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On an Important Remark Concerning Some MHD Motions of Second-Grade Fluids through Porous Media and Its Applications

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Abstract: In this work it is proven that the governing equations for the fluid velocity and non-trivial shear stress corresponding to some isothermal MHD unidirectional motions of incompressible second-grade fluids through a porous medium have identical forms. This important remark is used to provide exact steady-state solutions for motions with shear stress on the boundary when similar solutions of some motions with velocity on the boundary are known. Closed-form expressions are provided both for the fluid velocity and the corresponding shear stress and Darcy's resistance. As a check of the results that are obtained here, the solutions corresponding to motions over an infinite flat plate are presented in different forms whose equivalence is graphically proven. In the case of the motions between infinite parallel plates, the fluid behavior is symmetric with respect to the median plane due to the boundary conditions.

Keywords: second-grade fluids; isothermal MHD motions; porous media; steady-state solutions

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1. Introduction

Fluids' motions over an infinite plate or between two infinite parallel plates have been extensively studied in time. They are some of the most important motion problems near moving bodies having multiple applications in engineering and science in general. The exact solutions for the governing equations of these motions are important for many reasons. First of all, they describe the behavior of respective fluids in different circumstances. Secondly, they can be used as tests to verify numerical schemes which are used to study more complex motion problems. Although the numerical integration of the governing equations corresponding to such motions of fluids can be realized using computers, the accuracy of results can be established by comparing with exact solutions. The first exact solutions for unsteady motions of incompressible second-grade fluids seem to be those of Ting [1]. Other interesting solutions for the same fluids have been obtained by Siddiqui et al. [2] and Hayat et al. [3].

The magnetohydrodynamic (MHD) motions of fluids have many applications in hydrology, polymer technology, petroleum industry, nuclear reactors, and MHD generators. The interaction between the magnetic field and the electrical conducting fluid produces effects with important applications in physics, chemistry, engineering, horticulture, and hydrology. The MHD steady Couette flow of incompressible viscous fluids between parallel plates was studied by Kiema et al. [4] using the Sumudu transform. The same problem was also studied by Onyango et al. [5] when the magnetic-field lines are fixed relative to the moving upper plate. At the same time, the motions of incompressible fluids through porous media have received special attention due to their practical applications in geophysical and astrophysical studies, agricultural engineering, petroleum industries, and oil reservoir technology. Some interesting studies on porosity can be found, for instance, in the book of Vafai [6]. The most recent results regarding motions of non-Newtonian fluids through porous media seem to be those of Fetecau et al. [7,8]. Viscoelastic MHD flow between porous parallel plates has been studied by Dash and Ojha [9] in the presence of a sinusoidal pressure gradient. The combined magnetic and porous effects have been taken into consideration by Hayat et al. [10] and Khan et al. [11] for unsteady motions of second-grade and generalized Burgers fluids, respectively. However, the first exact general solution for isothermal MHD unsteady motions of viscous fluids over an infinite plate or between infinite parallel plates embedded in a porous medium have been established by Fetecau et al. [12,13] using a marvelous remark regarding the governing equations of the fluid velocity and the corresponding non-trivial shear stress.

The main purpose of this work is to show that the above-mentioned remark is also valid for incompressible second-grade fluids performing the same MHD motions through porous media. More precisely, the governing equations for velocity and shear stress corresponding to some MHD motions of incompressible second-grade fluids through a porous medium are identical in form. On the basis of this remark, the solutions corresponding to some unidirectional MHD motions of incompressible second-grade fluids through a porous medium with velocity or shear stress on the boundary can be immediately obtained if similar solutions for MHD motions of the same fluids through a porous medium are known when the shear stress or the fluid velocity is given on the boundary, respectively. In order to bring to light the advantages of this remark, some significant examples are provided, and new exact solutions for MHD motions of incompressible second-grade fluids through porous media are easily determined.

2. Statement of the Problem

The Cauchy stress tensor $\boldsymbol{T}$ for incompressible second-grade fluids is given by the relation [1–3]

$$\boldsymbol{T} = -p\boldsymbol{I} + \mu\boldsymbol{A}_1 + \alpha_1\boldsymbol{A}_2 + \alpha_2\boldsymbol{A}_1^2, \tag{1}$$

where $-p\boldsymbol{I}$ is the spherical stress due to the constraint of incompressibility, μ is the fluid viscosity, α_1 and α_2 are material constants also called normal stress moduli [14], while the first two Rivlin–Ericksen tensors $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are defined as

$$\boldsymbol{A}_1 = \boldsymbol{L} + \boldsymbol{L}^T,\ \boldsymbol{A}_2 = \frac{d\boldsymbol{A}_1}{dt} + \boldsymbol{A}_1\boldsymbol{L} + \boldsymbol{L}^T\boldsymbol{A}_1. \tag{2}$$

In the last relation, $\boldsymbol{L}$ is the gradient of the velocity vector $\boldsymbol{v}$. The Clausius–Duhem inequality and the assumption that the Helmholtz free energy is minimum when the fluid is at rest imply the following restrictions [3]

$$\mu \geq 0,\ \ \alpha_1 \geq 0,\ \text{and}\ \alpha_1 + \alpha_2 = 0. \tag{3}$$

If $\alpha_1 = \alpha_2 = 0$, Equation (1) represents the constitutive equation of incompressible viscous fluids. Because incompressible fluids undergo isochoric motions only, the continuity equation

$$\mathrm{div}\boldsymbol{v} = 0\ \text{ or }\ \text{equivalent}\ \mathrm{tr}\boldsymbol{A}_1 = 0, \tag{4}$$

has to be satisfied. In addition, in the absence of body forces, the balance of linear momentum for MHD unsteady fluid motions through porous media is given by the following relation [10]

$$\rho\frac{d\boldsymbol{v}}{dt} = \mathrm{div}\boldsymbol{T} + \boldsymbol{R} + \boldsymbol{J}\times\boldsymbol{B}. \tag{5}$$

In the above equation, ρ is the fluid density and $\boldsymbol{R}$ is the Darcy's resistance defined by

$$\boldsymbol{R} = -\frac{\mu\varphi}{k}\left(1 + \frac{\alpha_1}{\mu}\frac{\partial}{\partial t}\right)\boldsymbol{v}, \tag{6}$$

where φ $(0 < \varphi < 1)$ is the porosity and $k > 0$ is the permeability of the porous medium. The last term from the right part of Equation (5) represents the Lorentz force due to the interaction between the current density $\boldsymbol{J}$ and the magnetic induction $\boldsymbol{B}$. We also assume that the fluid is finitely conducting so that the Joule heat due to the presence of magnetic field is negligible. In addition, there is no surplus electric charge distribution present in the fluid and the magnetic Reynolds number is small enough. Consequently, the induced magnetic field can be neglected. In these conditions, we can obtain [10]

$$\boldsymbol{J} \times \boldsymbol{B} = -\sigma B^2 \boldsymbol{v}, \tag{7}$$

where σ is the electrical conductivity of the fluid and B is the magnitude of the applied magnetic field.

In the following, we shall consider isothermal unidirectional motions whose velocity field $\boldsymbol{v}$ reports to a suitable Cartesian coordinate system x, y, and z, and is of the form

$$\boldsymbol{v} = \boldsymbol{v}(y,t) = u(y,t)\mathbf{e}_x, \tag{8}$$

where $\mathbf{e}_x$ is the unit vector along the x-direction. Substituting the fluid velocity $\boldsymbol{v}(y,t)$ from Equation (8) in (1), making use of the equalities (6) and (7), and assuming that there exists no pressure gradient in the flow direction, the equality (5) becomes

$$\rho\frac{\partial u(y,t)}{\partial t} = \frac{\partial \tau(y,t)}{\partial y} - \sigma B^2 u(y,t) - \frac{\mu\varphi}{k}\left(1+\alpha\frac{\partial}{\partial t}\right)u(y,t), \tag{9}$$

where the non-trivial component $\tau(y,t) = T_{xy}(y,t)$ of $\boldsymbol{T}$ is given by the next relation

$$\tau(y,t) = \mu\left(1+\alpha\frac{\partial}{\partial t}\right)\frac{\partial u(y,t)}{\partial y}, \tag{10}$$

and $\alpha = \alpha_1/\mu$.

Eliminating $\tau(y,t)$ between Equations (9) and (10), one obtains the governing equation

$$\rho\frac{\partial u(y,t)}{\partial t} = \mu\left(1+\alpha\frac{\partial}{\partial t}\right)\frac{\partial^2 u(y,t)}{\partial y^2} - \sigma B^2 u(y,t) - \frac{\mu\varphi}{k}\left(1+\alpha\frac{\partial}{\partial t}\right)u(y,t), \tag{11}$$

for the dimensional velocity field $u(y,t)$. Deriving Equation (9) with respect to y, we can obtain

$$\rho\frac{\partial}{\partial t}\left[\frac{\partial u(y,t)}{\partial y}\right] = \frac{\partial^2 \tau(y,t)}{\partial y^2} - \sigma B^2\frac{\partial u(y,t)}{\partial y} - \frac{\mu\varphi}{k}\left(1+\alpha\frac{\partial}{\partial t}\right)\frac{\partial u(y,t)}{\partial y}. \tag{12}$$

Further, deriving Equation (12) with respect to the temporal variable t and multiplying the obtained result with α, one finds that

$$\rho\alpha\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial t}\left[\frac{\partial u(y,t)}{\partial y}\right]\right\} = \alpha\frac{\partial}{\partial t}\left[\frac{\partial^2\tau(y,t)}{\partial y^2}\right] - \sigma B^2\alpha\frac{\partial}{\partial t}\left[\frac{\partial u(y,t)}{\partial y}\right] - \frac{\mu\varphi}{k}\alpha\frac{\partial}{\partial t}\left[\left(1+\alpha\frac{\partial}{\partial t}\right)\frac{\partial u(y,t)}{\partial y}\right]. \tag{13}$$

Finally, adding the equalities (12) and (13) and bearing in mind the equality (10), one obtains for the dimensional shear stress $\tau(y,t)$ the following partial differential equation

$$\rho\frac{\partial \tau(y,t)}{\partial t} = \mu\left(1+\alpha\frac{\partial}{\partial t}\right)\frac{\partial^2 \tau(y,t)}{\partial y^2} - \sigma B^2 \tau(y,t) - \frac{\mu\varphi}{k}\left(1+\alpha\frac{\partial}{\partial t}\right)\tau(y,t), \tag{14}$$

which is identical in form to that of the velocity field $u(y,t)$.

3. Applications

In the previous section, it was proven that the governing equations of the fluid velocity $u(y,t)$ and the adequate non-trivial shear stress $\tau(y,t)$ corresponding to a large class of isothermal MHD unidirectional motions (whose velocity field is given by Equation (8)) of incompressible second-grade fluids through a porous medium are identical in form.

In order to bring to light the power of this important remark, some applications will be provided in the next section.

3.1. Motions over an Infinite Flat Plate

3.1.1. Stokes' Second Problem (Motions with Velocity on the Boundary)

Let us consider the isothermal MHD unsteady unidirectional motion of an electrically conducting incompressible second-grade fluid (ECISGF) over an infinite flat plate embedded in a porous medium. The fluid motion is due to the plate that moves in its plane according to one of the relations

$$\boldsymbol{v} = U\cos(\omega t)\mathbf{e}_x \ \text{ or } \ \boldsymbol{v} = U\sin(\omega t)\mathbf{e}_x, \tag{15}$$

where U and ω are the amplitude and the frequency of the oscillations, respectively. The fluid velocity $u(y,t)$ and the corresponding shear stress $\tau(y,t)$ have to satisfy the governing Equations (9) and (10) from the previous section. The next boundary conditions

$$u(0,t) = U\cos(\omega t), \ \lim_{y\to\infty} u(y,t) = 0, \tag{16}$$

or

$$u(0,t) = U\sin(\omega t), \ \lim_{y\to\infty} u(y,t) = 0, \tag{17}$$

have to be satisfied. The second condition from Equations (16) and (17) indicates the fact that the fluid is quiescent far away from the plate. We also assume that there is no shear in the free stream, i.e.,

$$\lim_{y\to\infty} \tau(y,t) = 0. \tag{18}$$

The dimensionless forms of the governing Equations (9) and (10) and of the boundary conditions (16) and (17), namely

$$\tau(y,t) = \left(1 + \alpha\frac{\partial}{\partial t}\right)\frac{\partial u(y,t)}{\partial y}, \tag{19}$$

$$\frac{\partial u(y,t)}{\partial t} = \left(1 + \alpha\frac{\partial}{\partial t}\right)\frac{\partial^2 u(y,t)}{\partial y^2} - Mu(y,t) - K\left(1 + \alpha\frac{\partial}{\partial t}\right)u(y,t) \tag{20}$$

and

$$u(0,t) = \cos(\omega t), \ \lim_{y\to\infty} u(y,t) = 0, \tag{21}$$

or

$$u(0,t) = \sin(\omega t), \ \lim_{y\to\infty} u(y,t) = 0, \tag{22}$$

are obtained using the next non-dimensional variables, functions, and parameters

$$y^* = \frac{U}{\nu}y, \ t^* = \frac{U^2}{\nu}t, \ u^* = \frac{u}{U}, \ \tau^* = \frac{1}{\rho U^2}\tau, \ \omega^* = \frac{\nu}{U^2}\omega, \ \alpha^* = \frac{U^2}{\nu}\alpha. \tag{23}$$

For simplicity, the star notation has been eliminated. In addition, in the above relations, $\nu = \mu/\rho$ is the kinematic viscosity of the fluid and the magnetic and porous parameters M and K, respectively, which are defined by the next relations

$$M = \frac{\sigma B^2}{\rho}\frac{\nu}{U^2}, \ K = \frac{\varphi}{k}\left(\frac{\nu}{U}\right)^2. \tag{24}$$

From Equations (6) and (8), we can see that Darcy's resistance $\boldsymbol{R}$ has only one component, R, different from zero, which in non-dimensional form is given by the equation

$$R(y,t) = K\left(1 + \alpha\frac{\partial}{\partial t}\right)u(y,t). \tag{25}$$

In the following, in order to avoid confusion, we denote by $u_c(y,t)$, $\tau_c(y,t)$, $R_c(y,t)$, and $u_s(y,t)$, $\tau_s(y,t)$, $R_s(y,t)$ the dimensionless starting solutions of the partial differential Equations (19), (20), and (25) with the boundary conditions (21) and (22), respectively. The initial condition $u(y,0) = 0$ also has to be satisfied. These solutions can be written as the sum of steady-state (permanent or long time) and transient components, namely

$$u_c(y,t) = u_{cp}(y,t) + u_{ct}(y,t),\ \ \tau_c(y,t) = \tau_{cp}(y,t) + \tau_{ct}(y,t),\ \ R_c(y,t) = R_{cp}(y,t) + R_{ct}(y,t), \tag{26}$$

$$u_s(y,t) = u_{sp}(y,t) + u_{st}(y,t),\ \ \tau_s(y,t) = \tau_{sp}(y,t) + \tau_{st}(y,t),\ \ R_s(y,t) = R_{sp}(y,t) + R_{st}(y,t). \tag{27}$$

The dimensionless steady-state solutions $u_{cp}(y,t)$ and $u_{sp}(y,t)$ are independent of the initial condition $u(y,0) = 0$, but they have to satisfy the governing Equation (20) and the boundary conditions (21) and (22), respectively.

Direct computations show that the dimensionless steady-state velocity fields $u_{cp}(y,t)$ and $u_{sp}(y,t)$ corresponding to these motions can be presented in the simple forms

$$u_{cp}(y,t) = \mathrm{e}^{-my}\cos(\omega t - ny),\ u_{sp}(y,t) = \mathrm{e}^{-my}\sin(\omega t - ny), \tag{28}$$

or equivalently

$$u_{cp}(y,t) = \Re e\left\{\mathrm{e}^{-\delta y + i\omega t}\right\},\ u_{sp}(y,t) = \mathrm{Im}\left\{\mathrm{e}^{-\delta y + i\omega t}\right\}, \tag{29}$$

where $\Re e$ and Im are the real and the imaginary part of that which follows, respectively.

The real constants m and n are defined by the equalities

$$m = \sqrt{\frac{\omega}{2}}\sqrt{\frac{a\omega + \sqrt{(a\omega)^2 + b^2}}{1 + (\alpha\omega)^2}},\ n = \sqrt{\frac{\omega}{2}}\sqrt{\frac{-a\omega + \sqrt{(a\omega)^2 + b^2}}{1 + (\alpha\omega)^2}}, \tag{30}$$

where

$$a = \alpha(1 + \alpha K) + \frac{M + K}{\omega^2},\ b = 1 - \alpha M, \tag{31}$$

while $\delta = \sqrt{\frac{M + i\omega + K(1 + i\alpha\omega)}{1 + i\alpha\omega}}$. Equivalence of the expressions of $u_{cp}(y,t)$ and $u_{sp}(y,t)$ given by Equations (28) and (29) is graphically proven by Figure 1. In addition, taking $\alpha = 0$ in Equations (28) and neglecting magnetic and porous effects, steady-state solutions obtained by Erdogan [15] are recovered. Furthermore, coming back to the dimensional variables, functions, and parameters and, again, neglecting the magnetic and porous effects, the present solutions (28) become identical to those obtained by Rajagopal [14].

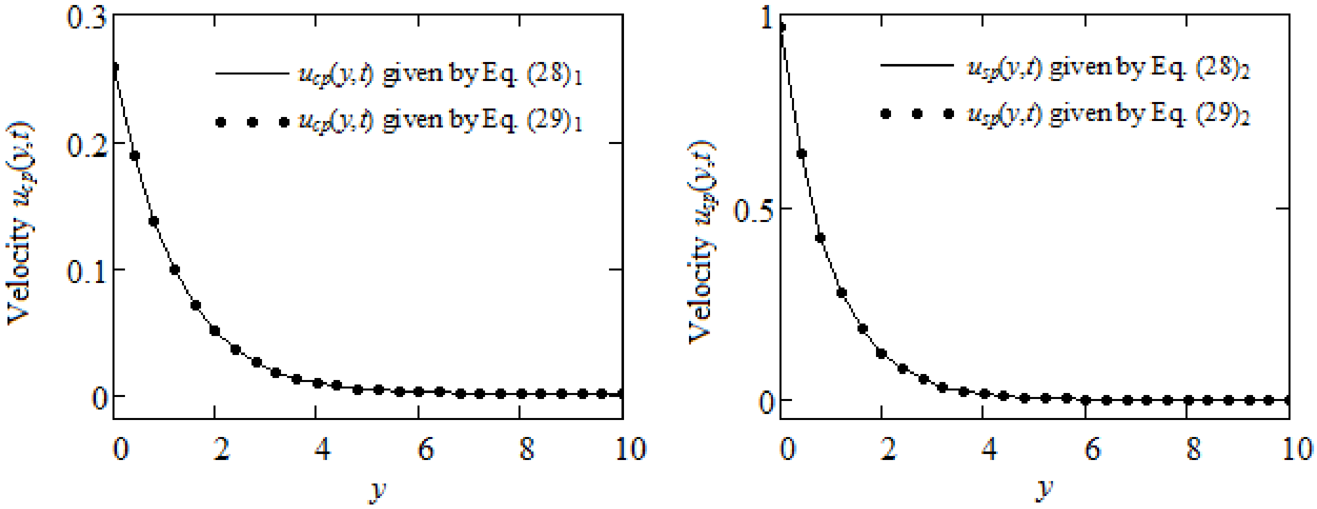

Figure 1. Profiles of velocities $u_{cp}(y,t)$ and $u_{sp}(y,t)$ given by Equations $(28)_1$ and $(29)_1$, respectively: $(28)_2$ and $(29)_2$ for $\alpha = 0.8,\ \omega = \pi/12,\ M = 0.6,\ K = 0.4$, and $t = 5$.

The corresponding dimensionless non-trivial steady-state shear stresses $\tau_{cp}(y,t)$ and $\tau_{sp}(y,t)$, as obtained from the equalities (19), (28), and (29), are given by the relations

$$\tau_{cp}(y,t) = \frac{1}{\sqrt{p^2+q^2}}\mathrm{e}^{-my}\cos(\omega t - ny - \psi),\ \tau_{sp}(y,t) = \frac{1}{\sqrt{p^2+q^2}}\mathrm{e}^{-my}\sin(\omega t - ny - \psi), \tag{32}$$

or equivalently

$$\tau_{cp}(y,t) = -\mathrm{Re}\left\{(1+i\alpha\omega)\delta \mathrm{e}^{-\delta y + i\omega t}\right\},\ \tau_{sp}(y,t) = -\mathrm{Im}\left\{(1+i\alpha\omega)\delta \mathrm{e}^{-\delta y + i\omega t}\right\}, \tag{33}$$

where $p = \alpha\omega m + n,\ q = \alpha\omega n - m$, and $\psi = \mathrm{arctg}(p/q)$.

Introducing $u_{cp}(y,t)$ and $u_{sp}(y,t)$ from Equations (28) and (29) in (25), one obtains the dimensionless expressions for the corresponding Darcy's resistances, namely

$$\begin{aligned} R_{cp}(y,t) &= K\sqrt{1+(\alpha\omega)^2}\,\mathrm{e}^{-my}\cos(\omega t - ny + \beta)\,, \\ R_{sp}(y,t) &= K\sqrt{1+(\alpha\omega)^2}\,\mathrm{e}^{-my}\sin(\omega t - ny + \beta)\,, \end{aligned} \tag{34}$$

or equivalently

$$R_{cp}(y,t) = K\,\Re e\left\{(1+i\alpha\omega)\mathrm{e}^{-\delta y + i\omega t}\right\},\ R_{sp}(y,t) = K\mathrm{Im}\left\{(1+i\alpha\omega)\mathrm{e}^{-\delta y + i\omega t}\right\}, \tag{35}$$

where $\beta = \mathrm{arctg}(\alpha\omega)$.

The equivalence of the expressions of $R_{cp}(y,t)$ and $R_{sp}(y,t)$ from Equations (34) and (35) is shown in Figure 2.

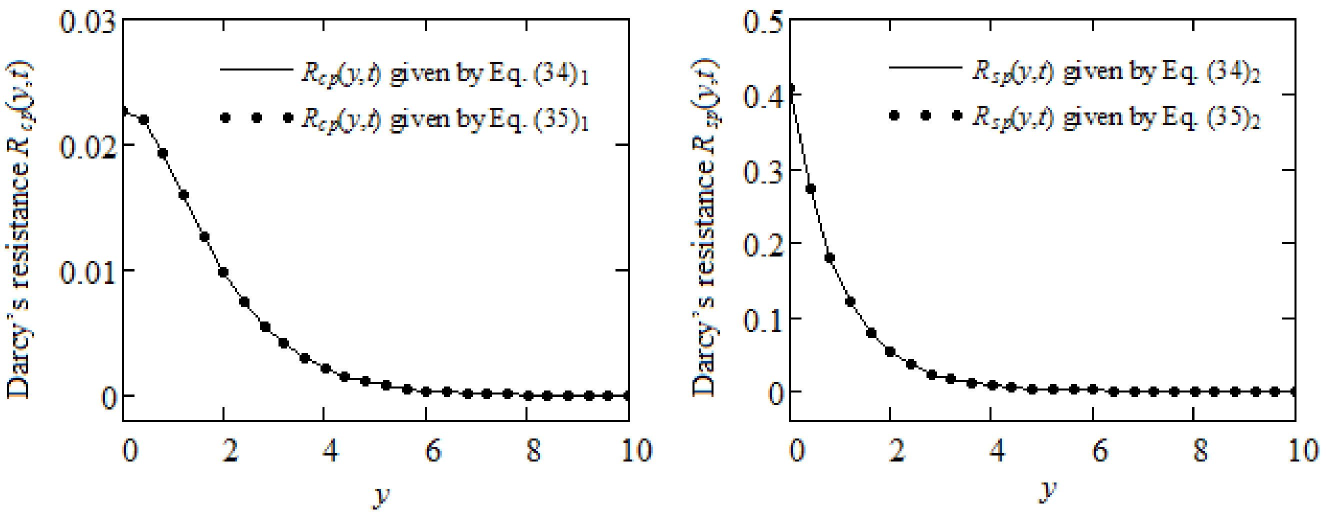

Figure 2. Profiles of Darcy's resistances $R_{cp}(y,t)$ and $R_{sp}(y,t)$ given by Equations (34)$_1$ and (35)$_1$, respectively: (34)$_2$ and (35)$_2$ for $\alpha = 0.8,\ \omega = \pi/12,\ M = 0.6,\ K = 0.4$, and $t = 5$.

3.1.2. Motions with Shear Stress on the Boundary

Let us now assume that the motion of the fluid is induced by the plate that applies a shear stress $S\cos(\omega t)$ or $S\sin(\omega t)$ to the fluid. It is clear that its velocity vector is also given by Equation (8) and the dimensional governing equations corresponding to this motion are identical to those from the previous section. Instead, the boundary conditions are given by the relations

$$\tau(0,t) = S\cos(\omega t),\ \lim_{y\to\infty}\tau(y,t) = 0, \tag{36}$$

or

$$\tau(0,t) = S\sin(\omega t),\ \lim_{y\to\infty}\tau(y,t) = 0. \tag{37}$$

Introducing the next non-dimensional variables, functions, and parameters

$$y^* = \frac{y}{\nu}\sqrt{\frac{S}{\rho}},\ \ t^* = \frac{S}{\mu}t,\ \ u^* = u\sqrt{\frac{\rho}{S}},\ \ \tau^* = \frac{\tau}{S},\ \ \omega^* = \frac{\mu}{S}\omega,\ \ \alpha^* = \frac{S}{\mu}\alpha \tag{38}$$

and, again, giving up the star notation, one obtains for the dimensionless velocity and shear stress fields $\tau(y,t)$, $u(y,t)$ and the corresponding Darcy' resistance $R(y,t)$ the same governing Equations (19), (20), and (25). However, as we have to solve problems with shear stress on the boundary, the non-dimensional form of the governing Equation (14), namely

$$\frac{\partial\tau(y,t)}{\partial t} = \left(1+\alpha\frac{\partial}{\partial t}\right)\frac{\partial^2\tau(y,t)}{\partial y^2} - M\tau(y,t) - K\left(1+\alpha\frac{\partial}{\partial t}\right)\tau(y,t), \tag{39}$$

will be used. As expected, its form is identical to Equation (20) for velocity. In the above equation, the new magnetic and porous parameters M and K are defined by the relations

$$M = \frac{\sigma B^2}{\rho}\frac{\mu}{S},\ K = \frac{\varphi}{k}\frac{\mu\nu}{S}. \tag{40}$$

The corresponding dimensionless boundary conditions are

$$\tau(0,t) = \cos(\omega t),\ \lim_{y\to\infty}\tau(y,t) = 0, \tag{41}$$

or

$$\tau(0,t) = \sin(\omega t),\ \lim_{y\to\infty}\tau(y,t) = 0. \tag{42}$$

Consequently, keeping the same notations for the dimensionless solutions corresponding to these new motion problems and bearing in mind the results of the previous section, the dimensionless shear stresses $\tau_{cp}(y,t)$ and $\tau_{sp}(y,t)$ are given by the relations

$$\tau_{cp}(y,t) = \mathrm{e}^{-my}\cos(\omega t - ny),\ \tau_{sp}(y,t) = \mathrm{e}^{-my}\sin(\omega t - ny), \tag{43}$$

or equivalently

$$\tau_{cp}(y,t) = \Re e\left\{\mathrm{e}^{-\delta y + i\omega t}\right\},\ \tau_{sp}(y,t) = \mathrm{Im}\left\{\mathrm{e}^{-\delta y + i\omega t}\right\}, \tag{44}$$

in which m, n, and δ have been defined in the previous section.

The velocity fields $u_{cp}(y,t)$ and $u_{sp}(y,t)$ corresponding to the shear stresses given by Equation (43), namely

$$u_{cp}(y,t) = -\sqrt{p_1^2 + q_1^2}\mathrm{e}^{-my}\cos(\omega t - ny - \gamma),\ u_{sp}(y,t) = -\sqrt{p_1^2 + q_1^2}\mathrm{e}^{-my}\sin(\omega t - ny - \gamma), \tag{45}$$

where $\gamma = \mathrm{arctg}(q_1/p_1)$ and

$$p_1 = \frac{\alpha\omega n - m}{(\alpha\omega n - m)^2 + (\alpha\omega m + n)^2},\ q_1 = -\frac{\alpha\omega m + n}{(\alpha\omega n - m)^2 + (\alpha\omega m + n)^2}, \tag{46}$$

are determined using the relations (10) and (43). Their equivalent forms

$$u_{cp}(y,t) = -\Re e\left\{\frac{1}{(1 + i\alpha\omega)\delta}\mathrm{e}^{-\delta y + i\omega t}\right\},\ u_{sp}(y,t) = -\mathrm{Im}\left\{\frac{1}{(1 + i\alpha\omega)\delta}\mathrm{e}^{-\delta y + i\omega t}\right\}, \tag{47}$$

are obtained using Equations (10) and (44). The equivalence of dimensionless velocity fields given by the equalities (45) and (47) is graphically proven in Figure 3.

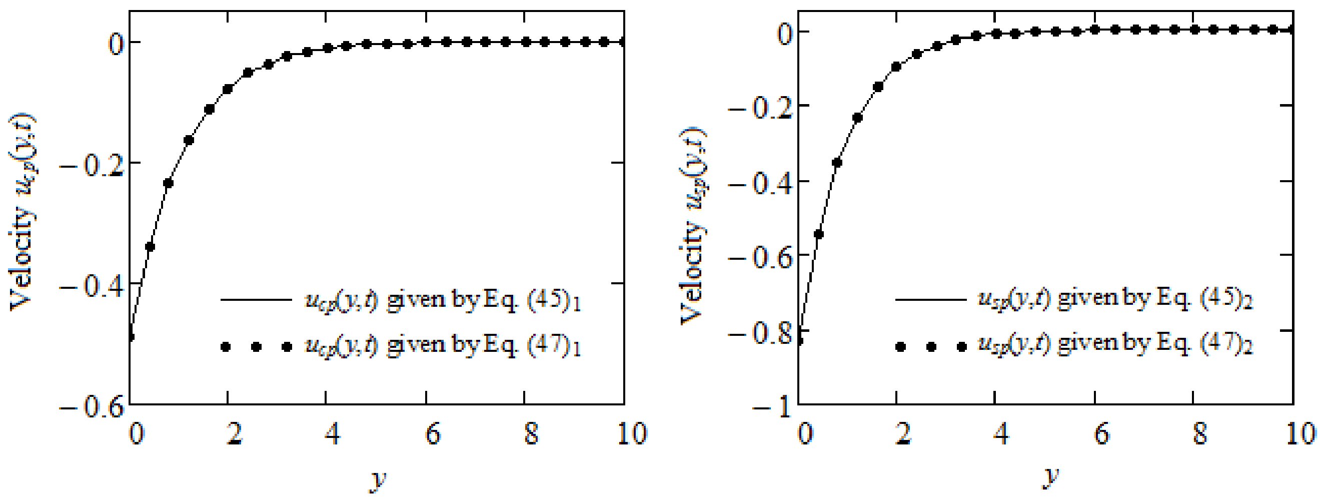

Figure 3. Profiles of velocities $u_{cp}(y,t)$ and $u_{sp}(y,t)$ given by Equations $(45)_1$ and $(47)_1$, respectively: $(45)_2$ and $(47)_2$ for $\alpha = 0.8,\ \omega = \pi/12,\ M = 0.6,\ K = 0.4$, and $t = 5$.

Introducing $u_{cp}(y,t)$ and $u_{sp}(y,t)$ from Equations (45) and (47) in (25), the expressions of the Darcy's resistances, namely

$$\begin{aligned} R_{cp}(y,t) &= -K\sqrt{(p_1^2 + q_1^2)[1 + (\alpha\omega)^2]}\,\mathrm{e}^{-my}\cos(\omega t - ny + \beta - \gamma), \\ R_{sp}(y,t) &= -K\sqrt{(p_1^2 + q_1^2)[1 + (\alpha\omega)^2]}\,\mathrm{e}^{-my}\sin(\omega t - ny + \beta - \gamma), \end{aligned} \tag{48}$$

or equivalent

$$R_{cp}(y,t) = -K\Re e\left\{\frac{1}{\delta}e^{-\delta y+i\omega t}\right\},\ R_{sp}(y,t) = -K\mathrm{Im}\left\{\frac{1}{\delta}e^{-\delta y+i\omega t}\right\}, \tag{49}$$

corresponding to this motion of incompressible second-grade fluids are obtained. The angle β from Equations (48) is equal with $\mathrm{arctg}(\alpha\omega)$, and the equivalence of the expressions of $R_{cp}(y,t)$ and $R_{sp}(y,t)$ from Equations (48) and (49) is proven in Figure 4. In conclusion, the steady-state solutions corresponding to motions with shear stress on the boundary are easily obtained using the steady-state solutions of some motions with velocities on the boundary.

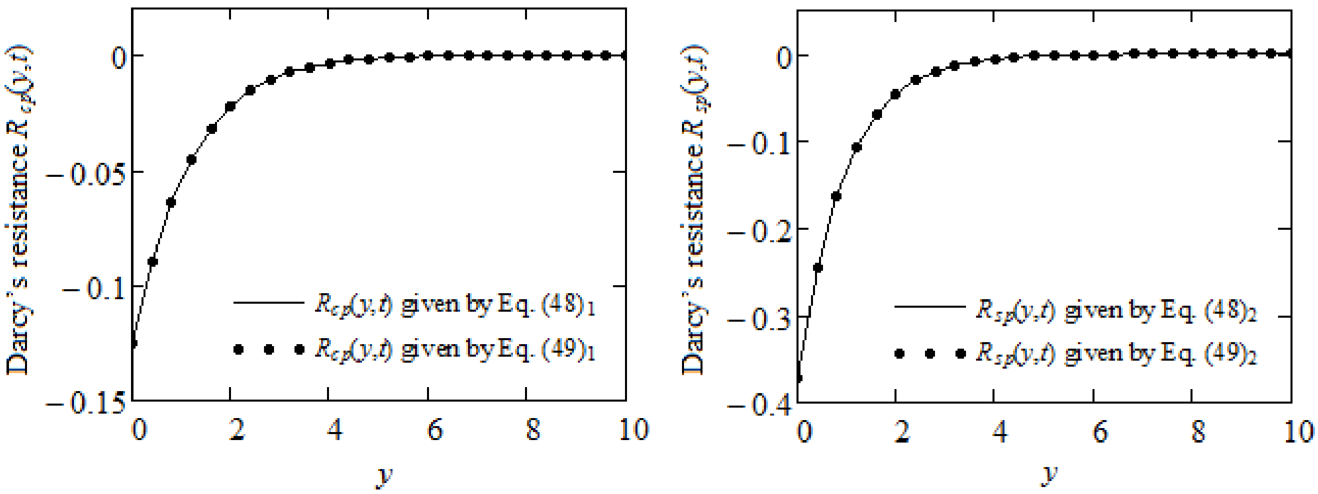

Figure 4. Profiles of Darcy's resistances $R_{cp}(y,t)$ and $R_{sp}(y,t)$ given by Equations $(48)_1$ and $(49)_1$, respectively: $(48)_2$ and $(49)_2$ for $\alpha = 0.8,\ \omega = \pi/12,\ M = 0.6,\ K = 0.4$, and $t = 5$.

3.2. Motions between Infinite Parallel Plates

Let us now consider isothermal MHD steady motions of an ECISGF between two infinite horizontal parallel flat plates embedded in a porous medium. The velocity field of such a motion, which can be generated by both plates, which move in their own planes with the same velocity $U\cos(\omega t)$ or $U\sin(\omega t)$, or applies a shear stress $S\cos(\omega t)$ or $S\sin(\omega t)$ to the fluid, is also of the form (8). In both cases, the corresponding dimensional governing equations are identical to those from Section 2. Exact solutions for velocity and pressure fields corresponding to isothermal steady flows of second-grade fluids in a plane channel have been recently provided by Baranovskii and Artemov [16].

3.2.1. Motions with Velocity on the Boundary

Keeping the same notations as in the previous sections, the dimensional steady-state velocity and shear stress fields $u_{cp}(y,t)$, $\tau_{cp}(y,t)$, and $u_{sp}(y,t)$, $\tau_{sp}(y,t)$, corresponding to these motions have to satisfy the governing Equations (9) and (10) with the boundary conditions

$$u(0,t) = U\cos(\omega t),\ u(d,t) = U\cos(\omega t), \tag{50}$$

or

$$u(0,t) = U\sin(\omega t),\ u(d,t) = U\sin(\omega t), \tag{51}$$

where d is the distance between plates. Introducing the following non-dimensional variables, functions, and parameters

$$y^* = \frac{y}{d},\ \ t^* = \frac{U}{d}t,\ \ u^* = \frac{u}{U},\ \ \tau^* = \frac{1}{\rho U^2}\tau,\ \ \omega^* = \frac{d}{U}\omega,\ \ \alpha^* = \frac{U}{d}\alpha, \tag{52}$$

in Equations (9), (10), (50), and (51) and dropping out the star notation, one obtains for the dimensionless velocity field $u(y,t)$ and the corresponding shear stress $\tau(y,t)$ the following partial differential equations

$$\mathrm{Re}\frac{\partial u(y,t)}{\partial t} = \left(1+\alpha\frac{\partial}{\partial t}\right)\frac{\partial^2 u(y,t)}{\partial y^2} - Mu(y,t) - K\left(1+\alpha\frac{\partial}{\partial t}\right)u(y,t), \tag{53}$$

$$\tau(y,t) = \frac{1}{\mathrm{Re}}\left(1+\alpha\frac{\partial}{\partial t}\right)\frac{\partial u(y,t)}{\partial y}, \tag{54}$$

with the boundary conditions

$$u(0,t) = \cos(\omega t),\ u(1,t) = \cos(\omega t), \tag{55}$$

respectively

$$u(0,t) = \sin(\omega t),\ u(1,t) = \sin(\omega t). \tag{56}$$

The Reynolds number Re and magnetic and porous parameters seen in the above equations, M and K, respectively, are defined by the next relations

$$\mathrm{Re} = \frac{Ud}{\nu},\ M = \frac{\sigma B^2}{\rho}\frac{d^2}{\nu},\ K = \frac{\varphi}{k}d^2. \tag{57}$$

The dimensionless Darcy's resistance $R(y,t)$ corresponding to these motions satisfies Equation (25) in which the porous parameter K is given by the last equality from (57).

Direct computations show that the dimensionless steady-state velocity fields $u_{cp}(y,t)$ and $u_{sp}(y,t)$ corresponding to these motions are given by the following relations

$$\begin{aligned} u_{cp}(y,t) &= \Re e\left\{\frac{\sinh(\widetilde{\delta}y)+\sinh[\widetilde{\delta}(1-y)]}{\sinh(\widetilde{\delta})}\mathrm{e}^{i\omega t}\right\}, \\ u_{sp}(y,t) &= \mathrm{Im}\left\{\frac{\sinh(\widetilde{\delta}y)+\sinh[\widetilde{\delta}(1-y)]}{\sinh(\widetilde{\delta})}\mathrm{e}^{i\omega t}\right\}, \end{aligned} \tag{58}$$

where $\widetilde{\delta} = \sqrt{\frac{M+i\omega\mathrm{Re}+K(1+i\alpha\omega)}{1+i\alpha\omega}}$. It is worth pointing out the fact that for $\omega = 0$, when both plates moves in their planes with the same constant velocity U, the dimensional form of the steady-state solution $u_{cp}(y,t)$ from Equation $(58)_1$ tends to the steady solution obtained by Erdogan ([17], Equation (12)) if $\alpha \to 0$ and magnetic and porous effects are neglected.

The corresponding shear stresses, namely

$$\begin{aligned} \tau_{cp}(y,t) &= \frac{1}{\mathrm{Re}}\Re e\left\{\frac{\cosh(\widetilde{\delta})-\cosh[\widetilde{\delta}(1-y)]}{\sinh(\widetilde{\delta})}\widetilde{\delta}(1+\mathrm{i}\alpha\omega)\mathrm{e}^{i\omega t}\right\}, \\ \tau_{sp}(y,t) &= \frac{1}{\mathrm{Re}}\mathrm{Im}\left\{\frac{\cosh(\widetilde{\delta})-\cosh[\widetilde{\delta}(1-y)]}{\sinh(\widetilde{\delta})}\widetilde{\delta}(1+\mathrm{i}\alpha\omega)\mathrm{e}^{i\omega t}\right\}, \end{aligned} \tag{59}$$

are obtained when introducing $u_{cp}(y,t)$ and $u_{sp}(y,t)$ from Equation (58) in (54).

Furthermore, introducing the expressions of $u_{cp}(y,t)$ and $u_{sp}(y,t)$ from Equation (58) in (25), one obtains the dimensionless expressions of corresponding Darcy's resistances, i.e.,

$$\begin{aligned} R_{cp}(y,t) &= K\Re e\left\{\frac{\sinh(\widetilde{\delta}y)+\sinh[\widetilde{\delta}(1-y)]}{\sinh(\widetilde{\delta})}(1+\mathrm{i}\alpha\omega)\mathrm{e}^{i\omega t}\right\}, \\ R_{sp}(y,t) &= K\mathrm{Im}\left\{\frac{\sinh(\widetilde{\delta}y)+\sinh[\widetilde{\delta}(1-y)]}{\sinh(\widetilde{\delta})}(1+\mathrm{i}\alpha\omega)\mathrm{e}^{i\omega t}\right\}. \end{aligned} \tag{60}$$

Simple computations show that the obtained solutions satisfy the dimensionless governing Equations (25), (53), (54), and the boundary conditions (55) and (56).

3.2.2. Motions Due to Shear Stresses on the Boundary

Let us now assume that the two plates apply the same shear stress $S\cos(\omega t)$ or $S\sin(\omega t)$ to the fluid. Bearing in mind the results of the second section, the dimensional shear stresses corresponding to this problem, $\tau_{cp}(y,t)$ and $\tau_{sp}(y,t)$, have to satisfy the governing Equation (14) with the boundary conditions

$$\tau(0,t) = S\cos(\omega t),\ \tau(d,t) = S\cos(\omega t), \tag{61}$$

or

$$\tau(0,t) = S\sin(\omega t),\ \tau(d,t) = S\sin(\omega t). \tag{62}$$

The corresponding velocity fields $u_{cp}(y,t)$ and $u_{sp}(y,t)$ have to satisfy Equations (9) and (10). Using the non-dimensional variables, functions, and parameters

$$y^* = \frac{y}{d},\ t^* = \frac{S}{\mu}t,\ u^* = u\sqrt{\frac{\rho}{S}},\ \tau^* = \frac{\tau}{S},\ \omega^* = \frac{\mu}{S}\omega,\ \alpha^* = \frac{S}{\mu}\alpha, \tag{63}$$

Equations (10) and (14) take the non-dimensional forms

$$\tau(y,t) = \frac{1}{\sqrt{\mathrm{Re}}}\left(1+\alpha\frac{\partial}{\partial t}\right)\frac{\partial u(y,t)}{\partial y}, \tag{64}$$

respectively,

$$\mathrm{Re}\frac{\partial \tau(y,t)}{\partial t} = \left(1+\alpha\frac{\partial}{\partial t}\right)\frac{\partial^2 \tau(y,t)}{\partial y^2} - M\tau(y,t) - K\left(1+\alpha\frac{\partial}{\partial t}\right)\tau(y,t), \tag{65}$$

while the boundary conditions (61) and (62) become

$$\tau(0,t) = \cos(\omega t),\ \tau(1,t) = \cos(\omega t), \tag{66}$$

or

$$\tau(0,t) = \sin(\omega t),\ \tau(1,t) = \sin(\omega t). \tag{67}$$

Into above relations the Reynolds number Re and magnetic and porous parameters M and K, respectively, are given by the relations

$$\mathrm{Re} = \frac{Sd^2}{\mu\nu} = \frac{Vd}{\nu},\ M = \frac{\sigma B^2}{\rho}\frac{d^2}{\nu},\ K = \frac{\varphi}{k}d^2, \tag{68}$$

where $V = Sd/\mu$ is a characteristic velocity. Bearing in mind the results of the last subsection, the dimensionless shear stresses $\tau_{cp}(y,t)$ and $\tau_{sp}(y,t)$ corresponding to this problem are given by the next relations

$$\tau_{cp}(y,t) = \Re e\left\{\frac{\sinh(\widetilde{\delta}y)+\sinh[\widetilde{\delta}(1-y)]}{\sinh(\widetilde{\delta})}e^{i\omega t}\right\},\ \tau_{sp}(y,t) = \mathrm{Im}\left\{\frac{\sinh(\widetilde{\delta}y)+\sinh[\widetilde{\delta}(1-y)]}{\sinh(\widetilde{\delta})}e^{i\omega t}\right\}. \tag{69}$$

The corresponding velocity fields, as it can be proven using Equation (64), have the forms

$$\begin{aligned} u_{cp}(y,t) &= \sqrt{\mathrm{Re}}\ \Re e\left\{\frac{1}{(1+i\alpha\omega)\widetilde{\delta}}\frac{\cosh(\widetilde{\delta}y)-\cosh[\widetilde{\delta}(1-y)]}{\sinh(\widetilde{\delta})}e^{i\omega t}\right\}, \\ u_{sp}(y,t) &= \sqrt{\mathrm{Re}}\mathrm{Im}\left\{\frac{1}{(1+i\alpha\omega)\widetilde{\delta}}\frac{\cosh(\widetilde{\delta}y)-\cosh[\widetilde{\delta}(1-y)]}{\sinh(\widetilde{\delta})}e^{i\omega t}\right\}, \end{aligned} \tag{70}$$

while the expressions of $R_{cp}(y,t)$ and $R_{sp}(y,t)$ are given by the next relations

$$R_{cp}(y,t) = K\sqrt{\text{Re}}\,\Re e\left\{\frac{\cosh(\widetilde{\delta}y) - \cosh[\widetilde{\delta}(1-y)]}{\sinh(\widetilde{\delta})}\frac{e^{i\omega t}}{\widetilde{\delta}}\right\},$$
$$R_{sp}(y,t) = K\sqrt{\text{Re}}\,\text{Im}\left\{\frac{\cosh(\widetilde{\delta}y) - \cosh[\widetilde{\delta}(1-y)]}{\sinh(\widetilde{\delta})}\frac{e^{i\omega t}}{\widetilde{\delta}}\right\}. \tag{71}$$

Direct computations clearly show that the dimensionless steady-state solutions from Equations (69)–(71) satisfy the governing Equations (25), (53), and (54).

4. Conclusions

The establishment problem of exact solutions for motions of non-Newtonian fluids, especially in the presence of magnetic and porous effects, is very important and still open. In order to facilitate the possibility of obtaining new exact solutions for isothermal MHD unidirectional motions of incompressible second-grade fluids through a porous medium, a very important remark regarding such motions has been brought to light. More precisely, it was proven that the governing equations for velocity and non-trivial shear stress corresponding to the motions of fluids in discussion have identical forms. Then, in order to stand out the power of this remark, some isothermal MHD steady or permanent motions of incompressible second-grade fluids through a porous medium were taken into consideration and studied. More exactly, the steady-state solutions for motions of these fluids with velocity on the boundary were used to provide exact solutions for motions of the same fluids with shear stress on the boundary. For results validation, some of these solutions have been presented in different forms and their equivalence was graphically proven. Of course, using the same remark, solutions for fluid motions with velocity on the boundary can also be determined if similar solutions for motions with shear stress on the boundary are known.

Finally, we mention the fact that all solutions that are presented here can be easily particularized to give similar solutions for the incompressible Newtonian fluids performing the same motions. The dimensionless velocity fields $u_{cp}(y,t)$ and $u_{sp}(y,t)$ given by Equation (45) and the corresponding Darcy's resistances $R_{cp}(y,t)$ and $R_{sp}(y,t)$ from Equation (48), for instance, take the simplified forms

$$u_{Ncp}(y,t) = -\frac{1}{\sqrt[4]{K_{eff}^2+\omega^2}}e^{-fy}\cos(\omega t - gy - \chi),$$
$$u_{Nsp}(y,t) = -\frac{1}{\sqrt[4]{K_{eff}^2+\omega^2}}e^{-fy}\sin(\omega t - gy - \chi), \tag{72}$$

respectively,

$$R_{Ncp}(y,t) = -\frac{K}{\sqrt[4]{K_{eff}^2+\omega^2}}e^{-fy}\cos(\omega t - gy - \chi),$$
$$R_{Nsp}(y,t) = -\frac{K}{\sqrt[4]{K_{eff}^2+\omega^2}}e^{-fy}\sin(\omega t - gy - \chi), \tag{73}$$

where

$$\chi = \text{arctg}\left(\frac{\sqrt{K_{eff}^2+\omega^2} - K_{eff}}{\omega}\right),\ f = \sqrt{\frac{K_{eff} + \sqrt{K_{eff}^2+\omega^2}}{2}},\ g = \sqrt{\frac{-K_{eff} + \sqrt{K_{eff}^2+\omega^2}}{2}} \tag{74}$$

and $K_{eff} = M + K$ is the effective permeability [18]. Interesting solutions for motions of Newtonian fluids can be also obtained from the recent work of Baranovskii et al. [19].

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Nomenclature

Nomenclature	
$\boldsymbol{T}$	Cauchy stress tensor
$\boldsymbol{A}_1$, $\boldsymbol{A}_2$	First two Rivlin–Ericksen tensors
$\boldsymbol{L}$	Velocity gradient
$\boldsymbol{I}$	Identity tensor
$\boldsymbol{p}$	Hydrostatic pressure
$\boldsymbol{v}$	Velocity vector
$R(y,t)$	Darcy's resistance
$u(y,t)$	Fluid velocity
M	Magnetic parameter
K	Porous parameter
k	Permeability of porous medium
B	Magnitude of the applied magnetic field
K_{eff}	Effective permeability
Greek Symbols	
ν	Kinematic viscosity
μ	Dynamic viscosity
ρ	Fluid density
φ	Porosity
σ	Electrical conductivity
ω	Frequency of oscillations
$\tau(y,t)$	Non-trivial shear stress
$\alpha_1,\ \alpha_2$	Material constants

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