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# Fractional Series Solution Construction for Nonlinear Fractional Reaction-Diffusion Brusselator Model Utilizing Laplace Residual Power Series

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**Abstract:** This article investigates different nonlinear systems of fractional partial differential equations analytically using an attractive modified method known as the Laplace residual power series technique. Based on a combination of the Laplace transformation and the residual power series technique, we achieve analytic and approximation results in rapid convergent series form by employing the notion of the limit, with less time and effort than the residual power series method. Three challenges are evaluated and simulated to validate the suggested method's practicability, efficiency, and simplicity. The analysis of the acquired findings demonstrates that the method mentioned above is simple, accurate, and appropriate for investigating the solutions to nonlinear applied sciences models.

**Keywords:** Laplace transform; residual power series technique; system of partial differential equations; analytical solution



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## 1. Introduction

Fractional calculus (FC) is a fast-developing area of mathematics that mathematicians use to explore the integrals and derivatives of any order of functions. Due to the excellent outcomes produced when various tools from fractional calculus are used to simulate some aspects of a problem, it has been gaining appeal among scientists working on multiple areas. FC was a hot topic at the end of the 17th century. Since its inception almost 324 years ago, FC has remained firmly rooted in mathematical ideas. The core and essential conclusions of the solution of fractional differential equations (FDEs) are in [1–5]. The integer-order derivatives are local, while the fractional-order derivatives are non-local. Integer-order derivatives can investigate variations in a point's immediate vicinity, whereas fractional-order derivatives can investigate variations throughout the interval. Systems with an arbitrary order have recently gained considerable attention and recognition for classical-order system generalization. Researchers are now aware of the value of fractional calculus and how well it may simulate various natural processes. In particular, given that fractional-order and integral operators are non-local, in contrast to integer-order and integral operators, the system's future state depends on its present and past conditions [6–9]. Therefore, when studying partial differential equations, and more particularly when studying equations from the mathematics of finance, symmetry analysis is useful. Symmetry is the key to nature; however, the majority of observations in the natural world lack symmetry. The phenomena of spontaneous symmetry-breaking is a profound method of symmetry concealment. Symmetries come in two varieties: finite and infinitesimal. Discrete or continuous finite symmetries are possible. While space is a continuous transformation, parity and temporal reversal are discrete symmetries of nature. Patterns have always captivated mathematicians. In the 19th century, classifications of spatial and planar patterns got off to a serious start. Unfortunately, solving non-linear fractional differential equations precisely has proven to be rather challenging.

Fractional partial differential equations (FPDEs) are used to represent the majority of complicated events mathematically. These non-linear FPDEs' dynamic processes are crucial for both scientific research and production, and they should be investigated using a method that can manage non-linear issues. FPDEs have been used to describe various complicated events in physics, engineering, chemistry, and other fields of study, such as elasticity, fluid flow, electrostatics, electrodynamics, signal processing, control theory, and the transmission of sound or heat [10–16]. An effective, dependable, and accurate numerical technique is needed to solve FPDEs because most real-world problems are complicated to solve precisely. The usefulness and significance of numerical approaches in physics, engineering, and mathematics have substantially expanded with the development of effective and quick computers [17–19]. As a result, researchers have suggested various numerical and analytical techniques to resolve FPDEs: the Adomian decomposition method [20], the He–Laplace variational iteration method [21], the pseudo-spectral method [22], the inverse scattering method [23], the Backlund transformation method [24], the sine-cosine and tanh methods [25], the tanh–coth methods [26], the homotopy perturbation method [27], the residual power series (RPS) method [28], the meshless method [29], the generalized differential transform method (GDTM) [30], and the fractional complex transform [31] have all been used in recent years to construct solutions to the FPDEs.

The Laplace residual power series method (LRPSM) [32] was first introduced and validated by El-Ajou [33] for research on precise solutions for nonlinear FPDEs [34–38]. LRPSM is a novel, efficient, and straightforward method. It is the focus of this study. Our new strategy is based on the LT and the power series method [39,40], which involves translating non-linear differential equations into Laplace space. Then, using the RPS method, an appropriate expansion is utilized to solve the new equation obtained in Laplace space. This involves putting an expansion that shows the equation's solution in the Laplace space. To calculate the expansion coefficients, the RPS method is utilized, but with a new theory and mechanism that makes it simpler to use than the traditional RPS approach. This paper is organized as follows: Some notions and theorems are provided in Section 2, while the LRPSM algorithm is explained in Section 3. In Section 4, a few problems are solved. In the final section, we provide the conclusion.

## 2. Preliminaries

In this section, we will discuss several basic definitions and conclusions relating to the Caputo fractional derivative, as well as the fractional Laplace transform.

**Definition 1.** *The fractional derivative of a function $u(x,t)$ of order $\alpha$ in the Caputo sense is defined as*

$$^{C}D_t^{\alpha}u(x,t) = J_t^{m-\alpha}u^{m}(x,t), \;\; m-1 < \zeta \leq m, \;\; t > 0. \tag{1}$$

*where $m \in N$, and $J_t^{\gamma}$ is the Riemann-Liouville (RL) fractional integral (FI) of $u(x,t)$ of fractional-order $\gamma$, which is defined as*

$$J_t^{\gamma}u(x,t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-\kappa)u(x,t)dt, \gamma > 0, t > 0, \kappa \geq 0, \tag{2}$$

*assuming that the given integral exists.*

**Definition 2.** *Suppose that $u(x,t)$ is continuous piecewise and has $\alpha$ as its exponential order. This can be explained as follows:*

$$u(x,s) = \mathcal{L}_t[u(x,t)] = \int_0^{\infty} e^{-st}u(x,t)dt, \;\; s > \alpha, \tag{3}$$

*where the inverse LT is given as*

$$u(x,t) = \mathcal{L}_s^{-1}[u](x,s)] = \int_{l-i\infty}^{l+i\infty} e^{st} u(x,s)ds, \;\; l = Re(s) > l_0, \tag{4}$$

*where $c_0$ is in the right half-plane of the Laplace integral's absolute convergence [41].*

**Lemma 1.** *Assume that $u(x,t)$ is a continuous piecewise function of exponential order $\psi$, and $u(x,s) = \mathcal{L}_t[u(x,t)]$. Then, we have*

1. $\mathcal{L}_t[J_t^{\beta} u(x,t)] = \frac{u(x,s)}{s^{\beta}}, \;\; \beta > 0.$
2. $\mathcal{L}_t[D_t^{\psi} u(x,t)] = s^{\psi} u(x,s) - \sum_{k=0}^{m-1} s^{\psi-k-1} u^k(x,0), \;\; m-1 < \psi \le m.$
3. $\mathcal{L}_t[D_t^{n\psi} u(x,t)] = s^{n\psi} u(x,s) - \sum_{k=0}^{n-1} s^{(n-k)\psi-1} D_t^{k\psi} u(x,0), \;\; 0 < \psi \le 1.$

**Proof.** The proofs are in [1–3,33]. □

**Theorem 1.** *Let us assume that $u(x,t)$ is a continuous piecewise function on $\mathbf{I} \times [0,\infty)$. Consider that $u(x,s) = \mathcal{L}_t[u(x,t)]$ has a fractional power series (FPS) representation:*

$$u(x,s) = \sum_{i=0}^{\infty} \frac{f_i(x)}{s^{1+i\alpha}}, \;\; 0 < \zeta \le 1, x \in \mathbf{I}, s > \psi. \tag{5}$$

*Then, $f_i(x) = D_t^{n\alpha} u(x,0)$.*

**Proof.** Consider that $u(x,s)$ has an FPS representation, as in Equation (5). Then, we have $f_i(x) = \lim_{s\to\infty} su(x,s) = u(x,0)$. Then, Equation (5) becomes:

$$u(x,s) = \frac{u(x,0)}{s} + \sum_{i=1}^{\infty} \frac{f_i(x)}{s^{1+i\alpha}}. \tag{6}$$

Multiplying Equation (6) by $s^{1+\alpha}$, we get

$$f_1(x) = s^{\alpha+1} u(x,s) - s^{\alpha} u(x,0) - \frac{f_2(x)}{s^{\alpha}} - \frac{f_3(x)}{s^{2\zeta}} - \frac{f_4(x)}{s^{3\alpha}} - \cdots. \tag{7}$$

Using part (2) of Lemma 1, we get

$$\begin{aligned} f_1(x) &= \lim_{s\to\infty} (s^{\alpha+1} u(x,s) - s^{\alpha} u(x,0)) = \lim_{s\to\infty} s(s^{\alpha} u(x,s) - s^{\alpha-1} u(x,0)). \\ &= \lim_{s\to\infty} s(\mathcal{L}_t[D_t^{\alpha} u(x,t)](s)) = D_t^{\alpha} u(x,0). \end{aligned} \tag{8}$$

Similarly, if we multiply Equation (6) by $s^{1+2\alpha}$, again using part (2) of Lemma 1, we get

$$\begin{aligned} f_2(x) &= \lim_{s\to\infty} (s^{2\alpha+1} u(x,s) - s^{2\alpha} u(x,0) - s^{\alpha} D_t^{\alpha} u(x,0)). \\ &= \lim_{s\to\infty} s(s^{2\alpha} u(x,s) - s^{2\alpha-1} u(x,0) - s^{\alpha-1} D_t^{\alpha} u(x,0)), \\ &= \lim_{s\to\infty} s\left(\mathcal{L}_t\left[D_t^{2\alpha} u(x,t)\right](s)\right) = D_t^{2\alpha} u(x,0). \end{aligned} \tag{9}$$

The mathematical induction principle is used to complete the proof. Assume that $f_{i-1}(x) = D_t^{(i-1)\alpha} u(x,0)$. Multiplying Equation (6) by $s^{1+i\alpha}$ and using part (2) of Lemma 1, we get

$$
\begin{aligned}
f_i(x) &= \lim_{s\to\infty}(s^{i\alpha+1}u(x,s) - s^{i\alpha}u(x,0) - s^{(i-1)\alpha}D_t^{\alpha}u(x,0) - \cdots - s^{\alpha}D_t^{(i-1)\alpha}u(x,0)). \\
&= \lim_{s\to\infty} s\Big(s^{i\alpha}u(x,s) - s^{i\alpha-1}u(x,0) - s^{1-(i-1)\alpha}D_t^{\alpha}u(x,0) - \cdots - s^{\alpha-1}D_t^{(i-1)\alpha}u(x,0)\Big). \\
&= \lim_{s\to\infty} s\Big(\mathcal{L}_t\Big[D_t^{i\alpha}u(x,t)\Big](s)\Big) = D_t^{i\alpha}u(x,0).
\end{aligned} \tag{10}
$$

The proof is completed. □

**Remark 1.** *The inverse LT of Equation (5) is represented as:*

$$u(x,t) = \sum_{i=0}^{\infty} \frac{D_t^{\psi}u(x,0)}{\Gamma(1+i\psi)} t^{i(\psi)}, \;\; 0 < \psi \le 1, t \ge 0. \tag{11}$$

*which is equivalent to the fractional Taylor's formula shown in [42].*

The convergence of the FPS in Theorem 1 is determined in the following Theorem.

**Theorem 2.** *Assume that $u(x,t)$ is a continuous piecewise function on $\mathbf{I} \times [0,\infty)$ and of order $\psi$, and as shown in Theorem 1, $u(x,s) = \mathcal{L}_t[u(x,t)]$ can be written as the new form of a fractional Taylor's formula. If $\Big|s\mathcal{L}_t[D_t^{i\alpha+1}u(x,t)]\Big| \le M(x)$, on $\mathbf{I} \times (\psi,\gamma]$ where $0 < \alpha \le 1$,, then $R_i(x,s)$, and the remainder of the new form of fractional Taylor's formula in Theorem 1 satisfies the following inequality*

$$|R_i(x,s)| \le \frac{M(x)}{S^{1+(i+1)\alpha}}, \;\; x \in \mathbf{I}, \;\; \psi < s \le \gamma. \tag{12}$$

**Proof.** Let us consider that $\mathcal{L}_t\Big[D_t^{k\alpha}u(x,t)\Big](s)$ on interval $\mathbf{I} \times (\psi,\gamma]$ for $k = 0,1,2,3,\cdots,i+1$, and suppose that

$$\Big|s\mathcal{L}_t[D_t^{i\alpha+1}u(x,t)]\Big| \le M(x), \;\; x \in \mathbf{I}, \;\; \psi < s \le \gamma. \tag{13}$$

Using the definition of the remainder, $R_i(x,s) = u(x,s) - \sum_{k=0}^{i} \frac{D_t^{k\alpha}u(x,0)}{s^{1+k\alpha}}$, we can obtain:

$$
\begin{aligned}
S^{1+(i+1)\alpha}R_i(x,s) =& s^{1+(i+1)\alpha}u(x,s) - \sum_{k=0}^{i} s^{(i+1-k)\alpha}D_t^{k\alpha}u(x,0), \\
=& s\left(s^{(i+1)\alpha}u(x,s) - \sum_{k=0}^{i} s^{(i+1-k)\alpha-1}D_t^{k\alpha}u(x,0)\right), \\
=& s\mathcal{L}_t\Big[D_t^{(n+1)\zeta}u(x,t)\Big].
\end{aligned} \tag{14}
$$

From Equations (13) and (14), we can obtain that $|s^{1+(i+1)\alpha}R_i(x,s)| \le M(x)$. Thus,

$$-M(x) \le s^{1+(i+1)\alpha}R_i(x,s) \le M(x), x \in \mathbf{I}, \;\; \psi < s \le \gamma. \tag{15}$$

The proof of Theorem 2 is completed. □

## 3. LRPS Methodology

In this section, we will go through the LRPS methodology for the nonlinear system of fractional-order partial differential equations.

$$
\begin{aligned}
&D_t^{\alpha} u(x,t) + v^2(x,t)\left(\frac{\partial u(x,t)}{\partial x}\right)^2 + u(x,t) = h(x),\\
&D_t^{\alpha} v(x,t) + u^2(x,t)\left(\frac{\partial v(x,t)}{\partial x}\right)^2 - v(x,t) = h(x), \ \ 0 < \alpha \leq 1,
\end{aligned} \tag{16}
$$

subject to the initial conditions

$$
u(x,0) = f_0(x), \ \ v(x,0) = g_0(x). \tag{17}
$$

Using Laplace transform on Equation (16) and the initial conditions of Equation (17), we get

$$
\begin{aligned}
&U(x,s) - \frac{f_0(x)}{s} + \frac{1}{s^{\alpha}}\mathcal{L}_t\left[\mathcal{L}_t^{-1}\left(V^2(x,s)\right)\mathcal{L}_t^{-1}\left(\left(\frac{\partial U(x,s)}{\partial x}\right)^2\right) + \mathcal{L}_t^{-1}(U(x,s))\right] = \frac{H(x)}{s^{\alpha+1}},\\
&V(x,s) - \frac{g_0(x)}{s} + \frac{1}{s^{\alpha}}\mathcal{L}_t\left[\mathcal{L}_t^{-1}\left(U^2(x,s)\right)\mathcal{L}_t^{-1}\left(\left(\frac{\partial V(x,s)}{\partial x}\right)^2\right) - \mathcal{L}_t^{-1}(V(x,s))\right] = \frac{h(x)}{s^{\alpha+1}}.
\end{aligned} \tag{18}
$$

The solution of Equation (18) has the following series:

$$
\begin{aligned}
U(x,s) &= \sum_{n=0}^{\infty} \frac{f_n(x)}{s^{n\alpha+1}},\\
V(x,s) &= \sum_{n=0}^{\infty} \frac{g_n(x)}{s^{n\alpha+1}}.
\end{aligned} \tag{19}
$$

and the $k^{th}$-truncated term series are

$$
\begin{aligned}
U(x,s) &= \frac{f_0(x)}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}},\\
V(x,s) &= \frac{g_0(x)}{s} + \sum_{n=1}^{k} \frac{g_n(x)}{s^{n\alpha+1}}.
\end{aligned} \tag{20}
$$

The Laplace residual functions are

$$
\begin{aligned}
&\mathcal{L}_t Res_u(x,s) = U_k(x,s) - \frac{g_0(x)}{s} + \frac{1}{s^{\alpha}}\mathcal{L}_t\left[\mathcal{L}_t^{-1}\left(V^2(x,s)\right)\mathcal{L}_t^{-1}\left(\left(\frac{\partial U(x,s)}{\partial x}\right)^2\right) + \mathcal{L}_t^{-1}(U(x,s))\right] - \frac{h(x)}{s^{\alpha+1}},\\
&\mathcal{L}_t Res_v(x,s) = V(x,s) - \frac{g_0(x)}{s} + \frac{1}{s^{\alpha}}\mathcal{L}_t\left[\mathcal{L}_t^{-1}\left(U^2(x,s)\right)\mathcal{L}_t^{-1}\left(\left(\frac{\partial V(x,s)}{\partial x}\right)^2\right) - \mathcal{L}_t^{-1}(V(x,s))\right] - \frac{h(x)}{s^{\alpha+1}}.
\end{aligned} \tag{21}
$$

and the $k^{th}$-Laplace residual functions are

$$
\begin{aligned}
&\mathcal{L}_t Res_{u,k}(x,s) = U_k(x,s) - \frac{g_0(x)}{s} + \frac{1}{s^{\alpha}}\mathcal{L}_t\left[\mathcal{L}_t^{-1}\left(V_k^2(x,s)\right)\mathcal{L}_t^{-1}\left(\left(\frac{\partial U_k(x,s)}{\partial x}\right)^2\right) + \mathcal{L}_t^{-1}(U_k(x,s))\right] - \frac{h(x)}{s^{\alpha+1}},\\
&\mathcal{L}_t Res_{v,k}(x,s) = V_k(x,s) - \frac{g_0(x)}{s} + \frac{1}{s^{\alpha}}\mathcal{L}_t\left[\mathcal{L}_t^{-1}\left(U_k^2(x,s)\right)\mathcal{L}_t^{-1}\left(\left(\frac{\partial V_k(x,s)}{\partial x}\right)^2\right) - \mathcal{L}_t^{-1}(V_k(x,s))\right] - \frac{h(x)}{s^{\alpha+1}}.
\end{aligned} \tag{22}
$$

Here are some properties that arise in the LRPSM [43], to point out some facts:

$$(i)\mathcal{L}_t Res(x,s) = 0 \ \text{ and } \lim_{j\to\infty} \mathcal{L}_t Res_{u,k}(x,s) = \mathcal{L}_t Res_u(x,s) \ \text{ for each } \ s > 0.$$

$$(ii)\mathcal{L}_t Res(x,s) = 0 \ \text{ and } \lim_{j\to\infty} \mathcal{L}_t Res_{v,k}(x,s) = \mathcal{L}_t Res_v(x,s) \ \text{ for each } \ s > 0.$$

$$(iii) \lim_{s\to\infty} s\mathcal{L}_t Res_u(x,s) = 0 \Rightarrow \lim_{s\to\infty} s\mathcal{L}_t Res_{u,k}(x,s) = 0.$$

$$(iv) \lim_{s\to\infty} s\mathcal{L}_t Res_v(x,s) = 0 \Rightarrow \lim_{s\to\infty} s\mathcal{L}_t Res_{v,k}(x,s) = 0.$$

$$(v) \lim_{s\to\infty} s^{k\alpha+1}\mathcal{L}_t Res_{u,k}(x,s) = \lim_{s\to\infty} s^{k\alpha+1}\mathcal{L}_t Res_{u,k}(x,s) = 0, \ \ 0 < \alpha \le 1, \ \ k = 1,2,3,\cdots.$$

$$(vi) \lim_{s\to\infty} s^{k\alpha+1}\mathcal{L}_t Res_{v,k}(x,s) = \lim_{s\to\infty} s^{k\alpha+1}\mathcal{L}_t Res_{v,k}(x,s) = 0, \ \ 0 < \alpha \le 1, \ \ k = 1,2,3,\cdots.$$

To find the coefficients $f_n(x)$ and $g_n(x)$, we recursively solve the following system:

$$\begin{aligned} &\lim_{s\to\infty} s^{k\alpha+1}\mathcal{L}_t Res_{u,k}(x,s) = 0, \ \ k = 1,2,\cdots, \\ &\lim_{s\to\infty} s^{k\alpha+1}\mathcal{L}_t Res_{v,k}(x,s) = 0, \ \ k = 1,2,\cdots, \end{aligned} \tag{23}$$

At last, we use inverse LT on Equation (20), to get the $k^{th}$ approximate solutions of $u_k(x,t)$, $v_k(x,t)$.

## 4. Numerical Problem

In this section, we consider a system of equations that describes the unsteady flow of a polytropic gas of fractional order in order to validate the applicability and accuracy of the proposed technique.

**Problem 1.** *Consider the following system of inhomogeneous FPDEs:*

$$\begin{aligned} &D_t^{\alpha} u(x,t) + v^2(x,t)\left(\frac{\partial u(x,t)}{\partial x}\right)^2 + u(x,t) = 1, \\ &D_t^{\alpha} v(x,t) + u^2(x,t)\left(\frac{\partial v(x,t)}{\partial x}\right)^2 - v(x,t) = 1, \ \ 0 < \alpha \le 1, \end{aligned} \tag{24}$$

*subject to the initial conditions*

$$u(x,0) = e^x, \ \ v(x,0) = e^{-x}. \tag{25}$$

*Using Laplace transform on Equation (24) and the initial conditions of Equation (25), we get*

$$\begin{aligned} &U(x,s) - \frac{e^x}{s} + \frac{1}{s^{\alpha}}\mathcal{L}_t\left[\mathcal{L}_t^{-1}\left(V^2(x,s)\right)\mathcal{L}_t^{-1}\left(\left(\frac{\partial U(x,s)}{\partial x}\right)^2\right) + \mathcal{L}_t^{-1}(U(x,s))\right] = \frac{1}{s^{\alpha+1}}, \\ &V(x,s) - \frac{e^{-x}}{s} + \frac{1}{s^{\alpha}}\mathcal{L}_t\left[\mathcal{L}_t^{-1}\left(U^2(x,s)\right)\mathcal{L}_t^{-1}\left(\left(\frac{\partial V(x,s)}{\partial x}\right)^2\right) - \mathcal{L}_t^{-1}(V(x,s))\right] = \frac{1}{s^{\alpha+1}}. \end{aligned} \tag{26}$$

*The $K^{th}$-truncated term series are*

$$\begin{aligned} U(x,s) &= \frac{e^x}{s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}}, \\ V(x,s) &= \frac{e^{-x}}{s} + \sum_{n=1}^{k} \frac{g_n(x)}{s^{n\alpha+1}}. \end{aligned} \tag{27}$$

*The $k^{th}$-Laplace residual functions are*

$$\mathcal{L}_t Res_{u,k}(x,s) = U_k(x,s) - \frac{e^x}{s} + \frac{1}{s^\alpha}\mathcal{L}_t\left[\mathcal{L}_t^{-1}\left(V_k^2(x,s)\right)\mathcal{L}_t^{-1}\left(\left(\frac{\partial U_k(x,s)}{\partial x}\right)^2\right) + \mathcal{L}_t^{-1}(U_k(x,s))\right] - \frac{1}{s^{\alpha+1}},$$
$$\mathcal{L}_t Res_{u,k}(x,s) = V_k(x,s) - \frac{e^{-x}}{s} + \frac{1}{s^\alpha}\mathcal{L}_t\left[\mathcal{L}_t^{-1}\left(U_k^2(x,s)\right)\mathcal{L}_t^{-1}\left(\left(\frac{\partial V_k(x,s)}{\partial x}\right)^2\right) - \mathcal{L}_t^{-1}(V_k(x,s))\right] - \frac{1}{s^{\alpha+1}}. \quad (28)$$

*Now, to find $f_k(x)$ and $g_k(x)$, $k = 1,2,3,\cdots$, we substitute the $k^{th}$-truncated series Equation (27) into the $k^{th}$-Laplace residual function Equation (28), multiply the obtained results by $s^{k\alpha+1}$, and then recursively solve the relation $\lim_{s\to\infty}(s^{k\alpha+1}\mathcal{L}_t Res_{u,k}(x,s)) = 0$, and $\lim_{s\to\infty}(s^{k\alpha+1}\mathcal{L}_t Res_{v,k}(x,s))$ $= 0$, $k = 1,2,3,\cdots$. The first few terms are as follows:*

$$\begin{aligned} &f_0 = e^x, \;\; g_0 = e^{-x},\\ &f_1 = -e^x, \;\; g_1 = e^{-x},\\ &f_2 = e^x, \;\; g_2 = e^{-x},\\ &f_3 = -e^x, \;\; g_3 = e^{-x},\\ &f_4 = e^x, \;\; g_4 = e^{-x},\\ &f_5 = -e^x, \;\; g_5 = e^{-x},\\ &\vdots \end{aligned} \quad (29)$$

*Substituting the values of $f_k(x)$ and $g_k(x)$, $k = 1,2,3,\cdots$ into Equation (27), we have*

$$\begin{aligned} U(x,s) &= \frac{e^x}{s} + \frac{f_1(x)}{s^{\alpha+1}} + \frac{f_2(x)}{s^{2\alpha+1}} + \frac{f_3(x)}{s^{3\alpha+1}} + \frac{f_4(x)}{s^{4\alpha+1}} + \frac{f_5(x)}{s^{5\alpha+1}} + \cdots,\\ V(x,s) &= \frac{e^{-x}}{s} + \frac{g_1(x)}{s^{\alpha+1}} + \frac{g_2(x)}{s^{2\alpha+1}} + \frac{g_3(x)}{s^{3\alpha+1}} + \frac{g_4(x)}{s^{4\alpha+1}} + \frac{g_5(x)}{s^{5\alpha+1}} + \cdots,\\ U(x,s) &= \frac{e^x}{s} - \frac{e^x}{s^{\alpha+1}} + \frac{e^x}{s^{2\alpha+1}} - \frac{e^x}{s^{3\alpha+1}} + \frac{e^x}{s^{4\alpha+1}} - \frac{e^x}{s^{5\alpha+1}} + \cdots,\\ V(x,s) &= \frac{e^{-x}}{s} + \frac{e^{-x}}{s^{\alpha+1}} + \frac{e^{-x}}{s^{2\alpha+1}} + \frac{e^{-x}}{s^{3\alpha+1}} + \frac{e^{-x}}{s^{4\alpha+1}} + \frac{e^{-x}}{s^{5\alpha+1}} + \cdots. \end{aligned} \quad (30)$$

*Applying inverse Laplace transform, we get*

$$\begin{aligned} u(x,t) &= e^x\left(1 - \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} - \frac{t^{5\alpha}}{\Gamma(5\alpha+1)} + \cdots\right),\\ v(x,t) &= e^{-x}\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{t^{5\alpha}}{\Gamma(5\alpha+1)} + \cdots\right). \end{aligned} \quad (31)$$

*Taking $\alpha = 1$, we get the exact solutions*

$$\begin{aligned} u(x,t) &= e^{x-t},\\ v(x,t) &= e^{-x+t}. \end{aligned} \quad (32)$$

In Figure 1, show that the different fractional order graph (a) represent with respect to $u(x,(t))$ and (b) represent with respect to $v(x,(t))$ of Problem 1. Figure 2, (a) represent fractional order $\alpha = 0.5$ and (b) fractional order $\alpha = 0.75$ of $u(x,(t))$. Figure 2 shows (c) the exact and (d) LRPSM solutions for $u(x,t)$ at $k = \frac{2}{3}$, $c = 1$ of integer order $\alpha$ for Problem 1. Figure 3, (a) represent fractional order $\alpha = 0.5$ and (b) fractional order $\alpha = 0.75$ of $u(x,(t))$. Figure 3 shows (c) the exact and (d) LRPSM solutions for $u(x,t)$ at $k = \frac{2}{3}$, $c = 1$ of integer order $\alpha$ for Problem 1. In Tables 1 and 2, error analysis is shown for the LRPSM solution of $u(x,t)$ and $v(x,t)$ for the proposed Problem 1 with various values of $x$ and $t$.

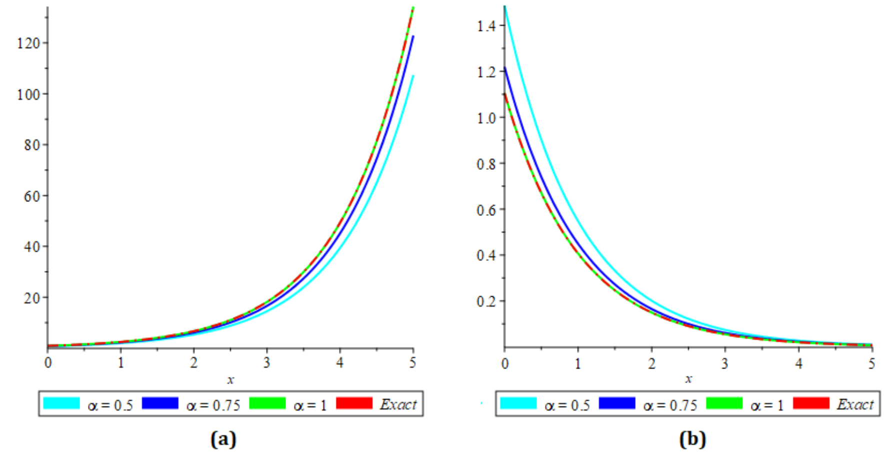


**Figure 1.** Subplot (**a**) presents the approximate solution $u(x,(t))$ and subplot (**b**) presents the approximate solution $v(x,(t))$ of Problem 1.

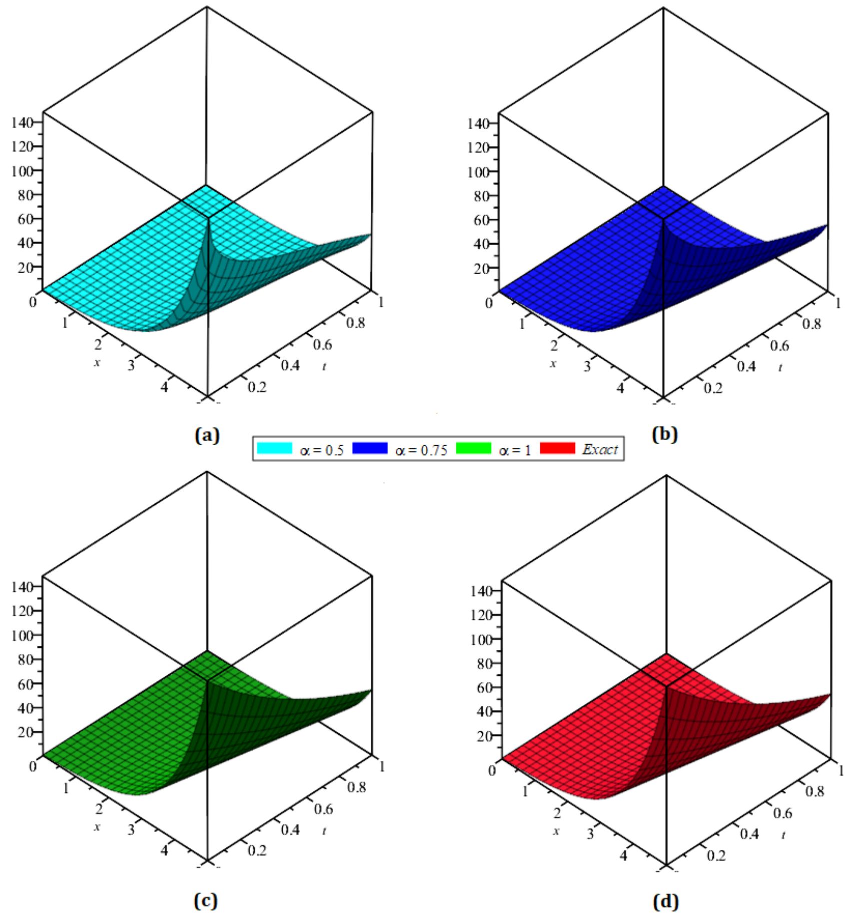


**Figure 2.** Exact and LRPSM solutions for $u(x,t)$ at $k = \frac{2}{3}$, $c = 1$, and distinct values of $\alpha$ for Problem 1.

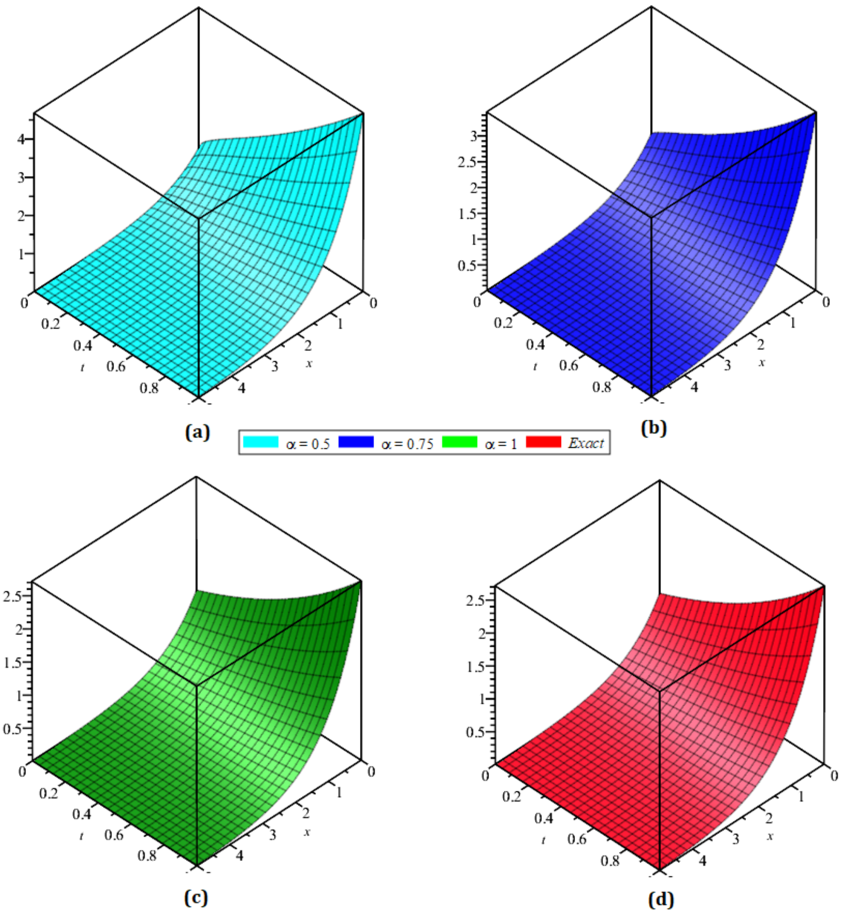


**Figure 3.** Exact and LRPSM solutions for $v(x,t)$ at $k = \frac{2}{3}$, $c = 1$, and distinct values of $\alpha$ for Problem 1.

**Table 1.** Error analysis for the LRPSM solution of $u(x,t)$ for the proposed Problem 1 with various values of $x$ and $t$.

| $t$ | $x$ | $AE$ at $\alpha = 0.5$ | $AE$ at $\alpha = 0.75$ | $AE$ at $\alpha = 1$ |
|---|---|---|---|---|
| 0.1 | 0.2 | 0.2215650201 | 0.093544132 | $2.00 \times 10^{-9}$ |
| | 0.4 | 0.270620127 | 0.114255061 | $2.00 \times 10^{-9}$ |
| | 0.6 | 0.330536170 | 0.139551447 | $3.00 \times 10^{-9}$ |
| | 0.8 | 0.403717788 | 0.170448522 | $3.00 \times 10^{-9}$ |
| | 1 | 0.493102021 | 0.208186295 | $4.00 \times 10^{-9}$ |
| 0.25 | 0.2 | 0.2017453516 | 0.1032370436 | $4.00 \times 10^{-7}$ |
| | 0.4 | 0.2464123288 | 0.126094010 | $4.88 \times 10^{-7}$ |
| | 0.6 | 0.300968699 | 0.154011572 | $5.97 \times 10^{-7}$ |
| | 0.8 | 0.367603998 | 0.188110158 | $7.29 \times 10^{-7}$ |
| | 1 | 0.448992538 | 0.229758266 | $8.91 \times 10^{-7}$ |

**Table 2.** Error analysis for the LRPSM solution of $v(x,t)$ for the proposed Problem 1 with various values of $x$ and $t$.

| $t$ | $x$ | AE at $\alpha = 0.5$ | AE at $\alpha = 0.75$ | AE at $\alpha = 1$ |
|---|---|---|---|---|
| 0.1 | 0.2 | 0.3122588000 | 0.0937357454 | $8.0 \times 10^{-10}$ |
| | 0.4 | 0.2556558826 | 0.0767443373 | $8.0 \times 10^{-10}$ |
| | 0.6 | 0.2093133334 | 0.0628329492 | $6.0 \times 10^{-10}$ |
| | 0.8 | 0.1713712630 | 0.0514432677 | $5.0 \times 10^{-10}$ |
| | 1 | 0.1403069234 | 0.0421181854 | $3.0 \times 10^{-10}$ |
| 0.25 | 0.2 | 0.544332261 | 0.176001462 | $2.88 \times 10^{-7}$ |
| | 0.4 | 0.4456615616 | 0.1440978086 | $2.358 \times 10^{-7}$ |
| | 0.6 | 0.3648768263 | 0.1179773077 | $1.930 \times 10^{-7}$ |
| | 0.8 | 0.2987358784 | 0.0965916499 | $1.581 \times 10^{-7}$ |
| | 1 | 0.2445842508 | 0.0790825544 | $1.293 \times 10^{-7}$ |

**Problem 2.** *Let us consider a coupled system of reaction-diffusion equations [28]:*

$$\begin{aligned} D_t^{\alpha} u &= u - u^2 - uv + u_{xx}, \\ D_t^{\alpha} v &= v_{xx} - uv, \ \ 0 < \alpha \leq 1, \ \ t > 0, \end{aligned} \tag{33}$$

*with initial conditions*

$$\begin{aligned} u(x,0) &= \frac{e^{px}}{\left(1 + e^{0.5px}\right)^2}, \\ v(x,0) &= \frac{1}{1 + e^{0.5px}}, \end{aligned} \tag{34}$$

*where $p$ is constant.*

*Applying Laplace transform on Equation (33) and using Equation (34), we have*

$$\begin{aligned} U(x,s) &= \frac{e^{px}}{\left(1 + e^{0.5px}\right)^2 s} + \frac{1}{s^{\alpha}} \mathcal{L}_t \Big[ \mathcal{L}_t^{-1}(U(x,s)) - \mathcal{L}_t^{-1}\Big(U^2(x,s)\Big) - \mathcal{L}_t^{-1}(U(x,s))\mathcal{L}_t^{-1}(V(x,s)) + \mathcal{L}_t^{-1}(U_{xx}(x,s)) \Big], \\ V(x,s) &= \frac{1}{\left(1 + e^{0.5px}\right) s} + \frac{1}{s^{\alpha}} \mathcal{L}_t \Big[ \mathcal{L}_t^{-1}(V_{xx}(x,s)) - \mathcal{L}_t^{-1}(U(x,s))\mathcal{L}_t^{-1}(V(x,s)) \Big]. \end{aligned} \tag{35}$$

*The $K^{th}$-truncated term series are*

$$\begin{aligned} U(x,s) &= \frac{e^{px}}{\left(1 + e^{0.5px}\right)^2 s} + \sum_{n=1}^{k} \frac{f_n(x)}{s^{n\alpha+1}}, \\ V(x,s) &= \frac{1}{\left(1 + e^{0.5px}\right) s} + \sum_{n=1}^{k} \frac{g_n(x)}{s^{n\alpha+1}}, \end{aligned} \tag{36}$$

*The $k^{th}$-Laplace residual functions are*

$$\begin{aligned} \mathcal{L}_t Res_{u,k}(x,s) =& U_k(x,s) - \frac{e^{px}}{\left(1 + e^{0.5px}\right)^2 s} - \frac{1}{s^{\alpha}} \mathcal{L}_t \Big[ \mathcal{L}_t^{-1}(U_k(x,s)) - \mathcal{L}_t^{-1}\Big(U_k^2(x,s)\Big) \\ & - \mathcal{L}_t^{-1}(U_k(x,s))\mathcal{L}_t^{-1}(V_k(x,s)) + \mathcal{L}_t^{-1}(U_{k,xx}(x,s)) \Big], \\ \mathcal{L}_t Res_{u,k}(x,s) =& V_k(x,s) - \frac{1}{\left(1 + e^{0.5px}\right) s} - \frac{1}{s^{\alpha}} \mathcal{L}_t \Big[ \mathcal{L}_t^{-1}(V_{k,xx}(x,s)) - \mathcal{L}_t^{-1}(U_k(x,s))\mathcal{L}_t^{-1}(V_k(x,s)) \Big]. \end{aligned} \tag{37}$$

*Now, for finding $f_k(x)$, and $g_k(x)$, $k = 1,2,3,\cdots$, we substitute the $k^{th}$-truncated series Equation (36) into the $k^{th}$-Laplace residual function Equation (37), multiply the obtained* *results by $s^{k\alpha+1}$, and then recursively solve the relation $\lim_{s\to\infty}(s^{k\alpha+1}\mathcal{L}_t Res_{u,k}(x,s)) = 0$, and $\lim_{s\to\infty}(s^{k\alpha+1}\mathcal{L}_t Res_{v,k}(x,s)) = 0$, $k = 1,2,3,\cdots$. The first few terms are*

$$\begin{aligned}
f_0(x) =& \frac{e^{px}}{\left(1+e^{0.5px}\right)^2},\\
g_0(x) =& \frac{1}{1+e^{0.5px}},\\
f_1(x) =& \frac{(p^2e^{0.5px}-2p^2-2e^{0.5px})e^{px}}{(1+e^{0.5px})^4},\\
g_1(x) =& -\frac{0.25(4e^{px}+p^2e^{0.5px}-p^2e^{p*x})}{(1+e^{0.5px})^3},\\
f_2(x) =& \frac{1}{8(1+e^{0.5px})^6}\Big(-32p^2e^{2px}+16e^{2px}+28e^{1.5px}p^2-33p^4e^{1.5px}+4p^2e^{2.5px}+18p^4e^{2px}-p^4e^{2.5px}\\
&+8p^4e^{px}\Big),\\
g_2(x) =& \frac{1}{16(1+e^{0.5px})^5}\Big(p^4e^{2px}+16e^{2px}-8p^2e^{2px}+11p^2e^{px}-11p^4e^{1.5px}+40p^2e^{1.5px}-p^4e^{0.5px}\\
&-32p^2e^{px}-16e^{1.5px}\Big).
\end{aligned} \tag{38}$$

*Putting the values of $f_k(x)$ and $g_k(x)$, $k = 1,2,3,\cdots$ in Equation (36), we get*

$$\begin{aligned}
U(x,s) =& \frac{e^{px}}{\left(1+e^{0.5px}\right)^2 s}+\Big(\frac{(p^2e^{0.5px}-2p^2-2e^{0.5px})e^{px}}{(1+e^{0.5px})^4}\Big)\frac{1}{s^{\alpha+1}}+\Big(\frac{1}{8(1+e^{0.5px})^6}\Big(-32p^2e^{2px}+16e^{2px}+28e^{1.5px}p^2\\
&-33p^4e^{1.5px}+4p^2e^{2.5px}+18p^4e^{2px}-p^4e^{2.5px}+8p^4e^{px}\Big)\Big)\frac{1}{s^{2\alpha+1}},\\
V(x,s) =& \frac{1}{(1+e^{0.5px})s}-\Big(\frac{0.25(4e^{px}+p^2e^{0.5px}-p^2e^{p*x})}{(1+e^{0.5px})^3}\Big)\frac{1}{s^{\alpha+1}}+\Big(\frac{1}{16(1+e^{0.5px})^5}(p^4e^{2px}+16e^{2px}-8p^2e^{2px}\\
&+11p^2e^{px}-11p^4e^{1.5px}+40p^2e^{1.5px}-p^4e^{0.5px}-32p^2e^{px}-16e^{1.5px})\Big)\frac{1}{s^{2\alpha+1}}.
\end{aligned} \tag{39}$$

*Using inverse Laplace transform, we get*

$$\begin{aligned}
u(x,t) =& \frac{e^{px}}{\left(1+e^{0.5px}\right)^2}+\Big(\frac{(p^2e^{0.5px}-2p^2-2e^{0.5px})e^{px}}{(1+e^{0.5px})^4}\Big)\frac{t^\alpha}{\Gamma(\alpha+1)}+\Big(\frac{1}{8(1+e^{0.5px})^6}\Big(-32p^2e^{2px}+16e^{2px}\\
&+28e^{1.5px}p^2-33p^4e^{1.5px}+4p^2e^{2.5px}+18p^4e^{2px}-p^4e^{2.5px}+8p^4e^{px}\Big)\Big)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\
v(x,t) =& \frac{1}{\left(1+e^{0.5px}\right)}-\Big(\frac{0.25(4e^{px}+p^2e^{0.5px}-p^2e^{p*x})}{(1+e^{0.5px})^3}\Big)\frac{t^\alpha}{\Gamma(\alpha+1)}+\Big(\frac{1}{16(1+e^{0.5px})^5}(p^4e^{2px}+16e^{2px}-8p^2e^{2px}\\
&+11p^2e^{px}-11p^4e^{1.5px}+40p^2e^{1.5px}-p^4e^{0.5px}-32p^2e^{px}-16e^{1.5px})\Big)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}.
\end{aligned} \tag{40}$$

In Figure 4, show that the different fractional order graph (a) represent with respect to $u(x,(t))$ and (b) represent with respect to $v(x,(t))$ of Problem 2. Figure 5, (a) represent fractional order $\alpha = 0.5$ and (b) fractional order $\alpha = 0.75$ of $u(x,(t))$. Figure 5 shows (c) the exact and (d) LRPSM solutions for $u(x,t)$ at $p = \frac{2}{3}$, $c = 1$ of integer order $\alpha$ for Problem 2. Figure 6, (a) represent fractional order $\alpha = 0.5$ and (b) fractional order $\alpha = 0.75$ of $u(x,(t))$. Figure 6 shows (c) the exact and (d) LRPSM solutions for $u(x,t)$ at $p = \frac{2}{3}$, $c = 1$ of integer order $\alpha$ for Problem 2. In Tables 3 and 4, error analysis is shown for the LRPSM solution of $u(x,t)$ and $v(x,t)$ for the proposed Problem 2 with various values of $x$ and $t$.

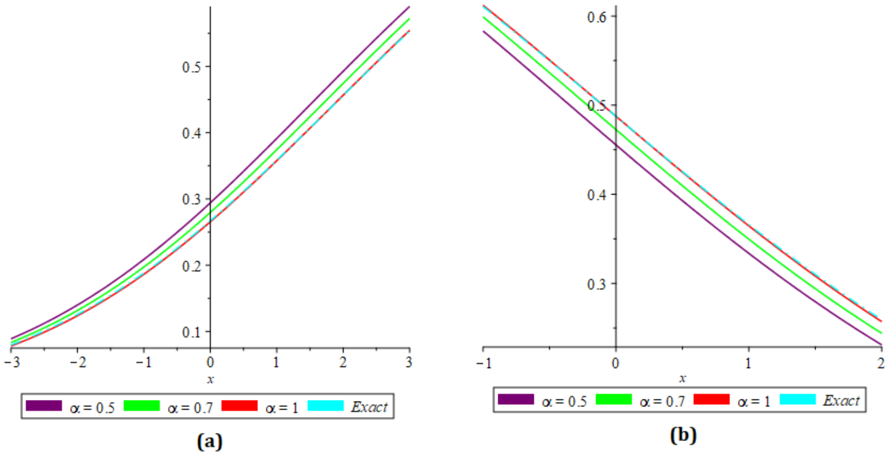


**Figure 4.** Subplot (**a**) presents the approximate solution $u(x,(t))$ and subplot (**b**) presents the approximate solution $v(x,(t))$ of Problem 2.

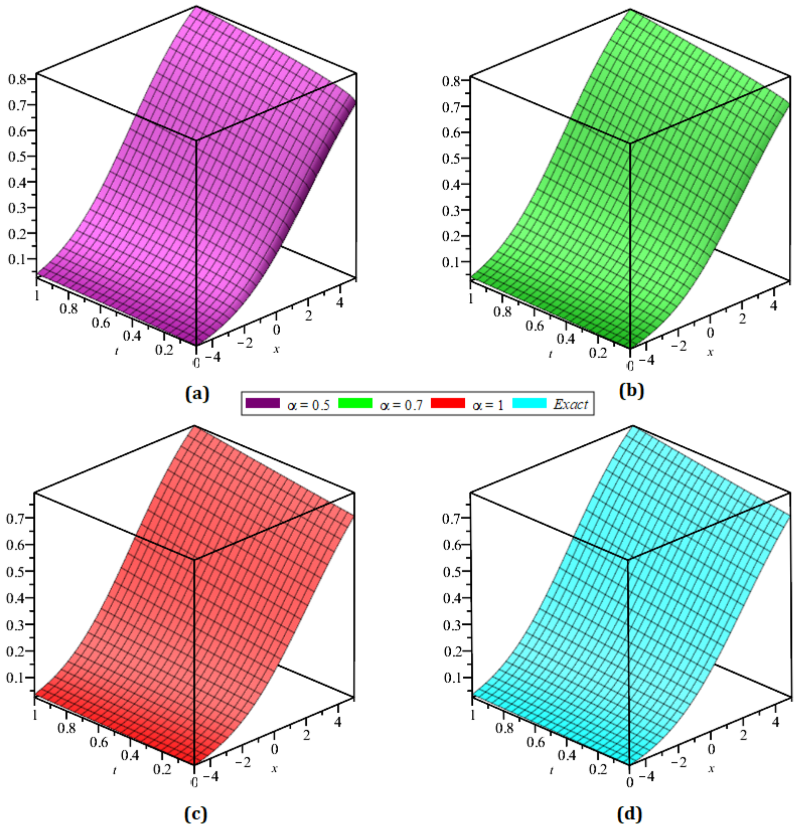


**Figure 5.** Exact and LRPSM solutions for $u(x,t)$ at $k = \frac{2}{3}$, $c = 1$, and distinct values of $\alpha$ for Problem 2.

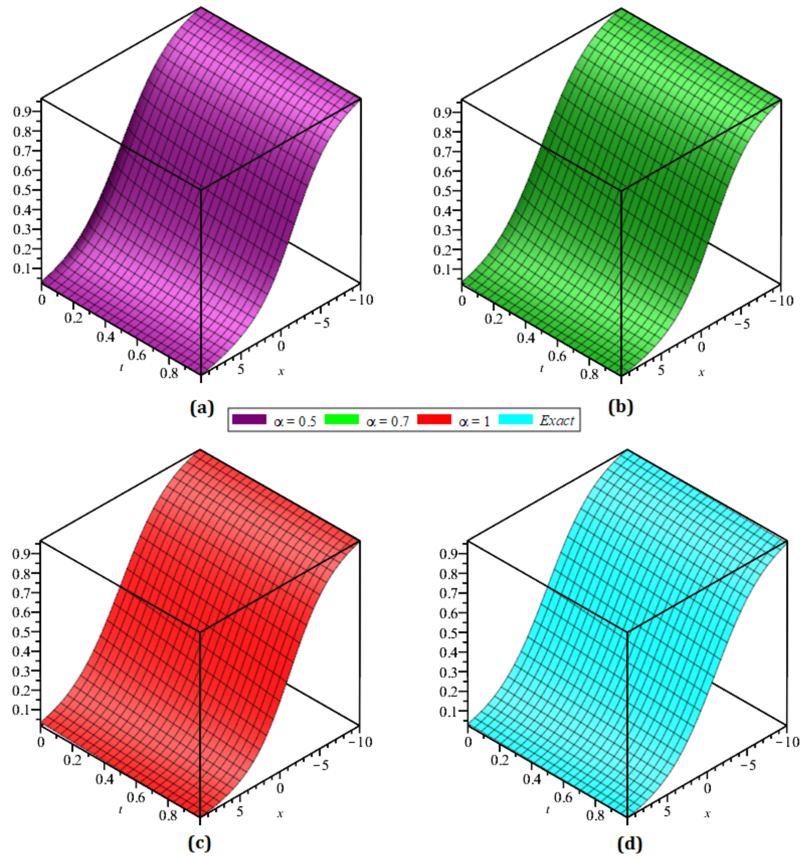


**Figure 6.** Exact and LRPSM solutions for $v(x,t)$ at $p = \frac{2}{3}$, $c = 1$, and distinct values of $\alpha$ for Problem 2.

**Table 3.** Error analysis for the LRPSM solution of $u(x,t)$ for the proposed Problem 2 with various values of $x$, $t$, and $p = \frac{2}{3}$.

| $t$ | $x$ | $AE\ at\ \alpha = 0.5$ | $AE\ at\ \alpha = 0.75$ | $AE\ at\ \alpha = 1$ |
|---|---|---|---|---|
| 0.1 | 0.2 | 0.0805629452 | 0.0502204062 | 0.0325020875 |
| | 0.4 | 0.0825454718 | 0.0515331039 | 0.0334241734 |
| | 0.6 | 0.0839111844 | 0.0524787587 | 0.0341170570 |
| | 0.8 | 0.0846437383 | 0.0530448606 | 0.0345706478 |
| | 1 | 0.0847452771 | 0.0532295274 | 0.0347813764 |
| 0.25 | 0.2 | 0.1386150391 | 0.1064055404 | 0.0816294703 |
| | 0.4 | 0.1421027846 | 0.1091805284 | 0.0838571430 |
| | 0.6 | 0.1445063572 | 0.1111623891 | 0.0855012205 |
| | 0.8 | 0.1457962927 | 0.1123251454 | 0.0865387309 |
| | 1 | 0.1459764695 | 0.1126662559 | 0.0869633834 |

**Table 4.** Error analysis for the LRPSM solution of $v(x,t)$ for the proposed Problem 2 with various values of $x$, $t$, and $p = \frac{2}{3}$.

| $t$ | $x$ | AE at $\alpha = 0.5$ | AE at $\alpha = 0.75$ | AE at $\alpha = 1$ |
|---|---|---|---|---|
| 0.1 | 0.2 | 0.0365409370 | 0.0162647517 | 0.0044485940 |
| | 0.4 | 0.0372349264 | 0.0167576585 | 0.0047433264 |
| | 0.6 | 0.0378161928 | 0.0172017544 | 0.0050225392 |
| | 0.8 | 0.0382802895 | 0.0175933608 | 0.0052838629 |
| | 1 | 0.0386242276 | 0.0179294469 | 0.0055251728 |
| 0.25 | 0.2 | 0.0493801887 | 0.0277102702 | 0.0109897476 |
| | 0.4 | 0.0502167121 | 0.0285674155 | 0.0116808654 |
| | 0.6 | 0.0508790642 | 0.0293301937 | 0.0123303104 |
| | 0.8 | 0.0513628857 | 0.0299925619 | 0.0129326754 |
| | 1 | 0.0516662223 | 0.0305498127 | 0.0134832205 |

**Problem 3.** *Next, we consider the two-dimensional reaction-diffusion Brusselator model [30,31]:*

$$\begin{aligned} D_t^{\alpha} u(x,y,t) =& u^2(x,y,t)v(x,y,t) - u(x,y,t)(A+1) + \frac{1}{500}\big(u_{xx}(x,y,t) + u_{yy}(x,y,t)\big) + B, \\ D_t^{\alpha} v(x,y,t) =& -u^2(x,y,t)v(x,y,t) + Au(x,y,t) + \frac{1}{500}\big(u_{xx}(x,y,t) + u_{yy}(x,y,t)\big), \end{aligned} \tag{41}$$

*subject to the initial conditions*

$$\begin{aligned} u(x,y,0) =& 2 + \frac{1}{4}y, \\ v(x,y,0) =& 1 + \frac{4}{5}x, \end{aligned} \tag{42}$$

*where $u(x,y,t)$ and $v(x,y,t)$ represent the chemical concentrations [30] of intermediate reaction products, and $A$ and $B$ are constant concentrations of input reagents, where $A = \frac{17}{5}$ and $B = 1$.*

*Applying Laplace transform on Equation (41) and using Equation (42), we have*

$$\begin{aligned} U(x,y,s) =& \frac{2 + \frac{1}{4}y}{s} + \frac{1}{s^{\alpha}}\mathcal{L}_t\Big[\mathcal{L}_t^{-1}\Big(U^2(x,y,s)\Big)\mathcal{L}_t^{-1}(V(x,y,s)) - \mathcal{L}_t^{-1}(U(x,y,s))(A+1) \\ &+ \frac{1}{500}\Big(\mathcal{L}_t^{-1}(U_{xx}(x,y,s)) + \mathcal{L}_t^{-1}(U_{yy}(x,y,s))\Big) + B\Big], \\ V(x,y,s) =& \frac{1 + \frac{4}{5}x}{s} + \frac{1}{s^{\alpha}}\mathcal{L}_t\Big[-\mathcal{L}_t^{-1}\Big(U^2(x,y,s)\Big)\mathcal{L}_t^{-1}(V(x,y,s)) + A\mathcal{L}_t^{-1}(U(x,y,s)) \\ &+ \frac{1}{500}\Big(\mathcal{L}_t^{-1}(U_{xx}(x,y,s)) + \mathcal{L}_t^{-1}(U_{yy}(x,y,s))\Big)\Big]. \end{aligned} \tag{43}$$

*The $k^{th}$ truncated term series are*

$$\begin{aligned} U(x,s) &= \frac{2 + \frac{1}{4}y}{s} + \sum_{n=1}^{k} \frac{f_n(x,y)}{s^{n\alpha+1}}, \\ V(x,s) &= \frac{1 + \frac{4}{5}x}{s} + \sum_{n=1}^{k} \frac{g_n(x,y)}{s^{n\alpha+1}}, \end{aligned} \tag{44}$$

*The $k^{th}$-Laplace residual functions are*

$$\begin{aligned}
\mathcal{L}_t Res_{u,k}(x,y,s) =& U_k(x,y,s) - \frac{2+\frac{1}{4}y}{s} - \frac{1}{s^\alpha}\mathcal{L}_t\Big[\mathcal{L}_t^{-1}\Big(U_k^2(x,y,s)\Big)\mathcal{L}_t^{-1}(V_k(x,y,s)) - \mathcal{L}_t^{-1}(U_k(x,y,s))(A+1) \\
&+ \frac{1}{500}\Big(\mathcal{L}_t^{-1}(U_{k,xx}(x,y,s)) + \mathcal{L}_t^{-1}\Big(U_{k,yy}(x,y,s)\Big)\Big) + B\Big], \\
\mathcal{L}_t Res_{v,k}(x,y,s) =& V_k(x,y,s) - \frac{1+\frac{4}{5}x}{s} - \frac{1}{s^\alpha}\mathcal{L}_t\Big[-\mathcal{L}_t^{-1}\Big(U_k^2(x,y,s)\Big)\mathcal{L}_t^{-1}(V_k(x,y,s)) + A\mathcal{L}_t^{-1}(U_k(x,y,s)) \\
&+ \frac{1}{500}\Big(\mathcal{L}_t^{-1}(U_{k,xx}(x,y,s)) + \mathcal{L}_t^{-1}\Big(U_{k,yy}(x,y,s)\Big)\Big)\Big].
\end{aligned} \tag{45}$$

*Now, to determine $f_k(x,y)$ and $g_k(x,y)$, $k = 1,2,3,\cdots$, we substitute the $k^{th}$-truncated series Equation (44) into the $k^{th}$-Laplace residual function Equation (45), multiply the resulting equation by $s^{k\alpha+1}$, and then recursively solve the relation $\lim_{s\to\infty}(s^{k\alpha+1}\mathcal{L}_t Res_{u,k}(x,\psi,s)) = 0$, and $\lim_{s\to\infty}(s^{k\alpha+1}\mathcal{L}_t Res_{v,k}(x,\psi,s)) = 0$, $k = 1,2,3,\cdots$. Following are the first few terms:*

$$\begin{aligned}
f_0(x,y) =& 2 + \frac{y}{4}, \\
g_0(x,y) =& 1 + \frac{4x}{5}, \\
f_1(x,y) =& -\frac{1899}{500} + \frac{16x}{5} - \frac{y}{10} + \frac{4xy}{5} + \frac{1y^2}{16} + \frac{y^2x}{20}, \\
g_1(x,y) =& \frac{1401}{500} - \frac{16x}{5} - \frac{3y}{20} - \frac{4yx}{5} - \frac{y^2}{16} - \frac{y^2x}{20}, \\
f_2(x,y) =& \frac{66261}{5000} - \frac{36667x}{1250} + \frac{256y^2}{251} - \frac{13y^3}{320} - \frac{y^4}{256} - \frac{y^3x}{20} + \frac{y}{2} + \frac{96x^2y}{25} + \frac{16x}{5} + \frac{12x^2y^2}{25} \\
&+ \frac{y^3x^2}{50} - \frac{y^4x}{320} - \frac{33y^2x}{50} - \frac{9199yx}{1250} - \frac{2}{5} - \frac{2399y^2}{8000} - \frac{157y}{1000} + \frac{2yx}{5}, \\
g_2(x,y) =& -\frac{44021}{5000} + \frac{y^3x}{20} - \frac{96yx^2}{25} - \frac{12y^2x^2}{25} + \frac{y^4x}{320} - \frac{y^3 * x^2}{50} + \frac{28917x}{1250} - \frac{256x^2}{25} + \frac{13y^3}{320} \\
&+ \frac{y^4}{256} + \frac{1899y^2}{8000} + \frac{7699yx}{1250} + \frac{61y^2x}{100} - \frac{243y}{1000}, \\
&\vdots
\end{aligned} \tag{46}$$

*Putting the values of $f_k(x,y)$ and $g_k(x,y)$, $k = 1,2,3,\cdots$ into Equation (44), we get*

$$\begin{aligned}
U(x,y,s) =& \frac{2+\frac{1}{4}y}{s} + \Big[-\frac{1899}{500} + \frac{16x}{5} - \frac{y}{10} + \frac{4xy}{5} + \frac{1y^2}{16} + \frac{y^2x}{20}\Big]\frac{1}{s^{\alpha+1}} + \Big[\frac{66261}{5000} - \frac{36667x}{1250} + \frac{256y^2}{251} \\
&- \frac{13y^3}{320} - \frac{y^4}{256} - \frac{y^3x}{20} + \frac{y}{2} + \frac{96x^2y}{25} + \frac{16x}{5} + \frac{12x^2y^2}{25} + \frac{y^3x^2}{50} - \frac{y^4x}{320} - \frac{33y^2x}{50} - \frac{9199yx}{1250} \\
&- \frac{2}{5} - \frac{2399y^2}{8000} - \frac{157y}{1000} + \frac{2yx}{5}\Big]\frac{1}{s^{2\alpha+1}} + \cdots, \\
V(x,y,s) =& \frac{1+\frac{4}{5}x}{s} + \Big[\frac{1401}{500} - \frac{16x}{5} - \frac{3y}{20} - \frac{4yx}{5} - \frac{y^2}{16} - \frac{y^2x}{20}\Big]\frac{1}{s^{\alpha+1}} + \Big[-\frac{44021}{5000} + \frac{y^3x}{20} - \frac{96yx^2}{25} \\
&- \frac{12y^2x^2}{25} + \frac{y^4x}{320} - \frac{y^3x^2}{50} + \frac{28917x}{1250} - \frac{256x^2}{25} + \frac{13y^3}{320} + \frac{y^4}{256} + \frac{1899y^2}{8000} + \frac{7699yx}{1250} \\
&+ \frac{61y^2x}{100} - \frac{243y}{1000}\Big]\frac{1}{s^{2\alpha+1}} + \cdots.
\end{aligned} \tag{47}$$

*Now, applying inverse Laplace transform, we get*

$$
\begin{aligned}
u(x,y,t) =& 2+\frac{1}{4}y+\Big[-\frac{1899}{500}+\frac{16x}{5}-\frac{y}{10}+\frac{4xy}{5}+\frac{y^2}{16}+\frac{y^2x}{20}\Big]\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\Big[\frac{66261}{5000}-\frac{36667x}{1250}+\frac{256y^2}{251}\\
&-\frac{13y^3}{320}-\frac{y^4}{256}-\frac{y^3x}{20}+\frac{y}{2}+\frac{96x^2y}{25}+\frac{16x}{5}+\frac{12x^2y^2}{25}+\frac{y^3x^2}{50}-\frac{y^4x}{320}-\frac{33y^2x}{50}-\frac{9199yx}{1250}\\
&-\frac{2}{5}-\frac{2399y^2}{8000}-\frac{157y}{1000}+\frac{2yx}{5}\Big]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots,\\
v(x,y,t) =& 1+\frac{4}{5}x+\Big[\frac{1401}{500}-\frac{16x}{5}-\frac{3y}{20}-\frac{4yx}{5}-\frac{y^2}{16}-\frac{y^2x}{20}\Big]\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\Big[-\frac{44021}{5000}+\frac{y^3x}{20}-\frac{96yx^2}{25}\\
&-\frac{12y^2x^2}{25}+\frac{y^4x}{320}-\frac{y^3x^2}{50}+\frac{28917x}{1250}-\frac{256x^2}{25}+\frac{13y^3}{320}+\frac{y^4}{256}+\frac{1899y^2}{8000}+\frac{7699yx}{1250}\\
&+\frac{61y^2x}{100}-\frac{243y}{1000}\Big]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots.
\end{aligned}
\tag{48}
$$

In Figure 7, show that the different fractional order graph (a) represent with respect to $u(x,(t))$ and (b) represent with respect to $v(x,(t))$ of Problem 3. Figure 8, (a) represent fractional order $\alpha = 0.5$ and (b) fractional order $\alpha = 0.75$ of $u(x,(t))$. Figure 8 shows (c) the exact and (d) LRPSM solutions for $u(x,t)$ at $A = \frac{17}{5}$, $B = 1$, and $y = 1$ of integer order $\alpha$ for Problem 3. Figure 9, (a) represent fractional order $\alpha = 0.5$ and (b) fractional order $\alpha = 0.75$ of $u(x,(t))$. Figure 9 shows (c) the exact and (d) LRPSM solutions for $u(x,t)$ at $A = \frac{17}{5}$, $B = 1$, and $y = 1$, of integer order $\alpha$ for Problem 3. In Tables 5 and 6, error analysis is shown for the LRPSM solution of $u(x,t)$ and $v(x,t)$ for the proposed Problem 3 with various values of $x$ and $t$.

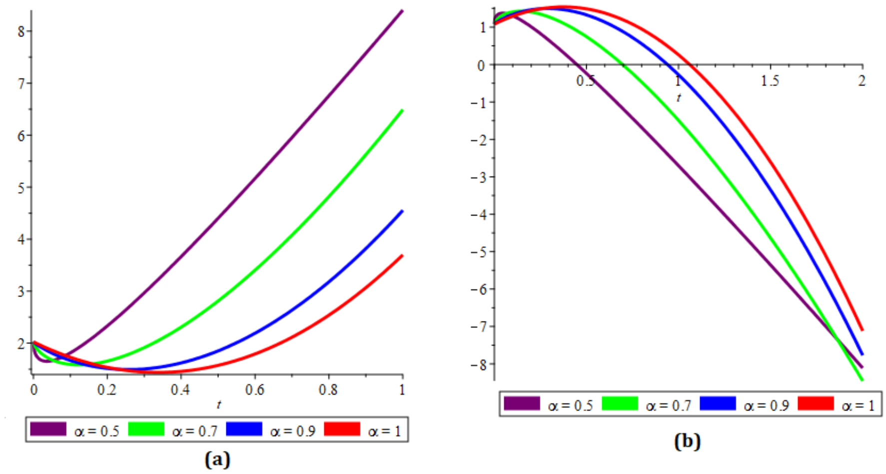


**Figure 7.** Subplot (**a**) presents the approximate solution $u(x,(t))$ and subplot (**b**) presents the approximate solution $v(x,(t))$ at $A = \frac{17}{5}$, $B = 1$, $x = 0.1$, and $y = 0.1$, respectively, of Problem 3.

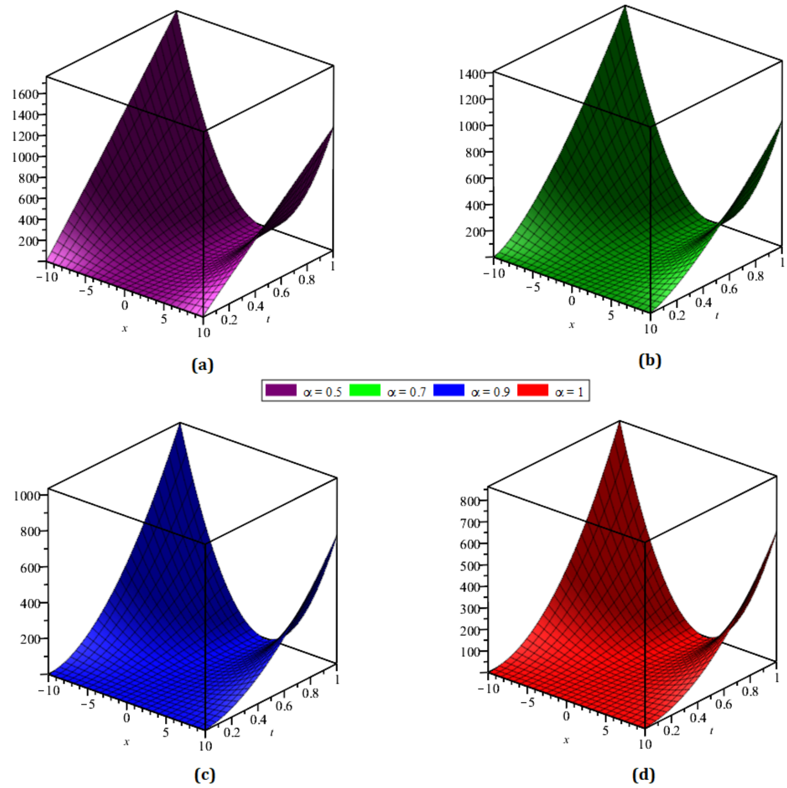


**Figure 8.** LRPSM solutions for $u(x,t)$ at $A = \frac{17}{5}$, $B = 1$, $y = 1$, and distinct values of $\alpha$ for Problem 3.

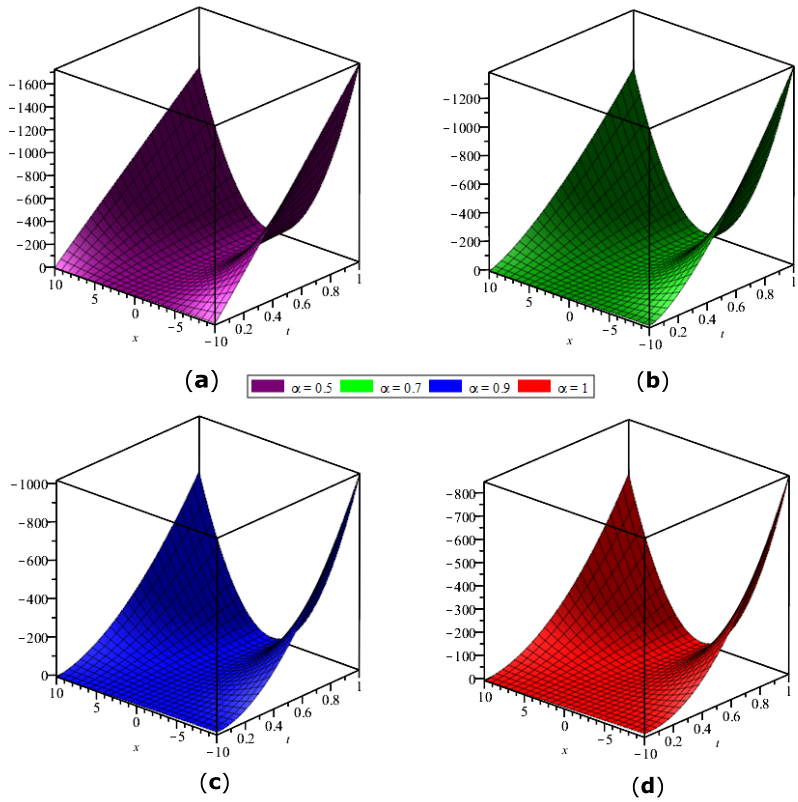


**Figure 9.** LRPSM solutions for $v(x,t)$ at $A = \frac{17}{5}$, $B = 1$, $y = 1$, and distinct values of $\alpha$ for Problem 3.

**Table 5.** Numerical simulation for the LRPSM solution of $u(x,t)$ for the proposed Problem 2 with various values of $x$, $t$, and $p = \frac{2}{3}$.

| $t$ | $x$ | Numerical Simulation at $\alpha = 0.5$ | Numerical Simulation at $\alpha = 0.75$ | Numerical Simulation at $\alpha = 1$ |
|---|---|---|---|---|
| | 0.2 | 1.837707373 | 1.917052547 | 1.980814044 |
| | 0.4 | 1.625576979 | 1.975701657 | 2.036756119 |
| 0.1 | 0.6 | 1.530086586 | 2.045377487 | 2.098530194 |
| | 0.8 | 1.551236193 | 2.126080036 | 2.166136269 |
| | 1 | 1.689025800 | 2.217809305 | 2.239574344 |
| | 0.2 | 1.713577815 | 1.674859093 | 1.702150273 |
| | 0.4 | 1.553928691 | 1.670189950 | 1.748038242 |
| 0.25 | 0.6 | 1.503958022 | 1.722897876 | 1.830376211 |
| | 0.8 | 1.563665809 | 1.832982869 | 1.949164180 |
| | 1 | 1.733052051 | 2.000444931 | 2.104402148 |

**Table 6.** Numerical simulation for the LRPSM solution of $v(x,t)$ for the proposed Problem 2 with various values of $x$, $t$, and $p = \frac{2}{3}$.

| $t$ | $x$ | Numerical Simulation at $\alpha = 0.5$ | Numerical Simulation at $\alpha = 0.75$ | Numerical Simulation at $\alpha = 1$ |
|---|---|---|---|---|
| | 0.2 | 1.459238898 | 1.424447874 | 1.321163456 |
| | 0.4 | 1.754369291 | 1.528622455 | 1.421371381 |
| 0.1 | 0.6 | 1.932859684 | 1.605050337 | 1.515747306 |
| | 0.8 | 1.994710077 | 1.653731523 | 1.604291231 |
| | 1 | 1.939920471 | 1.674666009 | 1.687003156 |
| | 0.2 | 1.324648176 | 1.528861254 | 1.499959102 |
| | 0.4 | 2.088050864 | 1.776106048 | 1.590008633 |
| 0.25 | 0.6 | 2.559853552 | 1.913672387 | 1.643608164 |
| | 0.8 | 2.740056238 | 1.941560270 | 1.660757695 |
| | 1 | 2.628658925 | 1.859769698 | 1.641457227 |

## 5. Conclusions

In this article, significant nonlinear fractional partial differential equations are solved by utilizing a combination of the residual power series and the Laplace transformation. The advantage of the new method is that it reduces the amount of computational effort required to obtain a solution in a power series form, whose coefficients must be computed in successive algebraic steps. The suggested method is employed to solve three distinct physical models, and its capacity to address fractional nonlinear equations with high precision and simple computation steps has been demonstrated. Finally, we can conclude that the regarded technique is better than its competitors and highly effective, and it can be utilized to study the various classes of nonlinear problems that arise in real life.

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