^{(\theta_k^*,\theta_m^*)} J_k(t,z_k,l,v) = 0,$$

and the jump-uncertain stochastic differential Equation (15) admits a unique solution, then $J_k = W_k$, (π_k^*, π_m^*) are equilibrium strategies.

Proof of Theorem 4. This proof is similar to [22] of Theorem 1, we omit it here. □

3.2.1. Equilibrium Solution for the Exponential Utility Function

Because the exponential utility function was proposed, it has been widely used in the optimization pricing and other fields. It not only has excellent properties such as additive/smoothness, but also can well describe its utility changes in the case of large fluctuations in wealth. This section discusses how the investors are CARA agents, i.e., each agent has an exponential utility function, and we can obtain the explicit value function and equilibrium strategies for each investor.

Assume that each investor has an exponential utility function, i.e.,

$$U_k(z_k) = -\frac{e^{\kappa_k z_k}}{\kappa_k},$$

where $\kappa_k > 0$ is the risk aversion coefficient of investor k.

Next, we give the equilibrium investment strategies and value functions of the non-zero-sum game for jump-uncertain stochastic systems under exponential utility, as shown in the following Theorem 5.

Theorem 5. *The equilibrium investment strategies and value functions take the form*

$$\pi_1^*(t) = \frac{(\mu - r) + \frac{3(3-r_1-r_2)}{4}\lambda}{\sigma_1^2}\left(\frac{1}{\kappa_1 f_1(t)} + \frac{\lambda_1}{\kappa_2 f_2(t)}\right),$$

$$\pi_2^*(t) = \frac{(\mu - r) + \frac{3(3-r_1-r_2)}{4}\lambda}{\sigma_1^2(1-\lambda_2\lambda_1)}\left(\frac{\lambda_2}{\kappa_1 f_1(t)} + \frac{1}{\kappa_2 f_2(t)}\right),$$

$$J_k(z_k) = -\frac{e^{\kappa_k[z_k f_k(t) + P_k(t)]}}{\kappa_k}e^{-\beta t},$$

where

$$f_k(t) = e^{r(T-t)}, P_k(t) = \frac{\kappa_k m}{\beta}\left(e^{\frac{\beta}{\kappa_k}(T-t)} - 1\right), m = \frac{1}{2}\frac{\left((\mu - r) + \frac{3(3-r_1-r_2)}{4}\lambda\right)^2}{\sigma_1^2\kappa_k}.$$

Proof of Theorem 5. The HJB equation that satisfies the jump-uncertain stochastic differential game of Problem 1 is

$$\begin{aligned}0 = \frac{\partial J_k}{\partial t} + \sup_{\pi_k \in \Pi_k}\Bigg\{&[rz_k + (\mu - r)(\pi_k(t) - \lambda_k\pi_m^*(t))]\frac{\partial J_k}{\partial z_k} \\ &+ \frac{3(3-r_1-r_2)}{4}(\pi_k(t) - \lambda_k\pi_m^*(t))\lambda\frac{\partial J_k}{\partial z_k} + \frac{1}{2}(\pi_k(t) - \lambda_k\pi_m^*(t))^2\sigma_1^2\frac{\partial^2 J_k}{\partial z_k^2}\Bigg\}.\end{aligned} \tag{17}$$

By using the first-order conditions of (17), we have

$$\pi_k(t) - \lambda_k\pi_m^*(t) = -\frac{(\mu - r)\frac{\partial J_k}{\partial z_k} + \frac{3(3-r_1-r_2)}{4}\lambda\frac{\partial J_k}{\partial z_k}}{\sigma_1^2\frac{\partial^2 J_k}{\partial z_k^2}}. \tag{18}$$

Substituting (18) into (17), we obtain

$$\frac{\partial J_k}{\partial t} + rz_k\frac{\partial J_k}{\partial z_k} - \frac{1}{2}\frac{\left((\mu - r)\frac{\partial J_k}{\partial z_k} + \frac{3(3-r_1-r_2)}{4}\lambda\frac{\partial J_k}{\partial z_k}\right)^2}{\sigma_1^2\frac{\partial^2 J_k}{\partial z_k^2}} = 0. \tag{19}$$

To solve the Equation (19), we conjecture that

$$J_k(z_k) = -\frac{e^{\kappa_k[z_k f_k(t) + P_k(t)]}}{\kappa_k}e^{-\beta t},$$

where β is utility discount rate.

Then we obtain

$$\frac{\partial J_k}{\partial t} = J_k\kappa_k\left[z_k\dot{f}_k(t) + \dot{P}_k(t)\right] - J_k\beta, \frac{\partial J_k}{\partial z_k} = \kappa_k f_k(t)J_k, \frac{\partial^2 J_k}{\partial z_k^2} = \kappa_k^2 f_k^2(t)J_k. \tag{20}$$

Inserting (20) into (17) and simplifying gives

$$z_k\left[\dot{f}_k(t) + rf_k(t)\right] + \dot{P}_k(t) - \frac{\beta}{\kappa_k} - \frac{1}{2}\frac{\left((\mu - r) + \frac{3(3-r_1-r_2)}{4}\lambda\right)^2}{\sigma_1^2\kappa_k} = 0,$$

with the boundary condition $f_k(T) = 0$ and $P_k(T) = 0$.

Thus, we have to solve the following ordinary differential equations

$$\dot{f}_k(t) + rf_k(t) = 0,$$
$$\dot{P}_k(t) - \frac{\beta}{\kappa_k}P_k(t) - \frac{1}{2}\frac{\left((\mu - r) + \frac{3(3-r_1-r_2)}{4}\lambda\right)^2}{\sigma_1^2\kappa_k} = 0.$$

Inserting (20) into (18) and simplifying gives

$$\pi_k(t) - \lambda_k\pi_m^*(t) = \frac{(\mu - r) + \frac{3(3-r_1-r_2)}{4}\lambda}{\kappa_k f_k(t)\sigma_1^2}.$$

Then Theorem 5 is proved. □

3.2.2. Equilibrium Solution for the Power Utility Function

In the portfolio of securities, each investor has his own degree of risk and preference for return. The power utility function is an important and typical risk-aversion function. Therefore, it is necessary to obtain the equilibrium investment strategy under the power utility function.

Theorem 6. *In the jump-uncertain stochastic portfolio game system under the power utility function, the equilibrium investment strategies and value functions are*

$$\pi_1^*(t) = -\frac{(\mu - r) + \frac{3(3-r_1-r_2)}{4}\lambda}{\sigma_1^2(1 - \kappa_1\kappa_2)}\left(\frac{z_1}{(\kappa_1 - 1)} + \frac{\kappa_1\kappa_2}{(\kappa_2 - 1)}\right),$$
$$\pi_2^*(t) = -\frac{(\mu - r) + \frac{3(3-r_1-r_2)}{4}\lambda}{\sigma_1^2(1 - \kappa_1\kappa_2)}\left(\frac{z_1\kappa_2}{(\kappa_1 - 1)} + \frac{z_2}{(\kappa_2 - 1)}\right),$$

$$J_k(z_k) = e^{-\beta t}\frac{z_k^{\kappa_k}}{\kappa_k}A_k(t),$$

where

$$A_k(t) = e^{m(T-t)},$$
$$m = \kappa_k r - \beta - \frac{\left((\mu - r) + \frac{3(3-r_1-r_2)}{4}\lambda\right)\kappa_k}{2\sigma_1^2(\kappa_k - 1)}.$$

Proof of Theorem 6. Assume that the value function is

$$J_k(z_k) = e^{-\beta t}\frac{z_k^{\kappa_k}}{\kappa_k}A_k(t),$$

with boundary condition given by $A_k(T) = 0$. Further, the partial derivatives of $J_k(z_k)$ with respect to t and z_k are as follows

$$\frac{\partial J_k}{\partial t} = -\beta e^{-\beta t}\frac{z_k^{\kappa_k}}{\kappa_k}A_k(t) + e^{-\beta t}\frac{z_k^{\kappa_k}}{\kappa_k}\dot{A}_k(t), \frac{\partial J_k}{\partial z_k} = e^{-\beta t}{z_k}^{\kappa_k - 1}A_k(t),$$
$$\frac{\partial^2 J_k}{\partial z_k^2} = (\kappa_k - 1)e^{-\beta t}{z_k}^{\kappa_k - 2}A_k(t). \tag{21}$$

Plugging (21) into (17), after some simple calculations, we derive

$$\dot{A}_k(t) + \left[\kappa_k r - \beta - \frac{\left((\mu - r) + \frac{3(3-r_1-r_2)}{4}\lambda\right)^2 \kappa_k}{2\sigma_1^2(\kappa_k - 1)}\right] A_k(t) = 0. \quad (22)$$

Equation (22) can be decomposed into the following equation

$$A_k(t) = e^{m(T-t)},$$
$$m = \kappa_k r - \beta - \frac{\left((\mu - r) + \frac{3(3-r_1-r_2)}{4}\lambda\right)\kappa_k}{2\sigma_1^2(\kappa_k - 1)}.$$

Putting (21) into (18), we get

$$\pi_k(t) - \kappa_k \pi_m^*(t) = -\frac{(\mu - r) + \frac{3(3-r_1-r_2)}{4}\lambda}{\sigma_1^2(\kappa_k - 1)} z_k.$$

Therefore, the proof is completed. □

4. Conclusions

Based on the concepts of standard Brownian motion, canonical process, and the jump-uncertain process, this paper provides an uncertain stochastic optimal control model with jump. The principle of optimality and the equation of optimality for uncertain stochastic optimal control with jump are obtained. As the applications of equation of optimality, two investors' game model was discussed. This paper formulates a framework where investors have relative wealth concerns, i.e., investors derive utility not only from maximizing their wealth but also from performing well relative to their peers. According to the dynamic programming principle, the equilibrium strategies and value functions are analytically derived. The results show that the equilibrium investment strategies under exponential utility have nothing to do with the state of the system, but only with the parameters of the system state, and the equilibrium investment strategies under power utility have something to do with the state. The subjective uncertainty and objective randomness of investors will appear in a system at the same time, so the equilibrium investment strategies will be more realistic and better than the portfolio model under the stochastic system.

Author Contributions: Conceptualization, C.W. and L.Y.; methodology, C.Z.; validation, L.Y.; formal analysis, C.W.; writing—original draft preparation, C.W. and L.Y.; writing—review and editing, C.Z.; visualization, C.W., C.Z. and L.Y.; supervision, C.W., C.Z. and L.Y. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the National Natural Science Foundation of China grant number 71571053.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Zadeh, L.A. Fuzzy sets. *Inf. Control.* **1965**, *8*, 338–353. [CrossRef]
2. Zadeh, L.A. Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets Syst.* **1978**, *1*, 3–28. [CrossRef]
3. Liu, B.; Liu, Y.K. Expected value of fuzzy variable and fuzzy expected value models. *IEEE Trans. Fuzzy Syst.* **2002**, *4*, 445–450.
4. Liu, B.; Peng, J. Trustworthiness theory-axiomatic foundation of fuzzy sets. In Proceedings of the Second Annual Conference on Uncertain Systems, Busan, Korea, 2–6 November 2004; pp. 7–17.
5. Liu, B. *Uncertainty Theory*, 2nd ed.; Springer: Berlin/Heidelberg, Germany, 2007.
6. Liu, B. Some research problems in uncertainty theory. *J. Uncertain Syst.* **2009**, *3*, 3–10.
7. Zhu, Y. Uncertain optimal control with application to a portfolio selection mode. *Cybern. Syst.* **2010**, *41*, 535–547. [CrossRef]

8. Yao, K. Uncertain calculus with the renewal process. *Fuzzy Optim. Decis. Mak.* **2012**, *11*, 285–297. [CrossRef]
9. Yu, X. A stock model with jumps for uncertain markets. *Int. J. Uncertain. Fuzziness Knowl.-Based Syst.* **2012**, *20*, 421–432. [CrossRef]
10. Deng, L; Zhu Y. Uncertain optimal control of linear quadratic models with jump. *Math. Comput. Model.* **2012**, *10*, 2432–2441. [CrossRef]
11. Liu, Y. Uncertain random variables: A mixture of uncertainty and randomness. *Soft Comput.* **2013** *17*, 625–634. [CrossRef]
12. Liu, Y. Uncertain random programming with applications. *Fuzzy Optim. Decis. Mak.* **2013**, *12*, 153–169. [CrossRef]
13. Fei, W. On existence and uniqueness of solutions to uncertain backward stochastic differential equations. *Appl. Math. J. Chin. Univ.* **2014**, *29*, 53–66. [CrossRef]
14. Fei, W. Optimal control of uncertain stochastic systems with Markov switching and Its applications to portfolio decisions. *Cybern. Syst.* **2014**, *45*, 69–88. [CrossRef]
15. Gao, J.; Wu, Y. Optimal pension investment strategy problem with uncertain jump diffusion process. *J. Inn. Mong. Norm. Univ. (Nat. Sci. Ed.)* **2018**, *47*, 108–113.
16. Matenda, F.; Chikodza, E. A stock model with jumps for Itô–Liu financial markets. *Soft Comput.* **2019**, *23*, 4065–4080. [CrossRef]
17. Liu, Y.; Ahmadzade, H.; Farahikia, M. Portfolio selection of uncertain random returns based on value at risk. *Soft Comput.* **2021**, *25*, 6339–6346. [CrossRef]
18. Deng, L.; Zhu, Y. Uncertain Optimal Control with Jump. *ICIC Express Lett. Int. J. Res. Surv. Part Appl.* **2012**, *9*, 1453–1468.
19. Yang, L.; Zhu, H.; Zhang, C. Portfolio game with liabilities based on the Heston stochastic volatility model. *Oper. Res. Manag. Sci.* **2020**, *29*, 27–34.
20. Wu, H.; Ma, C. Non-zero-sum stochastic differential portfolio game under a constant elastic of variance model. *Syst. Eng.* **2015**, *33*, 738181.
21. Bensoussan, A.; Siu, C.C.; Yam, S.C.P.; Yang, H. A class of non-zero-sum stochastic differential investment and reinsurance games. *Automatica* **2014**, *50*, 2025–2037. [CrossRef]
22. Zeng X.; Taksar, M. A stochastic volatility model and optimal portfolio selection. *Quant. Financ.* **2013**, *13*, 1547–1558. [CrossRef]