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The *g*-Extra Connectivity of the Strong Product of Paths and Cycles

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Abstract: Let G be a connected graph and g be a non-negative integer. A vertex set S of graph G is called a g-extra cut if $G - S$ is disconnected and each component of $G - S$ has at least $g + 1$ vertices. The g-extra connectivity of G is the minimum cardinality of a g-extra cut of G if G has at least one g-extra cut. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the strong product $G_1 \boxtimes G_2$ is defined as follows: its vertex set is $V_1 \times V_2$ and its edge set is $\{(x_1, x_2)(y_1, y_2)|\ x_1 = x_2 \text{ and } y_1 y_2 \in E_2;\ \text{or } y_1 = y_2 \text{ and } x_1 x_2 \in E_1;\ \text{or } x_1 x_2 \in E_1 \text{ and } y_1 y_2 \in E_2\}$, where $(x_1, x_2), (y_1, y_2) \in V_1 \times V_2$. In this paper, we obtain the g-extra connectivity of the strong product of two paths, the strong product of a path and a cycle, and the strong product of two cycles.

Keywords: conditional connectivity; g-extra connectivity; strong product; paths; cycles

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *minimum degree* of G is denoted by $\delta(G)$. A *vertex cut* in G is a set of vertices whose deletion makes G disconnected. The *connectivity* $\kappa(G)$ of the graph G is the minimum order of a vertex cut in G if G is not a complete graph; otherwise $\kappa(G) = |V(G)| - 1$. Usually, the topology structure of an interconnection network can be modeled by a graph G, where $V(G)$ represents the set of nodes and $E(G)$ represents the set of links connecting nodes in the network. Connectivity is used to measure the reliability the network, while it always underestimates the resilience of large networks.

To overcome this deficiency, Harary [1] generalized the concept of the classical connectivity $\kappa(G)$ as follows. Let $\mathcal{P}$ be a graph-theoretic property. A vertex set $S \subseteq V(G)$ is a $\mathcal{P}$-*cut* if $G - S$ is disconnected and each component of $G - S$ has property $\mathcal{P}$. The *conditional connectivity* $\kappa(G; \mathcal{P})$ is the minimum cardinality of a $\mathcal{P}$-cut if G has at least one $\mathcal{P}$-cut. Later, Fàbrega and Fiol [2] introduced the concept of g-extra connectivity, which is a kind of conditional connectivity. Let g be a non-negative integer. If the vertex set $S \subseteq V(G)$ satisfies $G - S$ is disconnected and each component of $G - S$ has at least $g + 1$ vertices, then S is called a g-*extra cut*. If G has at least one g-extra cut, then the g-*extra connectivity* of G, denoted by $\kappa_g(G)$, is the the minimum cardinality of a g-extra cut. Otherwise, define $\kappa_g(G) = \infty$. If S is a g-extra cut in G with order $\kappa_g(G)$, then we call S a κ_g-*cut*. Since $\kappa_0(G) = \kappa(G)$ for any connected graph G that is not a complete graph, the g-extra connectivity can be seen as a generalization of the traditional connectivity. The authors in [3] pointed out that there is no polynomial-time algorithm for computing κ_g for a general graph. Consequently, much of the work has been focused on the computing of the g-extra connectivity of some given graphs, see [3–20] for examples.

Graph product is used to produce large graphs from small ones. There are many kinds of products, such as Cartesian product, direct product, strong product and lexicographic product. The *Cartesian product* of two graphs G_1 and G_2, denoted by $G_1 \square G_2$, is defined on the vertex sets $V(G_1) \times V(G_2)$, and $(x_1, y_1)(x_2, y_2)$ is an edge in $G_1 \square G_2$ if and only if one of the following is true: (*i*) $x_1 = x_2$ and $y_1 y_2 \in E(G_2)$; (*ii*) $y_1 = y_2$ and $x_1 x_2 \in E(G_1)$.

For two graphs G_1 and G_2, the strong product $G_1 \boxtimes G_2$ is defined as follows: its vertex set is $V(G_1) \times V(G_2)$ and its edge set is $\{(x_1, x_2)(y_1, y_2) \mid x_1 = x_2 \text{ and } y_1 y_2 \in E(G_2); \text{ or } y_1 = y_2 \text{ and } x_1 x_2 \in E(G_1); \text{ or } x_1 x_2 \in E(G_1) \text{ and } y_1 y_2 \in E(G_2)\}$, where $(x_1, x_2), (y_1, y_2) \in V(G_1) \times V(G_2)$.

Špacapan [21] proved that for any nontrivial graphs G_1 and G_2, $\kappa(G_1 \square G_2) = \min \{\kappa(G_1)|V(G_2)|, \kappa(G_2)|V(G_1)|, \delta(G_1 \square G_2)\}$. Lü, Wu, Chen and Lv [22] provided bounds for the 1-extra connectivity of the Cartesian product of two connected graphs. Tian and Meng [23] determined the exact values of the 1-extra connectivity of the Cartesian product for some class of graphs. In [24], Chen, Meng, Tian and Liu further studied the 2-extra connectivity and the 3-extra connectivity of the Cartesian product of graphs.

Brešar and Špacapan [25] determined the edge-connectivity of the strong products of two connected graphs. For the connectivity of the strong product graphs, Špacapan [26] obtained *Theorem* 1 in the following. Let S_i be a vertex cut in G_i for $i = 1, 2$, and let A_i be a component of $G_i - S_i$ for $i = 1, 2$. Following the definitions in [26], $I = S_1 \times V_2$ or $I = V_1 \times S_2$ is called an *I*-set in $G_1 \boxtimes G_2$, and $L = (S_1 \times A_2) \cup (S_1 \times S_2) \cup (A_1 \times S_2)$ is called an *L*-set in $G_1 \boxtimes G_2$.

Theorem 1 ([26]). *Let G_1 and G_2 be two connected graphs. Then every minimum vertex cut in $G_1 \boxtimes G_2$ is either an I-set or an L-set in $G_1 \boxtimes G_2$.*

Motivated by the results above, we will study the g-extra connectivity of the strong product graphs. In the next section, we introduce some definitions and lemmas. In Section 3, we will give the g-extra connectivity of the strong product of two paths, the strong product of a path and a cycle, and the strong product of two cycles. Conclusion will be given in Section 4.

2. Preliminary

For graph-theoretical terminology and notations not defined here, we follow [27]. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *neighborhood* of a vertex u in G is $N_G(u) = \{v \in V(G) \mid v \text{ is adjacent to the vertex } u\}$. Let A be a subset of $V(G)$, the neighborhood of A in G is $N_G(A) = \{v \in V(G) \backslash A \mid v \text{ is adjacent to a vertex in } A\}$. The subgraph induced by A in G is denoted by $G[A]$. We use P_n to denote the path with order n and C_n to denote the cycle with order n.

Let G_1 and G_2 be two graphs. Define two natural projections p_1 and p_2 on $V(G_1) \times V(G_2)$ as follows: $p_1(x, y) = x$ and $p_2(x, y) = y$ for any $(x, y) \in V(G_1) \times V(G_2)$. The subgraph induced by $\{(u, y) \mid u \in V(G_1)\}$ in $G_1 \boxtimes G_2$, denoted by G_{1y}, is called a G_1-*layer* in $G_1 \boxtimes G_2$ for each vertex $y \in V(G_2)$. Analogously, the subgraph induced by $\{(x, v) \mid v \in V(G_2)\}$ in $G_1 \boxtimes G_2$, denoted by $_xG_2$, is called a G_2-*layer* in $G_1 \boxtimes G_2$ for each vertex $x \in V(G_1)$. Clearly, a G_1-layer in $G_1 \boxtimes G_2$ is isomorphic to G_1, and a G_2-layer in $G_1 \boxtimes G_2$ is isomorphic to G_2.

Let $S \subseteq V(G_1 \boxtimes G_2)$. For any $x \in V(G_1)$, denote $S \cap V(_xG_2)$ by $_xS$, and analogously, for any $y \in V(G_2)$, denote $S \cap V(G_{1y})$ by S_y. Furthermore, we use $\overline{_xS} = V(_xG_2) \setminus {_xS}$ and $\overline{S_y} = V(G_{1y}) \setminus S_y$. By almost the same argument as the proof of the second paragraph of *Theorem* 3.2 in [26], we can obtain the following lemma. For completeness, we also address the proof here.

Lemma 2 ([26]). *Let G be the strong product $G_1 \boxtimes G_2$ of two connected graphs G_1 and G_2, and let g be a non-negative integer. Assume G has g-extra cuts and S is a κ_g-cut of G.*
(i) If $_xS \neq \varnothing$ for some $x \in V(G_1)$, then $|_xS| \geq \kappa(G_2)$.
(ii) If $S_y \neq \varnothing$ for some $y \in V(G_2)$, then $|S_y| \geq \kappa(G_1)$.

Proof. (*i*) Suppose $_xS \neq \varnothing$ for some $x \in V(G_1)$. Note that this is obviously true if $_xS = V(_xG_2)$. If $\overline{_xS}$ is not contained in one component of $G - S$, then clearly the induced subgraph $G[\overline{_xS}]$ is not connected, and hence $|_xS| \geq \kappa(G_2)$. If $\overline{_xS}$ is contained in one

component of $G - S$, then choose an arbitrary fixed vertex (x, y) from $_x S$. Let H_1 be the component of $G - S$ such that $\overline{_x S} \subseteq V(H_1)$ and let $H_2 = G - S \cup V(H_1)$. Since S is a κ_g-cut, we find that the vertex $(x, y) \in {_x S}$ has a neighbor $(x_1, y_1) \in V(H_2)$. Since $(x_1, y_1) \in V(H_2)$, we find that $(x, y_1) \in {_x S}$, moreover, for any $(x, u) \in \overline{_x S}$, we find that (x, u) is not adjacent to (x, y_1), otherwise, (x, u) would be adjacent to (x_1, y_1), which is not true since those two vertices are in different components of $G - S$. Thus if $R = {_x S} \setminus \{(x, y_1)\}$, then $p_2(R)$ is a vertex cut in G_2 and one component of $G_2 - p_2(R)$ is $\{y_1\}$. Thus $|_x S| = |R| + 1 \geq \kappa(G_2) + 1$. Analogously, we can get $|S_y| \geq \kappa(G_1)$ if (*ii*) holds. □ □

3. Main Results

Let H be a subgraph of $G_1 \boxtimes G_2$. For the sake of simplicity, we use $_x H$ instead of $_x V(H)$ to represent $V(H) \cap V(_x G_2)$ for any $x \in V(G_1)$ and H_y to represent $V(H) \cap V(G_{1y})$ for any $y \in V(G_2)$. Since $\kappa_g(P_1 \boxtimes P_n) = 1$ for $g \leq \lfloor \frac{n-1}{2} \rfloor - 1$ and $\kappa_g(P_2 \boxtimes P_n) = 2$ for $g \leq 2\lfloor \frac{n-1}{2} \rfloor - 1$, we assume $m, n \geq 3$ in the following theorem.

Theorem 3. *Let g be a non-negative integer and $G = P_m \boxtimes P_n$, where $m, n \geq 3$. If $g \leq min\{n\lfloor \frac{m-1}{2} \rfloor - 1, m\lfloor \frac{n-1}{2} \rfloor - 1\}$, then $\kappa_g(G) = min\{m, n, \lceil 2\sqrt{g+1} \rceil + 1\}$.*

Proof. Denote $P_m = x_1 x_2 \ldots x_m$ and $P_n = y_1 y_2 \ldots y_n$. Let $S_1 = V(P_m) \times \{y_{\lfloor \frac{n-1}{2} \rfloor + 1}\}$ and $S_2 = \{x_{\lfloor \frac{m-1}{2} \rfloor + 1}\} \times V(P_n)$. Since $g \leq min\{n\lfloor \frac{m-1}{2} \rfloor - 1, m\lfloor \frac{n-1}{2} \rfloor - 1\}$, we verify that S_1 and S_2 are two g-extra cuts of G. Thus $\kappa_g(G) \leq min\{m, n\}$. If $\lceil 2\sqrt{g+1} \rceil + 1 \geq min\{m, n\}$, then $\kappa_g(G) \leq min\{m, n, \lceil 2\sqrt{g+1} \rceil + 1\}$. If $\lceil 2\sqrt{g+1} \rceil + 1 < min\{m, n\}$, then let $S_3 = (J_1 \times K_2) \cup (J_1 \times J_2) \cup (K_1 \times J_2)$, where $J_1 = \{x_{\lceil \sqrt{g+1} \rceil + 1}\}$, $K_1 = \{x_1, x_2, \ldots, x_{\lceil \sqrt{g+1} \rceil}\}$, $J_2 = \{y_{\lceil \frac{g+1}{\lceil \sqrt{g+1} \rceil} \rceil + 1}\}$ and $K_2 = \{y_1, y_2, \ldots, y_{\lceil \frac{g+1}{\lceil \sqrt{g+1} \rceil} \rceil}\}$. It is routine to verify that S_3 is a g-extra cut of G. By $|S_3| = \lceil \sqrt{g+1} \rceil + \lceil \frac{g+1}{\lceil \sqrt{g+1} \rceil} \rceil + 1 = \lceil 2\sqrt{g+1} \rceil + 1$, we have $\kappa_g(G) \leq \lceil 2\sqrt{g+1} \rceil + 1$. Therefore, $\kappa_g(G) \leq min\{m, n, \lceil 2\sqrt{g+1} \rceil + 1\}$ holds. □

Now, it is sufficient to prove $\kappa_g(G) \geq min\{m, n, \lceil 2\sqrt{g+1} \rceil + 1\}$. Assume S is a κ_g-cut of G. We consider two cases in the following.

Case 1. $_x S \neq \emptyset$ for all $x \in V(P_m)$, or $S_y \neq \emptyset$ for all $y \in V(P_n)$.

Assume $_x S \neq \emptyset$ for all $x \in V(P_m)$. By Lemma 2.1, $|S| = \sum_{x \in V(P_m)} |_x S| \geq \kappa(P_n)|V(P_m)| = m$. Analogously, if $S_y \neq \emptyset$ for all $y \in V(P_n)$, then $|S| = \sum_{y \in V(P_n)} |S_y| \geq \kappa(P_m)|V(P_n)| = n$.

Case 2. There exist a vertex $x_a \in V(P_m)$ and a vertex $y_b \in V(P_n)$ such that $_{x_a} S = S_{y_b} = \emptyset$.

By the assumption $_{x_a} S = S_{y_b} = \emptyset$, we know $V(_{x_a} G_2)$ and $V(G_{1y_b})$ are contained in a component H' of $G - S$. Let H be another component of $G - S$. Let $p_1(V(H)) = \{x_{s+1}, x_{s+2}, \ldots, x_{s+k}\}$ and $p_2(V(H)) = \{y_{t+1}, y_{t+2}, \ldots, y_{t+h}\}$. Without loss of generality, assume $s + k < a$ and $t + h < b$. Clearly, $|V(H)| \leq kh$. Since S is a κ_g-cut, we have $N_G(V(H)) = S$ and $|V(H)| \geq g + 1$. If we can prove $|N_G(V(H))| \geq k + h + 1$, then $\kappa_g(G) = |S| = |N_G(V(H))| \geq k + h + 1 \geq 2\sqrt{kh} + 1 \geq 2\sqrt{g+1} + 1$ and the theorem holds. Thus, we only need to show that $|N_G(V(H))| \geq k + h + 1$ in the remaining proof.

Let (x_{s+i}, y_{d_i}) be the vertex in $_{x_{s+i}} H$ such that d_i is maximum for $i = 1, \ldots, k$, and let (x_{r_j}, y_{t+j}) be the vertex in $H_{y_{t+j}}$ such that r_j is maximum for $j = 1, \ldots, h$. Denote $D = \{(x_{s+1}, y_{d_1}), \ldots, (x_{s+k}, y_{d_k})\}$ and $R = \{(x_{r_1}, y_{t+1}), \ldots, (x_{r_h}, y_{t+h})\}$. For the convenience of counting, we will construct an injective mapping f from $D \cup R$ to $N_G(V(H) \setminus \{(x_{s+k+1}, y_{d_k+1})\})$. Although D and R may have common elements, we consider the elements in D and R to be different in defining the mapping f below.

First, the mapping f on D is defined as follows.

$$f((x_{s+i}, y_{d_i})) = (x_{s+i}, y_{d_i+1}) \text{ for } i = 1, \ldots, k.$$

Denote $F_1 = \{(x_{s+1}, y_{d_1+1}), \ldots, (x_{s+k}, y_{d_k+1})\}$.

Second, for each vertex (x_{r_j}, y_{t+j}) satisfying $(x_{r_j+1}, y_{t+j}) \notin F_1$, define $f((x_{r_j}, y_{t+j})) = (x_{r_j+1}, y_{t+j})$.

If (x_{r_j}, y_{t+j}) satisfies $(x_{r_j+1}, y_{t+j}) \notin F_1$ for any $j \in \{1, \ldots, h\}$, then we are done. Otherwise, for each $(x_{r_{j'}}, y_{t+j'})$ satisfying $(x_{r_{j'}+1}, y_{t+j'}) \in F_1$, we give the definition as follows. By the definitions of D and R, we have $(x_{r_{j'}+1+i}, y_{t+j'+j}) \notin V(H)$ for all $i, j \geq 0$, and $\{(x_{r_{j'}}, y_{t+j'}), \ldots, (x_{r_{j'}}, y_{d_{r_{j'}-s}})\} \subseteq R$ (see Figure 1 for an illustration). Now, we define $f((x_{r_{j'}}, y_{t+j'})) = (x_{r_{j'}+1}, y_{t+j'+1})$ and change the images of $(x_{r_{j'}}, y_{t+j'+1}), \ldots, (x_{r_{j'}}, y_{d_{r_{j'}-s}})$ to $(x_{r_{j'}+1}, y_{t+j'+2}), \ldots, (x_{r_{j'}+1}, y_{d_{r_{j'}-s}+1})$, respectively. The images of f on R are well-defined.

Finally, we have an injective mapping f from $D \cup R$ to $N_G(V(H) \setminus \{(x_{s+k+1}, y_{d_k+1})\})$. Then $\kappa_g(G) = |S| = |N_G(V(H))| \geq |D| + |R| + 1 \geq k + h + 1 \geq 2\sqrt{kh} + 1 \geq 2\sqrt{g+1} + 1$. The proof is thus complete.
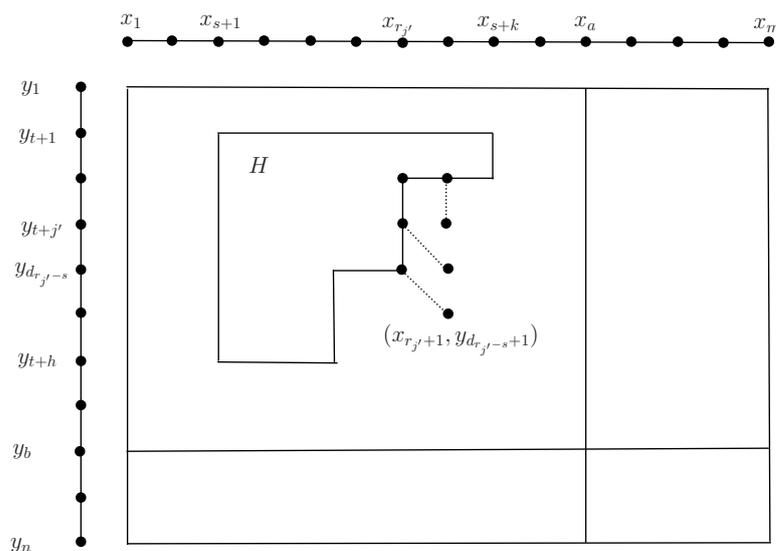

Figure 1. An illustration for the proof of Theorem 3.

Since $\kappa_g(C_3 \boxtimes P_n) = 3$ for $g \leq 3\lfloor \frac{n-1}{2} \rfloor - 1$, we assume $m \geq 4$ in the following theorem.

Theorem 4. *Let g be a non-negative integer and $G = C_m \boxtimes P_n$, where $m \geq 4$, $n \geq 3$. If $g \leq \min\{n\lfloor \frac{m-2}{2} \rfloor - 1, m\lfloor \frac{n-1}{2} \rfloor - 1\}$, then $\kappa_g(G) = \min\{m, 2n, \lceil 2\sqrt{2(g+1)} \rceil + 2\}$.*

Proof. Denote $C_m = x_0 x_1 \ldots x_{m-1} x_m$ (where $x_0 = x_m$) and $P_n = y_1 y_2 \ldots y_n$. The addition of the subscripts of x in the proof is modular m arithmetic. Let $S_1 = V(C_m) \times \{y_{\lfloor \frac{n-1}{2} \rfloor + 1}\}$ and $S_2 = \{x_0, x_{\lfloor \frac{m-2}{2} \rfloor + 1}\} \times V(P_n)$. Since $g \leq \min\{n\lfloor \frac{m-2}{2} \rfloor - 1, m\lfloor \frac{n-1}{2} \rfloor - 1\}$, it is routine to check that S_1 and S_2 are two g-extra cuts of G. Thus $\kappa_g(G) \leq \min\{m, 2n\}$. If $\lceil 2\sqrt{2(g+1)} \rceil + 2 \geq \min\{m, 2n\}$, then $\kappa_g(G) \leq \min\{m, 2n, \lceil 2\sqrt{2(g+1)} \rceil + 2\}$. If $\lceil 2\sqrt{2(g+1)} \rceil + 2 < \min\{m, 2n\}$, then let $S_3 = (J_1 \times K_2) \cup (J_1 \times J_2) \cup (K_1 \times J_2)$, where $J_1 = \{x_0, x_{\lceil \sqrt{2(g+1)} \rceil + 1}\}$, $K_1 = \{x_1, x_2, \ldots, x_{\lceil \sqrt{2(g+1)} \rceil}\}$, $J_2 = \{y_{\lceil \frac{2(g+1)}{\lceil \sqrt{2(g+1)} \rceil} \rceil + 1}\}$ and $K_2 = \{y_1, y_2, \ldots, y_{\lceil \frac{2(g+1)}{\lceil \sqrt{2(g+1)} \rceil} \rceil}\}$. It is routine to verify that S_3 is a g-extra cut of G. By $|S_3| = \lceil \sqrt{2(g+1)} \rceil + \lceil \frac{2(g+1)}{\lceil \sqrt{2(g+1)} \rceil} \rceil + 2 = \lceil 2\sqrt{2(g+1)} \rceil + 2$, we have $\kappa_g(G) \leq \lceil 2\sqrt{2(g+1)} \rceil + 2$. Therefore, $\kappa_g(G) \leq \min\{m, 2n, \lceil 2\sqrt{2(g+1)} \rceil + 2\}$. $\square$

Now, it is sufficient to prove $\kappa_g(G) \geq \min\{m, 2n, \lceil 2\sqrt{2(g+1)} \rceil + 2\}$. Assume S is a κ_g-cut of G. We consider two cases in the following.
Case 1. $_xS \neq \varnothing$ for all $x \in V(C_m)$, or $S_y \neq \varnothing$ for all $y \in V(P_n)$.

Assume $_xS \neq \varnothing$ for all $x \in V(C_m)$. By *Lemma* 2.1, $|S| = \sum_{x \in V(C_m)} |_xS| \geq \kappa(P_n)|V(C_m)| = m$. Analogously, if $S_y \neq \varnothing$ for all $y \in V(P_n)$, then $|S| = \sum_{y \in V(P_n)} |S_y| \geq \kappa(C_m)|V(P_n)| = 2n$.

Case 2. There exist a vertex $x_a \in V(C_m)$ and a vertex $y_b \in V(P_n)$ such that $_{x_a}S = S_{y_b} = \emptyset$.

By the assumption $_{x_a}S = S_{y_b} = \emptyset$, we know $V(_{x_a}G_2)$ and $V(G_{1y_b})$ are contained in a component H' of $G - S$. Let H be another component of $G - S$. Let $p_1(V(H)) = \{x_{s+1}, x_{s+2}, \ldots, x_{s+k}\}$ and $p_2(V(H)) = \{y_{t+1}, y_{t+2}, \ldots, y_{t+h}\}$. Without loss of generality, assume $s + k < a$ and $t + h < b$. Clearly, $|V(H)| \leq kh$. Since S is a κ_g-cut, we have $N_G(V(H)) = S$ and $|V(H)| \geq g + 1$. If we can prove $|N_G(V(H))| \geq k + 2h + 2$, then $\kappa_g(G) = |S| = |N_G(V(H))| \geq k + 2h + 2 \geq 2\sqrt{2kh} + 2 \geq 2\sqrt{2(g+1)} + 2$ and the theorem holds. Thus, we only need to show that $|N_G(V(H))| \geq k + 2h + 2$ in the remaining proof.

Let (x_{s+i}, y_{d_i}) be the vertex in $_{x_{s+i}}H$ such that d_i is maximum for $i = 1, \ldots, k$, and let (x_{l_j}, y_{t+j}) and (x_{r_j}, y_{t+j}) be the vertices in $H_{y_{t+j}}$ such that l_j and r_j are listed in the foremost and in the last along the sequence $(a+1, \ldots, m-1, 0, 1, \ldots, a-1)$, respectively, for $j = 1, \ldots, h$. Denote $D = \{(x_{s+1}, y_{d_1}), \ldots, (x_{s+k}, y_{d_k})\}$, $L = \{(x_{l_1}, y_{t+1}), \ldots, (x_{l_h}, y_{t+h})\}$ and $R = \{(x_{r_1}, y_{t+1}), \ldots, (x_{r_h}, y_{t+h})\}$. For the convenience of counting, we will construct an injective mapping f from $D \cup L \cup R$ to $N_G(V(H) \setminus \{(x_s, y_{d_1+1}), (x_{s+k+1}, y_{d_k+1})\}$. Although D, L and R may have common elements, we consider the elements in D, L and R to be different in defining the mapping f below.

First, the mapping f on D is defined as follows.

$$f((x_{s+i}, y_{d_i})) = (x_{s+i}, y_{d_i+1}) \text{ for } i = 1, \ldots, k.$$

Denote $F_1 = \{(x_{s+1}, y_{d_1+1}), \ldots, (x_{s+k}, y_{d_k+1})\}$.

Second, for each vertex (x_{r_j}, y_{t+j}) satisfying $(x_{r_j+1}, y_{t+j}) \notin F_1$, define $f((x_{r_j}, y_{t+j})) = (x_{r_j+1}, y_{t+j})$.

If (x_{r_j}, y_{t+j}) satisfies $(x_{r_j+1}, y_{t+j}) \notin F_1$ for any $j \in \{1, \ldots, h\}$, then we are done. Otherwise, for each $(x_{r_{j'}}, y_{t+j'})$ satisfying $(x_{r_{j'}+1}, y_{t+j'}) \in F_1$, we give the definition as follows. By the definitions of D and R, we have $\{(x_{r_{j'}}, y_{t+j'}), \ldots, (x_{r_{j'}}, y_{d_{r_{j'}-s}})\} \subseteq R$. Now, we define $f((x_{r_{j'}}, y_{t+j'})) = (x_{r_{j'}+1}, y_{t+j'+1})$ and change the images of $(x_{r_{j'}}, y_{t+j'+1}), \ldots, (x_{r_{j'}}, y_{d_{r_{j'}-s}})$ to $(x_{r_{j'}+1}, y_{t+j'+2}), \ldots, (x_{r_{j'}+1}, y_{d_{r_{j'}-s}+1})$, respectively. The mapping f on R is defined well .

Third, for each vertex (x_{l_j}, y_{t+j}) satisfying $(x_{l_j-1}, y_{t+j}) \notin F_1$, define $f((x_{l_j}, y_{t+j})) = (x_{l_j-1}, y_{t+j})$.

If (x_{l_j}, y_{t+j}) satisfies $(x_{l_j-1}, y_{t+j}) \notin F_1$ for any $j \in \{1, \ldots, h\}$, then we are done. Otherwise, for each $(x_{l_{j'}}, y_{t+j'})$ satisfying $(x_{l_{j'}-1}, y_{t+j'}) \in F_1$. By the definitions of D and L, we have $\{(x_{l_{j'}}, y_{t+j'}), \ldots, (x_{l_{j'}}, y_{d_{l_{j'}-s}})\} \subseteq L$. Now, we define $f((x_{l_{j'}}, y_{t+j'})) = (x_{l_{j'}-1}, y_{t+j'+1})$ and change the images of $(x_{l_{j'}}, y_{t+j'+1}), \ldots, (x_{l_{j'}}, y_{d_{l_{j'}-s}})$ to $(x_{r_{j'}-1}, y_{t+j'+2}), \ldots, (x_{l_{j'}-1}, y_{d_{r_{j'}-s}+1})$, respectively. The definition of f on L is complete.

Finally, we construct an injective mapping f from $D \cup L \cup R$ to $N_G(V(H) \setminus \{(x_s, y_{d_1+1}), (x_{s+k+1}, y_{d_k+1})\}$. Then $\kappa_g(G) = |S| = |N_G(V(H))| \geq |D| + |L| + |R| + 2 \geq k + 2h + 1 \geq 2\sqrt{2kh} + 2 \geq 2\sqrt{2(g+1)} + 2$. The proof is thus complete. $\square$

Since $\kappa_g(C_3 \boxtimes C_n) = 6$ for $g \leq 3\lfloor \frac{n-2}{2} \rfloor - 1$, we assume $m, n \geq 4$ in the following theorem.

Theorem 5. *Let g be a non-negative integer and $G = C_m \boxtimes C_n$, where $m, n \geq 4$. If $g \leq min\{n\lfloor \frac{m-2}{2} \rfloor - 1, m\lfloor \frac{n-2}{2} \rfloor - 1\}$, then $\kappa_g(G) = min\{2m, 2n, \lceil 4\sqrt{g+1} \rceil + 4\}$.*

Proof. Denote $C_m = x_0 x_1 \ldots x_{m-1} x_m$ (where $x_0 = x_m$) and $C_n = y_0 y_1 \ldots y_n$ (where $y_0 = y_n$). The addition of the subscripts of x in the proof is modular m arithmetic, and the addition of the subscripts of y in the proof is modular n arithmetic. Let $S_1 = V(C_m) \times \{y_0, y_{\lfloor \frac{n-2}{2} \rfloor+1}\}$ and $S_2 = \{x_0, x_{\lfloor \frac{m-2}{2} \rfloor+1}\} \times V(C_n)$. Since $g \leq min\{n\lfloor \frac{m-2}{2} \rfloor - 1, m\lfloor \frac{n-2}{2} \rfloor - 1\}$, we can check that S_1 and S_2 are two g-extra cuts of G. Thus $\kappa_g(G) \leq min\{2m, 2n\}$. If $\lceil 4\sqrt{g+1} \rceil + 4 \geq min\{2m, 2n\}$, then $\kappa_g(G) \leq min\{2m, 2n, \lceil 4\sqrt{g+1} \rceil + 4\}$. If $\lceil 4\sqrt{g+1} \rceil + 4 < min\{2m, 2n\}$, then let $S_3 = (J_1 \times K_2) \cup (J_1 \times J_2) \cup (K_1 \times J_2)$,

where $J_1 = \{x_0, x_{\lceil\sqrt{g+1}\rceil+1}\}$, $K_1 = \{x_1, x_2, \ldots, x_{\lceil\sqrt{g+1}\rceil}\}$, $J_2 = \{y_0, y_{\lceil\frac{g+1}{\lceil\sqrt{g+1}\rceil}\rceil+1}\}$ and
$K_2 = \{y_1, y_2, \ldots, y_{\lceil\frac{g+1}{\lceil\sqrt{g+1}\rceil}\rceil}\}$. It is routine to verify that S_3 is a g-extra cut of G. By
$|S_3| = 2\lceil\sqrt{g+1}\rceil + 2\lceil\frac{g+1}{\lceil\sqrt{g+1}\rceil}\rceil + 4 = \lceil 4\sqrt{g+1}\rceil + 4$, we have $\kappa_g(G) \le \lceil 4\sqrt{g+1}\rceil + 4$.
Therefore, $\kappa_g(G) \le \min\{2m, 2n, \lceil 4\sqrt{g+1}\rceil + 4\}$. □

Now, it is sufficient to prove $\kappa_g(G) \ge \min\{2m, 2n, \lceil 4\sqrt{g+1}\rceil + 4\}$. Assume S is a
κ_g-cut of G. We consider two cases in the following.

Case 1. $_xS \ne \varnothing$ for all $x \in V(C_m)$, or $S_y \ne \varnothing$ for all $y \in V(C_n)$.

Assume $_xS \ne \varnothing$ for all $x \in V(C_m)$. By *Lemma* 2.1, $|S| = \sum_{x\in V(C_m)} |_xS| \ge \kappa(C_n)|V(C_m)| = 2m$. Analogously, if $S_y \ne \varnothing$ for all $y \in V(C_n)$, then $|S| = \sum_{y\in V(C_n)} |S_y| \ge \kappa(C_m)|V(C_n)| = 2n$.

Case 2. There exist a vertex $x_a \in V(C_m)$ and a vertex $y_b \in V(C_n)$ such that $_{x_a}S = S_{y_b} = \varnothing$.

By the assumption $_{x_a}S = S_{y_b} = \varnothing$, we know $V(_{x_a}G_2)$ and $V(G_{1y_b})$ are contained
in a component H' of $G - S$. Let H be another component of $G - S$. Let $p_1(V(H)) = \{x_{s+1}, x_{s+2}, \ldots, x_{s+k}\}$ and $p_2(V(H)) = \{y_{t+1}, y_{t+2}, \ldots, y_{t+h}\}$. Without loss of generality,
assume $s + k < a$ and $t + h < b$. Clearly, $|V(H)| \le kh$. Since S is a κ_g-cut, we have
$N_G(V(H)) = S$ and $|V(H)| \ge g + 1$. If we can prove $|N_G(V(H))| \ge 2k + 2h + 4$, then
$\kappa_g(G) = |S| = |N_G(V(H))| \ge 2k + 2h + 4 \ge 4\sqrt{kh} + 4 \ge 4\sqrt{g+1} + 4$ and the theorem
holds. Thus, we only need to show that $|N_G(V(H))| \ge 2k + 2h + 4$ in the remaining proof.

Let (x_{s+i}, y_{t_i}) and (x_{s+i}, y_{d_i}) be the vertices in $_{x_{s+i}}H$ such that t_i and d_i are listed in
the foremost and in the last along the sequence $(b+1, \ldots, n-1, 0, 1, \ldots, b-1)$, respec-
tively, for $i = 1, \ldots, k$, and let (x_{l_j}, y_{t+j}) and (x_{r_j}, y_{t+j}) be the vertices in $H_{y_{t+j}}$ such that
l_j and r_j are listed in the foremost and in the last along the sequence $(a+1, \ldots, m-1, 0, 1, \ldots, a-1)$, respectively, for $j = 1, \ldots, h$. Denote $D = \{(x_{s+1}, y_{d_1}), \ldots, (x_{s+k}, y_{d_k})\}$,
$T = \{(x_{s+1}, y_{t_1}), \ldots, (x_{s+k}, y_{t_k})\}$, $L = \{(x_{l_1}, y_{t+1}), \ldots, (x_{l_h}, y_{t+h})\}$ and $R = \{(x_{r_1}, y_{t+1}), \ldots, (x_{r_h}, y_{t+h})\}$. For the convenience of counting, we will construct an injective mapping f
from $D \cup T \cup L \cup R$ to $N_G(V(H) \setminus \{(x_s, y_{d_1+1}), (x_s, y_{t_1-1}), (x_{s+k+1}, y_{d_k+1}), (x_{s+k+1}, y_{t_k-1})\}$.
Although D, T, L and R may have common elements, we consider the elements in D, T, L
and R to be different in defining the mapping f below.

First, the mapping f on D is defined as follows.

$$f((x_{s+i}, y_{d_i})) = (x_{s+i}, y_{d_i+1}) \text{ for } i = 1, \ldots, k.$$

Denote $F_1 = \{(x_{s+1}, y_{d_1+1}), \ldots, (x_{s+k}, y_{d_k+1})\}$.

Second, the mapping f on T is defined as follows.

$$f((x_{s+i}, y_{t_i})) = (x_{s+i}, y_{t_i-1}) \text{ for } i = 1, \ldots, k.$$

Denote $F_2 = \{(x_{s+1}, y_{t_1-1}), \ldots, (x_{s+k}, y_{t_k-1})\}$.

Third, for each vertex (x_{r_j}, y_{t+j}) satisfying $(x_{r_j+1}, y_{t+j}) \notin F_1$, define $f((x_{r_j}, y_{t+j})) = (x_{r_j+1}, y_{t+j})$.

If (x_{r_j}, y_{t+j}) satisfies $(x_{r_j+1}, y_{t+j}) \notin F_1$ for any $j \in \{1, \ldots, h\}$, then we are done. Oth-
erwise, for each $(x_{r_{j'}}, y_{t+j'})$ satisfying $(x_{r_{j'}+1}, y_{t+j'}) \in F_1$, we define as follows. By the
definitions of D and R, we have $\{(x_{r_{j'}}, y_{t+j'}), \ldots, (x_{r_{j'}}, y_{d_{r_{j'}-s}})\} \subseteq R$. Now, we define
$f((x_{r_{j'}}, y_{t+j'})) = (x_{r_{j'}+1}, y_{t+j'+1})$ and change the images of $(x_{r_{j'}}, y_{t+j'+1}), \ldots, (x_{r_{j'}}, y_{d_{r_{j'}-s}})$
to $(x_{r_{j'}+1}, y_{t+j'+2}), \ldots, (x_{r_{j'}+1}, y_{d_{r_{j'}-s}+1})$, respectively.

Fourth, for each vertex (x_{r_j}, y_{t+j}) satisfying $(x_{r_j+1}, y_{t+j}) \notin F_2$, define $f((x_{r_j}, y_{t+j})) = (x_{r_j+1}, y_{t+j})$.

If (x_{r_j}, y_{t+j}) satisfies $(x_{r_j+1}, y_{t+j}) \notin F_2$ for any $j \in \{1, \ldots, h\}$, then we are done. Oth-
erwise, for each $(x_{r_{j'}}, y_{t+j'})$ satisfying $(x_{r_{j'}+1}, y_{t+j'}) \in F_2$, we define as follows. By the
definitions of T and R, we have $\{(x_{r_{j'}}, y_{t+j'}), \ldots, (x_{r_{j'}}, y_{d_{t_{j'}-s}})\} \subseteq R$. Now, we define

$f((x_{r_{j'}}, y_{t+j'})) = (x_{r_{j'}+1}, y_{t+j'-1})$ and change the images of $(x_{r_{j'}}, y_{t+j'-1}), \ldots, (x_{r_{j'}}, y_{d_{t_{j'}-s}})$ to $(x_{r_{j'}+1}, y_{t+j'-2}), \ldots, (x_{r_{j'}+1}, y_{d_{r_{j'}-s}-1})$, respectively.

Note that the proof of four paragraphs above gives the definition of the mapping f on R. In the following proof, we will give the definition of the mapping f on L.

Fifth, for each vertex (x_{l_j}, y_{t+j}) satisfying $(x_{l_j-1}, y_{t+j}) \notin F_1$, define $f((x_{l_j}, y_{t+j})) = (x_{l_j-1}, y_{t+j})$.

If (x_{l_j}, y_{t+j}) satisfies $(x_{l_j-1}, y_{t+j}) \notin F_1$ for any $j \in \{1, \ldots, h\}$, then we are done. Otherwise, for each $(x_{l_{j'}}, y_{t+j'})$ satisfying $(x_{l_{j'}-1}, y_{t+j'}) \in F_1$, we define as follows. By the definitions of D and L, we have $\{(x_{l_{j'}}, y_{t+j'}), \ldots, (x_{l_{j'}}, y_{d_{l_{j'}-s}})\} \subseteq L$. Now, we define $f((x_{l_{j'}}, y_{t+j'})) = (x_{l_{j'}-1}, y_{t+j'+1})$ and change the images of $(x_{l_{j'}}, y_{t+j'+1}), \ldots, (x_{l_{j'}}, y_{d_{l_{j'}-s}})$ to $(x_{r_{j'}-1}, y_{t+j'+2}), \ldots, (x_{l_{j'}-1}, y_{d_{r_{j'}-s}+1})$, respectively.

Sixth, for each vertex (x_{l_j}, y_{t+j}) satisfying $(x_{l_j-1}, y_{t+j}) \notin F_2$, define $f((x_{l_j}, y_{t+j})) = (x_{l_j-1}, y_{t+j})$.

If (x_{l_j}, y_{t+j}) satisfies $(x_{l_j-1}, y_{t+j}) \notin F_2$ for any $j \in \{1, \ldots, h\}$, then we are done. Otherwise, for each $(x_{l_{j'}}, y_{t+j'})$ satisfying any $(x_{t_{j'}-1}, y_{t+j'}) \in F_2$, we define as follows. By the definitions of L and T, we have $\{(x_{l_{j'}}, y_{t+j'}), \ldots, (x_{l_{j'}}, y_{d_{t_{j'}-s}})\} \subseteq L$. Now, we define $f((x_{l_{j'}}, y_{t+j'})) = (x_{l_{j'}-1}, y_{t+j'-1})$ and change the images of $(x_{l_{j'}}, y_{t+j'-1}), \ldots, (x_{l_{j'}}, y_{d_{t_{j'}-s}})$ to $(x_{l_{j'}-1}, y_{t+j'-2}), \ldots, (x_{l_{j'}-1}, y_{d_{t_{j'}-s}-1})$, respectively.

Finally, we construct an injective mapping f from $D \cup T \cup L \cup R$ to $N_G(V(H) \setminus \{(x_s, y_{d_1+1}), (x_s, y_{t_1-1}), (x_{s+k+1}, y_{d_k+1}), (x_{s+k+1}, y_{t_k-1})\}$. Then $\kappa_g(G) = |S| = |N_G(V(H))| \geq |D| + |T| + |L| + |R| + 4 \geq 2k + 2h + 4 \geq 4\sqrt{kh} + 4 \geq 4\sqrt{g+1} + 4$. The proof is thus complete. $\square$

4. Conclusions

Graph products are used to construct large graphs from small ones. Strong product is one of the most studied four graph products. As a generalization of traditional connectivity, g-extra connectivity can be seen as a refined parameter to measure the reliability of interconnection networks. There is no polynomial-time algorithm to compute the g (≥ 1)-extra connectivity for a general graph. In this paper, we determined the g-extra connectivity of the strong product of two paths, the strong product of a path and a cycle, and the strong product of two cycles. In the future work, we would like to investigate the g-extra connectivity of the strong product of two general graphs.

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