symmetry MDPI

*Article*

# Big Bang and Topology

**Torsten Asselmeyer-Maluga [1,*], Jerzy Król [2] and Alissa Wilms [3]**

[1] German Aerospace Center (DLR), Rosa-Luxemburg-Str. 2, 10178 Berlin, Germany
[2] Cognitive Science and Mathematical Modelling Chair, University of Information Technology and Management, ul. Sucharskiego 2, 35-225 Rzeszów, Poland
[3] Physics Department, Freie Universität Berlin, Arnimallee 14, 14195 Berlin, Germany
* Correspondence: torsten.asselmeyer-maluga@dlr.de

**Abstract:** In this paper, we discuss the initial state of the universe at the Big Bang. By using the ideas of Freedman in the proof of the disk embedding theorem for 4-manifolds, we describe the corresponding spacetime as a gravitational instanton. The spatial space is a fractal space (wild embedded 3-sphere). Then, we construct the quantum state from this fractal space. This quantum state is part of the string algebra of Ocneanu. There is a link between the Jones polynomial and Witten's topological field theory. Using this link, we are able to determine the physical theory (action) as the Chern–Simons functional. The gauge fixing of this action determines the foliation of the spacetime and the smoothness properties. Finally, we determine the quantum symmetry of the quantum state to be the enveloped Lie algebra $U_q(sl_2(\mathbb{C}))$, where $q$ is the fourth root of unity.

**Keywords:** topological quantum state of the Big Bang; smooth exotic K3 and $R^4$; TQFT; operator algebras

check for updates

**Citation:** Asselmeyer-Maluga, T.; Król, J.; Wilms, A. Big Bang and Topology. *Symmetry* **2022**, *14*, 1887. https://doi.org/10.3390/sym14091887

Academic Editor: Abraham A. Ungar

Received: 13 July 2022
Accepted: 6 September 2022
Published: 9 September 2022



## 1. Introduction

What was the initial state of the universe? This question is fundamental to understanding the further development of the universe. However, the usual extrapolation techniques fail here. Therefore, an answer to this question seems to be of global nature. As a mathematical method, therefore, the field of topology would immediately suggest its usefulness here, precisely because such questions of global nature are answered for spaces within the field.

From this motivation, we have therefore awoken the construction of spacetime by means of topological methods as a starting point. Here, of course, one can only start with rather general assumptions. All the more astonishing is the result that only a few quite natural assumptions are sufficient to arrive at an unambiguous result. The discussion of this exact approach can be found in the next section of the paper. Thereby, the part of spacetime can also be identified, which belongs to the Big Bang itself. In the search for this part, topology again plays an important role. Special solutions, e.g., gravitational instantons, represent the tunnel transition to the initial state of the universe (see [1,2]). Finally, we obtain the rather obvious solution that the 4-disk represents this formation of the initial state and the 3-sphere is the initial state. However, one would expect that such a state would be a quantum state and not just a "classical" 3-manifold. Using our work on quantization by introducing wild embeddings [3], we simply obtain the quantum state by the transition to the wildly embedded 3-sphere (see Appendix A for a short description of this work). Here, we have to explain the concept of a wild embedding. In general, an embedding is a map $f : N \to M$ so that $N$ and $f(N)$ are topologically equivalent (homeomorphic). The difference between a tame and wild embedding is given by the description of the image. If the image $f(N)$ can be described by a finite amount of information (polygons, triangulation etc.), then the embedding is tame. Examples are the usual knots (as embeddings $S^1 \to \mathbb{R}^3$). In contrast, a wild embedding consists of an infinite collection of substructures. Examples are the Fox–Artin wild knot or Alexander's horned sphere. A wild embedding is also given by iterating a structure as in the case of a fractal. In [3], we showed that a wild embedding is a geometric/topological expression for a quantum state. Therefore, we will identify quantum states with wild embeddings and call it a fractal space. Here, we will consider the quantum 3-sphere as a wildly embedded 3-sphere (or a fractal 3-sphere). The description of the wildly embedded 3-sphere is given in Section 3 and its formation in Section 4, using Freedman's idea, which he used in solving the disk embedding problem in dimension four [4–6]. Here, the 4-disk is covered by special manifolds (Casson handle) with a tree-like structure. The description of this structure leads to the string algebra of Ocneanu [7], closely related to the Jones polynomial [8,9] of knot theory. It is, of course, a stroke of luck that Witten [10,11] has developed a topological quantum field theory exactly for this invariant. Thus, we obtain exactly the physical theory that describes the formation of the quantum state. The underlying action is the Chern–Simons invariant, and the observable is the Wilson line along the knot. In a special gauge (axial gauge for the case of a light cone directed into the future), we obtain a relation to the foliation of spacetime and later to the Seiberg–Witten theory. These many interrelations to known theories and approaches show the complexity of the approach. In our forthcoming work, we will turn to the exact description of the initial state and the implications for the initial distribution of matter and dark matter.

## 2. A (Coherent) Model for the Spacetime

In this section, we will describe the model of the spacetime seen as the space of spacetime events $\mathcal{M}$. At first, we start with three (more or less) obvious assumptions to restrict the class of spaces $\mathcal{M}$: smooth 4-manifold (we can use the concept of a differential equation for the dynamics), compactness (every sequence of events is an event) and simply-connectedness (every time-like loop can be contracted to maintain causality at least in principle). Then, the spacetime is an open submanifold of $\mathcal{M}$, including examples such as $S^3 \times \mathbb{R}$. To determine $\mathcal{M}$ completely, we need the realization of Ricci-flatness in $\mathcal{M}$, representing the vacuum state (no matter) of general relativity. Together with the other assumptions, $\mathcal{M}$ is the K3 surface, a Calabi–Yau space of two complex dimensions (i.e., a 4-dimensional real manifold). In the following, we will discuss the consequences of this approach. In particular, the K3 surface is a gravitational instanton and using the ideas of Hartle and Hawking, the Big Bang can be understood as a tunneling event induced from a gravitational instanton. We will argue below that the Big Bang is represented by the 4-disk $D^4$ with the initial state $S^3$. Then, the corresponding quantum state must be a fractal space with $S^3-$topology. In our previous work, we obtained a relation between the quantum state and a so-called wildly embedded 3-sphere as fractal space. It is the main result of our argumentation in this section: *the initial state of the universe is the fractal 3-sphere*. The reader who is willing to accept this assumption can proceed to the next section.

There are infinitely many suitable topologies for the spacetime, seen as a 4-manifold, and for the space, seen as a 3-manifold. Of course, there are some heuristics, but they are usually not sufficient for the unique determination of spacetime. Here, we will take a different approach. Why not try to determine the space $\mathcal{M}$ of all possible spacetime-events? Therefore, we start with a definition: let $\mathcal{M}$ be the space of all possible spacetime events, i.e., the set of all spacetime events carrying a manifold structure. Then, a specific physical system or configuration is an embedding of a 3-manifold into $\mathcal{M}$, and a dynamics is an embedding of a cobordism between 3-manifolds (representing the configuration at the initial and end points) into $\mathcal{M}$. Here, we assume implicitly that everything can be geometrically/topologically expressed as submanifolds (see [12,13]). In the following, we will try to discuss this approach and how far one can go. Some heuristic arguments are rather obvious:

1. $\mathcal{M}$ is a smooth 4-manifold,
2. Any sequence of spacetime events has to converge to a spacetime event and
3. Any loop (time-like or not) must be contracted.

A dynamics is known to be a mapping of a spacetime event to a new spacetime event. It is usually a smooth map (differential equations) motivating the first argument. The second argument expresses the fact that any initial spacetime event must converge to a final spacetime event, or the limit of any sequence of spacetime events must converge to a spacetime event. Then, $\mathcal{M}$ is a compact, smooth 4-manifold. The usual or actual spacetime is an open subset of $\mathcal{M}$. The third argument above is motivated to neglect time-like loops in principle. If the underlying spacetime is multiple-connected, then there are loops in the spacetime that cannot be contracted to a point, leading to potential time-like loops. Therefore, a simple-connected spacetime is a necessary condition to avoid closed time-like loops. However, compact spacetime always admits closed time-like loops, see [14]. Therefore, this condition is not sufficient, but the usual (or actual) spacetime is an open subset of $\mathcal{M}$, or the usual spacetime is embedded in $\mathcal{M}$. Then, if the usual spacetime is also simply-connected because of the non-compactness, see [14] again, there are no time-like loops. However, to understand the property 'simple-connectedness', we consider a loop in the spacetime. If this loop cannot contract, then there are two ways or two different curves connecting two different events. By changing the embedding of the curves via a diffeomorphism (this procedure is called isotopy), we can deform one curve to agree with the other curve, or every loop formed by the two curves can be contracted. Therefore, this argument implies that there are no time-like loops, and the non-compactness of the open subset implies causality. Finally, $\mathcal{M}$ is a compact, simply connected, smooth 4-manifold.

The following restrictions of $\mathcal{M}$ will determine the spacetime completely. For that reason, we demand that the equations of general relativity are valid without any restrictions. Then, the vacuum equations are given by

$$R_{\mu\nu} = 0$$

so that we obtain Ricci-flatness. However, as shown in [15,16] and in recent years in [12,13,17], the coupling to matter can be described by a change in the smoothness structure. Therefore, the modification of the smoothness structure will produce matter (or sources of gravity). However, at the same time, we need a smoothness structure that can be interpreted as a vacuum given by a Ricci-flat metric. Therefore, we will demand that

4. $\mathcal{M}$ has to admit a smoothness structure with a Ricci-flat metric representing the vacuum.

Interestingly, these four demands are restrictive enough to determine the topology of $\mathcal{M}$ completely. With the help of Yau's seminal work [18], the K3 surface is the unique compact, simply connected Ricci-flat 4-manifold, and we will obtain that
*$\mathcal{M}$ is topologically equivalent (homeomorphic) to the K3 surface.*
However, it is known by the work of LeBrun [19] that there are non-Ricci-flat smoothness structures. Therefore, in the next step, we will determine the smoothness structure of $\mathcal{M}$. For that purpose, we will present some known results in the differential topology of 4-manifolds (see [20] for details and the construction of the $E_8-$manifold):

- There is a compact, contractible submanifold $A \subset \mathcal{M}$ (called Akbulut cork) so that cutting out $A$ and regluing it (by an involution) will produce a new smoothness structure,
- $\mathcal{M}$ splits topologically into

$$|E_8 \oplus E_8| \# \underbrace{\left(S^2 \times S^2\right) \# \left(S^2 \times S^2\right) \# \left(S^2 \times S^2\right)}_{3(S^2 \times S^2)} = 2|E_8| \# 3(S^2 \times S^2) \tag{1}$$

  two copies of the $E_8-$manifold and three copies of $S^2 \times S^2$ and
- The 3-sphere $S^3$ is a submanifold of $A$.

In [21], we already discussed this case. From the topological point of view, any sum of $E_8-$manifolds and $S^2 \times S^2$ is realized by a closed, simply-connected, topological 4-manifold but not all topological 4-manifolds are smooth manifolds. To clarify this point, let us consider the 4-manifold, which splits topologically into $p$ copies of the $|E_8|$ manifold and $q$ copies of $S^2 \times S^2$ or

$$p|E_8|\#q\left(S^2 \times S^2\right).$$

Then, this 4-manifold is smoothable for every $q$ but $p = 0$. The first combination for $p \neq 0$ is the pair of numbers $p = 2, q = 3$ (which is the K3 surface). Any other combination ($p = 2, q < 3$ or every $q$ and $p = 1$) is forbidden, as shown by Donaldson [22]. Therefore, the simplest combination of $|E_8|$ and $S^2 \times S^2$ is realized by the K3 surface.

Now we consider the smooth K3 surface, which is Ricci-flat, simply connected and smooth. A main part of the following discussion will be the usage of the smoothness condition. As discussed above, the smoothness structure is determined by the Akbulut cork $A$. Furthermore, as argued above, the smoothness structure is strongly related to the appearance of matter (see [12,13,17]), and this process is strongly connected to the evolution of our cosmos (see [23,24]). This process is known as reheating after the inflationary phase. Therefore, the Akbulut cork (including its embedding) should represent the inflationary phase with reheating. We have already partly discussed this in our works (see [17,25] for the first results in this direction).

The central submanifold determining the smoothness structure is the Akbulut cork $A$, a contractible submanifold with boundary $\partial A$. As shown by Freedman [5], the Akbulut cork is build from a homology 3-sphere, which will become the boundary $\partial A$. The difference to a usual 3-sphere $S^3$ is given by the so-called fundamental group, the equivalence class of closed loops up to deformation (homotopy) with concatenation as the group operation. In principle, one constructs a cobordism between $S^3$ and the homology 3-sphere $\partial A$. All elements of the fundamental group will be killed by adding appropriate disks. In the end, one can add a 4-disk to obtain the full contractible cork $A$. The topology of $\partial A$ depends strongly on the topology of $\mathcal{M}$. In the case of the K3 surface, $\partial A$ is known to be a Brieskorn sphere, precisely the 3-manifold

$$\partial A = \Sigma(2,5,7) = \left\{x, y, z \in \mathbb{C} \,|\, x^2 + y^5 + z^7 = 0 \,|x|^2 + |y|^2 + |z|^2 = 1\right\}.$$

The construction of the smoothness structures is based on the work [26,27]. The smoothness structure depends on the Casson handle (used to construct an exotic $\mathbb{R}^4$ in the cited work). A Casson handle is uniquely determined by a branched tree. Then, the simplest Casson handle is given by an unbranched tree, and we will choose this smoothness structure in the following. The corresponding K3 surface is constructed in [27].

The embedding of the Akbulut cork is essential for the following results. In [23], it was shown that the embedded cork admits a hyperbolic geometry if the underlying K3 surface has an exotic smoothness structure. Additionally, the open neighborhood $N(A)$ of the Akbulut cork in the K3 surface is an exotic $\mathbb{R}^4$, i.e., a space homeomorphic to the Euclidean space $\mathbb{R}^4$ but not diffeomorphic to it. In the following, we will denote this exotic $\mathbb{R}^4$ as $R^4$. One of the characterizing properties of an exotic $\mathbb{R}^4$ (all known examples) is the existence of a compact subset $K \subset R^4$, which cannot be surrounded by any smoothly embedded 3-sphere (and homology 3-sphere bounding a contractible, smooth 4-manifold). However, there is always a topologically embedded 3-sphere, i.e., this 3-sphere is wildly embedded. In [17], we described this wildly embedded 3-sphere explicitly (denoted as $Y_\infty$), and we showed in [3] that this wildly embedded 3-sphere can be understood as a quantum state, i.e., it is the deformation quantization of a tame (or usual) embedding. The notation *wildly embedded* or *wild* is purely mathematical. Instead, we will denote this wild 3-sphere as a *fractal 3-sphere*. However, at first, we will look at the Akbulut cork $A$, which can be decomposed as

$$A = D^4 \cup_{S^3} W(S^3, \partial A) \tag{2}$$

where $W(S^3, \partial A)$ describes a cobordism between the 3-sphere and the boundary $\partial A = \Sigma(2,5,7)$. In [23], we discussed this cobordism $W(S^3, \partial A)$ as the first (inflationary) transition $S^3 \to \partial A$ from the initial state (the 3-sphere) to a non-trivial space (containing matter).

Then, by using the embedding of $A$ into the K3 surface, we identify the 3-sphere (boundary of $D^4$) with the wild 3-sphere $Y_\infty$ (from the open neighborhood $N(A)$), or the initial state of our model of the universe is a fractal 3-sphere (which is a quantum state, see [3,17]). With this identification in mind, we are able to interpret the first transition $W(S^3, \partial A)$ (from the wild 3-sphere to the (classical) non-trivial state $\partial A$) as a decoherence process, see [28]. In [23], we discussed a second transition leading to a cosmological constant. Finally, we have the two transitions

$$S^3 \xrightarrow{cork} \partial A = \Sigma(2,5,7) \xrightarrow{gluing} P\#P \tag{3}$$

where $P$ denotes the Poincare sphere. In this paper, we are interested in the formation of the initial state (the fractal 3-sphere), also called the Big Bang. Using the decomposition (2), this formation is expressed in spacetime via the 4-manifold $D^4$ with the boundary $\partial D^4 = S^3$, the (fractal) 3-sphere. Again, the embedding of $D^4$ into the K3 surface is important, otherwise one will never obtain the fractal 3-sphere as a boundary. Therefore, many properties of the K3 surface go over to $D^4$ by using the embedding.

To describe this embedding, we need the following fact: the K3 surface is a gravitational instanton. We implicitly used this fact above when we constructed a simply-connected, Ricci-flat spacetime (uniquely given by the K3 surface). In general, an instanton is a field configuration, which is interpreted as a tunneling effect between topologically in-equivalent sectors of the vacuum. The term "gravitational instanton" is usually used for 4-manifolds whose Weyl tensor is self-dual and fulfills the Einstein condition $Ric = \Lambda g$. Usually, it is assumed that the metric is asymptotic to the standard metric of Euclidean 4-space. In the case of the K3 surface, there is the phenomenon where gravitational instantons are created by bubbling off a subspace. Here, we recommend the recent publication [29] for the description of this process. To state it more precisely, there is a family of hyperkähler metrics $g_\beta$ on a K3 surface, which collapse to an interval $[0,1]$ in the Gromov–Hausdorff limit ($\beta \to \infty$ with metrics $dt^2$) with Taub-NUT bubbles in the interior and Tian–Yau metrics at the endpoints. For the embedding of $D^4$, we choose the Taub-NUT metric in the (open) neighborhood of the boundary. However, what about the interior of $D^4$? Here, we have to use the elliptic fibration of the K3 surface (as torus bundle over the $S^2$ with singular fibers, see [30]). Then, we can describe the embedded $D^4$ by the Eguchi–Hanson metrics (a gravitational instanton). This metric is a Riemannian metric. Here, the signature of the metrics changes from the Riemannian signature (for $D^4$) to the Lorentzian signature (for $\partial D^4 \times (0,1)$). In a recent publication [31], a gravitational instanton with these properties is constructed. The construction explicitly used the hyperkähler structure ($SU(2)$ holonomy group). The gluing of the instanton solutions can be performed by using the work in [29].

As explained above, the boundary $\partial D^4$ is identified with the wild (or fractal) 3-sphere. Then, the signature change in the metric can be identified with the formation of this fractal 3-sphere. Here, we follow the usual interpretation (Hartle–Hawking and Hawking–Turok see [1,2]) that the gravitational instanton $D^4$ represents the Big Bang (via a tunneling event) leading to the quantum state of the universe. In [3], we showed that a quantum state can be topologically understood as a wildly embedded 3-sphere or a fractal 3-sphere for short. Therefore, we will argue accordingly that the quantum state of the universe (as initial state) is represented by the fractal 3-sphere. In the next section, we will describe this fractal 3-sphere explicitly.

## 3. The Construction of the Fractal 3-Sphere as a Quantum State

In [23–25,32], we described a model for the cosmic evolution, which is in good agreement with current measurements [33,34]. Amazingly, as discussed above, we are able to extrapolate the state at the Big Bang [17,32]: a fractal 3-sphere as a boundary of a 4-disk $D^4$, i.e., a gravitational instanton as a transition (tunneling) to a fractal 3-sphere representing the quantum state [3]. Furthermore, as explained above, this fractal 3-sphere is part of $R^4$, an exotic $\mathbb{R}^4$. Before we start with the construction of the fractal 3-sphere, we will describe the physical ideas behind the construction. In the introduction, we explained the concept of a wild embedding (or fractal space). In short, a wild embedding is a submanifold (image of an embedding map), which must be decomposed into infinitely many substructures (polygons etc.). Therefore, it contains an infinite amount of information. In our previous work, we showed that the wild embedding is an expression for a quantized geometry. In the case of a fractal 3-sphere (as wildly embedded 3-sphere), one decomposes the 3-sphere into similar-looking pieces with constant curvature. Every piece has a different curvature so that the whole fractal 3-sphere represents the set of possible curvatures. These structures appear at all scales. Because of this property, we have to use the methods of noncommutative geometry to obtain a rigorous definition of this procedure. The following construction of the fractal 3-sphere is directly motivated by the exotic smoothness structure. The basic structure is a tree (used to define the Casson handle). Every part of the tree-like edge or vertex is associated with a 3-manifold. For the whole tree, one obtains an infinitely complicated 3- manifold, which is topologically equivalent to a 3-sphere. This fractal 3-sphere is the boundary of a 4- disk or 4-ball, described in the next section, and represents the Big Bang as a gravitational instanton (via a tunneling event).

In [17], we described this fractal 3-sphere as a sequence of 3-manifolds

$$Y_0 \to Y_1 \to \cdots \to Y_\infty$$

with increasing complexity. At first, we want to comment on the uniqueness of the construction. The sequence of 3-manifolds is determined by the smoothness structure or, better, by the Casson handle, which is used to construct this structure. Every Casson handle is represented by a tree. This tree is translated into a link: every $n$-branching point (vertex of the tree) is given by a Whitehead link with $n$ circles, and every line (edge of the tree) is given by the circle of the Whitehead link. In the previous section, we introduced the smoothness structure as given by the unbranched tree. Obviously, the unbranched tree is a subtree for any other more complex tree. It is a fundamental property of Casson handles (see [5]) that a Casson handle $CH_1$ embeds into another Casson handle $CH_2$, say $CH_1 \subset CH_2$, iff the tree of $CH_2$ embeds into the tree of $CH_1$. Therefore, any other Casson handle embeds into the Casson handle represented by the unbranched tree. This property is unique for the smoothness structure and the construction of the fractal 3-sphere.

For completeness, we will shortly explain the construction. The 3-manifold $Y_0$ is given by surgery ($0-$framed) along the pretzel knot $(-3, 3, -3)$ (or the knot $9_{46}$ in Rolfson notation), $Y_1$ is constructed by $0-$framed surgery along the Whitehead double of the pretzel knot $(-3, 3, -3)$, and finally, $Y_n$ is constructed by $0-$framed surgery along the $n$th Whitehead double of the pretzel knot $(-3, 3, -3)$. In the limit $n \to \infty$, we obtained $Y_\infty$ as a $0-$framed surgery along the $\infty$th Whitehead double of the pretzel knot $(-3, 3, -3)$ (a so-called wild knot). This 3-manifold $Y_\infty$ is the fractal 3-sphere (it has the topology of a 3-sphere by a theorem of Freedman [5]). The whole process can be seen as an iteration process at the level of 3-manifolds: we start with $Y_0$ and end with $Y_\infty$, the fractal 3-sphere.

To understand this abstract construction (via Dehn surgery or Kirby calculus [30]), we have to describe the construction of the first 3-manifold $Y_0$ more carefully. For that purpose, we have to describe Dehn surgery or surgery along a knot. If we remove a thicken knot $N(K) = K \times D^2$ (so-called tubular neighborhood) from the 3-sphere $S^3$, then one obtains the knot complement $C(K) = S^3 \setminus N(K)$. Now we glue in one solid torus $D^2 \times S^1$ to $C(K)$ by a mapping of the boundary $\phi : \partial C(K) = T^2 \to \partial(D^2 \times S^1) = T^2$ so that we obtain

$$M_{K,\phi} = C(K) \cup_\phi \left( D^2 \times S^1 \right).$$

All closed curves on a torus can be generated by the two possible non-contracting curves $m, \ell$ the meridian and longitude, respectively. In principle, any closed curve $\gamma$ on a torus $T^2$ is given by two numbers with $[\gamma] = [a\ell + bm]$ (for the homotopy classes). Then the map $\phi$ is characterized by a mapping of the meridian $m$ of one torus to the curve $\gamma$ determined by the ratio $r = b/a$ (including $\infty$ for $a = 0$) called the frame number. As a warm-up example, we consider the 0-framed surgery along the unknot $S^1$ in $S^3$. The knot complement of the unknot $C(S^1) = D^2 \times S^1$ is glued to another solid torus $D^2 \times S^1$ (along its boundary $\partial(D^2 \times S^1) = S^1 \times S^1$) with framing 0, which means that the meridian of $\partial C(S^1)$ is mapped to the meridian of $\partial(D^2 \times S^1)$. However, that means that $D^2 \times S^1$ is glued to $D^2 \times S^1$ along the boundary, i.e., $(D^2 \cup_{\partial D^2} D^2) \times S^1 = S^2 \times S^1$. Therefore, the $0-$framed surgery along the unknot gives $S^2 \times S^1$. Interestingly, $0-$framed surgery along any knot produces a 3-manifold, which is very similar to $S^2 \times S^1$ (having the same homology). Every $Y_n$ in the sequence above is produced by $0-$framed surgery along a knot of increasing complexity. One starts for $n = 0$ with the knot $9_{46}$ (in Rolfson notation) producing $Y_0$, then $n = 1$ with $Y_1$ is produced by the Whitehead double $Wh_1(9_{46})$ of this knot, $Y_2$ is given by the second iterated Whitehead double $Wh_2(9_{46})$ and so on. In the limit $n \to \infty$, one obtains $Y_\infty$ as $0-$framed surgery along the $\infty-$iterated Whitehead double $Wh_\infty(9_{46})$ of $9_{46}$ (a so-called wild knot). However, this limit changes the topology of $Y_\infty$. For every finite $n \geq 0$, $Y_n$ has the same homology as $S^2 \times S^1$ but $Y_\infty$ is topologically equivalent to $S^3$ (by a theorem of Freedman [5]).

In [3,17], we constructed a quantum state from a wild embedding. The main idea is to develop a description of the wild embedding by using operator algebra in the spirit of noncommutative geometry. This relation is strict: the wild embedding has a one-to-one relation to a foliation with leaf space of factor *III* von Neumann algebra known as the observable algebra of a quantum field theory. To understand this relation from a geometrical point of view, we will use the decomposition of factor *III* into factor *II* and a one-parameter group of automorphisms. We remark that this decomposition was used by Rovelli and Connes [35] to introduce a time variable in quantum gravity. This decomposition means that in some sense, the intractable factor *III* can be reduced to the easier accessible factor *II* (operators of finite trace).

For completeness, we will also present the construction (see [3]) of the $C^*-$algebra from the wild embedded 3-sphere. Let $I : S^3 \to \mathbb{R}^4$ be a wild embedding of codimension-one so that $I(S^3) = S^3_\infty = Y_\mathcal{T}$. Now we consider the complement $\mathbb{R}^4 \setminus I(S^3)$, which is non-trivial, i.e., $\pi_1(\mathbb{R}^4 \setminus I(S^3)) = \pi \neq 1$. Now we define the $C^*-$algebra $C^*(\mathcal{G}, \pi)$ associated with the complement $\mathcal{G} = \mathbb{R}^4 \setminus I(S^3)$ with group $\pi = \pi_1(\mathcal{G})$. If $\pi$ is non-trivial, then this group is not finitely generated. From an abstract point of view, we have a decomposition of $\mathcal{G}$ by an infinite union

$$\mathcal{G} = \bigcup_{i=0}^{\infty} C_i$$

of 'level sets' $C_i$. Then every element $\gamma \in \pi$ lies (up to homotopy) in a finite union of levels.

The basic elements of the $C^*-$algebra $C^*(\mathcal{G}, \pi)$ are smooth half-densities with compact supports on $\mathcal{G}$, $f \in C_c^\infty(\mathcal{G}, \Omega^{1/2})$, where $\Omega^{1/2}_\gamma$ for $\gamma \in \pi$ is the one-dimensional complex vector space of maps from the exterior power $\Lambda^k L$ ($\dim L = k$), of the union of levels $L$ representing $\gamma$, to $\mathbb{C}$ such that

$$\rho(\lambda\nu) = |\lambda|^{1/2}\rho(\nu) \qquad \forall \nu \in \Lambda^2 L, \lambda \in \mathbb{R}\,.$$

For $f, g \in C_c^\infty(\mathcal{G}, \Omega^{1/2})$, the convolution product $f * g$ is given by the equality

$$(f * g)(\gamma) = \int_{\gamma_1 \circ \gamma_2 = \gamma} f(\gamma_1) g(\gamma_2)$$

with the group operation $\gamma_1 \circ \gamma_2$ in $\pi$. Then we define via $f^*(\gamma) = \overline{f(\gamma^{-1})}$ a $*$operation making $C_c^\infty(\mathcal{G}, \Omega^{1/2})$ into a $*$algebra. Each level set $C_i$ consists of simple pieces (in the case of Alexanders horned sphere, we will explain it below) denoted by $T$. For these pieces, one has a natural representation of $C_c^\infty(\mathcal{G}, \Omega^{1/2})$ on the $L^2$ space over $T$. Then, one defines the representation

$$(\pi_x(f)\xi)(\gamma) = \int\limits_{\gamma_1 \circ \gamma_2 = \gamma} f(\gamma_1)\xi(\gamma_2) \qquad \forall \xi \in L^2(T), \forall x \in \gamma.$$

The completion of $C_c^\infty(\mathcal{G}, \Omega^{1/2})$ with respect to the norm

$$||f|| = \sup_{x \in \mathcal{G}} ||\pi_x(f)||$$

makes it into a $C^*$algebra $C_c^\infty(\mathcal{G}, \pi)$. Finally, we are able to define the $C^*-$algebra associated to the wild embedding. Using a result in [3], one can show that the corresponding von Neumann algebra is the factor $III_1$. This algebra is the observable algebra of a free (algebraic) quantum field theory with one vacuum vector [36]. Here we will discuss an alternative way to construct factor $III_1$. For that purpose, we look again at the construction of the wild 3-sphere $Y_\infty$. The $\infty-$iterated Whitehead double $Wh_\infty(9_{46})$ of the knot $9_{46}$ gives a wild knot $\mathcal{K}$, and $Y_\infty$ can be constructed by

$$Y_\infty = C(\mathcal{K}) \cup \left(D^2 \times S^1\right)$$

the $0-$framed surgery. In [3], we discussed the known result that the (deformation) quantization of the geometric structures (space of constant curvature) is given by the Kauffman bracket skein module. For $Y_\infty$, it means that we have to consider the Kauffman bracket skein module $K_h(C(\mathcal{K}))$ of $C(\mathcal{K})$. Here, it is known that $K_h(C(\mathcal{K}))$ is a module over the noncommutative torus, which is related (for $h = 0$) to the boundary $\partial C(\mathcal{K}) = T^2$. The noncommutative torus defines a factor $II_\infty$ algebra, and we will show in our forthcoming work that the whole $K_h(C(\mathcal{K}))$ gives the factor $III_1$.

## 4. The Quantum Spacetime at the Big Bang

In Section 2, we described the Big Bang as gravitational instanton $D^4$ (induced from spacetime, the K3 gravitational instanton). The initial state of the universe is given as the boundary $\partial D^4 = S^3$, a wild 3-sphere, via a tunneling process (Hartle–Hawking). Usually, nothing is known about the formation of the initial state via the tunneling process. In contrast, we have here the comfortable situation that there is a relation between the boundary—the wild 3-sphere—and the interior of the 4-disk. There is a process for the formation of the wild 3-sphere, which is divided into an infinite number of subprocesses, called Casson handles. This structure is called the design and was developed for the classification of 4-manifolds [4,5]. All subprocesses can be parameterized by all paths in a binary tree. The detailed construction of these Casson handles is unimportant for the following (but see [5]). Again before we start with the construction, we will discuss the physics behind it. As in the case of the fractal 3-sphere, the design is a geometric/topological expression for the quantum state of the spacetime. Here, it is the formation of the fractal 3-sphere seen as the boundary of the 4-disk. The design is a summation of all possible formation processes. It is an expression for the functional integral. As for the construction of the fractal 3-sphere, we also obtain complicated substructures at all scales, so we need the methods of the noncommutative geometry again. Here, the formation process is parametrized by a binary tree, where every path is a particular process. However, we need all processes or paths of the binary. Therefore, we associate to every path an operator, which consists of a sum of elementary operators (projection operators). Then one directly obtains an operator algebra (Temperley–Lieb algebra) which can be interpreted as an algebra of field operators. Here, we use the fact that we consider paths of a binary tree: the operator algebra is the algebra of fermion field operators. Interestingly, the expectation value in this algebra is related to a structure (Jones polynomial), which is well-known for three-dimensional manifolds and knots. Now we argue backwards: the expectation value is defined by a functional integral with Chern–Simons action in agreement with our previous work. The Chern–Simons action in the light cone gauge is interpreted as an invariant of the underlying foliation of the spacetime. Again with the help of noncommutative geometry, we are able to obtain a kind of quantum action (the so-called flow of weights). We remark that at the topological level, we have a kind of duality between the design (4D) and links (3D), which will be further investigated in our forthcoming work.

The design $S(Q)$ is a structure to label all Casson handles that embed in a given Casson handle $Q$. In our case, this Casson handle $Q$ is represented by an unbranched tree. Then, this Casson handle $Q$ represents (in some sense) all Casson handles. We will define this design $S(Q)$ to be the quantum state of $Q$. Below, we will determine the operator algebra associated with $Q$, and we will show that this algebra is a von Neumann algebra of finite trace as well as with one vacuumB vector (factor $II_1$). However, at first, we will describe the construction of the design $S(Q)$. In [37], we also described this construction but in a different context. For completeness, we will present this construction again.

According to Freedman ([5] p. 393), a Casson handle is represented by a labeled finitely-branching tree $Q$ with basepoint $\star$, having all edge paths infinitely extendable away from $\star$. Each edge should be given a label $+$ or $-$. The tree $Q$ is fixed, generating the wild 3-sphere (as the boundary of $D^4$). Then Freedman ([5] p. 398) constructs another labeled tree $S(Q)$ from the tree $Q$. There is a base point from which a single edge (called "decimal point") emerges. The tree is binary: one edge enters and two edges leave a vertex. The edges are named by initial segments of infinite base 3-decimals, representing numbers in the standard "middle third" Cantor set $C.s. \subset [0,1]$. This kind of Cantor set is given by the following construction: start with the unit Interval $S_0 = [0,1]$ and remove from that set the middle third and set $S_1 = S_0 \setminus (1/3, 2/3)$. Continue in this fashion, where $S_{n+1} = S_n \setminus \{\text{middle thirds of subintervals of } S_n\}$. Then the Cantor set $C.s.$ is defined as $C.s. = \cap_n S_n$. In other words, if we are using a ternary system (a number system with base 3), then we can write the Cantor set as $C.s. = \{x : x = (0.a_1a_2a_3\ldots) \text{ where each } a_i = 0 \text{ or } 2\}$. Each edge $e$ of $S(Q)$ carries a label $\tau_e$, where $\tau_e$ is an ordered finite disjoint union of 6-level-subtrees. There are three constraints on the labels, which leads to the correspondence between the $\pm$-labeled tree $Q$ and the (associated) $\tau$-labeled tree $S(Q)$.

Every path in $S(Q)$ represents one tree leading to a Casson handle. Any subtree represents a Casson handle, which embeds in $Q$ (see above). Now we will introduce an (operator) algebra structure on $S(Q)$. For that purpose, we have to consider pairs of paths in the (dual) tree of $S(Q)$. Thus, we have to concentrate on the so-called string algebra, according to Ocneanu [7]. For that purpose, we define a non-negative function $\mu : Edges \to \mathbb{C}$ together with the adjacency matrix $\triangle$ acting on $\mu$ by

$$\triangle\mu(x) = \sum_{\substack{v \in Edges \\ s(v)=x \\ r(v)=y}} \mu(y)$$

where $s(v)$ and $r(v)$ denote the source and the range of an edge $v$. A path in the tree is a succession of edges $\xi = (v_1, v_2, \ldots, v_n)$, where $r(v_i) = s(v_{i+1})$, and we write $\tilde{v}$ for the edge $v$ with the reversed orientation. Then, a string on the tree is a pair of paths $\rho = (\rho_+, \rho_-)$, with $s(\rho_+) = s(\rho_-)$, $r(\rho_+) \sim r(\rho_-)$, which means that $r(\rho_+)$ and $r(\rho_-)$ ending on the same level in the tree and $\rho_+, \rho_-$ have equal lengths, i.e., $|\rho_+| = |\rho_-|$ expressing the previously described property $r(\rho_+) \sim r(\rho_-)$ too. Now we define an algebra $String^{(n)}$ with the linear basis of the $n$-strings, i.e., strings with length $n$ and the additional operations:

$$\begin{aligned} (\rho_+, \rho_-) \cdot (\eta_+, \eta_-) &= \delta_{\rho_-, \eta_+}(\rho_+, \eta_-) \\ (\rho_+, \rho_-)^* &= (\rho_-, \rho_+) \end{aligned}$$

where $\cdot$ can be seen as the concatenation of paths. We normalize the function $\mu$ by $\mu(root) = 1$. Now we choose a function $\mu$ in such a manner that

$$\triangle\mu = \beta\mu \tag{4}$$

for a complex number $\beta$. Then we can construct elements $e_n$ in the algebra $String^{(n+1)}$ by

$$e_n = \sum_{\substack{|\alpha|=n-1 \\ |v|=|w|=1}} \frac{\sqrt{\mu(r(v))\mu(r(w))}}{\mu(r(\alpha))}(\alpha \cdot v \cdot \tilde{v}, \alpha \cdot w \cdot \tilde{w}) \tag{5}$$

which are the generators of the so-called Temperley–Lieb algebra. A *Temperley–Lieb algebra* is an algebra with unit element **1** over a number field $K$ generated by a countable set of generators $\{e_1, e_2, \ldots\}$ with the defining relations

$$\begin{gathered} e_i^2 = \tau \cdot e_i\,, \quad e_i e_j = e_j e_i \,:\, |i-j| > 1, \\ e_i e_{i+1} e_i = \tau e_i\,, \quad e_{i+1} e_i e_{i+1} = \tau e_{i+1}\,, \, e_i^* = e_i \end{gathered} \tag{6}$$

where $\tau$ is a real number in $(0,1]$. By [8], the Temperley–Lieb algebra has a uniquely defined trace $Tr$, which is normalized to lie in the interval $[0,1]$. The generators (5) also fulfill these algebraic relations (6) where $\tau = \beta^{-2}$. The trace of the string algebra is given by

$$tr(\rho) = \delta_{\rho_+,\rho_-} \beta^{-|\rho|} \mu(r(\rho)) \tag{7}$$

and defines on $A_\infty = (\bigcup_n String^{(n)}, tr)$ an inner product by $\langle x, y \rangle = tr(xy^*)$ given after completion the Hilbert space $L^2(A_\infty, tr)$.

Now we will determine the parameter $\tau$. Originally, Ocneanu introduces its string algebra to classify the splittings of modules over operator algebra (see also [38]). Thus, to determine this parameter, we look for the simplest generating structure in the tree. The simplest structure in the binary tree $S(Q)$ is one edge, which is connected with two other edges. This graph is represented by the following adjacency matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

with eigenvalues $0, \sqrt{2}, -\sqrt{2}$. According to our definition above, $\beta$ is given by the greatest eigenvalue of this adjacency matrix, i.e., $\beta = \sqrt{2}$ and thus $\tau = \beta^{-2} = \frac{1}{2}$. Then, without proof, we state that the algebra $R$ is given by the Clifford algebra on $\mathbb{R}^\infty$. The coefficients of this algebra are given by a map $\mu : Edges \to \mathbb{C}$.

The definition of the trace (7) (or better, the inner product) has a strong link to knot theory. This algebra (6) was used by Jones [8,9] to define a new knot invariant. Therefore, we can interpret every expectation value as the knot/link invariant of a certain knot/link (represented by a braid, see [39]) or a sum of these invariants. However, before we have to map the projectors $e_i$ to the generators $g_i$ so that $e_k = \frac{1}{1+i} + \frac{1}{\sqrt{2}i} g_k$ (for the special value $\tau = \frac{1}{2}$), see [9]. Then every generator $b_i$ of the braid group $B_n$ is mapped to $g_i$ (and vice versa). Therefore, the expectation value is associated with a (formal sum) of braids. The closure of these braids are links or every string $\rho$ defines a (formal) sum of links $L_\rho$. Then, $tr(\rho)$ must be equal (by definition) to the Jones polynomial $V_{L_\rho}(t = i)$ for the link $L_\rho$ for the special value $t = i$ (in general $t = \exp(i\pi\tau)$). The value of the Jones polynomial for $t = i$ is known to be

$$V_{L_\rho}(i) = -\left(\sqrt{2}\right)^{\ell-1} (-1)^{Arf(L_\rho)}$$

where $\ell$ is the number of components for $L_\rho$ and $Arf(L)$ is the Arf-invariant of the link (see [40] for the proof of the result and the definitions).

By this chain of arguments, we are able to derive a further link to understand the underlying action for calculating the expectation value. In [11], Witten constructed a topological quantum field theory (TQFT) for the Jones polynomial. This theory has its home on a 3-manifold $\Sigma$, and we will discuss this 3-manifold below. Let $A$ be a connection of a $SU(2)$ principle bundle over $\Sigma$. The Chern–Simons action is given by

$$CS(A) = \frac{1}{2\pi} \int_{\Sigma} tr\left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \tag{8}$$

then from [11], one has the relation

$$tr(\rho) = V_{L_\rho}(i) = \int DA \, \exp(i \cdot CS(A)) W_A(L_\rho)$$

between the trace $tr(\rho)$ and the functional integral over the action (8), where $W_A(L)$ is the Wilson loop along the link $L$ for the connection $A$. With this trick, we obtain the action functional (Chern–Simons action) and the observable (Wilson loop) for the underlying physical theory. The Jones polynomial is known to be intricately connected with the quantum enveloping algebra of the Lie algebra of the group $SL(2, \mathbb{C})$, see [41]. In our case, the parameter $q = t = i$ is the fourth root of unity, and it is known that this quantum q-deformation of the Lie algebra $sl_2(\mathbb{C})$ yields a finite-dimensional modular Hopf algebra. Therefore, we have determined the underlying quantum symmetry (of the initial state at the Big Bang) as the enveloped algebra $U_q(sl_2(\mathbb{C}))$. Furthermore, in [11], a relation between the $(2+1)$-dimensional Chern–Simons theory and a $(1+1)-$dimensional conformal field theory is also discussed. In particular, it was shown that the Hilbert space of pure Chern–Simons theories is isomorphic to the space of conformal blocks of an underlying Conformal Field Theory. This link seems to imply that there is an underlying $(1+1)-$dimensional theory. We discussed a similar mechanism in [32] using the Morgan–Shalen compactification and will study the relation between the two approaches in our forthcoming work.

Now we have to determine the 3-manifold $\Sigma$ in the definition of the Chern–Simons theory. At the first view, we identify $\Sigma$ with the wild 3-sphere. Then, this theory is stationary, i.e., it contains no time variable. However, as explained above, the formation of the wild 3-sphere can be seen as a process where the 3-manifold is growing by attaching three-dimensional pieces along surfaces. In the definition of the string algebra, we used Casson handles to define the generators $e_i$. However, Casson handles have an inherent 2-dimensional definition (neighborhood of immersed disks), which is used to define the construction of the wild 3-sphere (see [17] for a detailed construction). Then we can see the 3-manifold $\Sigma$ as a non-trivial cobordism between surfaces (used to define the wild 3-sphere), i.e., we define the Chern–Simons theory as a $(2+1)$-dimensional theory right in the sense of Witten [11]. The 3-manifold is foliated by the surfaces. To construct this foliation, we introduce light cone coordinates $(x^+ = x^0 + x^1, x^- = x^0 - x^1, x^2)$ together with the connection 1-form

$$A(x) = A_+(x)dx^+ + A_-(x)dx^- + A_2(x)dx^2.$$

(following ([42], sec. 4)). Now we choose the gauge $A_- = 0$ (axial gauge) so that we have a non-zero gauge field for the future light cone (seen from the Big Bang). Then the Chern–Simons action simplifies to

$$CS(A, A_- = 0) = \frac{1}{2\pi} \int_{\Sigma} tr(A \wedge dA)$$

and the restriction of the $SU(2)$ bundle to the surface leads to a bundle reduction from $SU(2)$ to $U(1)$ bundle with an abelian connection $a$ and Chern–Simons form

$$CS_{U(1)}(a) = \frac{1}{2\pi} \int_{\Sigma} a \wedge da$$

This form has a different interpretation in foliation theory: it is the Godbillon–Vey invariant [43]. Recall that a foliation $(M, F)$ of a manifold $M$ is an integrable subbundle $F \subset TM$ of the tangent bundle $TM$. The leaves $L$ of the foliation $(M, F)$ are the maximal connected submanifolds $L \subset M$ with $T_x L = F_x \, \forall x \in L$. A codimension-1 foliation on a 3-manifold $\Sigma$ can be constructed by a smooth 1-form $\omega$, fulfilling the integrability condition $d\omega \wedge \omega = 0$. Now one defines another one-form $\eta$ by $d\omega = -\eta \wedge \omega$ and the integral over the expression $gv = \eta \wedge d\eta$ is the Godbillon–Vey invariant. Then the Chern–Simons invariant in the axial gauge defines a codimension-1 foliation of $\Sigma$, where the Chern–Simons invariant is the Godbillon–Vey invariant. The critical values of the functional $CS_{U(1)}(a)$ are given by $da = 0$, and we obtain a foliation by vanishing Godbillon–Vey invariant. These foliations are rather trivial (such as surface $\times$ line or Reeb foliation). As shown in [44,45], foliations are really complicated. In the language of noncommutative geometry, the leaf space of a foliation with non-vanishing Godbillon–Vey invariants is a von-Neumann algebra, which contains a factor *III* subalgebra. As shown by Connes [46,47], the Godbillon–Vey class $GV$ can be expressed as a cyclic cohomology class (the so-called flow of weights)

$$GV_{HC} \in HC^2(C_c^\infty(G))$$

of the $C^*-$algebra for the foliation. Then, we define an expression

$$S = Tr_\omega(GV_{HC})$$

uniquely associated with the foliation ($Tr_\omega$ is the Dixmier trace). The expression $S$ generates the action on the factor by

$$\Delta_\omega^{it} = exp(i\,S)$$

so that $S$ is the action or the Hamiltonian multiplied by the time. We have evaluated this expression for some cases in [17], and we interpret it as quantum action. A detailed analysis will be shifted to our forthcoming work.

However, this action is partly satisfactory. In noncommutative geometry, one introduces a spectral triple with a Dirac operator as the main ingredient. Therefore, let us consider a Dirac operator $D^\Sigma$ on $\Sigma$. As a second ingredient, we introduced a codimension-1 foliation along the 1-form $a$, which is interpreted as an abelian gauge field. To take this foliation into account, we couple the abelian gauge field $a$ and the spinor $\psi$ to the Dirac–Chern–Simons action functional on the 3-manifold

$$\int_\Sigma \left( \bar{\psi}\, D_a^\Sigma \psi \,\sqrt{h} d^3x + a \wedge da \right)$$

with the critical points at the solution

$$D_a^\Sigma \psi = 0 \quad d\eta = \tau(\psi, \psi)$$

where $\tau(\psi, \psi)$ is the unique quadratic form for the spinors locally given by $\bar{\psi}\gamma^\mu\psi$. Now we consider a spacetime $\Sigma \times I$, so that the solution is translationally invariant. Expressed differently, we choose a spacetime with foliation induced by the foliation of $\Sigma$ extended by a translation. An alternative description for this choice is by considering the gradient flow of these equations

$$\begin{aligned} \frac{d}{dt} a &= da - \tau(\psi, \psi) \\ \frac{d}{dt} \psi &= D_a^\Sigma \psi \end{aligned}$$

However, it is known that this system is equivalent to the Seiberg–Witten equation for $\Sigma \times I$ by using an appropriate choice of the $Spin_C$ structure [48,49]. Then, this $Spin_C$ structure is directly related to the foliation. Therefore a non-trivial foliation together with a spectral triple (Dirac operator) induces a non-trivial solution of the gradient system, which results in a non-trivial solution of the Seiberg–Witten equations. However, this non-trivial solution (i.e., $\psi \neq 0, a \neq 0$) is a necessary condition for the existence of an exotic smoothness structure. Therefore, we have a closed circle: we started with a smooth spacetime at the Big Bang forming the initial state. If this state is a wild 3-sphere, we obtain a non-trivial foliation (=non-vanishing Godbillon–Vey invariant), which produces a non-trivial solution of the Seiberg–Witten equations.

Before closing this section, we will discuss the dynamical interpretation of the string algebra above and the observable. The design $S(Q)$ relative to a Casson handle $Q$ (in our case, the unbranched tree) is the sum over all Casson handles leading to the quantum state (the fractal 3-sphere as constructed from $Q$). The string algebra for the binary tree (representing the design) is the Clifford algebra of the Hilbert space. From the physics point of view, it is the algebra of fermion field operators. Every field operator is given by a path in the binary tree (weighted by some coefficients). A combination of the results in [12,37] showed that the fermion field operators (as elements of the Ocneanu string algebra) can also be interpreted as the leaf space of a type *III* foliation (see [44]) seen as a crossed product of the string algebra and its modular automorphism group. This product with the automorphism group is a time-dependent representation of the field operators (see [35]). Therefore, the foliation of type *III* (having a non-zero Godbillon–Vey invariant) is the dynamical interpretation of string algebra. However, we know more because the design was seen as the formation of the fractal 3-sphere as given by a sequence of 3-manifolds. This process is given by a sequence of 3-manifold topology change, which was described in [25]. It leads to an inflationary behavior, which is approximately described by a de Sitter space (see [23]). In [50], the algebra of an observable for a de Sitter space is described to be a von Neumann algebra of type $II_1$. Here we conjecture that there must be a relation between our string algebra and this algebra of observables.

## 5. Conclusions

In this paper, we have worked out a model of the Big Bang driven by topological considerations. The starting point was the construction of a spacetime as a global expression of the evolution of the universe. However, the real core of the paper is the construction of the initial state as a wildly embedded or fractal 3-sphere. Here, the construction of a corresponding operator algebra was the decisive step to understanding this state, and many interrelations with other theories came to light. Thus, the expectation value in the operator algebra can always be reinterpreted as a knot invariant (Jones polynomial). The action of the theory is the Chern–Simons invariant, which already appears in the description of a $(2+1)$-dimensional gravitational theory. In general, these relations to conformal fields and Seiberg–Witten theory are the real strength of this approach. This paper only prepares the groundwork for further approaches to understanding the initial state at the Big Bang. In our next work, we will interpret and calculate the dark matter density as a topological quantity.

**Author Contributions:** Conceptualization, T.A.-M. and J.K.; formal analysis, T.A.-M. and A.W.; investigation, T.A.-M., J.K. and A.W.; writing—original draft, T.A.-M.; writing—review and editing, J.K. and A.W. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** We acknowledge the remarks and questions of the referees, improving the argumentation in the paper and increasing the readability.

**Conflicts of Interest:** The authors declare no conflict of interest.

## Appendix A

In this appendix, we will describe the methods and results in [3] to make the paper as self-contained as possible. There, we showed that the deformation quantization of a tame embedding is a wild embedding.

At first, we start with some definitions. A map $f : N \to M$ between two topological manifolds is an embedding if $N$ and $f(N) \subset M$ are homeomorphic to each other. An embedding $i : N \hookrightarrow M$ is *tame* if $i(N)$ is represented by a finite polyhedron homeomorphic to $N$. Otherwise, we call the embedding *wild*. Let $I : K^n \to \mathbb{R}^{n+k}$ be a wild embedding of codimension $k$ with $k = 0, 1, 2$. Now, we assume that the complement $\mathbb{R}^{n+k} \setminus I(K^n)$ is non-trivial, i.e., $\pi_1(\mathbb{R}^{n+k} \setminus I(K^n)) = \pi \neq 1$. Wild embeddings are usually characterized by this property, but if the group $\pi_1(\mathbb{R}^{n+k} \setminus I(K^n)) = 1$ is trivial, then the group $\pi_1(I(K^n))$ must be non-trivial for wild embeddings. In Section 3, we defined the $C^*-$algebra $C^*(\mathcal{G}, \pi)$ associated to the complement $\mathcal{G} = \mathbb{R}^{n+k} \setminus I(K^n)$ with group $\pi = \pi_1(\mathcal{G})$. Therefore, the methods of noncommutative geometry are applicable.

For the relation between the tame and wild embedding, we consider the space of geometric structures on the embedded manifold. In [3], we perform the calculations for Alexander's horned sphere. The space of geometric structures with isometry group $SL(2, \mathbb{C})$ admits a Poisson structure. The deformation quantization of this Poisson structure is known as the Drinfeld–Turaev quantization. In a series of papers, it was shown that the deformation quantization of the space of geometric structures with isometry group $SL(2, \mathbb{C})$ is the Kauffman bracket skein algebra. In the case of Alexander's horned sphere, we showed that the Kauffman bracket skein algebra is the factor $II_1$ algebra isomorphic to the enveloping von Neumann algebra of the $C^*$ algebra defined by the wild embedding. In particular, for a tame embedding, the skein algebra is trivial (it is only a 1-dimensional algebra, the center).

## References


1. Hartle, J.B.; Hawking, S.W. Wave function of the universe. *Phys. Rev. D* **1983**, *28*, 2960. [CrossRef]
2. Hawking, S.W.; Turok, N. Open inflation without false vacua. *Phys. Lett. B* **1998**, *425*, 25–32. [CrossRef]
3. Asselmeyer-Maluga, T.; Król, J. Quantum geometry and wild embeddings as quantum states. *Int. J. Geom. Methods Mod. Phys.* **2013**, *10*, 1350055. [CrossRef]
4. Freedman, M.; Quinn, F. *Topology of 4-Manifolds*; Princeton Mathematical Series; Princeton University Press: Princeton, NJ, USA, 1990.
5. Freedman, M.H. The topology of four-dimensional manifolds. *J. Diff. Geom.* **1982**, *17*, 357–454. [CrossRef]
6. Freedman, M.H. The disk problem for four-dimensional manifolds. *Proc. Internat. Cong. Math. Warzawa* **1983**, *17*, 647–663.
7. Ocneanu, A. Quantized groups, string algebras and Galois theory for algebras. In *Operator Algebras and Applications*; Evans, D.E., Takesaki, M., Eds.; Cambridge University Press: Cambridge, UK, 1988; pp. 119–172.
8. Jones, V. Index of subfactors. *Invent. Math.* **1983**, *72*, 1–25. [CrossRef]
9. Jones, V.F.R. A polynomial invariant for knots via von Neumann algebras. *BAMS* **1985**, *12*, 103. [CrossRef]
10. Witten, E. 2 + 1 dimensional gravity as an exactly soluble system. *Nucl. Phys. B* **1988**, *311*, 46–78/89. [CrossRef]
11. Witten, E. Quantum field theory and the Jones polynomial. *Commun. Math. Phys.* **1989**, *121*, 351–400. [CrossRef]
12. Asselmeyer-Maluga, T.; Brans, C.H. How to include fermions into general relativity by exotic smoothness. *Gen. Relativ. Grav.* **2015**, *47*, 30. [CrossRef]
13. Asselmeyer-Maluga, T.; Rosé, H. On the geometrization of matter by exotic smoothness. *Gen. Rel. Grav.* **2012**, *44*, 2825–2856. [CrossRef]
14. Hawking, S.W.; Ellis, G.F.R. *The Large Scale Structure of Space-Time*; Cambridge University Press: Cambridge, UK, 1994.
15. Brans, C.H. Exotic smoothness and physics. *J. Math. Phys.* **1994**, *35*, 5494–5506. [CrossRef]
16. Brans, C.H. Localized exotic smoothness. *Class. Quant. Grav.* **1994**, *11*, 1785–1792. [CrossRef]
17. Asselmeyer-Maluga, T. Smooth quantum gravity: Exotic smoothness and Quantum gravity. In *At the Frontiers of Spacetime: Scalar-Tensor Theory, Bell's Inequality, Mach's Principle, Exotic Smoothness*; Asselmeyer-Maluga, T., Ed.; Springer: Cham, Switzerland, 2016.
18. Yau, S.-T. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. *Commun. Pure Appl. Math.* **1978**, *31*, 339–411. [CrossRef]
19. LeBrun, C. Four-manifolds without Einstein metrics. *Math. Res. Lett.* **1996**, *3*, 133–147. [CrossRef]
20. Asselmeyer-Maluga, T.; Brans, C.H. *Exotic Smoothness and Physics*; World Scientific: Singapore, 2008.
21. Asselmeyer-Maluga, T.; Król, J. On topological restrictions of the spacetime in cosmology. *Mod. Phys. Lett. A* **2012**, *27*, 1250135. [CrossRef]

22. Donaldson, S. An application of gauge theory to the topology of 4-manifolds. *J. Diff. Geom.* **1983**, *18*, 269–316.
23. Asselmeyer-Maluga, T.; Krol, J. How to obtain a cosmological constant from small exotic $\mathbb{R}^4$. *Phys. Dark Universe* **2018**, *19*, 66–77. [CrossRef]
24. Asselmeyer-Maluga, T.; Krol, J. A topological approach to neutrino masses by using exotic smoothness. *Mod. Phys. Lett. A* **2019**, *34*, 1950097. [CrossRef]
25. Asselmeyer-Maluga, T.; Krol, J. A topological model for inflation. *arXiv* **2018**, arXiv:1812.08158.
26. Bizaca, Z. An explicit family of exotic Casson handles. *Proc. AMS* **1995**, *123*, 1297–1302. [CrossRef]
27. Bižaca, Z.; Gompf, R. Elliptic surfaces and some simple exotic $\mathbb{R}^4$'s. *J. Diff. Geom.* **1996**, *43*, 458–504.
28. Asselmeyer-Maluga, T.; Król, J. Decoherence in quantum cosmology and the cosmological constant. *Mod. Phys. Lett. A* **2013**, *28*, 1350158. [CrossRef]
29. Hein, H.-J.; Sun, S.; Viaclovsky, J.; Zhang, R. Nilpotent structures and collapsing Ricci-flat metrics on K3 surfaces. *arXiv* **2022**, arXiv:1807.09367.
30. Gompf, R.E.; Stipsicz, A.I. *4-Manifolds and Kirby Calculus*; American Mathematical Society: Providence, RI, USA, 1999.
31. Hein, H.-J.; Sun, S.; Viaclovsky, J.; Zhang, R. Gravitational instantons and del Pezzo surfaces. *arXiv* **2021**, arXiv:2111.09287.
32. Asselmeyer-Maluga, T. Hyperbolic groups, 4-manifolds and quantum gravity. *J. Phys. Conf. Ser.* **2019**, *1194*, 012009. [CrossRef]
33. Ade, P.A.R.; Aghanim, N.; Armitage-Caplan, C.; Arnaud, M.; Ashdown, M.; Atrio-Barandela, F.; Aumont, J.; Baccigalupi, C.; Banday, A.J.; Barreiro, R.B.; et al. Planck 2013 results. XVI. cosmological parameters. *arXiv* **2013**, arXiv:1303.5076[astro-ph.CO].
34. Ade, P.A.R.; Aghanim, N.; Armitage-Caplan, C.; Arnaud, M.; Ashdown, M.; Atrio-Barandela, F.; Aumont, J.; Baccigalupi, C.; Banday, A.J.; Barreiro, R.B.; et al. Planck 2013 results. XXII. constraints on inflation. *arXiv* **2013**, arXiv:1303.5082[astro-ph.CO].
35. Connes, A.; Rovelli, C. Von Neumann algebra automorphisms and time-thermodynamics relation in generally covariant quantum theories. *Class. Quan. Grav.* **1994**, *11*, 2899–2917.
36. Borchers, H.J. On revolutionizing quantum field theory with Tomita's modular theory. *J. Math. Phys.* **2000**, *41*, 3604–3673.
37. Asselmeyer-Maluga, T.; Król, J. Exotic smooth $\mathbb{R}^4$, noncommutative algebras and quantization. *arXiv* **2010**, arXiv:1001.0882.
38. Goodman, F.; Harpe, P.d.; Jones, V. *Coxeter Graphs and Towers of Algebras*; MSRI Publications Edition; Springer: Berlin/Heidelberg, Germany, 1989; Volume 14.
39. Asselmeyer-Maluga, T. Braids, 3-Manifolds, Elementary Particles: Number Theory and Symmetry in Particle Physics. *Symmetry* **2019**, *11*, 1298. [CrossRef]
40. Murakami, H. A recursive calculation of the Arf invariant of a link. *J. Math. Soc. Jpn.* **1986**, *38*, 335. [CrossRef]
41. Reshetikhin, N.Y.; Turaev, V. Invariants of three-manifolds via link polynomials and quantum groups. *Inv. Math.* **1991**, *103*, 547–597. [CrossRef]
42. Kauffman, L.H. Functional integration and the Kontsevich integral. In *Yang-Baxter Systems, Nonlinear Models and Their Applications, Proceedings of the APCTP-Nankai Symposium, Singapore*; World Scientific: Singapore, 1999.
43. Godbillon, C.; Vey, J. Un invariant des feuilletages de codimension. *C. R. Acad. Sci. Paris Ser. A-B* **1971**, *273*, A92.
44. Hurder, S.; Katok, A. Secondary classes and transverse measure theory of a foliation. *BAMS* **1984**, *11*, 347–349. [CrossRef]
45. Thurston, W. Noncobordant foliations of $S^3$. *BAMS* **1972**, *78*, 511–514. [CrossRef]
46. Connes, A. A survey of foliations and operator algebras. *Proc. Symp. Pure Math.* **1984**, *38*, 521–628.
47. Connes, A. *Non-Commutative Geometry*; Academic Press: Cambridge, MA, USA, 1994.
48. Morgan, J.W.; Szabo, Z.; Taubes, C.H. A product formula for the Seiberg-Witten invariants and the generalized Thom conjecture. *J. Diff. Geom.* **1996**, *44*, 706–788. [CrossRef]
49. Morgan, J.W.; Szabo, Z.; Taubes, C.H. Product formulas along $t^3$ for Seiberg-Witten invariants. *Math. Res. Lett.* **1997**, *4*, 915–929. [CrossRef]
50. Chandrasekaran, V.; Longo, R.; Penington, G.; Witten, E. An Algebra of Observables for de Sitter Space. *arXiv* **2022**, arXiv:2206.10780.